Dualizing Complexes over Noncommutative Rings

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Here is the plan of my lecture:

1. Notation, and Review of Derived Categories
2. Dualizing Complexes
3. Existence of Dualizing Complexes
4. The Auslander Condition
5. Classification of Dualizing Complexes
6. Applications in Ring Theory

There will be a second talk about the geometric aspects of noncommutative duality.

Most of the work is joint with James Zhang (UW, Seattle).
1 Notation, and Review of Derived Categories

Let $A$ be a ring. We denote by $\text{Mod} \ A$ the category of left $A$-modules.

The objects of the derived category $\text{D}(\text{Mod} \ A)$ are complexes of $A$-modules

$$M = (\cdots \to M^{-1} \to M^0 \to M^1 \to \cdots).$$

Recall that a homomorphism of complexes $\phi : M \to N$ is a quasi-isomorphism if $H^i(\phi) : H^i M \to H^i N$ is an isomorphism for all $i$.

The morphisms $\psi : M \to N$ in $\text{D}(\text{Mod} \ A)$ are of the form $\psi = \phi_2^{-1} \circ \phi_1$ where $\phi_1 : M \to L$ is a homomorphism of complexes and $\phi_2 : N \to L$ is a quasi-isomorphism.
There is a full embedding

$$\operatorname{Mod} A \hookrightarrow \mathcal{D}(\operatorname{Mod} A)$$

which is gotten by viewing a module $M$ as a complex concentrated in degree 0.

Of utmost importance for us is the derived functor $\operatorname{RHom}$. Given complexes $M, N \in \mathcal{D}(\operatorname{Mod} A)$ there is a complex

$$\operatorname{RHom}_A(M, N) \in \mathcal{D}(\operatorname{Mod} \mathbb{Z})$$

depending functorially on $M$ and $N$.

If $N$ happens to be an $A$-bimodule then

$$\operatorname{RHom}_A(M, N) \in \mathcal{D}(\operatorname{Mod} A^{\text{op}}),$$

where $A^{\text{op}}$ is the opposite ring.
There’s a functorial isomorphism

$$H^i \mathbf{RHom}_A(M, N) \cong \text{Hom}_{D(\text{Mod} A)}(M, N[i])$$

where $N[i]$ is the shifted complex.

If $M, N \in \text{Mod} A$ then we recover the familiar Exts:

$$H^i \mathbf{RHom}_A(M, N) = \text{Ext}^i_A(M, N).$$
2 Dualizing Complexes

Dualizing complexes on (commutative) schemes were introduced by Grothendieck in the 1960’s, in the book [RD].

Let us recall the definition of a dualizing complex over a commutative noetherian ring $A$. It is a complex $R \in D^b_f(\text{Mod } A)$ such that the contravariant functor

$$R\text{Hom}_A(-, R) : D^b_f(\text{Mod } A) \to D^b_f(\text{Mod } A)$$

is a duality (i.e. a contravariant equivalence). (I am omitting some details.)

Here $D^b_f(\text{Mod } A)$ is the derived category of bounded complexes with finitely generated cohomology modules.
Example 2.1. Let $\mathbb{K}$ be a field. Then the complex $R := \mathbb{K}$ is a dualizing complex over $\mathbb{K}$. The duality $\text{RHom}_\mathbb{K}(-, \mathbb{K})$ extends the usual duality of linear algebra.

So far for the classical commutative picture. From now on $\mathbb{K}$ will be a field, and $A$ will be a noetherian, unital, associative $\mathbb{K}$-algebra (not necessarily commutative).
We shall write \( A^e := A \otimes_K A^{\text{op}} \), where \( A^{\text{op}} \) is the opposite ring. So \( \text{Mod} \ A^e \) is the category of \( A \)-bimodules.

**Definition 2.2.** ([Ye1]) A complex \( R \in D^b(\text{Mod} \ A^e) \) is called *dualizing* if the functor

\[
\text{RHom}_A(-, R) : D^b_f(\text{Mod} \ A) \rightarrow D^b_f(\text{Mod} \ A^{\text{op}})
\]

is a duality, with adjoint \( \text{RHom}_{A^{\text{op}}}(−, R) \).

(Again I’m suppressing some details.)

**Example 2.3.** The complex \( R := A \) is a dualizing complex over \( A \) iff \( A \) is a Gorenstein ring (i.e. \( A \) has finite injective dimension as left and right module over itself).
There is a graded version of dualizing complex. Suppose $A$ is a connected graded algebra, namely $A = \bigoplus_{i \geq 0} A_i$, with $A_0 = \mathbb{K}$ and each $A_i$ a finitely generated $\mathbb{K}$-module.

Consider the category $\text{GrMod} A$ of graded left $A$-modules. Similarly to Definition 2.2 we may define a graded dualizing complex $R \in D^b(\text{GrMod} A^e)$.

The augmentation ideal of $A$ is denoted by $m$, and the left (resp. right) $m$-torsion functor is denoted by $\Gamma_m$ (resp. $\Gamma_{m^{\text{op}}}$).

We let $A^* := \text{Hom}_{\mathbb{K}}^{\text{gr}}(A, \mathbb{K})$, the graded dual of $A$. 
**Definition 2.4.** ([Ye1]) Let $A$ be a connected graded $\mathbb{K}$-algebra. A graded dualizing complex $R$ is called *balanced* if

$$R \Gamma_m R \cong R \Gamma_{m^{\text{op}}} R \cong A^*$$

in $D^b(\text{GrMod } A^e)$. It is known that a balanced dualizing complex is unique up to isomorphism.
Again $A$ is any noetherian $\mathbb{K}$-algebra (not graded). Van den Bergh discovered the following condition on a dualizing complex $R$ that turns out to be extremely powerful.

**Definition 2.5. ([VdB])** Let $R$ be a dualizing complex over $A$. Suppose there is an isomorphism

$$\rho : R \xrightarrow{\sim} \text{RHom}_{A^e}(A, R \otimes_{\mathbb{K}} R)$$

in $D(\text{Mod } A^e)$. Then $R$ is called a **rigid dualizing complex** and $\rho$ is a **rigidifying isomorphism**.

**Theorem 2.6.** ([VdB], [YZ1]) A rigid dualizing complex $(R, \rho)$ is unique up to a unique isomorphism in $D(\text{Mod } A^e)$. 

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Example 2.7. If $A$ is a commutative finitely generated $\mathbb{K}$-algebra, $X := \text{Spec } A$ and $\pi : X \to \text{Spec } \mathbb{K}$ is the structural morphism, then the dualizing complex $R := R\Gamma(X, \pi^! \mathbb{K})$ from [RD] is rigid.

Example 2.8. If $A$ is finite over $\mathbb{K}$ then the bimodule $A^* := \text{Hom}_A(A, \mathbb{K})$ is a rigid dualizing complex over $A$. 

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The question of existence of rigid dualizing complexes is quite hard. The best existence criterion we know is due to Van den Bergh.

**Theorem 3.1.** ([VdB]) Suppose $A$ admits a nonnegative exhaustive filtration $F = \{F_i A\}_{i \in \mathbb{Z}}$ such that the graded algebra $\bar{A} := \text{gr}^F A$ is a connected graded, commutative, finitely generated $K$-algebra. Then $A$ has a rigid dualizing complex.
Here is an outline of the proof. Let

$$\tilde{A} := \bigoplus_i (F_i A)t^i \subset A[t]$$

be the Rees algebra, where $t$ is a central indeterminate of degree 1. So $\bar{A} \cong \tilde{A}/(t)$ and $A \cong \tilde{A}/(t - 1)$.

Since $\bar{A}$ is commutative it follows that $\tilde{A}$ satisfies the $\chi$ condition of [AZ]. This implies that the local duality functor $\tilde{M} \mapsto (R\Gamma_{\tilde{m}} \tilde{M})^*$ is represented by a balanced dualizing complex $\tilde{R}$ over $\tilde{A}$. Then

$$R_A := A \otimes_{\tilde{A}} \tilde{R}[-1] \otimes_{\tilde{A}} A$$

is a rigid dualizing complex over $A$.

One should think of the filtration $F$ as a “compactification of Spec $A$”. Indeed if $A$ is commutative then Proj $\tilde{A}$ is a projective $\mathbb{K}$-scheme, $\{t = 0\}$ is an ample divisor, and its complement is isomorphic to Spec $A$. 

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In practice often an algebra $A$ comes equipped with a filtration $G$ that satisfies the conditions of the next definition, but is not connected (i.e. $\operatorname{gr}^G A$ is not a connected graded $\mathbb{K}$-algebra).

**Definition 3.2.** A nonnegative exhaustive filtration $G = \{G_i A\}_{i \in \mathbb{Z}}$ such that $\operatorname{gr}^G A$ is finite over its center $Z(\operatorname{gr}^G A)$, and $Z(\operatorname{gr}^G A)$ is a finitely generated $\mathbb{K}$-algebra, is called a *differential filtration of finite type.*

If $A$ admits such a filtration then it is called a *differential $\mathbb{K}$-algebra of finite type.*
We call the next result the “Theorem on the Two Filtrations”. A slightly weaker result appeared in [MS].

**Theorem 3.3.** ([YZ5]) Assume the ring $A$ has a differential filtration of finite type $G$. Then there exists a differential filtration of finite type $F$ on $A$ such that the graded algebra $\text{gr}^F A$ is connected and commutative.
The prototypical example is:

**Example 3.4.** Let \( \text{char } \mathbb{K} = 0 \). Consider the first Weyl algebra

\[
A := \mathbb{K} \langle x, y \rangle / (yx - xy - 1).
\]

It is of course isomorphic to the ring of differential operators \( \mathcal{D}(\mathbb{A}^1) \) on the affine line \( \mathbb{A}^1 = \text{Spec } \mathbb{K}[x] \), via \( y \mapsto \frac{\partial}{\partial x} \).

The first filtration of \( A \) is the filtration \( G \) by order of operator, namely \( \deg^G(x) = 0 \) and \( \deg^G(y) = 1 \). The filtration \( G \) has the benefit of localizing to a filtration of the sheaf of differential operators \( \mathcal{D}_{\mathbb{A}^1} \). However \( \text{gr}_0^G A = \mathbb{K}[\bar{x}] \), so \( \text{gr}^G A \) is not connected.

The second filtration of \( A \) is the filtration \( F \) in which \( \deg^F(x) = \deg^F(y) = 1 \). Here \( \text{gr}^F A \) is a polynomial algebra in the variables \( \bar{x}, \bar{y} \), both of degree 1, so it is connected.
More examples of differential $\mathbb{K}$-algebras of finite type are:

**Example 3.5.** The ring $\mathcal{D}(X)$ of differential operators on a smooth affine variety $X$ in characteristic 0. The rigid dualizing complex is $\mathcal{D}(X)[2n]$ where $n := \dim X$.

**Example 3.6.** The universal enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$. The rigid dualizing complex is $U(\mathfrak{g}) \otimes (\wedge^n \mathfrak{g})[n]$ where $n := \dim \mathfrak{g}$.

**Example 3.7.** Generalizing the previous two examples, the universal enveloping algebroid $U_C(L)$, where $C$ is a f.g. commutative $\mathbb{K}$-algebra and $L$ is a f.g. Lie algebroid over $C$. 
Example 3.8. Any quotient ring $A/I$ or any matrix ring $M_n(A)$ of a differential $\mathbb{K}$-algebra of finite type $A$.

By combining Van den Bergh’s existence result with the Theorem on the Two Filtrations, and some more work, we get:
**Theorem 3.9.** ([YZ5]) *Let* $A$ *be a differential* $\mathbb{K}$-*algebra of finite type.*

1. $A$ *has a rigid dualizing complex* $R_A$, which is unique up to a unique rigid isomorphism.
2. Suppose $A'$ is a localization of $A$ such that each bimodule $H^i R_A$ is evenly localizable to $A'$. Then $A'$ has a rigid dualizing complex $R_{A'}$, and there is a unique rigid localization morphism
   $$ q_{A/A'} : R_A \to R_{A'}.$$
3. Suppose $A \to B$ is a finite centralizing homomorphism. Then $B$ has a rigid dualizing complex $R_B$, and there is a unique rigid trace morphism
   $$ \text{Tr}_{B/A} : R_B \to R_A.$$
“Evenly localizable” is a variant of the Ore condition. Part (2) basically says that

\[ R_{A'} \cong A' \otimes_A R_A \otimes_A A' \]

in \( D(\text{Mod} \ A'^e) \).

And part (3) says that

\[ R_B \cong R\text{Hom}_A(B, R_A) \cong R\text{Hom}_{A^{\text{op}}}(B, R_A) \]

\( D(\text{Mod} \ A^e) \).
Remark 3.10. I wish to amplify the significance of part (3) of the theorem. Suppose $B = A/I$ and $M \in D(\text{Mod } A^e)$. Then $\text{Ext}^i_A(B, M)$ is a $B \otimes_K A^{\text{op}}$-module, but usually it is not a $B \otimes_K B^{\text{op}}$-module, i.e.

$$\text{Ext}^i_A(B, M) \cdot I \neq 0.$$ 

The existence of the rigid trace implies, among other things, that $\text{Ext}^i_A(B, R_A)$ is indeed a $B \otimes_K B^{\text{op}}$-module.

Applications of this theorem to ring theory will be discussed in Section 6. The geometric significance of part (2) will be explained in the second lecture.
4 The Auslander Condition

We continue with the hypothesis that $A$ is a noetherian algebra over a field $\mathbb{K}$.

**Definition 4.1.** ([Ye2], [YZ1]) Let $R$ be a dualizing complex over $A$. We say $R$ is Auslander if the two conditions below hold.

(i) For any finitely generated $A$-module $M$, any integers $p > q$ and any $A^{\text{op}}$-submodule $N \subset \text{Ext}^p_A(M, R)$ one has $\text{Ext}^q_{A^{\text{op}}}(M, R) = 0$.

(ii) The same after exchanging $A$ and $A^{\text{op}}$. 
This is a generalization of the classical notion of *Auslander-Gorenstein* ring. Indeed, a $\mathbb{K}$-algebra $A$ is called Auslander-Gorenstein precisely if it is Gorenstein, and the dualizing complex $R := A$ is Auslander in the sense of the definition above.

Auslander-Gorenstein rings were studied by Gabber, Levasseur and Björk, especially in the context of $\mathcal{D}$-modules.

However the Gorenstein condition is very restrictive (recall that unlike the commutative situation, a noncommutative noetherian ring $A$ is seldom a quotient of a “nice” noetherian ring).
On the other hand, Auslander dualizing complexes are relatively easy to find:

**Theorem 4.2.** ([YZ5]) *Suppose* \( A \) *is a differential* \( K \)-*algebra of finite type. Then its rigid dualizing complex* \( R_A \) *is Auslander.*

Applications of the theorem to ring theory will be discussed in Section 6. The geometric significance (the relation with perverse t-structures) will be explained in the second lecture.
5 Classification of Dualizing Complexes

In the commutative case the dualizing complexes are classified by the Picard group. Namely, given two dualizing complexes $R, R'$ over a commutative noetherian ring $A$, one has

$$R' \cong L[n] \otimes_A R$$

for some invertible $A$-module $L$ and some integer $n$. See [RD].

The noncommutative picture is much more complicated. Again let $A$ be a noetherian algebra over a field $\mathbb{K}$. A two-sided tilting complex over $A$ is a complex $P \in D^b(\text{Mod } A^e)$ such there exists some $Q \in D^b(\text{Mod } A^e)$ and isomorphisms

$$P \otimes_A^L Q \cong Q \otimes_A^L P \cong A$$

in $D(\text{Mod } A^e)$. 
Definition 5.1. ([Ye3]) The derived Picard group of $A$ is the group

\[
\text{DPic}(A) := \{\text{two-sided tilting complexes over } A\}\text{ isomorphism}.
\]

The derived Picard group classifies dualizing complexes in the following sense:

Theorem 5.2. ([Ye3]) Assume $A$ has at least one dualizing complex. Then the action of \(\text{DPic}(A)\) on the set

\[
\{\text{dualizing complexes over } A\}\text{ isomorphism},
\]

given by \((P, R) \mapsto P \otimes_A^L R\), is transitive with trivial stabilizers.
The group $\text{DPic}(A)$ always contains the subgroup $\text{Pic}(A) \times \mathbb{Z}$, where $\text{Pic}(A)$ is the noncommutative Picard group of $A$ (consisting of invertible bimodules), and $\mathbb{Z}$ is generated by the shift $\sigma$.

However when $A$ is neither commutative nor local, often $\text{DPic}(A)$ is bigger than $\text{Pic}(A) \times \mathbb{Z}$. 
Example 5.3. Let $A := \left[ \begin{array}{cc} K & K \\ 0 & K \end{array} \right]$, the algebra of upper triangular $2 \times 2$ matrices over $K$. The rigid dualizing complex $R_A = A^*$ turns out to be a two-sided tilting complex.

In fact the functor $R_A \otimes^L_A -$ is the Serre functor of $\mathcal{D}^b_f(\text{Mod } A)$, in the sense of [BK].

Here the group $\text{Pic}(A)$ is trivial, and $\text{DPic}(A) \cong \mathbb{Z}$, generated by the class $\nu$ of $R_A$. The shift satisfies $\sigma = \nu^3$. Thus

$$\text{Pic}(A) \times \mathbb{Z} \subsetneq \text{DPic}(A).$$

The relation $\sigma = \nu^3$ says that $A$ has “Calabi-Yau dimension $\frac{1}{3}$”, in the terminology of Kontsevich. See [MY] for details.
6 Applications in Ring Theory

Here are a few applications of the theory of dualizing complexes.

6.1 Left vs. Right Gorenstein

In [Jo1] Jørgensen used balanced dualizing complexes to prove that a connected graded algebra $A$ is left Gorenstein iff it is right Gorenstein.
6.2 Free Resolutions

Jørgensen [Jo2] used balanced dualizing complexes (implicitly) to establish a noncommutative version of Castelnuovo-Mumford regularity.

In [Jo3] he proceeded to show that if $A$ is a Koszul connected graded algebra with balanced dualizing complex, then any finitely generated $A$-module $M$, possibly after truncating low degrees, will admit a linear free resolution.
6.3 Duals of Verma Modules

Consider the universal enveloping algebra $A := \text{U}(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$. In [Ye4] we described the structure of the rigid dualizing complex of $A$ (this had been conjectured by Van den Bergh).

As a consequence, and using the functoriality of rigid dualizing complexes (the rigid trace) we extended results of Duflo, Brown and Levasseur [BL] regarding the Ext duals of Verma modules.
6.4 Multiplicities of Injectives

In [YZ4] we obtained several results regarding multiplicities of indecomposable injectives in the minimal injective resolution of a ring $A$.

These results extend work of previous authors (see Barou and Malliavin [BM], Brown and Levasseur [BL]).

Of particular interest is the case $A = U(\mathfrak{g})$, the universal enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$. Earlier papers on this topic tended to rely on localization; and this restricted their scope to solvable Lie algebras.

Since Auslander rigid dualizing complexes were used in [YZ4], we were able to obtain similar results for any Lie algebra (solvable or not).
6.5 Homological Transcendence Degree

In the paper [YZ6] we introduced a new notion of transcendence degree for division rings, called the homological transcendence degree, and denoted by $\text{Htr } D$.

This invariant seems to be better-behaved than other noncommutative invariants meant to generalize the commutative transcendence degree. For instance, if $D$ is the total ring of fractions of an Artin-Schelter regular algebra $A$ of global dimension $n$, then $\text{Htr } D = n$.

This, and some other good properties of the homological transcendence degree, were established with the aid of Auslander rigid dualizing complexes.
6.6 Catenarity

Recall that a noetherian ring $A$ is called catenary if given two prime ideals $p \subset q$, any saturated chain of prime ideals

$$p = p_0 \subset p_1 \subset \cdots \subset q$$

has the same length.

It is known that if $A$ is commutative and admits some dualizing complex then it is catenary (see [RD]).

In [YZ1] we proved that some rings of quantum type are catenary. This was extended by Goodearl-Zhang [GZ] to the case of the quantized coordinate rings $\mathcal{O}_q(G)$.
References


[VdB] M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative graded


