An Averaging Process for Unipotent Group Actions

Lecture Notes
written 19 June 2008

Amnon Yekutieli
Ben Gurion University, ISRAEL
http://www.math.bgu.ac.il/~amyekut/lectures

Here is the plan of my lecture:

1. Introduction
2. Exponential Map
3. First Approximation of the Average
4. Improved Formula for the Average
5. More Sophisticated Geometry
6. An Arithmetic Application
7. An Application to Deformation Quantization

1. Introduction

In this talk I will explain a geometric result that is a by-product of research on
deformation quantization [Ye1]. This could be useful for other applications. Full
details can be found in [Ye2].

Let \( K \) be a field of characteristic 0, and let \( G \) be a linear (i.e. affine) algebraic group
\( G \) over \( K \).

For a commutative \( K \)-algebra \( A \), we write \( G(A) \) for the group of \( A \)-valued points
of \( G \).

For instance if \( G = GL_n, K \), then \( G(A) = GL_n(A) \), the group of invertible matrices
with entries in \( A \).

Here is a naive definition of a \( G(\mathbb{K}) \)-torsor: it is a set \( Z \), endowed with an action
of the group \( G(\mathbb{K}) \), which is transitive and has trivial stabilizers.

Thus for every point \( z \in Z \) the map \( G(\mathbb{K}) \to Z, g \mapsto g \cdot z \), is bijective.

By a weight sequence we mean a sequence
\[ w = (w_0, \ldots, w_q) \in \mathbb{K}^{q+1} \]
such that
\[ \sum_{i=0}^{q} w_i = 1. \]
Suppose $G$ is a vector space, i.e. $G \cong \mathbb{A}_K^n$ with its additive group structure.

Let $Z$ be a $G(K)$-torsor.

Given a sequence $z = (z_0, \ldots, z_q)$ of points in $Z$, and a weight sequence $w$, we have the usual weighted average

$$\text{wav}_{G,w}(z) \in Z.$$

Let me recall the formula. To simplify notation, in this case we will use addition for the operation in $G$ and the action on $Z$.

Choose any base point $z \in Z$, and define $g_i \in G(K)$ to be the unique element such that

$$z_i = g_i + z.$$

Using the isomorphism $G \cong \mathbb{A}_K^n$ we get elements

$$w_ig_i \in G(K),$$

and then we let

$$\text{wav}_{G,w}(z) := \left(\sum_{i=0}^q w_ig_i\right) + z \in Z.$$

Since $\sum_{i=0}^q w_i = 1$, this is independent of the base point $z$.

We want to generalize this construction to unipotent groups.

Let $\bar{K}$ be an algebraic closure of $K$. An element $g \in G(\bar{K})$ is called unipotent if for any morphism of groups $\rho : G \to \text{GL}_n(\bar{K})$, the matrix $\rho(g) \in \text{GL}_n(\bar{K})$ is unipotent (i.e. its only eigenvalue is 1).

The group $G$ is called unipotent if all elements of $G(\bar{K})$ are unipotent. See [Ho] for details.

Here is equivalent characterization: $G$ is unipotent if and only if it is isomorphic to a closed subgroup of the upper triangular matrix group inside $\text{GL}_m(K)$, for some $m$.

Thus, for any $K$-algebra $A$ we get

$$G(A) \subset \begin{bmatrix} 1 & A & A & \cdots & A \\ 0 & 1 & A & \cdots & A \\ 0 & 0 & 1 & \cdots & A \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \subset \text{GL}_m(A).$$

**Example 1.1.** $G := \mathbb{A}_K^n$ is unipotent.

A possible embedding is

$$G(A) \cong \begin{bmatrix} 1 & A & A & \cdots & A \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \subset \text{GL}_m(A)$$

for $m = n + 1$.

In fact, any abelian unipotent group is isomorphic to $\mathbb{A}_K^n$, for some $n$. 
Remark 1.2. If \( \mathbb{K} = \mathbb{R} \) and \( G \) is unipotent, then \( G(\mathbb{R}) \) is a simply connected nilpotent Lie group. And vice versa.

Let \( G \) be a unipotent group, and let \( Z \) be a \( G(\mathbb{K}) \)-torsor.
Again we are given a sequence of points \( z = (z_0, \ldots, z_q) \) in \( Z \), and a weight sequence \( w \).
We want to construct a weighted average \( \text{wav}_{G,w}(z) \in Z \).
A couple of obvious requirements of such a construction are:

1. Functoriality: suppose \( G \rightarrow G' \) is a morphism of groups, with \( G' \) also unipotent; \( Z' \) is a torsor under \( G'(\mathbb{K}) \);
   \( f : Z \rightarrow Z' \)
   is a \( G(\mathbb{K}) \)-equivariant function; and
   \[ z' := (f(z_0), \ldots, f(z_q)). \]
   Then
   \[ f(\text{wav}_{G,w}(z)) = \text{wav}_{G',w}(z'). \]
2. Simpliciality. This will be defined later, but here is an instance: suppose \( w_q = 0 \).
   Then
   \[ \text{wav}_{G,w}(z) = \text{wav}_{G,(w_0,\ldots,w_{q-1})}(z_0,\ldots,z_{q-1}). \]
3. Symmetry: the average is invariant under simultaneous permutation of the sequences \( w \) and \( z \).

2. Exponential Map

As before, \( \mathbb{K} \) is a field of characteristic 0, and \( G \) is a unipotent linear algebraic group over \( \mathbb{K} \).
Let \( \mathfrak{g} \) be the Lie algebra of \( G \). This is a Lie algebra over \( \mathbb{K} \), and also a scheme: \( \mathfrak{g} \cong A^n_{\mathbb{K}} \), where \( n = \dim G \).
There is an isomorphism of schemes

\[ \exp_G : \mathfrak{g} \congto G \]

called the exponential map. Its inverse is \( \log_G \).

The exponential map is functorial; namely if \( G' \) is another unipotent group, and \( \phi : G \rightarrow G' \) is a morphism of groups, then the diagram

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\exp_G} & G \\
\downarrow{d\phi} & & \downarrow{\phi} \\
\mathfrak{g}' & \xrightarrow{\exp_{G'}} & G'
\end{array}
\]

is commutative.
The functor \( G \mapsto \mathfrak{g} \) is an equivalence from the category of unipotent groups to the category of nilpotent Lie algebras over \( \mathbb{K} \).
The multiplication in $G$ is expressible via the CBH formula:
\[
\exp_G(\gamma_1) \cdot \exp_G(\gamma_2) = \exp_G(F(\gamma_1, \gamma_2)),
\]
where
\[
F(\gamma_1, \gamma_2) = \gamma_1 + \gamma_2 + \frac{1}{2}[\gamma_1, \gamma_2] + \cdots
\]
is a universal “commutator power series”.
See [Ho] for details.

**Example 2.1.** For a unipotent closed subgroup $G \subset \text{GL}_n(K)$, the Lie algebra $\mathfrak{g}(K)$ is a subalgebra of $\mathfrak{gl}_n(K) = M_n(K)$ consisting of nilpotent matrices.

For any element $\gamma \in \mathfrak{g}(K)$, the formula for $\exp_G(\gamma)$ is
\[
\exp_G(\gamma) = \sum_{i \geq 0} \frac{1}{i!} \gamma^i \in G(K) \subset \text{GL}_n(K).
\]

3. **First Approximation of the Average**

With $G$ a unipotent group as before, let $Z$ be a $G(K)$-torsor.

Suppose $w = (w_0, \ldots, w_q)$ is a weight sequence, and $z = (z_0, \ldots, z_q)$ is a sequence of points in $Z$.

Choose some base point $z \in Z$, and define elements $g_i \in G(K)$ by the rule
\[
z_i = g_i \cdot z.
\]
For any $i$ let
\[
\gamma_i := \log_G(g_i).
\]
Using the vector space structure of $\mathfrak{g}(K)$ define
\[
\gamma := \sum_{i=0}^q w_i \gamma_i \in \mathfrak{g}(K).
\]

Next let
\[
g := \exp_G(\gamma) \in G(K).
\]
The first candidate for the weighted average is
\[
\text{wav}_{G,w}(z) := g \cdot z \in Z.
\]
But there is a problem here: this “average” depends on the choice of base point $z$!
A calculation shows that this dependence is in the form of commutators.

Observe that in the abelian case $G \cong \mathbb{A}^n_K$, the exponential map is actually an isomorphism of groups
\[
\exp_G : (\mathfrak{g}, +) \xrightarrow{\sim} (G, \cdot).
\]
This explains why the problem of defining the average does not arise in this case.
4. Improved Formula for the Average

Continuing with the previous setup, we will do something a little bit strange. There are \( q + 1 \) preferred base points \( z_0, \ldots, z_q \in Z \). We shall perform the averaging process with respect to each of these base points.

Thus, for fixed weight sequence \( w \), we will obtain \( q + 1 \) possibly distinct “averages” \( z'_0, \ldots, z'_q \).

The explicit formulas are

\[
    z'_i := \exp_G \left( \sum_{j=0}^{q} w_j \log_G(g_{i,j}) \right) \cdot z_i,
\]

where \( g_{i,j} \in G(\mathbb{K}) \) are the elements satisfying

\[
    g_{i,j} \cdot z_i = z_j.
\]

This can be viewed as an operation

\[
    \text{wsym}_{G,w} : \{ \mathbb{Z}^{q+1} \to \mathbb{Z}^{q+1} \}
\]

\[
    (z_0, \ldots, z_q) \mapsto (z'_0, \ldots, z'_q).
\]

I claim that if we iterate the operation \( \text{wsym}_{G,w} \) enough times, we end up with a constant sequence.

The reason is that the group \( G \) has a filtration by closed normal subgroups

\[
    G = G_0 \supset G_1 \supset \cdots \supset G_d = \{1\}
\]

such that \( G_i/G_{i+1} \) is abelian, and the conjugation action of \( G \) on \( G_i/G_{i+1} \) is trivial.

Therefore, if the points \( z_0, \ldots, z_q \) are all in the same \( G_i \)-orbit for some \( i \), then \( z'_0, \ldots, z'_q \) are in the same \( G_{i+1} \)-orbit.

Thus, starting with a sequence of points

\[
    z = (z_0, \ldots, z_q),
\]

we get

\[
    \text{wsym}^d_{G,w}(z) = (z, \ldots, z)
\]

for some \( z \in Z \).

Now we can define the correct weighted average

\[
    \text{wav}_{G,w}(z) := z.
\]

It is easy to see that this average enjoys the required properties (functoriality, simpliciality and symmetry).

5. More Sophisticated Geometry

The same idea can be dressed up in fancier language.

Let \( \mathbb{K} \) be a field, and let \( G \) be a linear algebraic group over \( \mathbb{K} \).

By \( G \)-torsor (or principal homogeneous space) we mean a finite type \( \mathbb{K} \)-scheme \( Z \), with group action...
\[ \mu : G \times Z \to Z, \]

such that
\[ \mu \times p_2 : G \times Z \to Z \times Z \]
is an isomorphism of schemes.

It follows that for any \( \mathbb{K} \)-algebra \( A \) for which \( Z(A) \neq \emptyset \), the set \( Z(A) \) is a \( G(A) \)-torsor in the naive sense.

For \( q \in \mathbb{N} \) let \( \Delta^q \) be the \textit{combinatorial} \( q \)-dimensional simplex.

Its set of \( p \)-dimensional faces \( \Delta^q_p \) consists of all sequences \( (i_0, \ldots, i_p) \) satisfying
\[
0 \leq i_0 \leq \cdots \leq i_p \leq q.
\]

In particular, for \( p = 0 \) we have the set of vertices
\[
\Delta^q_0 = \{0, \ldots, q\}.
\]

Hence the set
\[
Z(\mathbb{K})^{\Delta^q_0} = \text{Hom}_{\text{Set}}(\Delta^q_0, Z(\mathbb{K}))
\]
is nothing but the set of sequences \( (z_0, \ldots, z_q) \) in \( Z(\mathbb{K}) \).

The \textit{geometric} \( q \)-dimensional simplex is the scheme
\[
\Delta^q_k := \text{Spec} \mathbb{K}[t_0, \ldots, t_q] / (t_0 + \cdots + t_q - 1).
\]

A point in \( \mathbf{w} \in \Delta^q_k(\mathbb{K}) \) is precisely a weight sequence.

We denote by
\[
\text{Hom}_{\text{Sch}/\mathbb{K}}(\Delta^q_k, Z)
\]
the set of scheme morphisms.

Now assume \text{char} \( \mathbb{K} = 0 \) and \( G \) is unipotent.

The construction in the previous section can be geometrized, to yield a function
\[
(5.1) \quad \text{wav}_G : Z(\mathbb{K})^{\Delta^q_0} \to \text{Hom}_{\text{Sch}/\mathbb{K}}(\Delta^q_k, Z).
\]

The relation to the operation \( \text{wav}_{G, \mathbf{w}} \) of the previous section is as follows.

Take a sequence of points \( \mathbf{z} \in Z(\mathbb{K})^{\Delta^q_0} \). This gives rise to a morphism of \( \mathbb{K} \)-schemes
\[
\text{wav}_G(\mathbf{w}) : \Delta^q_k \to Z.
\]

Now take a weight sequence \( \mathbf{w} \in \Delta^q_k(\mathbb{K}) \). We get a point
\[
\text{wav}_G(\mathbf{z})(\mathbf{w}) \in Z(\mathbb{K}).
\]

It is easy to see that
\[
\text{wav}_G(\mathbf{z})(\mathbf{w}) = \text{wav}_{G, \mathbf{w}}(\mathbf{z}) \in Z(\mathbb{K}).
\]

We can make things even more sophisticated. Suppose \( X \) is some \( \mathbb{K} \)-scheme, and \( \pi : Z \to X \) is a \( G \)-torsor over \( X \).

For any \( X \)-scheme \( Y \), the set of \( Y \)-valued sections of \( \pi \) is of course
\[
\text{Hom}_{\text{Sch}/X}(Y, Z).
\]
The averaging operation now becomes a function

\[ \text{wav}_G : \text{Hom}_{\text{Sch}/X}(Y, Z)^{\Delta^d} \to \text{Hom}_{\text{Sch}/X}(\Delta^d_{K} \times Y, Z). \]

Here is an illustration with \( q = 1 \): we are given morphisms

\[ \sigma_0, \sigma_1 : Y \to Z, \]

The averaging process interpolates them into a 1-dimensional family of morphisms

\[ \sigma := \text{wav}_G(\sigma_0, \sigma_1) : \Delta^1_{K} \times Y \to Z. \]

The function \( \text{wav}_G \) enjoys the functoriality and symmetry explained earlier.

Regarding simpliciality: as \( q \) varies, the two sets appearing in equation (5.2) become simplicial sets, and the function \( \text{wav}_G \) becomes a map of simplicial sets.

Note that formula (5.1) above is the special case \( X = Y = \text{Spec} \, K \).

6. An Arithmetic Application

Here is an application of the averaging process to an arithmetic problem. The idea is due to Z. Reichstein.
Suppose $G$ is a linear algebraic group over a field $K$, and let $Z$ be a $G$-torsor. In case the set of rational points $Z(K)$ is nonempty, then any point $z \in Z(K)$ defines an isomorphism of $G$-schemes $G \cong Z$, namely $g \mapsto g \cdot z$. Hence one says that the torsor $Z$ is trivial.

It is easy to find nontrivial torsors. Here is an example.

**Example 6.1.** Take $K := \mathbb{R}$, and let $G$ be the non-split 1-dimensional torus.

Let’s recall that as an $\mathbb{R}$-scheme $G = \text{Spec} \, \mathbb{R}[s, t] / (s^2 + t^2 - 1)$.

Thus $G$ is a closed subscheme of the plane $\mathbb{A}^2_{\mathbb{R}}$. The set of points $G(\mathbb{R})$ is nothing but the unit circle inside the real plane: $G(\mathbb{R}) \subset \mathbb{A}^2(\mathbb{R}) = \mathbb{R}^2 \cong \mathbb{C}$.

The group structure on $G$ is such that $G(\mathbb{R})$ becomes a subgroup of the multiplicative group $\mathbb{C}^\times$.

For the torsor $Z$ we take the scheme $Z := \text{Spec} \, \mathbb{R}[s, t] / (s^2 + t^2 + 1)$, with an obvious $G$-action.

The set $Z(\mathbb{R})$, which is the circle of radius $-1$, is then empty. Hence $Z$ is a nontrivial torsor. (Of course the torsor $Z$ becomes trivial over $\mathbb{C}$.)

Note that the group $G$ in the example above is not unipotent.

There are also examples of nontrivial torsors under unipotent groups in positive characteristics.

However it is well-known that if $G$ is a unipotent group and $\text{char} \, K = 0$, then any $G$-torsor is trivial. This is usually proved using Galois cohomology.

Here is a new proof using averaging. So assume $\text{char} \, K = 0$, $G$ is a unipotent group over $K$, and $Z$ is a $G$-torsor. We want to show that $Z(K)$ is nonempty.

Choose any closed point $z \in Z$.

There is some finite Galois extension $L$ of $K$ such that $z \in Z(L)$.

Let us denote by $\Gamma$ the Galois group of $L/K$. Then $\Gamma$ acts on the set $Z(L)$, and $Z(L)^\Gamma = Z(K)$.

Let us denote by $z_0 = z, z_1, \ldots, z_q$ the distinct $\Gamma$-conjugates of $z$ inside $Z(L)$. We get a sequence $z = (z_0, \ldots, z_q) \in Z(L)^\Delta_q$.

Consider the weight sequence $w := (\frac{1}{\Delta_q}, \ldots, \frac{1}{\Delta_q}) \in \Delta_q^\Delta(K)$.

From formula (5.2), with $X := \text{Spec} \, K$ and $Y := \text{Spec} \, L$, we obtain an element $z' := \text{wav}_G(z)(w) \in Z(L)$. 

It is not hard to check, using the functoriality and symmetry of $\text{wav}_G$, that $z'$ is $\Gamma$-invariant. This implies that $z' \in Z(\mathbb{K})$, as desired.

7. AN APPLICATION TO DEFORMATION QUANTIZATION

Here is the original application.

Let $\mathbb{K}$ be a field of characteristic 0, let $X$ be a $\mathbb{K}$-scheme, and let $G$ be a pro-unipotent group.

As in Section 5, suppose $\pi : Z \to X$ is a $G$-torsor over $X$.

Assume

$$X = \bigcup_{i=0}^{m} U_i$$

is an open covering, and for every $i$ we are given a section

$$\sigma_i : U_i \to Z$$

of $\pi$.

Consider a sequence of indices

$$i = (i_0, \ldots, i_q) \in \Delta^m_q,$$

and let

$$U_i := U_{i_0} \cap \cdots \cap U_{i_q}.$$

Each of the sections

$$\sigma_{i_0}, \ldots, \sigma_{i_q} : U_i \to Z$$

is an element of $\text{Hom}_{\text{Sch}/X}(U_i, Z)$.

The averaging process of formula (5.2) works also for pro-unipotent groups (by passing to the limit), so we get a morphism

$$\sigma_i := \text{wav}_G(\sigma_{i_0}, \ldots, \sigma_{i_q}) : \Delta^q_{\mathbb{K}} \times U_i \to Z.$$

This is a family of sections $U_i \to Z$ interpolating between the various sections $\sigma_{i_0}, \ldots, \sigma_{i_q}$.

As $i$ varies the morphisms $\sigma_i$ form a simplicial section of $\pi : Z \to X$. 

Simplicial section, $q = 1$. We start with sections over two open sets $U_0$ and $U_1$ in the left picture; and we pass to a simplicial section $\sigma_{(0,1)}$ on the right.

Next consider the power series algebra
\[ \mathbb{K}[[t_1, \ldots, t_n]]. \]

Its automorphism group is a semi-direct product
\[ \text{GL}_n(\mathbb{K}) \ltimes G(\mathbb{K}), \]
where $G$ is a pro-unipotent group.

Suppose $X$ is a smooth $n$-dimensional scheme $X$ over $\mathbb{K}$. There is an infinite dimensional bundle
\[ \pi : Z \to X \]
called the \emph{bundle of formal coordinate systems}.

This construction goes back to Gelfand-Kazhdan, and is a key method in deformation quantization (after Fedosov and Kontsevich). Cf. [Ko] and [Ye1].

Now $Z$ turns out to be a torsor under the group $\text{GL}_n,\mathbb{K} \ltimes G$ above.

Let $\bar{Z}$ be the quotient bundle $Z/\text{GL}_n,\mathbb{K}$.

Locally the bundle $\bar{Z}$ has sections, and a modification of the averaging process described above yields a global simplicial section of $\bar{Z}$.

Using this simplicial section we can prove an algebro-geometric analogue of Kontsevich’s result on deformation quantization.

\textbf{References}


