0. Introduction

In this talk I will explain a geometric result that is a by-product of research on deformation quantization.

This result could be useful for other applications.

Here is the plan of the lecture:

1. Schemes and Points
2. Torsors and Averages
3. Unipotent Groups and Exponential Maps
4. First Approximation of the Average
5. Improved Formula for the Average
6. Geometric Torsors
7. Averaging for Geometric Torsors
8. An Arithmetic Application

1. Schemes and Points

Let $\mathbb{K}$ be a field. I will begin by recalling some basic facts about $\mathbb{K}$-schemes.

**Remark 1.1.** For those not comfortable with schemes: you may assume that $\mathbb{K}$ is algebraically closed (except in Section 8).

Whenever I mention “scheme”, you may replace this by “algebraic variety over $\mathbb{K}$”.

By default all rings in this talk are commutative.

For a $\mathbb{K}$-ring $A$, the corresponding affine algebraic variety is $\text{Spec} \ A$.

Let $X$ be a $\mathbb{K}$-scheme.

Given any $\mathbb{K}$-ring $B$, a morphism of schemes

$$f : \text{Spec} \ B \to X$$

is called a $B$-point of $X$.

The set of all $B$-points of $X$ is denoted by $X(B)$.

Thus

$$X(B) = \text{Hom}_{\mathbf{Sch}/\mathbb{K}}(\text{Spec} \ B, X),$$

where $\mathbf{Sch}/\mathbb{K}$ is the category of schemes over $\mathbb{K}$.
In this way we get a functor

\[ X : \text{Ring}/K \to \text{Set} \]

called the functor of points associated to \( X \).

A morphism of schemes \( X \to Y \) induces a natural transformation between the functors of points.

Observe that if \( X \) is an affine scheme, say \( X = \text{Spec} \ A \), then a point \( f \in X(B) \) is the same as a \( K \)-ring homomorphism

\[ f^* : A \to B. \]

So here

\[ X(B) \cong \text{Hom}_{\text{Ring}/K}(A, B). \]

Now consider a linear (i.e. affine) algebraic group \( G \) over \( K \).

For a \( K \)-ring \( A \), the set \( G(A) \) of \( A \)-points of \( G \) is then a group.

In particular we have the group \( G(K) \) of \( K \)-points of \( G \).

**Example 1.3.** If \( G = \text{GL}_{n,K} \), then \( G(A) = \text{GL}_{n}(A) \), the group of invertible matrices with entries in \( A \).

**2. Torsors and Averages**

Again \( K \) is a field and \( G \) is a linear algebraic group over \( K \). The group \( G(K) \) is just a group (no extra structure).

**Definition 2.1.** A \( G(K) \)-torsor is a set \( Z \), endowed with a simply transitive (left) action of the group \( G(K) \).

(Later we will see a more sophisticated definition of torsors.)

Note that given any point \( z \in Z \), the map

\[ G(K) \to Z, \ g \mapsto g \cdot z, \]

is bijective.

Thus a choice of base point \( z \in Z \) determines an isomorphism of torsors \( G(K) \xrightarrow{\sim} Z \).
**Definition 2.2.** By a **weight sequence** we mean a sequence
\[ w = (w_0, \ldots, w_q) \in \mathbb{K}^{q+1} \]
such that
\[ \sum_{i=0}^{q} w_i = 1. \]

Suppose \( G \) is an additive group, i.e. \( G \cong \mathbb{A}_\mathbb{K}^n \) with its additive group structure. So \( G(\mathbb{K}) \cong \mathbb{K}^n \) as groups.

Let \( Z \) be a \( G(\mathbb{K}) \)-torsor.

Given a sequence \( z = (z_0, \ldots, z_q) \) of points in \( Z \), and a weight sequence \( w = (w_0, \ldots, w_q) \), we have the usual **weighted average**
\[ \text{wav}_{G, w}(z) \in Z. \]

Let me recall the formula. Fix an isomorphism of groups \( G \cong \mathbb{A}_\mathbb{K}^n \).

To simplify notation, in this case we will use addition for the operation in \( G \) and the action on \( Z \).

Choose any base point \( z \in Z \), and define \( g_i \in G(\mathbb{K}) \) to be the unique element such that
\[ z_i = g_i + z. \]

Using the isomorphism \( G \cong \mathbb{A}_\mathbb{K}^n \) we get elements
\[ w_i \cdot g_i \in G(\mathbb{K}), \]
and then we let
\[ \text{wav}_{G, w}(z) := \left( \sum_{i=0}^{q} w_i \cdot g_i \right) + z \in Z. \quad (2.3) \]

Since \( \sum_{i=0}^{q} w_i = 1 \), this is independent of the base point \( z \).

We know that the weighted average has these properties:

1. **Functoriality**: Suppose \( G \to G' \) is a morphism of additive groups; \( Z' \) is a torsor under \( G'(\mathbb{K}) \); \( f : Z \to Z' \) is a \( G(\mathbb{K}) \)-equivariant function; and
\[ z' := (f(z_0), \ldots, f(z_q)). \]

Then
\[ f(\text{wav}_{G, w}(z)) = \text{wav}_{G', w}(z'). \]

2. **Symmetry**: \( \text{wav}_{G, w}(z) \) is invariant under simultaneous permutation of the sequences \( w \) and \( z \).

3. **Simpliciality**: This is another combinatorial property, that will be defined later.
We want to generalize this averaging construction to other groups \( G \).
We will see that this can be done for unipotent groups in characteristic 0; but it cannot be done for some other groups.

### 3. Unipotent Groups and Exponential Maps

Again \( \mathbb{K} \) is a field and \( G \) is a linear algebraic group over it.

Let \( \overline{\mathbb{K}} \) be an algebraic closure of \( \mathbb{K} \). Recall that an element \( g \in G(\overline{\mathbb{K}}) \) is called unipotent if for any morphism of groups \( \rho : G \to \text{GL}_n(\overline{\mathbb{K}}) \), the matrix \( \rho(g) \in \text{GL}_n(\overline{\mathbb{K}}) \) is unipotent (i.e. its only eigenvalue is 1).

The group \( G \) is called unipotent of all elements of \( G(\overline{\mathbb{K}}) \) are unipotent.

Here is equivalent characterization: \( G \) is unipotent iff it is isomorphic to a closed subgroup of the upper triangular matrix group inside \( \text{GL}_n(\overline{\mathbb{K}}) \), for some \( n \).

**Example 3.1.** An additive group \( G \cong \mathbb{A}_{\overline{\mathbb{K}}}^n \) is unipotent.

A possible embedding is

\[
G(A) \cong \begin{bmatrix} 1 & A & A & \cdots & A \\ 0 & 1 & A & \cdots & A \\ 0 & 0 & 1 & \cdots & A \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \subseteq \text{GL}_n(A),
\]

functorially in \( A \).

In fact, any abelian unipotent group is an additive group.

**Remark 3.2.** If \( \mathbb{K} \subseteq \mathbb{R} \) and \( G \) is unipotent, then the group \( G(\mathbb{R}) \), with its structure of real analytic manifold, is a simply connected nilpotent Lie group. And vice versa.

All results stated later can be interpreted in the differentiable setup, for \( G(\mathbb{R}) \)-torsors.
Now assume that \( \text{char } K = 0 \) and \( G \) is a unipotent group.

Let \( \mathfrak{g} = \text{Lie}(G) \) be the Lie algebra of \( G \). This is a Lie algebra over \( K \), and also a scheme: \( \mathfrak{g} \cong A^n_K \), where \( n = \dim G \).

There is an isomorphism of schemes

\[
\exp_G : \mathfrak{g} \to G
\]

called the exponential map. Its inverse is \( \log_G \).

The exponential map is functorial; namely if \( G' \) is another unipotent group, and \( \phi : G \to G' \) is a morphism of groups, then the diagram

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\exp_G} & G \\
\downarrow \text{Lie}(\phi) & & \downarrow \phi \\
\mathfrak{g}' & \xrightarrow{\exp_{G'}} & G'
\end{array}
\]

is commutative.

Example 3.3. For a unipotent closed subgroup \( G \subset \text{GL}_{n,K} \), the Lie algebra \( \mathfrak{g}(K) \) is a subalgebra of \( \mathfrak{gl}_n(K) = \text{Mat}_n(K) \) consisting of nilpotent matrices.

For any element \( \gamma \in \mathfrak{g}(K) \), the formula for \( \exp_G(\gamma) \) is

\[
\exp_G(\gamma) = \sum_{i=0}^{\infty} \frac{1}{i!} \gamma^i \in G(K) \subset \text{GL}_n(K).
\]

Since \( \gamma \) is a nilpotent matrix, this sum is finite.

4. First Approximation of the Average

Once more the field \( K \) has characteristic 0, and \( G \) is a unipotent group over it. Let \( Z \) be a \( G(K) \)-torsor.

Suppose \( w = (w_0, \ldots, w_q) \) is a weight sequence in \( K \), and \( z = (z_0, \ldots, z_q) \) is a sequence of points in \( Z \).

We want to construct a weighted average \( \text{wav}_{G,w}(z) \in Z \).

Choose some base point \( z \in Z \), and define elements \( g_i \in G(K) \) by the rule

\[
z_i = g_i \cdot z.
\]

For any \( i \) let

\[
\gamma_i := \log_G(g_i) \in \mathfrak{g}(K).
\]

Using the vector space structure of \( \mathfrak{g}(K) \) define

\[
\gamma := \sum_{i=0}^{q} w_i \cdot \gamma_i \in \mathfrak{g}(K).
\]

Next let

\[
g := \exp_G(\gamma) \in G(K).
\]

The first candidate for the weighed average is the element

\[
\text{wav}_{G,w}(z) := g \cdot z \in Z. \quad (4.1)
\]
But there is a problem here: this “average” depends on the choice of base point $z$!

A calculation shows that this dependence is in the form of commutators.

Note that in the additive case the exponential map is actually an isomorphism of groups

$$\exp_G : (\mathfrak{g} , +) \xrightarrow{\cong} (G , \cdot).$$

So the construction of the weighted average in equation (2.3) is actually the same as in (4.1).

Of course since the group is abelian the problem of base point does not arise in the additive case.

Is there a way to get around the dependence on the base point when $G$ is not abelian?

5. Improved Formula for the Average

Continuing with the previous setup, we will do something a little bit strange.

The sequence $z$ provides us with $q + 1$ preferred base points $z_0, \ldots, z_q \in Z$. We shall perform the averaging process with respect to each of these base points.

Thus, for fixed weight sequence $w$, we will obtain $q + 1$ possibly distinct “averages” $z'_0, \ldots, z'_q$.

Here is the explicit formula.

Let $g_{i,j} \in G(\mathbb{K})$ be the unique the elements satisfying $g_{i,j} \cdot z_i = z_j$.

Then

$$z'_i = \exp_G \left( \sum_{j=0}^q w_j \cdot \log_G (g_{i,j}) \right) \cdot z_i.$$

This can be viewed as an operation of weighted symmetrization

$$\text{wsym}_{G,w} : \left\{ \begin{array}{l} Z^{q+1} \to Z^{q+1} \\
(\mathfrak{z}_0, \ldots, \mathfrak{z}_q) \mapsto (\mathfrak{z}'_0, \ldots, \mathfrak{z}'_q) \end{array} \right..$$

I claim that if we iterate the operation $\text{wsym}_{G,w}$ enough times, we end up with a constant sequence.

The reason is that the group $G$ has a filtration by closed normal subgroups

$$G = G_0 \supset G_1 \supset \cdots \supset G_d = \{1\}$$

such that $G_i / G_{i+1}$ is abelian, and the conjugation action of $G$ on $G_i / G_{i+1}$ is trivial.

A little calculation shows that if the points $z_0, \ldots, z_q$ are all in the same $G_i$-orbit for some $i$, then the points $z'_0, \ldots, z'_q$ are in the same $G_{i+1}$-orbit.

Thus, starting with a sequence of points

$$z = (z_0, \ldots, z_q),$$

and repeating the operation $\text{wsym}_{G,w}$ $d$ times, we get

$$\text{wsym}_{G,w}^d(z) = (z, \ldots, z)$$

for some $z \in Z$.

Now we can define the weighted average

$$\text{wav}_{G,w}(z) := z.$$

It is easy to see that this average enjoys the required properties (functoriality, simpliciality and symmetry). See [Ye2] for details.
6. Geometric Torsors

The same idea can be dressed up in fancier language.
Let \( K \) be a field, and let \( G \) be a linear algebraic group over \( K \).
Take a finite type \( K \)-scheme \( X \).

A \( G \)-torsor (or principal homogeneous space) over \( X \) is a finite type \( K \)-scheme \( Z \),
with a flat surjective morphism
\[
\pi : Z \to X,
\]
and with a group action
\[
\mu : G \times Z \to Z
\]
that commutes with \( \pi \),
such that
\[
\mu \times p_2 : G \times Z \to Z \times_X Z
\]
is an isomorphism of schemes.

Suppose \( \pi : Z \to X \) is a \( G \)-torsor over \( X \), and \( U \to X \) is a morphism of schemes.
A morphism
\[
\sigma : U \to Z
\]
that lifts \( \pi \) will be called a \( U \)-section of \( Z \).

We denote the set of all \( U \)-sections of \( Z \) by \( Z(U) \).
In other words,
\[
Z(U) = \text{Hom}_{\text{Sch}/X}(U, Z).
\]
The torsor \( \pi : Z \to X \) is called \textit{trivial} if there is a section
\[ \sigma : X \to Z. \]

In other words, \( Z \) is a trivial torsor if
\[ Z \cong G \times X. \]

It is known that any \( G \)-torsor is locally trivial.
Namely there is a covering \( \{ U_i \} \) of \( X \) in a suitable topology, together with sections
\[ \sigma_i : U_i \to Z. \]

Let me give two examples.

**Example 6.1.** Suppose \( E \) is a rank \( n \) vector bundle over \( X \).
Consider the \textit{frame bundle} \( \pi : Z \to X \) of \( E \).
By definition, for every open set \( U \subset X \), the set of sections
\[ \sigma : U \to Z \]
of \( Z \) is the set of trivializations of the vector bundle \( E|_U \), namely the set of vector bundle isomorphisms
\[ E|_U \cong U \times \mathbb{A}^n_K. \]

Then \( \pi : Z \to X \) is a \( \text{GL}_{n,K} \)-torsor over \( X \), which is locally trivial in the Zariski topology.
The torsor \( Z \) is nontrivial iff the vector bundle \( E \) is a nontrivial.

**Example 6.2.** Suppose the group \( G \) is smooth over \( K \) (this is automatic in characteristic 0.)
Then any \( G \)-torsor \( \pi : Z \to X \) is trivial in the étale topology of \( X \).
(This is a theorem of Grothendieck; cf. [Mi, Section III.4].)
I will give a concrete example of this phenomenon in Section 8.
7. Averaging for Geometric Torsors

For \( q \in \mathbb{N} \) let \( \Delta^q \) be the \( q \)-dimensional combinatorial simplex.

The set of vertices of \( \Delta^q \) is the set

\[
\Delta^q_0 = \{ v_0, \ldots, v_q \}.
\]

The \( q \)-dimensional geometric simplex is the affine scheme

\[
\Delta^q_K := \text{Spec } k[t_0, \ldots, t_q] / (t_0 + \cdots + t_q - 1).
\]

A point \( w \in \Delta^q_K(k) \) is precisely a weight sequence.

We can view \( \Delta^q_0 \) as a subset of \( \Delta^q_K(k) \)

Now assume \( \text{char } k = 0 \) and \( G \) is unipotent group. Let \( \pi : Z \rightarrow X \) be a \( G \)-torsor over \( X \).

Let \( U \rightarrow X \) be a morphism of schemes, so we can talk about the set \( Z(U) \) of \( U \)-sections of the torsor \( Z \).

The construction in Section 5 can be geometrized, to yield a function

\[
\text{wav}_G : \text{Hom}_{\text{Set}}(\Delta^q_0, Z(U)) \rightarrow \text{Hom}_{\text{Sch}/X}(\Delta^q_K \times U, Z)
\]

which again we call the weighted average.

In other terms, the weighted average is a function

\[
\text{wav}_G : Z(U)^{q+1} \rightarrow Z(\Delta^q_K \times U).
\]

Here is what the operation \( \text{wav}_G \) actually does.

Suppose we are given \( U \)-sections

\[
\sigma_0, \ldots, \sigma_q : U \rightarrow Z.
\]

Then there is a morphism

\[
\sigma := \text{wav}_G(\sigma_0, \ldots, \sigma_q) : \Delta^q_K \times U \rightarrow Z.
\]

The morphism \( \sigma \) interpolates between \( \sigma_0, \ldots, \sigma_q \), in the following sense: for any \( i \), take the \( i \)-th vertex \( v_i \in \Delta^q_K(k) \). Then

\[
\sigma(v_i, x) = \sigma_i(x)
\]

for \( x \in U \).

See figure on next slide.
The function \( \text{wav}_G \) is functorial in \( G, X, Z \) and \( U \).

It also respects the action of the permutation group on \( \Delta^q_K \).

Regarding simpliciality: as \( q \) varies, the two sets appearing in equation (7.1) become simplicial sets, and the function \( \text{wav}_G \) becomes a map of simplicial sets.

See [Ye2] for details.

The operation \( \text{wav}_G \) was required to construct simplicial sections of torsors, in the papers [Ye1, Ye3].

8. An Arithmetic Application

Here is an application of the averaging process to an arithmetic problem. The idea is due to Z. Reichstein.

Suppose \( G \) is a linear algebraic group over a field \( K \), and let \( Z \) be a \( G \)-torsor over \( K \).

First let me give an example of a nontrivial torsor.
Example 8.1. Take $\mathbb{K} = \mathbb{R}$, and let $G$ be the non-split 1-dimensional torus.

Let’s recall that as an $\mathbb{R}$-scheme

$$G = \text{Spec} \mathbb{R}[t_1, t_2] / (t_1^2 + t_2^2 - 1).$$

Thus $G$ is a closed subscheme of the plane $\mathbb{A}^2_{\mathbb{R}}$. The set of points $G(\mathbb{R})$ is nothing but the unit circle inside the real plane:

$$G(\mathbb{R}) \subseteq \mathbb{A}^2(\mathbb{R}) = \mathbb{R}^2 \cong \mathbb{C}.$$

The group structure on $G$ is such that $G(\mathbb{R})$ becomes a subgroup of the multiplicative group $\mathbb{C}^\times$.

Note that the group $G$ in the example above is not unipotent.

There are examples of nontrivial torsors under unipotent groups in positive characteristics.

However it is well-known that if $G$ is a unipotent group and $\text{char} \, \mathbb{K} = 0$, then any $G$-torsor is trivial. This is usually proved using Galois cohomology.

I will now give a new proof using averaging.

(cont.) For the torsor $Z$ we take the scheme

$$Z := \text{Spec} \mathbb{R}[t_1, t_2] / (t_1^2 + t_2^2 + 1),$$

with an obvious $G$-action

$$\mu : G \times Z \to Z.$$  

To see that $Z$ is indeed a torsor, namely that $\mu \times p_2 : G \times Z \to Z \times Z$ is an isomorphism, we can check this after base change to $\mathbb{C}$.

Alternatively, one can show that the $\mathbb{R}$-ring homomorphism $\mu^* \otimes p_2^*$ is bijective.

The set $Z(\mathbb{R})$, which is the circle of radius $i = \sqrt{-1}$, is of course empty. Hence $Z$ is a nontrivial torsor. (Of course the torsor $Z$ becomes trivial over $\mathbb{C}$.)

So assume $\text{char} \, \mathbb{K} = 0$, $G$ is a unipotent group over $\mathbb{K}$, and $Z$ is a $G$-torsor. We want to show that the set $Z(\mathbb{K})$ is nonempty.

Choose any closed point $z_0 \in Z$.

The residue field $k(z_0)$ is a finite separable extension of $\mathbb{K}$, so there is a Galois extension $L$ of $\mathbb{K}$ that contains $k(z_0)$.

It follows that $z_0 \in Z(L)$.

Let us denote by $\Gamma$ the Galois group of $L/\mathbb{K}$.

Then $\Gamma$ acts on the set

$$Z(L) = \text{Hom}_{\text{Sch}/\mathbb{K}}(\text{Spec} L, Z),$$

and the set of invariants is

$$Z(L)^\Gamma = Z(\mathbb{K}).$$
Let us denote by $z_0, z_1, \ldots, z_q$ the distinct $\Gamma$-conjugates of $z_0$ inside $Z(L)$. We get a sequence

$$z = (z_0, \ldots, z_q) \in Z(L)^{q+1}.$$ 

Consider the weight sequence

$$w := \left(\frac{1}{q+1}, \ldots, \frac{1}{q+1}\right) \in \Delta_{\mathbb{K}}^q(\mathbb{K}).$$

From formula (7.2), with $X := \text{Spec} \mathbb{K}$ and $U := \text{Spec} L$, we obtain an element

$$z := \text{wav}_G(z)(w) \in Z(L).$$

We are going to show that $z$ is $\Gamma$-invariant.

So take $\gamma \in \Gamma$.

By the functoriality of $\text{wav}_G$ in $U = \text{Spec} L$, we get

$$\gamma(z) = \text{wav}_G(\gamma(z))(w),$$

where

$$\gamma(z) := (\gamma(z_0), \ldots, \gamma(z_q)).$$

But the sequence $\gamma(z)$ is a permutation of the sequence $z$; and the weight sequence $w$ is symmetric. Therefore, by the symmetry property of $\text{wav}_G$, we get

$$\text{wav}_G(\gamma(z))(w) = \text{wav}_G(z)(w) = z.$$ 

The argument above also tells us that there can be no averaging process for any group that admits nontrivial torsors.

- END -

References


