An Averaging Process for Unipotent Group Actions – in Differential Geometry

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0. Introduction

In this talk I will explain a geometric result that is a by-product of my research on deformation quantization.

The original work was in the context of algebraic geometry, but in this talk I will present it in the context of differential geometry.

Some of the results are very similar in the present context. But other results – those of arithmetic nature – do not have differentiable analogues.

The purpose of this talk is to expose the results to a wider audience, in particular to researchers working in topological dynamics and related areas.

1. Basic Facts on Lie Groups

In this talk we shall discuss *real Lie groups*.

Let me start be recalling some notions from differential geometry. This will help us establish a common language.

By *manifold* I mean a real differentiable manifold (of type C^∞).

Given manifolds X and Y, a *map of manifolds* $f : X \to Y$ is a continuous function that respects the differentiable structures.

Thus, for instance, a map of manifolds $f : X \to \mathbb{R}^1$ is just a differentiable function.

For a point x in a manifold X we have the tangent space $\mathrm{T}_x X$ to X at x. This is an n-dimensional vector space, where n is the dimension of the manifold X.

We will be interested in *Lie groups*.

Recall that a Lie group over $\mathbb{R}$ is a differentiable manifold G, equipped with a group structure, such that the operations of multiplication and inversion are maps of manifolds.

The unit element of a group G is denoted by e.

Suppose G and H are Lie groups. A *map of Lie groups* $f : G \to H$ is a map of manifolds which is also a group homomorphism.

An algebraic geometer would say that "Lie groups are group objects in the category Mfld of real manifolds".

In fact, I will use the language of categories and functors a bit, because it will make some concepts more lucid (I hope!).

Let G be a Lie group. The tangent space

$$\mathfrak{g} = \operatorname{Lie}(G) := \mathrm{T}_e G$$

has more structure than just a vector space – it is a *Lie algebra*.

A map of Lie groups $f : G \to H$ induces a Lie algebra homomorphism

$$\operatorname{Lie}(f) \ : \ \mathfrak{g} = \operatorname{Lie}(G) \ \to \ \mathfrak{h} = \operatorname{Lie}(H).$$

Let us denote the categories of real Lie groups and real Lie algebras by LieGrp and LieAlg respectively.

Then the Lie algebra of a group is a functor

$$\operatorname{Lie} : \mathsf{LieGrp} \to \mathsf{LieAlg}\,.$$

Given a Lie group G with Lie algebra $\mathfrak{g}$, the *exponential map*

$$\exp_G : \mathfrak{g} \to G$$

is a map of manifolds, such that $\exp_G(0) = e$, and it is a diffeomorphism near $0 \in \mathfrak{g}$.

The exponential map is functorial.

By functorial I mean that given a Lie group map $f : G \to H$, the diagram

$$\begin{array}{ccc} \mathfrak{g} = \operatorname{Lie}(G) & \xrightarrow{\exp_G} & G \\ {\scriptstyle \operatorname{Lie}(f)} \downarrow & & \downarrow {\scriptstyle f} \\ \mathfrak{h} = \operatorname{Lie}(H) & \xrightarrow{\exp_H} & H \end{array}$$

in Mfld is commutative.

Example 1.1. The Lie group $G = \mathrm{GL}_n(\mathbb{R})$ has as its Lie algebra $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$.

This is just the ring $\mathrm{Mat}_{n \times n}(\mathbb{R})$ of $n \times n$ matrices, with the commutator Lie bracket.

The exponential map here is the usual convergent matrix power series

$$\exp_G(\alpha) = \sum_{i \geq 0} \tfrac{1}{i!} \cdot \alpha^i.$$

Let G be a Lie group and Z a manifold. Recall that a (left) differentiable action of G on Z is a map of manifolds

$$G \times Z \to Z, \quad (g, z) \mapsto g \cdot z$$

such that $g_2 \cdot (g_1 \cdot z) = (g_2 \cdot g_1) \cdot z$ and $e \cdot z = z$.

Definition 1.2. Let G be a Lie group.

A *G-torsor*, or *principal homogeneous G-space*, is a manifold Z, endowed with a simply transitive differentiable action of the group G.

Note that given any *base point* $z \in Z$, the map

$$G \to Z, \quad g \mapsto g \cdot z, \tag{1.3}$$

is a diffeomorphism.

Thus a *choice of base point* $z \in Z$ determines an isomorphism of G-torsors $G \xrightarrow{\simeq} Z$.

2. Weighted Averages for Additive Lie Groups

Let me recall the usual weighed average, stated in a somewhat unusual way.

Definition 2.1. By a *weight sequence* we mean a sequence of numbers

$$\boldsymbol{w} = (w_0, \dots, w_q) \in \mathbb{R}^{q+1}$$

such that

$$\sum_{i=0}^{q} w_i = 1 \text{ and } w_i \geq 0.$$

Suppose G is an *additive Lie group*, i.e. $G \cong \mathbb{R}^n$ with its additive group structure.

Sometimes this is called a *vector group*.

Let Z be a G-torsor.

Given a sequence $\boldsymbol{z} = (z_0, \dots, z_q)$ of points in Z, and a weight sequence $\boldsymbol{w} = (w_0, \dots, w_q)$, we have the usual *weighted average*

$$\mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z}) \in Z.$$

I will give the formula. Fix an isomorphism of groups $G \cong \mathbb{R}^n$.

To simplify notation, in this case we will use addition for the operation in G and the action on Z.

Choose some base point $z \in Z$, and define $g_i \in G$ to be the unique group element (in this case vector) such that $z_i = g_i + z$.

Using the vector space structure on $G \cong \mathbb{R}^n$, we get a group element

$$g' := \sum\nolimits_{i=0}^{q} w_i \cdot g_i \in \mathbb{R}^n \cong G.$$

We then let

(2.2) $$\mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z}) = z' := g' + z \in Z.$$

Since $\sum_{i=0}^{q} w_i = 1$, the element $z' \in Z$ is independent of the base point z.

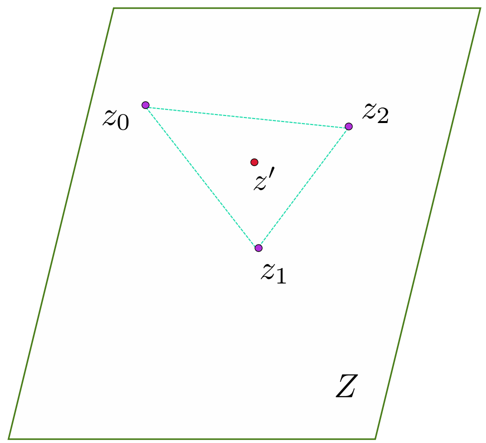


Figure: Weighted average for the additive group $G = \mathbb{R}^2$, with $q = 2$. We are given a sequence of points $\boldsymbol{z} = (z_0, z_1, z_2)$ in the torsor Z, and the weight sequence $\boldsymbol{w} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.
The weighted average z' is independent of the base point z.

We all know (or can easily prove) that:

Proposition 2.3. *The weighted average for additive groups has these properties:*

1. *Functoriality*. Suppose $G \to G'$ is a map of additive groups; Z' is a G'-torsor; $f : Z \to Z'$ is a G-equivariant map of manifolds; and
$$\boldsymbol{z}' := \big(f(z_0), \ldots, f(z_q)\big).$$
Then
$$f(\mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z})) = \mathrm{wav}_{G',\boldsymbol{w}}(\boldsymbol{z}').$$
2. *Symmetry*. $\mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z})$ is invariant under simultaneous permutation of the sequences $\boldsymbol{w}$ and $\boldsymbol{z}$.
3. *Simpliciality*. If $w_i = 0$ for some i, then deleting w_i and z_i does not change the average. And if $z_i = z_{i+1}$ for some i, then replacing w_i with $w_i + w_{i+1}$, and then deleting w_{i+1} and z_{i+1}, does not change the average.

We want to generalize this averaging process to other connected Lie groups G.

We will see that this can be done for *unipotent groups*; but it cannot be done for some other Lie groups (such as tori).

3. Unipotent Lie Groups

In algebraic geometry one talks about *unipotent algebraic groups*.

Let me recall the definition.

Then I will translate all to the language of Lie groups.

Let $\mathbb{K}$ be a field, and let G be an algebraic group over $\mathbb{K}$ (i.e. a finite type affine group scheme over $\mathbb{K}$).

One calls G a unipotent group if every nonzero finite dimensional algebraic representation of G over $\mathbb{K}$ has a nonzero fixed point.

Consider the algebraic group U_n, the algebraic group over $\mathbb{K}$ whose group of $\mathbb{K}$-points is

$$\mathrm{U}_n(\mathbb{K}) = \begin{bmatrix} 1 & \mathbb{K} & \mathbb{K} & \cdots & \mathbb{K} \\ 0 & 1 & \mathbb{K} & \cdots & \mathbb{K} \\ 0 & 0 & 1 & \cdots & \mathbb{K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \subseteq \mathrm{GL}_n(\mathbb{K}). \tag{3.1}$$

It is known (see [Mi, Theorem 14.5]) that an algebraic group G is unipotent iff it is isomorphic to a Zariski closed subgroup of U_n.

But we want to talk about Lie groups over $\mathbb{K} = \mathbb{R}$.

It turns out that if G is a unipotent algebraic group over $\mathbb{R}$, then its group of $\mathbb{R}$-points $G(\mathbb{R})$, with its structure of a Lie group – say as a closed subgroup of the Lie group $\mathrm{GL}_n(\mathbb{R})$ as in (3.1) – is *nilpotent and simply connected*.

In fact, there are equivalences between these three categories:

- Nilpotent Lie algebras over $\mathbb{R}$.
- Unipotent algebraic groups over $\mathbb{R}$.
- Nilpotent simply connected Lie groups over $\mathbb{R}$.

The secret behind these equivalences is this:

If G is a unipotent algebraic group over $\mathbb{R}$, with Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$ is a nilpotent Lie algebra, and there is an *isomorphism of algebraic varieties*

$$\mathrm{Exp}_G : \mathfrak{g} \to G.$$

The group structure of G is controlled by the Lie bracket of $\mathfrak{g}$, via the *Campbell-Baker-Haudorff formula*.

Likewise, if G is a nilpotent simply connected Lie group, with Lie algebra $\mathfrak{g}$, then $\mathfrak{g}$ is a nilpotent Lie algebra, and the exponential map

$$\mathrm{Exp}_G : \mathfrak{g} \to G$$

is an *isomorphism of manifolds*.

Again the group structure of G is controlled by the Lie bracket of $\mathfrak{g}$.

The unipotent algebraic groups have a much richer structure (mostly seen arithmetically).

But in this talk we only do differential geometry.

Thus we make a definition of convenience:

Definition 3.2. A real Lie group G is called *unipotent* if it is nilpotent and simply connected.

Let me emphasize the most important fact about these Lie groups: the exponential map $\exp_G$ is a diffeomorphism.

Its inverse is $\log_G$.

Example 3.3. An additive Lie group G is unipotent.

Indeed, if $G \cong \mathbb{R}^n$ as Lie groups, then

$$G \cong \begin{bmatrix} 1 & \mathbb{R} & \mathbb{R} & \cdots & \mathbb{R} \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \subseteq \mathrm{U}_{n+1}(\mathbb{R}) \subseteq \mathrm{GL}_{n+1}(\mathbb{R}).$$

In fact, *every abelian unipotent Lie group is an additive group*. This can be seen by the exponential map: here

$$\exp_G : (\mathfrak{g}, +) \to (G, \cdot) \tag{3.4}$$

is a Lie group isomorphism.

4. First Approximation of the Average

Let G be a unipotent Lie group and Z a G-torsor.

Suppose $\boldsymbol{w} = (w_0, \ldots, w_q)$ is a weight sequence in $\mathbb{R}$, and $\boldsymbol{z} = (z_0, \ldots, z_q)$ is a sequence of points in Z.

We want to construct a weighted average $\operatorname{wav}_{G,\boldsymbol{w}}(\boldsymbol{z}) \in Z$.

Choose some base point $z \in Z$.

Next, define elements $g_i \in G$ by the rule $z_i = g_i \cdot z$.

For any i let $\gamma_i := \log_G(g_i) \in \mathfrak{g}$.

Using the vector space structure of $\mathfrak{g}$ define

$$\gamma := \sum_{i=0}^{q} w_i \cdot \gamma_i \in \mathfrak{g}.$$

Next let

$$g' := \exp_G(\gamma) \in G.$$

The first candidate for the weighed average is the element

$$\operatorname{wav}_{G,\boldsymbol{w}}(\boldsymbol{z}) := g' \cdot z \in Z. \tag{4.1}$$

But there is a problem here: this "average" depends on the choice of base point z !

A calculation shows that this dependence is in the form of *commutators*.

In the additive case (i.e. the abelian case) all commutators vanish, so there is no dependence on the base point.

Another way to explain why all is fine in the additive case is this: here the exponential map an isomorphism of Lie groups, see formula (3.4). So the construction of the weighted average in (4.1) is the same as in Section 2, just written a bit differently.

The question is: how do we get around the dependence on the base point when the unipotent Lie group G is not abelian?

5. Improved Formula for the Average

Continuing with the previous setup, we will do something a little bit strange.

The sequence $\boldsymbol{z}$ provides us with $q+1$ preferred base points $z_0, \ldots, z_q \in Z$.

We shall perform the averaging process with respect to each of these base points.

Thus, for fixed weight sequence $\boldsymbol{w}$, we will obtain $q+1$ possibly distinct "averages" $z'_0, \ldots, z'_q$.

Here is the explicit formula.

Let $g_{i,j} \in G$ be the unique group elements satisfying $g_{i,j} \cdot z_i = z_j$.

Then

$$z'_i = \exp_G\Big(\sum_{j=0}^{q} w_j \cdot \log_G(g_{i,j})\Big) \cdot z_i \in Z.$$

This can be viewed as an operation of *weighted symmetrization*

$$\mathrm{wsym}_{G,w} : \begin{cases} Z^{q+1} \to Z^{q+1} \\ (z_0, \dots, z_q) \mapsto (z'_0, \dots, z'_q). \end{cases}$$

I claim that *if we iterate the operation* $\mathrm{wsym}_{G,w}$ *enough times, we end up with a constant sequence*.

The reason is this. Since G is a nilpotent simply connected Lie group, it has a filtration by closed normal Lie subgroups

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_d = \{1\}$$

such that each G_i / G_{i+1} is an additive Lie group, and the conjugation action of G on G_i / G_{i+1} is trivial.

The number d is at most the dimension of G.

A little calculation shows that if the points $z_0, \dots, z_q$ *are all in the same* G_i*-orbit for some* i*, then the points* $z'_0, \dots, z'_q$ *are in the same* G_{i+1}*-orbit.*

Thus, starting with an arbitrary sequence of points

$$\boldsymbol{z} = (z_0, \dots, z_q),$$

and repeating the operation $\mathrm{wsym}_{G,w}$ d times, we get

$$\mathrm{wsym}^d_{G,w}(\boldsymbol{z}) = (z, \dots, z)$$

for some $z \in Z$.

Now we can define the weighted average

$$\mathrm{wav}_{G,w}(\boldsymbol{z}) := z.$$

Theorem 5.1. *The averaging process* $\mathrm{wav}_{G,w}$ *described above has the properties functoriality, symmetry and simpliciality from Proposition 2.3.*

This was proved in [Ye2] for unipotent algebraic groups over a field $\mathbb{K}$ of characteristic 0.

The same proof also works for unipotent Lie groups over $\mathbb{R}$.

♦ ♦ ♦

The reason I needed this averaging process can only be seen in the more complicated setting of *geometric torsors*.

I probably don't have time to talk about this material. It is in the last part of the notes, and you are welcome to take a look later.

What about other connected Lie groups? Do they admit an averaging process? The answer is no in general.

Exercise 5.2. Consider the abelian Lie group

$$G := \mathbf{S}^1 = \mathbf{T}^1 = \mathrm{SO}_2(\mathbb{R}),$$

i.e. the unit circle in the plane.

Prove that *there does not exist* an averaging process for G-torsors, which satisfies the properties in Proposition 2.3.

- END -

(Material below is optional.)

6. Geometric Torsors

We are now going to upgrade our geometric setting.

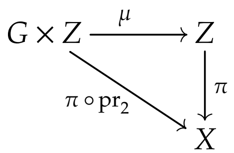
As before, G is a Lie group. Let X be a manifold.

A *G-manifold over X* is a manifold Z, with a map $\pi : Z \to X$, and an action

$$\mu : G \times Z \to Z$$

that respects the projection π.

In other words, the diagram



in the category Mfld is commutative.

Given an open set $U \subseteq X$, let

$$Z|_U := \pi^{-1}(U) \subseteq Z.$$

Then $Z|_U$ is a G-manifold over U.

Example 6.1. The *trivial G-manifold over X* is

$$Z := G \times X,$$

with projection $\pi := \mathrm{pr}_2$.

Definition 6.2. Let Z be a G-manifold over X, and let $U \subseteq X$ be an open set.

A *trivialization of Z over U* is an isomorphism

$$\psi : Z|_U \xrightarrow{\simeq} G \times U$$

of G-manifolds over U.

Definition 6.3. A *G-torsor* (or principal homogeneous space) over X is a G-manifold $\pi : Z \to X$ which is *locally trivial*.

Namely, there exists an open covering $X = \bigcup_i U_i$, and for every i there's a trivialization

$$\psi_i : Z|_{U_i} \xrightarrow{\simeq} G \times U_i$$

of G-manifolds.

Note that the projection $\pi : Z \to X$ is a submersion of manifolds.

The fibers of π are isomorphic to G as G-manifolds.

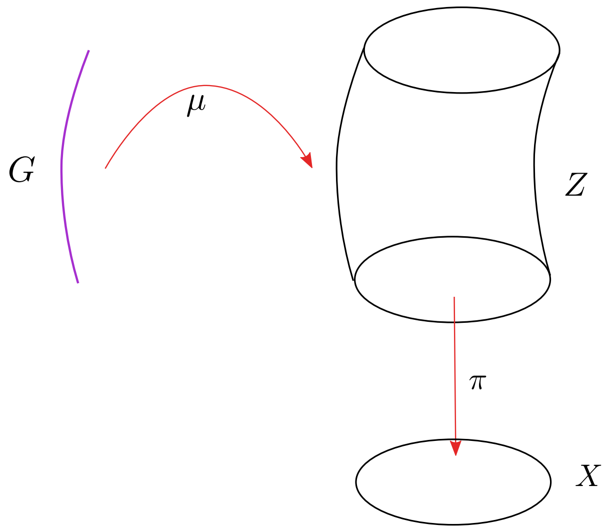
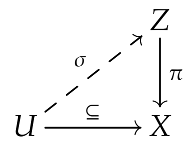


Figure: A G-torsor $\pi : Z \to X$. The action is μ.

Suppose $\pi : Z \to X$ is a G-torsor over X, and $U \subseteq X$ is an open set.

A map $\sigma : U \to Z$ that lifts π is called a *section of Z over U*.



We denote the set of all sections of Z over U by $Z(U)$.

In other words, letting Mfld/X be the category of manifolds over X, we have

$$Z(U) = \mathrm{Hom}_{\mathsf{Mfld}/X}(U, Z).$$

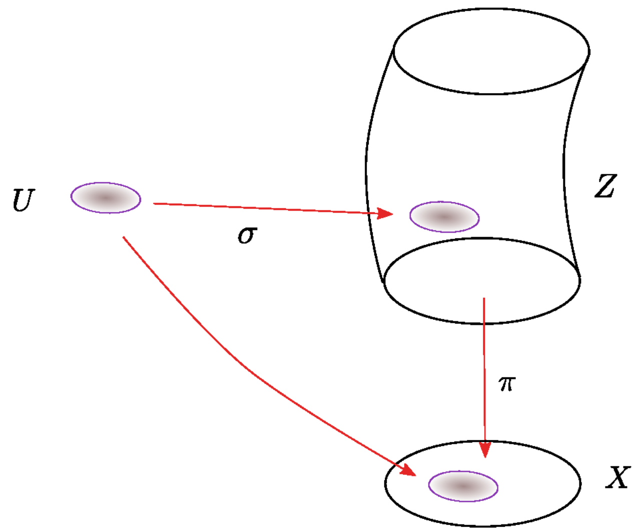


Figure: A section σ of the torsor $\pi : Z \to X$ over the open set U.

Proposition 6.4. *Let Z be a G-torsor over X, and let $U \subseteq X$ be an open set.*

There is a canonical bijection between the set of sections $\sigma : U \to Z$ and the set of trivializations

$$\psi : Z|_U \xrightarrow{\simeq} G \times U.$$

Exercise 6.5. Prove this Proposition, and find the formula for this canonical bijection. (Hint: see formula (1.3).)

For this reason, the G-torsor $\pi : Z \to X$ is called *trivial* if there is a global section $\sigma : X \to Z$.

Example 6.6. Suppose E is a rank n *vector bundle* over X.

Consider the *frame bundle* $\pi : Z \to X$ of E.

By definition, for every open set $U \subseteq X$, the set of sections $\sigma : U \to Z$ of Z is the set of trivializations of the vector bundle $E|_U$, namely the set of vector bundle isomorphisms

$$\psi : E|_U \xrightarrow{\simeq} U \times \mathbb{R}^n.$$

Then $\pi : Z \to X$ is a $\mathrm{GL}_n(\mathbb{R})$-*torsor* over X.

The torsor Z is nontrivial iff the vector bundle E is a nontrivial.

7. Geometric Averaging

For $q \in \mathbb{N}$ let $\boldsymbol{\Delta}^q$ be the *q-dimensional combinatorial simplex*.

The set of vertices of $\boldsymbol{\Delta}^q$ is the set

$$\boldsymbol{\Delta}^q_0 = \{v_0, \ldots, v_q\}.$$

The *q-dimensional geometric simplex* is the polyhedron

$$\boldsymbol{\Delta}^q(\mathbb{R}_{\geq 0}) := \big\{(a_0, \ldots, a_q) \mid a_0 + \cdots + a_q = 1 \text{ and } a_i \geq 0\big\} \subseteq \mathbb{R}^{q+1}.$$

A point $\boldsymbol{w} \in \boldsymbol{\Delta}^q(\mathbb{R}_{\geq 0})$ is precisely a weight sequence (Definition 2.1).

We can view $\boldsymbol{\Delta}^q_0$ as a subset of $\boldsymbol{\Delta}^q(\mathbb{R}_{\geq 0})$

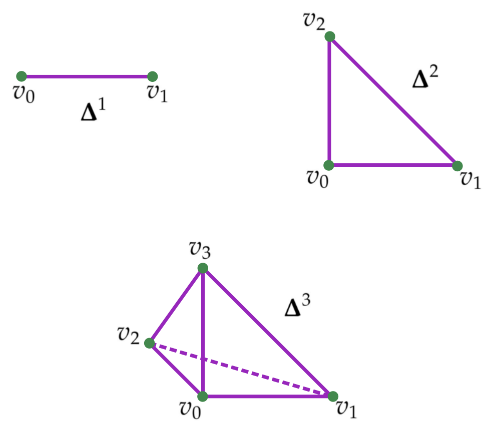


Figure: The geometric simplices $\boldsymbol{\Delta}^q(\mathbb{R}_{\geq 0})$ for $q = 1, 2, 3$.

The geometric simplex $\boldsymbol{\Delta}^q(\mathbb{R}_{\geq 0})$ is a compact *manifold with corners*.

Now consider a unipotent Lie group G, a manifold X, and a G-torsor $\pi : Z \to X$.

Let U be a manifold with corners, and let $g : U \to X$ be a map of manifolds. By a *U-section of the torsor Z* we mean a map of manifolds $\sigma : U \to Z$ such that $\pi \circ \sigma = g$.

In fancier words, $\sigma : U \to Z$ is a map of manifolds over X, i.e.

$$\sigma \in Z(U) = \mathrm{Hom}_{\mathsf{Mfld}/X}(U, Z).$$

The commutative diagram and the picture on slides 34 and 35 apply.

The construction of the weighed average $\mathrm{wav}_{G,\boldsymbol{w}}(z)$ from Section 5 can be geometrized. Namely, all the absolute constructions, including the exponential maps, can be made relative to the base manifold X.

We obtain a function

$$\text{(7.1)} \qquad \mathrm{wav}_G : Z(U)^{q+1} \to Z\big(\boldsymbol{\Delta}^q(\mathbb{R}_{\geq 0}) \times U\big),$$

which again we call the *weighted average*.

To be concrete, an element of the set $Z(U)^{q+1}$ is a sequence of U-sections

$$\sigma_0, \ldots, \sigma_q : U \to Z$$

of the torsor $\pi : Z \to X$.

The weighed average is a map of manifolds (with corners)

$$\sigma := \mathrm{wav}_G(\sigma_0, \ldots, \sigma_q) \ : \ \boldsymbol{\Delta}^q(\mathbb{R}_{\geq 0}) \times U \ \to \ Z.$$

The map σ *interpolates* between $\sigma_0, \ldots, \sigma_q$, in the following sense:

Take the i-th vertex $v_i \in \boldsymbol{\Delta}^q(\mathbb{R}_{\geq 0})$ and a point $x \in U$. Then

$$\sigma(v_i, x) = \sigma_i(x) \in Z.$$

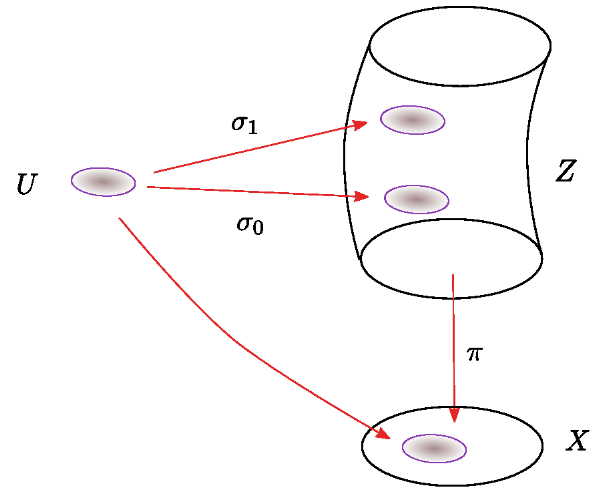


Figure: Given U-sections σ_0 and σ_1 of the G-torsor Z over X

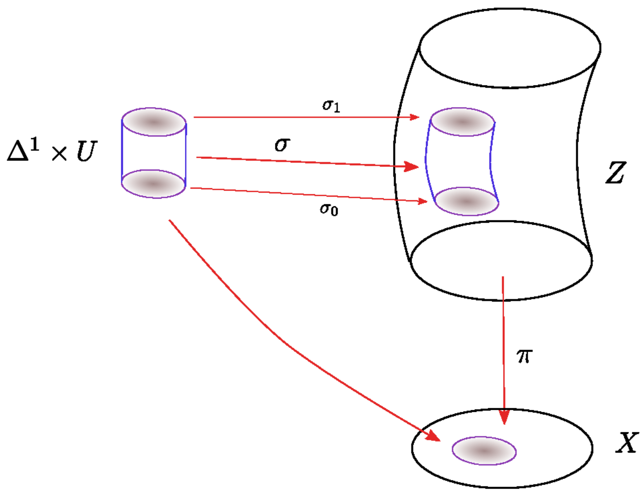


Figure: The map $\sigma := \mathrm{wav}_G(\sigma_0, \sigma_1)$ is a $\boldsymbol{\Delta}^1(\mathbb{R}_{\geq 0}) \times U$ -section σ of Z, interpolating between σ_0 and σ_1

The weighed average wav_G is functorial in G, X, Z and U.

It also respects the action of the permutation group on $\boldsymbol{\Delta}^q(\mathbb{R}_{\geq 0})$.

Regarding simpliciality: as q varies, the two sets appearing in equation (7.1) become simplicial sets, and wav_G becomes a map of simplicial sets.

If $X = U$ consists of one point, then we recover the weighed average from Section 5: for any $\boldsymbol{z} \in Z^{q+1}$ and $\boldsymbol{w} \in \boldsymbol{\Delta}^q(\mathbb{R}_{\geq 0})$ there is equality

$$\mathrm{wav}_G(\boldsymbol{w})(\boldsymbol{z}) = \mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z}) \in Z.$$

See [Ye2] for details.

The operation wav_G was required to construct *simplicial sections* of torsors, which we explain in the next section.

8. Simplicial Sections of Coordinate Bundles

Here is the reason we needed the unipotent averaging process.

An important ingredient in *deformation quantization* of Poisson manifolds (first used by Fedosov [Fe], and then by Kontsevich [Ko1]) is the *formal geometry* of Gelfand-Kazhdan. See also [CFT].

This is a method for globalizing various constructions in differential geometry. Crudely speaking, it enables *global Taylor expansions* of functions and of sections of canonical bundles.

Here is a slight modification of this story (to make the exposition easier).

One starts with a manifold X. There is an infinite dimensional bundle $\pi : Z \to X$, called the *coordinate bundle* of X.

A global section $\sigma : X \to Z$ is (i.e. can be suitably interpreted) as a global Taylor expansion of the differentiable functions on X.

The existence of such a section σ implies (after a lot of preliminary work) that every Poisson structure on X can be globally quantized (uniquely up to gauge transformations).

Now the bundle Z is a torsor under a *prounipotent Lie group* G (an inverse limit of unipotent Lie groups).

It follows that the fibers $\pi^{-1}(x) \subseteq Z$ are *contractible manifolds*.

A standard result in differential geometry says that a global differentiable section $\sigma : X \to Z$ exists.

I was interested in *algebraic manifolds*, namely *nonsingular algebraic varieties* over $\mathbb{R}$.

For such a variety X there is a coordinate bundle $\pi : Z \to X$, which is an infinite dimensional variety, and it is a torsor under the same prounipotent group G.

But in algebraic geometry it is very hard to find global sections of bundles.

In particular, *the coordinate bundle Z almost never has a global section*. (Even when X is an affine variety.)

The coordinate bundle Z does however admit sections on sufficiently small affine open sets.

So we can choose a finite affine open covering $\boldsymbol{U} = \{U_i\}_{i \in I}$ of X, and for each i a section $\sigma_i : U_i \to Z|_{U_i}$ of $\pi : Z \to X$.

The functoriality of the unipotent averaging lets us average also over prounipotent groups. Our bundle Z is a G-bundle, and G is prounipotent.

In this way, for every finite sequence $\boldsymbol{i} = (i_0, \dots, i_q)$ in I we obtain an algebraic section

$$\sigma_{\boldsymbol{i}} : \mathbf{\Delta}^q_{\mathbb{R}} \times_{\mathbb{R}} (U_{i_0} \cap \cdots \cap U_{i_q}) \to Z.$$

Here $\mathbf{\Delta}^q_{\mathbb{R}}$ is the algebro-geometric q-dimensional simplex, i.e. a q-dimensional linear variety with baricentric coordinates.

The sections $\boldsymbol{i}$ assemble into a *simplicial section* of Z.

This is sufficient to deduce the existence of a deformation quantization in the algebraic setting.

For details see my papers [Ye1], [Ye3] and [Ye4].

- END -

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