Rigid Dualizing Complexes via Differential Graded Algebras

Lecture Notes

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Here is the plan of my lecture:

1. Dualizing Complexes: Overview
2. Rigid Complexes and DG Algebras
3. Properties of Rigid Complexes
4. Rigid Dualizing Complexes
5. Rigid Complexes and CM Homomorphisms
1 Dualizing Complexes: Overview

Let $A$ be a noetherian commutative ring.
Denote by $D^b_f(\text{Mod } A)$ the derived category of bounded complexes of $A$-modules with finitely generated cohomology modules.

**Definition 1.** (Grothendieck [RD]) A *dualizing complex* over $A$ is a complex $R \in D^b_f(\text{Mod } A)$ satisfying the two conditions:

(i) $R$ has finite injective dimension.

(ii) The canonical morphism

$$A \to \text{RHom}_A(R, R)$$

is an isomorphism.

Condition (i) means that there is an integer $d$ such that $\text{Ext}^i_A(M, R) = 0$ for all $i > d$ and all modules $M$. 
**Example 2.** If $\mathbb{K}$ is a regular noetherian ring of finite Krull dimension (say a field, or the ring of integers $\mathbb{Z}$) then

$$R := \mathbb{K} \in \mathbf{D}^b_{\mathbb{f}}(\mathbf{Mod} \mathbb{K})$$

is a dualizing complex.

Dualizing complexes over commutative rings are part of Grothendieck’s duality theory in algebraic geometry, which was developed in [RD]. This duality theory deals with dualizing complexes on schemes and relations between them.

In this lecture I will explain a new approach to dualizing complexes over commutative rings, due to James Zhang and myself (see [YZ4] and [YZ5]). Specifically, I’ll talk about existence and uniqueness of rigid dualizing complexes.
The purpose of rigidity is to eliminate automorphisms, and to make the dualizing complexes functorial.

In a sequel paper [Ye2] we use the technique of \textit{perverse coherent sheaves} to construct rigid dualizing complexes on schemes, and we reproduce almost all of the geometric Grothendieck duality theory. But that’s a subject for a separate lecture.

Related work in noncommutative algebraic geometry (where rigid dualizing complexes were first introduced) can be found in [VdB, YZ1, YZ2, YZ3].
2 Rigid Complexes and DG Algebras

By default all rings considered in this talk are commutative.

Let me start with a discussion of rigidity for algebras over a field. Suppose $\mathbb{K}$ is a field, $B$ is a $\mathbb{K}$-algebra, and $M \in D(\text{Mod} \ B)$.

According to Van den Bergh [VdB] a rigidifying isomorphism for $M$ is an isomorphism

$$\rho : M \xrightarrow{\simeq} \text{RHom}_{B \otimes \mathbb{K}}(B, M \otimes \mathbb{K} M)$$

in $D(\text{Mod} \ B)$. 
Now suppose $A$ is any ring.

Trying to write $A$ instead of $\mathbb{K}$ in formula (1) does not make sense: instead of $M \otimes_A M$ we must take the derived tensor product $M \otimes^L_A M$; but then there is no obvious way to make $M \otimes^L_A M$ into a complex of $B \otimes_A B$-modules.

The problem is torsion: $B$ might fail to be a flat $A$-algebra.

This is where *differential graded algebras* (DG algebras) enter the picture.
A DG algebra is a graded ring $\tilde{A} = \bigoplus_{i \in \mathbb{Z}} \tilde{A}^i$, together with a graded derivation $d : \tilde{A} \to \tilde{A}$ of degree 1, satisfying $d \circ d = 0$.

A DG algebra quasi-isomorphism is a homomorphism $f : \tilde{A} \to \tilde{B}$ respecting degrees, multiplications and differentials, and such that $H(f) : H\tilde{A} \to H\tilde{B}$ is an isomorphism (of graded algebras).

We shall only consider super-commutative non-positive DG algebras. Super-commutative means that $ab = (-1)^{ij}ba$ and $c^2 = 0$ for all $a \in \tilde{A}^i$, $b \in \tilde{A}^j$ and $c \in \tilde{A}^{2i+1}$. Non-positive means that $\tilde{A} = \bigoplus_{i \leq 0} \tilde{A}^i$.

We view a ring $A$ as a DG algebra concentrated in degree 0. Given a DG algebra homomorphism $A \to \tilde{A}$ we say that $\tilde{A}$ is a DG $A$-algebra.
Let $A$ be a ring. A *semi-free* DG $A$-algebra is a DG $A$-algebra $\tilde{A}$, such that after forgetting the differential $\tilde{A}$ is isomorphic, as graded $A$-algebra, to a super-polynomial algebra on some graded set of variables.

**Definition 3.** Let $A$ be a ring and $B$ an $A$-algebra. A *semi-free DG algebra resolution of $B$ relative to $A$* is a quasi-isomorphism $\tilde{B} \to B$ of DG $A$-algebras, where $\tilde{B}$ is a semi-free DG $A$-algebra.

Such resolutions always exist, and they are unique up to quasi-isomorphism.

**Example 4.** Take $A := \mathbb{Z}$ and $B = \mathbb{Z}/(6)$. Define $\tilde{B}$ to be the super-polynomial algebra $\mathbb{Z}[\xi]$ on the variable $\xi$ of degree $-1$. So $\tilde{B} = \mathbb{Z} \oplus \mathbb{Z}\xi$ as free $\mathbb{Z}$-module, and $\xi^2 = 0$. Let $d(\xi) := 6$. Then $\tilde{B} \to \mathbb{Z}/(6)$ is a semi-free DG algebra resolution of $\mathbb{Z}/(6)$ relative to $\mathbb{Z}$. 
For a DG algebra $A$ one has the category $\text{DGMod} \hat{A}$ of DG $\hat{A}$-modules. It is analogous to the category of complexes of modules over a ring, and by a similar process of inverting quasi-isomorphisms we obtain the derived category $\tilde{\mathcal{D}}(\text{DGMod} \hat{A})$; see [Ke], [Hi].

For a ring $A$ (a DG algebra concentrated in degree 0) we have

$$\tilde{\mathcal{D}}(\text{DGMod} A) = \mathcal{D}(\text{Mod} A),$$

the usual derived category.

It is possible to derive functors of DG modules, again in analogy to $\mathcal{D}(\text{Mod} A)$.

An added feature is that for a quasi-isomorphism $\hat{A} \to \hat{B}$ the restriction of scalars functor

$$\tilde{\mathcal{D}}(\text{DGMod} \hat{B}) \to \tilde{\mathcal{D}}(\text{DGMod} \hat{A})$$

is an equivalence.
Getting back to our original problem, suppose $A$ is a ring and $B$ is an $A$-algebra. Choose a semi-free DG algebra resolution $\tilde{B} \to B$ relative to $A$. For $M \in D(\text{Mod } B)$ define

$$\text{Sq}_{B/A} M := \text{RHom}_{\tilde{B} \otimes_A \tilde{B}}(B, M \otimes_A^L M)$$

in $D(\text{Mod } B)$.

**Theorem 5.** ([YZ4]) *The functor* 

$$\text{Sq}_{B/A} : D(\text{Mod } B) \to D(\text{Mod } B)$$

*is independent of the resolution $\tilde{B} \to B$.***
The functor $\text{Sq}_{B/A}$, called the \textit{squaring operation}, is nonlinear. In fact, given a morphism $\phi : M \rightarrow M$ in $D(\text{Mod } B)$ and an element $b \in B$ one has

$$\text{Sq}_{B/A}(b\phi) = b^2 \text{Sq}_{B/A}(\phi) \quad (2)$$

in

$$\text{Hom}_{D(\text{Mod } B)}(\text{Sq}_{B/A} M, \text{Sq}_{B/A} M).$$
Definition 6. Let $B$ be a noetherian $A$-algebra, and let $M$ be a complex in $D^b_\mathfrak{f}(\text{Mod } B)$ that has finite flat dimension over $A$. Assume

$$\rho : M \xrightarrow{\sim} \text{Sq}_{B/A} M$$

is an isomorphism in $D(\text{Mod } B)$. Then the pair $(M, \rho)$ is called a rigid complex over $B$ relative to $A$.

Definition 7. Say $(M, \rho)$ and $(N, \sigma)$ are rigid complexes over $B$ relative to $A$. A morphism $\phi : M \to N$ in $D(\text{Mod } B)$ is called a rigid morphism relative to $A$ if the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\rho} & \text{Sq}_{B/A} M \\
\downarrow \phi & & \downarrow \text{Sq}_{B/A}(\phi) \\
N & \xrightarrow{\sigma} & \text{Sq}_{B/A} N
\end{array}
$$

is commutative.
We denote by $D^b_f(\text{Mod } B)_{\text{rig}/A}$ the category of rigid complexes over $B$ relative to $A$.

**Example 8.** Take $M = B := A$. Then

$$\text{Sq}_{A/A} A = \text{RHom}_{A \otimes_A A}(A, A \otimes_A A) = A,$$

and we interpret this as the tautological rigidifying isomorphism

$$\rho^{\text{tau}} : A \xrightarrow{\sim} \text{Sq}_{A/A} A.$$

The *tautological rigid complex* is

$$(A, \rho^{\text{tau}}) \in D^b_f(\text{Mod } A)_{\text{rig}/A}.$$
3 Properties of Rigid Complexes

The first property of rigid complexes explains their name.

**Theorem 9.** ([YZ4]) Let $A$ be a ring, $B$ a noetherian $A$-algebra, and

$$(M, \rho) \in D^b_f(\text{Mod } B)_{\text{rig}/A}.$$ 

Assume the canonical homomorphism

$$B \to \text{Hom}_{D(\text{Mod } B)}(M, M)$$

is bijective. Then the only automorphism of $(M, \rho)$ in

$$D^b_f(\text{Mod } B)_{\text{rig}/A}$$

is the identity $1_M$. 


The proof is very easy: an automorphism $\phi$ of $M$ has to be of the form $\phi = b \mathbf{1}_M$ for some invertible element $b \in B$. If $\phi$ is rigid then $b = b^2$ (cf. formula (2)), and hence $b = 1$.

We find it convenient to denote ring homomorphisms by $f^*$ etc. Thus a ring homomorphism $f^* : A \to B$ corresponds to the morphism of schemes

$$ f : \text{Spec} \, B \to \text{Spec} \, A. $$
Let $A$ be a noetherian ring. Recall that an $A$-algebra $B$ is called essentially finite type if it is a localization of some finitely generated $A$-algebra.

We say that $B$ is *essentially smooth* (resp. *essentially étale*) over $A$ if it is essentially finite type and formally smooth (resp. formally étale).

**Example 10.** If $A'$ is a localization of $A$ then $A \to A'$ is essentially étale. If $B = A[t_1, \ldots, t_n]$ is a polynomial algebra then $A \to B$ is smooth, and hence also essentially smooth.
Let $A$ be a noetherian ring and $f^* : A \to B$ an essentially smooth homomorphism. Then $\Omega^1_{B/A}$ is a finitely generated projective $B$-module.

Let

$$\text{Spec } B = \bigsqcup_i \text{Spec } B_i$$

be the decomposition into connected components, and for every $i$ let $n_i$ be the rank of $\Omega^1_{B_i/A}$. We define a functor

$$f^\#: D(\text{Mod } A) \to D(\text{Mod } B)$$

by

$$f^\# M := \bigoplus_i \Omega^n_{B_i/A} [n_i] \otimes_A M.$$
Recall that a ring homomorphism \( f^* : A \to B \) is called finite if \( B \) is a finitely generated \( A \)-module. Given such a finite homomorphism we define a functor

\[
f^b : \text{D}(\text{Mod } A) \to \text{D}(\text{Mod } B)
\]

by

\[
f^b M := \text{RHom}_A(B, M).
\]
Theorem 11. ([YZ4]) Let $A$ be a noetherian ring, let $B, C$ be essentially finite type $A$-algebras, let $f^* : B \to C$ be an $A$-algebra homomorphism, and let

$$(M, \rho) \in D^b_f(\text{Mod } B)_{\text{rig}/A}.$$ 

(1) If $f^*$ is finite and $f^b M$ has finite flat dimension over $A$, then $f^b M$ has an induced rigidifying isomorphism

$$f^b(\rho) : f^b M \xrightarrow{\sim} \text{Sq}_{C/A} f^b M.$$ 

(2) If $f^*$ is essentially smooth then $f^# M$ has an induced rigidifying isomorphism

$$f^#(\rho) : f^# M \xrightarrow{\sim} \text{Sq}_{C/A} f^# M.$$
4 Rigid Dualizing Complexes

Let $\mathbb{K}$ be a noetherian regular ring of finite Krull dimension. We denote by $\text{EFTAlg}/\mathbb{K}$ the category of essentially finite type $\mathbb{K}$-algebras.

**Definition 12.** A rigid dualizing complex over $A$ relative to $\mathbb{K}$ is a rigid complex $(R_A, \rho_A)$ such that $R_A$ is a dualizing complex.
Theorem 13. ([YZ5]) Let $\mathbb{K}$ be a regular finite dimensional noetherian ring, and let $A$ be an essentially finite type $\mathbb{K}$-algebra.

(1) The algebra $A$ has a rigid dualizing complex $(R_A, \rho_A)$, which is unique up to a unique rigid isomorphism.

(2) Given a finite homomorphism $f^*: A \to B$, there is a unique rigid isomorphism $f^\flat (R_A, \rho_A) \cong (R_B, \rho_B)$.

(3) Given an essentially smooth homomorphism $f^*: A \to B$, there is a unique rigid isomorphism $f^\sharp (R_A, \rho_A) \cong (R_B, \rho_B)$. 
Here is how the rigid dualizing complex $(R_A, \rho_A)$ is obtained. We begin with the tautological rigid complex

$$(\mathbb{K}, \rho^{\text{tau}}) \in D_f^b(\text{Mod} \mathbb{K})_{\text{rig}/\mathbb{K}},$$

which is dualizing. Now the structural homomorphism $\mathbb{K} \to A$ can be factored into

$$\mathbb{K} \xrightarrow{f^*} B \xrightarrow{g^*} C \xrightarrow{h^*} A,$$

where $f^*$ is smooth ($B$ is a polynomial algebra over $\mathbb{K}$); $g^*$ is finite (a surjection); and $h^*$ is also smooth (a localization). Then

$$(R_A, \rho_A) := h^\# g^b f^\# (\mathbb{K}, \rho^{\text{tau}}) \in D_f^b(\text{Mod} A)_{\text{rig}/\mathbb{K}}.$$
**Definition 14.** Given a homomorphism $f^*: A \to B$ in $EFT\text{Alg}/\mathbb{K}$, define the *twisted inverse image functor*

$$f^!: D^+_f(\text{Mod } A) \to D^+_f(\text{Mod } B)$$

by the formula

$$f^! M := R\text{Hom}_B(B \otimes^L_A R\text{Hom}_A(M, R_A), R_B).$$

It is not hard to show that the assignment $f^* \mapsto f^!$ is a 2-functor from the category $EFT\text{Alg}/\mathbb{K}$ to the 2-category $\text{Cat}$ of all categories.

One can show, using Theorem 13, that this operation has very good properties. For instance, when $f^*$ is finite, then there is a functorial, nondegenerate trace morphism

$$\text{Tr}_f : f^! M \to M.$$
5 Rigid Complexes and CM Homomorphisms

In this final section I’ll talk about the relation between rigid complexes and Cohen-Macaulay homomorphisms.

**Definition 15.** A ring $A$ is called *tractable* if there is an essentially finite type homomorphism $\mathbb{K} \to A$, for some regular noetherian ring of finite Krull dimension $\mathbb{K}$.

The homomorphism $\mathbb{K} \to A$ is *not* part of the structure – there is no preferred $\mathbb{K}$. “Most commutative noetherian rings we know” are tractable...
Theorem 16. ([YZ5], [Ye2]) Let $A$ be a tractable ring, and let $B$ be an essentially finite type $A$-algebra of finite flat dimension (e.g. $B$ is flat over $A$).

(1) There exists a unique (up to unique rigid isomorphism) rigid complex $R_{B/A}$ over $B$ relative to $A$, which is nonzero on each connected component of $	ext{Spec } B$.

(2) If $A$ is a Gorenstein ring (e.g. a regular ring) then $R_{B/A}$ is a dualizing complex over $B$. 
Let $f^*: A \to B$ be a finite type flat homomorphism of relative dimension $n$; namely the fibers of $f: \text{Spec } B \to \text{Spec } A$ are all equidimensional of dimension $n$.

Recall that $f^*$ is called a Cohen-Macaulay homomorphism if the fibers of $f$ are all $n$-dimensional Cohen-Macaulay schemes.

**Theorem 17.** ([Ye2]) Let $A$ be a tractable ring, and let $f^*: A \to B$ be a finite type flat homomorphism of relative dimension $n$. Then the following conditions are equivalent:

(i) $f^*$ is a Cohen-Macaulay homomorphism.

(ii) $\text{H}^iR_{B/A} = 0$ for all $i \neq -n$, and the $B$-module

$$\omega_{B/A} := \text{H}^{-n}R_{B/A}$$

is flat over $A$.  

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The module $\omega_{B/A}$ is the dualizing module of $B$ relative to $A$.

Note that the complex $\omega_{B/A}[n]$ is rigid relative to $A$, but in general it is not a dualizing complex (cf. part (2) of previous theorem.) Still the fibers of $\omega_{B/A}[n]$ are dualizing complexes – this can be seen by taking $A'$ to be a field in the next result.

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Let me end with a “rigid” version of Conrad’s base change theorem [Co]:
Theorem 18. ([Ye2]) Let

\[ \begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
A' & \longrightarrow & B'
\end{array} \]

be a cartesian diagram of rings, i.e.

\[ B' \cong A' \otimes_A B, \]

with A and A' tractable rings. Assume

A → B is a Cohen-Macaulay homomorphism. (There isn’t any restriction on the homomorphism A → A'.) Then:

(1) \( A' \to B' \) is a Cohen-Macaulay homomorphism.

(2) There is a unique isomorphism of \( B' \)-modules

\[ \omega_{B'/A'} \cong A' \otimes_A \omega_{B/A} \]

which respects rigidity.
References


[VdB] M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative


