Rigid Dualizing Complexes via Differential Graded Algebras

Lecture Notes

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Here is the plan of my lecture:

1. Dualizing Complexes: Overview
2. Rigid Complexes and DG Algebras
3. Properties of Rigid Complexes
4. Rigid Dualizing Complexes
5. Rigid Complexes and CM Homomorphisms

1. DUALIZING COMPLEXES: OVERVIEW

Let $A$ be a noetherian commutative ring. Denote by $D^b_{f}({\text{Mod}}A)$ the derived category of bounded complexes of $A$-modules with finitely generated cohomology modules.

**Definition 1.** (Grothendieck [RD]) A dualizing complex over $A$ is a complex $R \in D^b_{f}({\text{Mod}}A)$ satisfying the two conditions:

(i) $R$ has finite injective dimension.
(ii) The canonical morphism $A \rightarrow \text{RHom}_A(R, R)$ is an isomorphism.

Condition (i) means that there is an integer $d$ such that $\text{Ext}^i_A(M, R) = 0$ for all $i > d$ and all modules $M$.

**Example 2.** If $K$ is a regular noetherian ring of finite Krull dimension (say a field, or the ring of integers $\mathbb{Z}$) then

$$R := K \in D^b_{f}({\text{Mod}}K)$$

is a dualizing complex.

Dualizing complexes over commutative rings are part of Grothendieck’s duality theory in algebraic geometry, which was developed in [RD]. This duality theory deals with dualizing complexes on schemes and relations between them.

In this lecture I will explain a new approach to dualizing complexes over commutative rings, due to James Zhang and myself (see [YZ4] and [YZ5]). Specifically, I’ll talk about existence and uniqueness of rigid dualizing complexes.

The purpose of rigidity is to eliminate automorphisms, and to make the dualizing complexes functorial.

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In a sequel paper [Ye2] we use the technique of perverse coherent sheaves to construct rigid dualizing complexes on schemes, and we reproduce almost all of the geometric Grothendieck duality theory. But that’s a subject for a separate lecture.

Related work in noncommutative algebraic geometry (where rigid dualizing complexes were first introduced) can be found in [VdB, YZ1, YZ2, YZ3].

2. RIGID COMPLEXES AND DG ALGEBRAS

By default all rings considered in this talk are commutative.

Let me start with a discussion of rigidity for algebras over a field. Suppose \( K \) is a field, \( B \) is a \( K \)-algebra, and \( M \in D(\text{Mod} B) \).

According to Van den Bergh [VdB] a rigidifying isomorphism for \( M \) is an isomorphism

\[
\rho : M \cong \text{RHom}_{B \otimes_k B}(B, M \otimes_k M)
\]

in \( D(\text{Mod} B) \).

Now suppose \( A \) is any ring.

Trying to write \( A \) instead of \( K \) in formula (1) does not make sense: instead of \( M \otimes A M \) we must take the derived tensor product \( M \otimes^L_A M \); but then there is no obvious way to make \( M \otimes^L_A M \) into a complex of \( B \otimes A B \)-modules.

The problem is torsion: \( B \) might fail to be a flat \( A \)-algebra.

This is where differential graded algebras (DG algebras) enter the picture.

A DG algebra is a graded ring \( \hat{A} = \bigoplus_{i \in \mathbb{Z}} \hat{A}^i \), together with a graded derivation \( d : \hat{A} \rightarrow \hat{A} \) of degree 1, satisfying \( d \circ d = 0 \).

A DG algebra quasi-isomorphism is a homomorphism \( f : \hat{A} \rightarrow \hat{B} \) respecting degrees, multiplications and differentials, and such that \( H(f) : H\hat{A} \rightarrow H\hat{B} \) is an isomorphism (of graded algebras).

We shall only consider super-commutative non-positive DG algebras. Super-commutative means that \( ab = (-1)^{ij}ba \) and \( c^2 = 0 \) for all \( a \in \hat{A}^i \), \( b \in \hat{A}^j \) and \( c \in \hat{A}^{2i+1} \). Non-positive means that \( \hat{A} = \bigoplus_{i \leq 0} \hat{A}^i \).

We view a ring \( A \) as a DG algebra concentrated in degree 0. Given a DG algebra homomorphism \( A \rightarrow \hat{A} \) we say that \( \hat{A} \) is a DG \( A \)-algebra.

Let \( A \) be a ring. A semi-free DG \( A \)-algebra is a DG \( A \)-algebra \( \hat{A} \), such that after forgetting the differential \( \hat{A} \) is isomorphic, as graded \( A \)-algebra, to a super-polynomial algebra on some graded set of variables.

Definition 3. Let \( A \) be a ring and \( B \) an \( A \)-algebra. A semi-free DG algebra resolution of \( B \) relative to \( A \) is a quasi-isomorphism \( \hat{B} \rightarrow B \) of DG \( A \)-algebras, where \( \hat{B} \) is a semi-free DG \( A \)-algebra.

Such resolutions always exist, and they are unique up to quasi-isomorphism.

Example 4. Take \( A := \mathbb{Z} \) and \( B = \mathbb{Z}/(6) \). Define \( \hat{B} \) to be the super-polynomial algebra \( \mathbb{Z}[\xi] \) on the variable \( \xi \) of degree \(-1\). So \( \hat{B} = \mathbb{Z} \oplus \mathbb{Z} \xi \) as free \( \mathbb{Z} \)-module, and
$\xi^2 = 0$. Let $d(\xi) := 6$. Then $\tilde{B} \to \mathbb{Z}/(6)$ is a semi-free DG algebra resolution of $\mathbb{Z}/(6)$ relative to $\mathbb{Z}$.

For a DG algebra $A$ one has the category $\text{DGMod} \tilde{A}$ of DG $\tilde{A}$-modules. It is analogous to the category of complexes of modules over a ring, and by a similar process of inverting quasi-isomorphisms we obtain the derived category $\tilde{D}(\text{DGMod} A)$; see [Ke], [Hi].

For a ring $A$ (a DG algebra concentrated in degree 0) we have

$\tilde{D}(\text{DGMod} A) = D(\text{Mod} A),$

the usual derived category.

It is possible to derive functors of DG modules, again in analogy to $D(\text{Mod} A)$.

An added feature is that for a quasi-isomorphism $\tilde{A} \to \tilde{B}$ the restriction of scalars functor

$\tilde{D}(\text{DGMod} \tilde{B}) \to \tilde{D}(\text{DGMod} \tilde{A})$

is an equivalence.

Getting back to our original problem, suppose $A$ is a ring and $B$ is an $A$-algebra. Choose a semi-free DG algebra resolution $\tilde{B} \to B$ relative to $A$. For $M \in D(\text{Mod} B)$ define

$\text{Sq}_{B/A} M := \text{RHom}_{\tilde{B} \otimes A} \tilde{B}(B, M \otimes^L_A M)$

in $D(\text{Mod} B)$.

**Theorem 5.** ([YZ4]) The functor

$\text{Sq}_{B/A} : D(\text{Mod} B) \to D(\text{Mod} B)$

is independent of the resolution $\tilde{B} \to B$.

The functor $\text{Sq}_{B/A}$, called the squaring operation, is nonlinear. In fact, given a morphism $\phi : M \to M$ in $D(\text{Mod} B)$ and an element $b \in B$ one has

(2) $\text{Sq}_{B/A}(b\phi) = b^2 \text{Sq}_{B/A}(\phi)$

in

$\text{Hom}_{D(\text{Mod} B)}(\text{Sq}_{B/A} M, \text{Sq}_{B/A} M)$.

**Definition 6.** Let $B$ be a noetherian $A$-algebra, and let $M$ be a complex in $D^+_f(\text{Mod} B)$ that has finite flat dimension over $A$. Assume

$\rho : M \xrightarrow{\cong} \text{Sq}_{B/A} M$

is an isomorphism in $D(\text{Mod} B)$. Then the pair $(M, \rho)$ is called a rigid complex over $B$ relative to $A$.

**Definition 7.** Say $(M, \rho)$ and $(N, \sigma)$ are rigid complexes over $B$ relative to $A$. A morphism $\phi : M \to N$ in $D(\text{Mod} B)$ is called a rigid morphism relative to $A$ if the
diagram
\[
\begin{array}{c}
\begin{array}{ccc}
M & \xrightarrow{\rho} & \text{Sq}_{B/A}M \\
\downarrow{\phi} & & \downarrow{\text{Sq}_{B/A}(\phi)} \\
N & \xrightarrow{\sigma} & \text{Sq}_{B/A}N
\end{array}
\end{array}
\]

is commutative.

We denote by \(\text{D}^b_{\text{f}}(\text{Mod } B)_{\text{rig}/A}\) the category of rigid complexes over \(B\) relative to \(A\).

**Example 8.** Take \(M = B := A\). Then
\[
\text{Sq}_{A/A} A = \text{RHom}_{A\otimes_A A}(A, A \otimes_A A) = A,
\]
and we interpret this as the tautological rigidifying isomorphism
\[
\rho^{\text{tau}} : A \cong \text{Sq}_{A/A} A.
\]
The **tautological rigid complex** is
\[
(A, \rho^{\text{tau}}) \in \text{D}^b_{\text{f}}(\text{Mod } A)_{\text{rig}/A}.
\]

3. **Properties of Rigid Complexes**

The first property of rigid complexes explains their name.

**Theorem 9.** ([YZ4]) Let \(A\) be a ring, \(B\) a noetherian \(A\)-algebra, and
\[(M, \rho) \in \text{D}^b_{\text{f}}(\text{Mod } B)_{\text{rig}/A}.
\]
Assume the canonical homomorphism
\[
B \to \text{Hom}_{\text{D}(\text{Mod } B)}(M, M)
\]
is bijective. Then the only automorphism of \((M, \rho)\) in
\[
\text{D}^b_{\text{f}}(\text{Mod } B)_{\text{rig}/A}
\]
is the identity \(1_M\).

The proof is very easy: an automorphism \(\phi\) of \(M\) has to be of the form \(\phi = b 1_M\) for some invertible element \(b \in B\). If \(\phi\) is rigid then \(b = b^2\) (cf. formula (2)), and hence \(b = 1\).

We find it convenient to denote ring homomorphisms by \(f^*\) etc. Thus a ring homomorphism \(f^* : A \to B\) corresponds to the morphism of schemes
\[
f : \text{Spec } B \to \text{Spec } A.
\]

Let \(A\) be a noetherian ring. Recall that an \(A\)-algebra \(B\) is called essentially finite type if it is a localization of some finitely generated \(A\)-algebra.

We say that \(B\) is *essentially smooth* (resp. *essentially étale*) over \(A\) if it is essentially finite type and formally smooth (resp. formally étale).
Example 10. If $A'$ is a localization of $A$ then $A \to A'$ is essentially étale. If $B = A[t_1, \ldots, t_n]$ is a polynomial algebra then $A \to B$ is smooth, and hence also essentially smooth.

Let $A$ be a noetherian ring and $f^* : A \to B$ an essentially smooth homomorphism. Then $\Omega^1_{B/A}$ is a finitely generated projective $B$-module.

Let $\text{Spec } B = \bigsqcup_i \text{Spec } B_i$ be the decomposition into connected components, and for every $i$ let $n_i$ be the rank of $\Omega^1_{B_i/A}$. We define a functor

$$f^1 : \text{D}(\text{Mod } A) \to \text{D}(\text{Mod } B)$$

by

$$f^1 M := \bigoplus_i \Omega^1_{B_i/A}[n_i] \otimes_A M.$$

Recall that a ring homomorphism $f^* : A \to B$ is called finite if $B$ is a finitely generated $A$-module. Given such a finite homomorphism we define a functor $f^\flat : \text{D}(\text{Mod } A) \to \text{D}(\text{Mod } B)$ by

$$f^\flat M := \text{RHom}_A(B, M).$$

Theorem 11. ([YZ4]) Let $A$ be a noetherian ring, let $B, C$ be essentially finite type $A$-algebras, let $f^* : B \to C$ be an $A$-algebra homomorphism, and let $(M, \rho) \in \text{D}^b_{\text{rig}}(\text{Mod } B)_{\text{rig}/A}$.

(1) If $f^*$ is finite and $f^\flat M$ has finite flat dimension over $A$, then $f^\flat M$ has an induced rigidifying isomorphism

$$f^\flat(\rho) : f^\flat M \simeq_{\text{rig}} \text{Sq}_{C/A} f^\flat M.$$

(2) If $f^*$ is essentially smooth then $f^\sharp M$ has an induced rigidifying isomorphism

$$f^\sharp(\rho) : f^\sharp M \simeq_{\text{rig}} \text{Sq}_{C/A} f^\sharp M.$$

4. Rigid Dualizing Complexes

Let $\mathcal{K}$ be a noetherian regular ring of finite Krull dimension. We denote by $\text{EFTAlg } / \mathcal{K}$ the category of essentially finite type $\mathcal{K}$-algebras.

Definition 12. A rigid dualizing complex over $A$ relative to $\mathcal{K}$ is a rigid complex $(R_A, \rho_A)$ such that $R_A$ is a dualizing complex.

Theorem 13. ([YZ5]) Let $\mathcal{K}$ be a regular finite dimensional noetherian ring, and let $A$ be an essentially finite type $\mathcal{K}$-algebra.

(1) The algebra $A$ has a rigid dualizing complex $(R_A, \rho_A)$, which is unique up to a unique rigid isomorphism.
Given a finite homomorphism $f^*: A \rightarrow B$, there is a unique rigid isomorphism $f^*(R_A, \rho_A) \cong (R_B, \rho_B)$.

Given an essentially smooth homomorphism $f^*: A \rightarrow B$, there is a unique rigid isomorphism $f^!(R_A, \rho_A) \cong (R_B, \rho_B)$.

Here is how the rigid dualizing complex $(R_A, \rho_A)$ is obtained. We begin with the tautological rigid complex $(\mathbb{K}, \rho^{tau}) \in D^+_f(\text{Mod} \mathbb{K})_{\text{rig}/\mathbb{K}}$, which is dualizing. Now the structural homomorphism $\mathbb{K} \rightarrow A$ can be factored into $\mathbb{K} \xrightarrow{f^*} B \xrightarrow{g^*} C \xrightarrow{h^*} A$, where $f^*$ is smooth ($B$ is a polynomial algebra over $\mathbb{K}$); $g^*$ is finite (a surjection); and $h^*$ is also smooth (a localization). Then $(R_A, \rho_A) := h^* g^* f^!(\mathbb{K}, \rho^{tau}) \in D^+_f(\text{Mod} A)_{\text{rig}/\mathbb{K}}$.

Definition 14. Given a homomorphism $f^*: A \rightarrow B$ in $\text{EFTAlg}/\mathbb{K}$, define the twisted inverse image functor $f^!: D^+_f(\text{Mod} A) \rightarrow D^+_f(\text{Mod} B)$ by the formula

$$f^! M := \text{RHom}_B(B \otimes_A^L \text{RHom}_A(M, R_A), R_B).$$

It is not hard to show that the assignment $f^* \mapsto f^!$ is a 2-functor from the category $\text{EFTAlg}/\mathbb{K}$ to the 2-category $\text{Cat}$ of all categories.

One can show, using Theorem 13, that this operation has very good properties. For instance, when $f^*$ is finite, then there is a functorial, nondegenerate trace morphism $\text{Tr}_f : f^! M \rightarrow M$.

5. Rigid Complexes and CM Homomorphisms

In this final section I’ll talk about the relation between rigid complexes and Cohen-Macaulay homomorphisms.

Definition 15. A ring $A$ is called tractable if there is an essentially finite type homomorphism $\mathbb{K} \rightarrow A$, for some regular noetherian ring of finite Krull dimension $\mathbb{K}$.

The homomorphism $\mathbb{K} \rightarrow A$ is not part of the structure – there is no preferred $\mathbb{K}$. “Most commutative noetherian rings we know” are tractable...

Theorem 16. ([YZ5], [Ye2]) Let $A$ be a tractable ring, and let $B$ be an essentially finite type $A$-algebra of finite flat dimension (e.g. $B$ is flat over $A$).

1. There exists a unique (up to unique rigid isomorphism) rigid complex $R_{B/A}$ over $B$ relative to $A$, which is nonzero on each connected component of $\text{Spec} B$. 


(2) If $A$ is a Gorenstein ring (e.g. a regular ring) then $R_{B/A}$ is a dualizing complex over $B$.

Let $f^*: A \to B$ be a finite type flat homomorphism of relative dimension $n$; namely the fibers of $f: \text{Spec } B \to \text{Spec } A$ are all equidimensional of dimension $n$.

Recall that $f^*$ is called a Cohen-Macaulay homomorphism if the fibers of $f$ are all $n$-dimensional Cohen-Macaulay schemes.

**Theorem 17.** ([Ye2]) Let $A$ be a tractable ring, and let $f^*: A \to B$ be a finite type flat homomorphism of relative dimension $n$. Then the following conditions are equivalent:

(i) $f^*$ is a Cohen-Macaulay homomorphism.

(ii) $H^i R_{B/A} = 0$ for all $i \neq -n$, and the $B$-module

$$\omega_{B/A} := H^{-n} R_{B/A}$$

is flat over $A$.

The module $\omega_{B/A}$ is the dualizing module of $B$ relative to $A$.

Note that the complex $\omega_{B/A}[n]$ is rigid relative to $A$, but in general it is not a dualizing complex (cf. part (2) of previous theorem.) Still the fibers of $\omega_{B/A}[n]$ are dualizing complexes – this can be seen by taking $A'$ to be a field in the next result.

Let me end with a “rigid” version of Conrad’s base change theorem [Co]:

**Theorem 18.** ([Ye2]) Let

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
A' & \longrightarrow & B'
\end{array}
\]

be a cartesian diagram of rings, i.e.

$$B' \cong A' \otimes_A B,$$

with $A$ and $A'$ tractable rings. Assume $A \to B$ is a Cohen-Macaulay homomorphism. (There isn’t any restriction on the homomorphism $A \to A'$.) Then:

(1) $A' \to B'$ is a Cohen-Macaulay homomorphism.

(2) There is a unique isomorphism of $B'$-modules

$$\omega_{B'/A'} \cong A' \otimes_A \omega_{B/A}$$

which respects rigidity.

**References**


