Differential Graded Rings and Derived Categories of Bimodules

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1. Commutative vs. Noncommutative Ring Theory

In the theory of commutative rings, one of the important tools is localization at prime ideals.

Example 1.1. Let A be noetherian commutative ring.

Recall that A is called *regular* if all its local rings $A_{\mathfrak{p}}$ are regular local rings.

Namely

$$\dim A_{\mathfrak{p}} = \operatorname{rank}_{A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}}(\mathfrak{p}_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^2).$$

Now let us look at a noncommutative noetherian ring A.

It is very rare that A can be localized at a prime ideal $\mathfrak{p}$. The criterion for localizeabilty is the Ore condition, and it most often fails.

Furthermore, there is no good notion of "geometry" associated to A. There do exist some definitions of "spectrum of A", but they are all ad-hoc, and are only useful in special situations.

Fortunately, *homological methods* can sometimes replace geometry in noncommutative ring theory.

It is known that the local definition of regularity in the commutative case (Example 1.1) has an equivalent global homological definition.

The homological definition of regularity makes sense when A is noncommutative, with a small modification: we have to take care of left modules and right modules.

Let's indroduce the *opposite ring* A^{op}, which is the same abelian group as A, but with reversed multiplication.

The category of left A-modules is $\mathsf{Mod}\, A$, and the category of right A-modules is $\mathsf{Mod}\, A^{\mathrm{op}}$.

We say that A is *regular* if there is a number $n \in \mathbb{N}$, such that

$$\operatorname{Ext}^i_A(M, N) = 0 \quad \text{and} \quad \operatorname{Ext}^i_{A^{\mathrm{op}}}(M', N') = 0$$

for all $i > n$, all $M, N \in \mathsf{Mod}\, A$, and all $M', N' \in \mathsf{Mod}\, A^{\mathrm{op}}$.

The smallest such n is called the *global cohomological dimension of A*.

Some of the most important work on noncommutative rings since 1985 was by the Artin school of noncommutative algebraic geometry, with the participation of Schelter, Tate, Van den Bergh, Zhang, Stafford and others.

They considered *regular graded rings over a field* $\mathbb{K}$.

The main achievement was the classification of regular graded rings of dimension 3.

This classification used the *noncommutative projective scheme* $\operatorname{Proj} A$ associated to a graded ring A, that was introduced in [AZ].

This noncommutative projective geometry is a way to translate homological properties of the graded ring A to "geometric" properties of the "scheme" $\operatorname{Proj} A$.

See the survey paper [SV] for an exposition.

2. Derived Categories of Bimodules: over a Base Field

Derived categories greatly increase the scope of the homological approach to noncommutative ring theory.

Until Section 4, we work in this setting: $\mathbb{K}$ is a base field, and A is a noetherian central $\mathbb{K}$-ring.

The reason we need $\mathbb{K}$ to be a field is technical: it insures the existence of various kinds of resolutions. I will mention some instances as I go along.

Though "technical", this condition is crucial. Removing it is the motivation of our current research project!

Recall that the category of left A-modules is $\mathsf{Mod}\, A$. The category of complexes is $\mathsf{C}(\mathsf{Mod}\, A)$.

The *derived category* $\mathsf{D}(\mathsf{Mod}\, A)$ has the same objects as $\mathsf{C}(\mathsf{Mod}\, A)$.

There is a localization functor

$$\mathrm{Q} : \mathsf{C}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A),$$

which is the identity on objects, and it inverts quasi-isomorphisms.

Thus any morphism

$$\phi : M \to N$$

in $\mathsf{D}(\mathsf{Mod}\, A)$ can be written (not uniquely!) as a fraction

$$\phi = \mathrm{Q}(\phi_0) \cdot \mathrm{Q}(\phi_1)^{-1},$$

where ϕ_i are homomorphisms of complexes, and ϕ_1 is a quasi-isomorphism.

I already mentioned the opposite ring A^{op}.

The *enveloping ring* of A is

$$A^{\mathrm{en}} := A \otimes_{\mathbb{K}} A^{\mathrm{op}}.$$

Left modules over A^{en} are the same as A-bimodules.

If M and N are A-bimodules, then $\operatorname{Hom}_A(M, N)$ is an A-bimodule too.

Thus we obtain a functor

$$\operatorname{Hom}_A(-,-) : (\mathsf{Mod}\, A^{\mathrm{en}})^{\mathrm{op}} \times \mathsf{Mod}\, A^{\mathrm{en}} \to \mathsf{Mod}\, A^{\mathrm{en}}.$$

Notice that "op" also designates the opposite category, taking care of contravariance in the first argument.

We are interested in the *right derived functor*

$$\mathrm{RHom}_A(-,-) : \mathsf{D}(\mathsf{Mod}\,A^{\mathrm{en}})^{\mathrm{op}} \times \mathsf{D}(\mathsf{Mod}\,A^{\mathrm{en}}) \to \mathsf{D}(\mathsf{Mod}\,A^{\mathrm{en}}).$$

It is constructed using *K-injective resolutions*. These are a generalization of the injective resolutions in the classical sense.

Given complexes $M, N \in \mathsf{D}(\mathsf{Mod}\,A^{\mathrm{en}})$, we choose a K-injective resolution $N \to I$ over A^{en}. Such a resolution exists even if N is unbounded.

Now we use the fact that $\mathbb{K}$ is a field: the ring homomorphism $\mathbb{K} \to A^{\mathrm{op}}$ is flat; therefore $A \to A^{\mathrm{en}}$ is flat; and therefore I is also K-injective over A.

It follows that the complex

$$\mathrm{RHom}_A(M,N) := \mathrm{Hom}_A(M,I) \in \mathsf{D}(\mathsf{Mod}\,A^{\mathrm{en}})$$

is well-defined, up to a canonical isomorphism.

By changing the roles of A and A^{op} we get another functor

$$\mathrm{Hom}_{A^{\mathrm{op}}}(-,-) : (\mathsf{Mod}\,A^{\mathrm{en}})^{\mathrm{op}} \times \mathsf{Mod}\,A^{\mathrm{en}} \to \mathsf{Mod}\,A^{\mathrm{en}}.$$

This Hom functor can be right derived too, in the same way as above.

There are also tensor functors $M \otimes_A N$ and $N \otimes_A M$ for bimodules.

These have *left derived functors* $M \otimes^{\mathrm{L}}_A N$ and $N \otimes^{\mathrm{L}}_A M$, that are constructed using *K-flat resolutions*. These are a generalization of the flat resolutions in the classical sense.

We shall refer to these right derived Hom functors, and these left derived tensor functors, as the *package of standard derived functors* associated to A.

3. Dualizing Complexes

The following definition is taken from [Ye1]. It is a variation of the commutative definition by Grothendieck in [RD]. Recall that $\mathbb{K}$ is a field, A is a noetherian central $\mathbb{K}$-ring, and $A^{\mathrm{en}} = A \otimes_{\mathbb{K}} A^{\mathrm{op}}$.

Definition 3.1. A complex $R \in \mathsf{D}(\mathsf{Mod}\,A^{\mathrm{en}})$ is called a *dualizing complex* if these three conditions hold:

(i) The cohomology bimodules $\mathrm{H}^i(R)$ are finite over A and over A^{op}.
(ii) R has finite injective dimension over A and over A^{op}.
(iii) The canonical morphisms

$$A \to \mathrm{RHom}_A(R,R) \quad \text{and} \quad A \to \mathrm{RHom}_{A^{\mathrm{op}}}(R,R)$$

in $\mathsf{D}(\mathsf{Mod}\,A^{\mathrm{en}})$ are isomorphisms.

Let us denote by $\mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\,A)$ and $\mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\,A^{\mathrm{op}})$ the categories of complexes with finite cohomology modules.

The next result, which was proved by Grothendieck in [RD] for commutative rings, explains the name "dualizing".

Theorem 3.2. ([Ye1]) *Let R be a dualizing complex over A. The functor*

$$\mathrm{RHom}_A(-,R) : \mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\,A)^{\mathrm{op}} \to \mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\,A^{\mathrm{op}})$$

is an equivalence, with quasi-inverse $\mathrm{RHom}_{A^{\mathrm{op}}}(-,R)$.

There are usually many nonisomorphic dualizing complexes over A. However, some of them are "better" than others.

Here is a definition from [VdB1].

Definition 3.3. Let R be a dualizing complex over A.

1. A *rigidifying isomorphism* for R is an isomorphism
$$\rho : R \xrightarrow{\simeq} \mathrm{RHom}_{A^{\mathrm{en}}}(A, R \otimes_{\mathbb{K}} R)$$
in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{en}})$.
2. The pair (R, ρ) is called a *rigid dualizing complex* over A relative to $\mathbb{K}$.

Theorem 3.4. ([VdB1], [YZ2]) *If A has a rigid dualizing complex (R, ρ), then it is unique, up to a unique isomorphism.*

By a filtration F of A we mean an ascending exhaustive filtration $\{F_j(A)\}_{j \geq 0}$, that respects multiplication.

Such a filtration gives rise to a graded central $\mathbb{K}$-ring
$$\mathrm{gr}^F(A) = \bigoplus_{j \geq 0} \mathrm{gr}^F_j(A).$$

Let us call the filtration F a *Bernstein filtration* if $\mathrm{gr}^F(A)$ is commutative, connected (i.e. $\mathrm{gr}^F_0(A) = \mathbb{K}$), and finitely generated as $\mathbb{K}$-ring.

Theorem 3.5. (Van den Bergh Existence Theorem, [VdB1])

If A admits a Bernstein filtration, then it has a rigid dualizing complex.

This existence theorem is extremely powerful.

The idea of its proof also yields the next functoriality result.

Theorem 3.6. ([YZ2]) *Let $A \to B$ be a surjective ring homomorphism. Assume A admits a Bernstein filtration.*

1. The rigid dualizing complexes R_A and R_B exist.
2. There is a unique rigid trace morphism
$$\mathrm{Tr}_{B/A} : R_B \to R_A$$
in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{en}})$.

Often the ring A comes equipped with a filtration that is not as nice. This can be improved:

Theorem 3.7. ([MS], [YZ3]) *Assume A admits a filtration G, such that $\mathrm{gr}^G(A)$ is a finite module over its center $\mathrm{Z}(\mathrm{gr}^G(A))$, and $\mathrm{Z}(\mathrm{gr}^G(A))$ is finitely generated as $\mathbb{K}$-ring.*

Then A admits a Bernstein filtration F.

4. Examples and Applications of Dualizing Complexes

Rings of Differential Operators. Here we assume $\mathbb{K}$ has characteristic 0.

The n-th Weyl algebra A is the ring of differential operators of the polynomial ring $C := \mathbb{K}[t_1, \ldots, t_n]$.

It comes with two filtrations:

- The order filtration G, where $\mathrm{gr}^G_0(A) = C$.
- The Bernstein filtration F, where $\mathrm{gr}^F_0(A) = \mathbb{K}$.

When C is any smooth $\mathbb{K}$-ring (namely $\mathrm{Spec}\, A$ is smooth affine variety), the ring of differential operators $A := \mathcal{D}_C$ only comes with the order filtration G. Still, by Theorem 3.7, A has (noncanonical) Bernstein filtrations.

By the Van den Bergh Existence Theorem, this implies that $A = \mathcal{D}_C$ has a rigid dualizing complex R_A.

It turns out that $R_A \cong A[n]$, where $n = \dim C$. Namely R_A is the trivial bimodule A, sitting in cohomological degree $-n$.

In modern terminology (post 2005), this says that A is a *Calabi-Yau ring* of dimension n.

Lie Algebras. Suppose $\mathfrak{g}$ is a Lie algebra over $\mathbb{K}$, of finite rank n.

The universal enveloping algebra $A := \mathrm{U}(\mathfrak{g})$ is equipped with a canonical Bernstein filtration F, and $\mathrm{gr}^F(A)$ is a commutative polynomial ring in n variables.

Van den Bergh Existence Theorem tells us that A has a rigid dualizing complex R_A.

Theorem 4.1. ([VdB2], [Ye3])

The rigid dualizing complex R_A of $A := \mathrm{U}(\mathfrak{g})$ is isomorphic to

$$\big(A \otimes_{\mathbb{K}} {\textstyle\bigwedge}^n(\mathfrak{g})\big)[n].$$

Explanation: $A \otimes_{\mathbb{K}} \bigwedge^n(\mathfrak{g})$ is the bimodule A twisted by the character $\bigwedge^n(\mathfrak{g})$; and R_A is this bimodule sitting in cohomological degree $-n$.

The theorem says that A is a *twisted Calabi-Yau ring* of dimension n.

Note that the character $\bigwedge^n(\mathfrak{g})$ is trivial when $\mathfrak{g}$ is either semisimple or nilpotent; but it is nontrivial for many solvable Lie algebras.

Van den Bergh [VdB2] proved the theorem in the semisimple case. He used it to deduce what is now called *Van den Bergh Duality* for Hochschild (co)homology.

The general case was done in [Ye3], and the proof relies on the rigid trace (Theorem 3.6).

We applied it to the Borel subalgebra of a semisimple Lie algebra, to deduce facts on Verma modules.

More Applications.

- Derived Morita theory, and the derived Picard group $\mathrm{DPic}_{\mathbb{K}}(A)$, which is an important noncommutative invariant. See [Ri], [Ye2], [RZ], [MY].
- Structure of a Hopf algebra A, noetherian but not finite over $\mathbb{K}$. Here $\mathbb{K}$-linear duality is replaced by the duality $\mathrm{RHom}_A(-, R_A)$ and the rigid trace $\mathrm{Tr}_{A/\mathbb{K}} : \mathbb{K} \to R_A$. See [LWZ], [BZ], [RRZ] and their references.
- Noncommutative homological identities, see [YZ2] and [JZ].

5. The Arithmetic Setup

Now we drop the assumption that $\mathbb{K}$ is a field. It is still a commutative ring, and A is a noetherian central $\mathbb{K}$-ring.

We refer to this as the *arithmetic setup*.

As we shall see, the *difficulty lies in the fact that A might fail to be flat over* $\mathbb{K}$.

The next two examples should convince us that rigid dualizing complexes in the arithmetic setup are interesting.

Example 5.1. In (commutative) algebraic geometry, rigid dualizing complexes allow us to expand the range of Grothendieck duality.

For the first time, we can talk about global Grothendieck duality for DM stacks and proper maps between them.

Since it is important to include arithmetic schemes and stacks in the framework, we do not want to assume that our geometric objects are defined over a base field.

Example 5.2. Here $\mathbb{K} := \widehat{\mathbb{Z}}_p$, the p-adic integers.

Let G be a compact p-adic analytic group, and let A be the Iwasawa algebra

$$A := \varprojlim \widehat{\mathbb{Z}}_p[G/N],$$

where N ranges over the open normal subgroups of G. See [AB] and [AW].

Even though A itself is flat over $\mathbb{K}$, if we look at a 2-sided ideal I of A, the quotient ring $B := A/I$ need not be flat over $\mathbb{K}$.

We would like to have a theory of rigid dualizing complexes over A and its quotient rings, with *rigid traces between them*.

This should give us a better understanding of the structure and representations of A.

Now to the theory. Recall that the enveloping ring of A is $A^{\mathrm{en}} = A \otimes_{\mathbb{K}} A^{\mathrm{op}}$.

Trying to mimic Definition 3.1 naively, a dualizing complex over A should be a complex

$$R \in \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{en}}).$$

However, when we try to construct the complexes

$$\mathrm{RHom}_A(R,R),\ \mathrm{RHom}_{A^{\mathrm{op}}}(R,R) \ \in \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{en}})$$

that appear in Definition 3.1, we run into a technical problem.

We need a resolution $R \to I$, where I is a complex of A^{en}-modules that is K-injective over A and over A^{op}.

But when A is not flat over $\mathbb{K}$, we do not know how to find such resolutions!

When the ring A is commutative, the problem of K-injective resolutions that was discussed in the previous slide does not arise.

This is because in the commutative theory of dualizing complexes, we only work with central A-bimodules, so all the action takes place in $\mathsf{D}(\mathsf{Mod}\, A)$.

However, when we try to define rigid dualizing complexes, the bimodule resolution problem resurfaces, even when the ring A is commutative.

(I won't go into details.)

Can we overcome this difficulty?

Here is where DG rings enter the picture.

6. DG Rings

Definition 6.1. A (nonpositive) *differential graded ring* is a graded ring

$$\tilde{A} = \bigoplus_{i \leq 0} \tilde{A}^i,$$

together with a differential d of degree 1, that satisfies the graded Leibniz rule

$$\text{(6.2)} \qquad \mathrm{d}(a \cdot b) = \mathrm{d}(a) \cdot b + (-1)^i \cdot a \cdot \mathrm{d}(b)$$

for $a \in \tilde{A}^i$ and $b \in \tilde{A}^j$.

A homomorphism of DG rings $u : \tilde{A} \to \tilde{B}$ is a graded ring homomorphism that respects the differentials.

In cohomology, we get a graded ring homomorphism

$$\mathrm{H}(u) : \mathrm{H}(\tilde{A}) \to \mathrm{H}(\tilde{B}).$$

Definition 6.3. A left *DG A-module* is left graded A-module

$$\tilde{M} = \bigoplus_{i \in \mathbb{Z}} \tilde{M}^i,$$

together with a differential d of degree 1, that satisfies the graded Leibniz rule, like (6.2).

The DG $\tilde{A}$-modules form a category, denoted by $\mathsf{DGMod}\, \tilde{A}$.

Any ring A can be viewed as a DG ring concentrated in degree 0.

As in the case of $\mathsf{C}(\mathsf{Mod}\, A)$, we can invert quasi-isomorphisms in $\mathsf{DGMod}\, \tilde{A}$, and thus we obtain the derived category

$$\mathsf{D}(\tilde{A}) := \mathsf{D}(\mathsf{DGMod}\, \tilde{A}).$$

There are enough K-injective and K-flat resolutions in $\mathsf{DGMod}\, \tilde{A}$, and therefore the derived functors $\mathrm{RHom}_{\tilde{A}}(\tilde{M}, \tilde{N})$ and $\tilde{M} \otimes^{\mathrm{L}}_{\tilde{A}} \tilde{N}$ exist.

If $\tilde{A} \to \tilde{B}$ is a DG ring quasi-isomorphism, then the restriction functor

$$\mathsf{D}(\tilde{B}) \to \mathsf{D}(\tilde{A})$$

is an equivalence.

Definition 6.4. Let A be a central $\mathbb{K}$-ring.

A *K-flat DG ring resolution of A* (relative to $\mathbb{K}$) consists of a DG ring $\tilde{A}$,

together with a factorization $\mathbb{K} \to \tilde{A} \to A$ of the structural homomorphism $\mathbb{K} \to A$, such that:

- The DG ring homomorphism $\mathbb{K} \to \tilde{A}$ is central, and $\tilde{A}$ is K-flat as a DG $\mathbb{K}$-module.
- The DG ring homomorphism $\tilde{A} \to A$ is a quasi-isomorphism.

It is known that K-flat DG ring resolutions exist.

These resolutions are usually polynomial rings over $\mathbb{K}$ in infinitely many noncommuting variables.

If A is commutative, then there are also commutative K-flat DG ring resolutions.

7. Derived Categories of Bimodules

As before, $\mathbb{K}$ is a commutative base ring, and A is a noetherian central $\mathbb{K}$-ring.

Let $\tilde{A}$ be some K-flat DG ring resolution of A, and consider DG ring

$$\tilde{A}^{\mathrm{en}} := \tilde{A} \otimes_{\mathbb{K}} \tilde{A}^{\mathrm{op}}.$$

It has a derived category $\mathsf{D}(\tilde{A}^{\mathrm{en}})$. This is our candidate for the derived category of A-bimodules.

All the resolutions we need are available in $\mathsf{D}(\tilde{A}^{\mathrm{en}})$. Therefore the package of standard derived functors associated to A (see slide 10) exists.

There are canonical equivalences $\mathsf{D}(A) \xrightarrow{\approx} \mathsf{D}(\tilde{A})$ and $\mathsf{D}(A^{\mathrm{op}}) \xrightarrow{\approx} \mathsf{D}(\tilde{A}^{\mathrm{op}})$. Thus we do not change the derived categories of left and right modules.

We can now provide a good generalization of Definition 3.1:

Definition 7.1. A DG module $\tilde{R} \in \mathsf{D}(\tilde{A}^{\mathrm{en}})$ is called a *dualizing complex over A* if these three conditions hold:

(i) The cohomology bimodules $\mathrm{H}^i(\tilde{R})$ are finite over A and over A^{op}.
(ii) R has finite injective dimension over $\tilde{A}$ and over $\tilde{A}^{\mathrm{op}}$.
(iii) The canonical morphisms

$$\tilde{A} \to \mathrm{RHom}_{\tilde{A}}(\tilde{R}, \tilde{R}) \quad \text{and} \quad \tilde{A} \to \mathrm{RHom}_{\tilde{A}^{\mathrm{op}}}(\tilde{R}, \tilde{R})$$

in $\mathsf{D}(\tilde{A}^{\mathrm{en}})$ are isomorphisms.

Likewise we can extend Definition 3.3:

Definition 7.2. Let $\tilde{R}$ be a dualizing complex over A.

1. A *rigidifying isomorphism* for $\tilde{R}$ is an isomorphism

$$\rho : \tilde{R} \xrightarrow{\simeq} \mathrm{RHom}_{\tilde{A}^{\mathrm{en}}}(\tilde{A}, \tilde{R} \otimes^{\mathrm{L}}_{\mathbb{K}} \tilde{R})$$

in $\mathsf{D}(\tilde{A}^{\mathrm{en}})$.

2. The pair $(\tilde{R}, \rho)$ is called a *rigid dualizing complex over A relative to $\mathbb{K}$*.

Problem 7.3. *There are many K-flat DG ring resolutions $\tilde{A}$ of A.*

Do we get different answers (e.g. to the question of existence of rigid dualizing complex) when choosing different resolutions?

This problem is settled by the next new result.

Theorem 7.4. ([Ye5]) *Let $\mathbb{K}$ be a commutative ring, and let A be a central $\mathbb{K}$-ring.*

Suppose $\tilde{A}_0$ and $\tilde{A}_1$ are K-flat DG ring resolutions of A relative to $\mathbb{K}$.

There is a canonical equivalence of triangulated categories

$$\Phi : \mathsf{D}(\tilde{A}_0^{\mathrm{en}}) \to \mathsf{D}(\tilde{A}_1^{\mathrm{en}})$$

that respects the package of standard derived functors.

The theorem tells us that the notion of dualizing complex does not depend on the resolution $\tilde{A}$ chosen: a DG module $\tilde{R} \in \mathsf{D}(\tilde{A}_0^{\mathrm{en}})$ is dualizing iff the DG module $\Phi(\tilde{R}) \in \mathsf{D}(\tilde{A}_1^{\mathrm{en}})$ is dualizing.

Similarly for rigidity.

This justifies the following definition.

Definition 7.5. *The derived category of $\mathbb{K}$-central A-bimodules is the category* $\mathsf{D}(\tilde{A}^{\mathrm{en}})$, *where $\tilde{A}$ is any K-flat DG ring resolution of A over* $\mathbb{K}$.

Most of the results mentioned earlier on dualizing complexes, when $\mathbb{K}$ is a field, appear to hold in the arithmetic setup.

The big challenge for us is:

Problem 7.6. *Try to extend the Van den Bergh Existence Theorem to the arithmetic setup.*

This is the subject of an ongoing project with Rishi Vyas.

– END – ??

8. On the Proof of Theorem 7.4

Here is some extra material (not in the notes).

I know two ways to prove this theorem, and they complement each other.

Consider two K-flat resolutions $\tilde{A}_0 \to A$ and $\tilde{A}_1 \to A$.

We have enveloping DG rings

$$\tilde{A}_i^{\mathrm{en}} = \tilde{A}_i \otimes \tilde{A}_i^{\mathrm{op}}$$

for $i = 0, 1$.

The unadorned tensor product is over $\mathbb{K}$.

Look at the following DG ring:

$$\tilde{A}_1^{\mathrm{en}} \otimes (\tilde{A}_0^{\mathrm{en}})^{\mathrm{op}} = \tilde{A}_1 \otimes \tilde{A}_1^{\mathrm{op}} \otimes \tilde{A}_0^{\mathrm{op}} \otimes \tilde{A}_0.$$

There is a well defined object

$$P := A \otimes^{\mathrm{L}} A^{\mathrm{op}} \in \mathsf{D}\big(\tilde{A}_1^{\mathrm{en}} \otimes (\tilde{A}_0^{\mathrm{en}})^{\mathrm{op}}\big).$$

As always, we must keep track of the four actions involved; but let's leave this aside.

It turns out that P is a *tilting DG module*.

The functor

$$\Phi_{0,1}(M) := P \otimes^{\mathrm{L}}_{\tilde{A}_0^{\mathrm{en}}} M$$

is the equivalence

$$\Phi_{0,1} : \mathsf{D}(\tilde{A}_0^{\mathrm{en}}) \to \mathsf{D}(\tilde{A}_1^{\mathrm{en}})$$

that we want.

This construction has an obvious pseudofunctorial property: if $\tilde{A}_2$ is yet another resolution of A, then there is a canonical isomorphism

$$\Phi_{1,2} \circ \Phi_{0,1} \cong \Phi_{0,2}$$

of equivalences

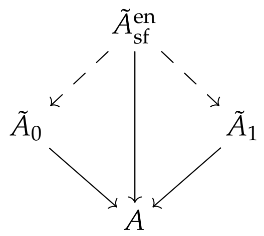
$$\mathsf{D}(\tilde{A}_0^{\mathrm{en}}) \to \mathsf{D}(\tilde{A}_2^{\mathrm{en}}).$$

However, it is not easy to see why these equivalences respect the package of standard derived functors.

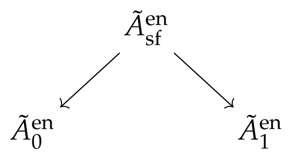
For this we turn to the second approach.

Let $\tilde{A}_{\mathrm{sf}} \to A$ be a *semi-free* DG ring resolution. This means that when forgetting the differential, $\tilde{A}_{\mathrm{sf}}$ is a noncommutative polynomial ring over $\mathbb{K}$ is nonpositive graded variables.

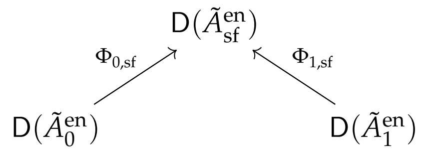
Then we can find homomorphisms of DG rings (the dashed arrows) that fit into a commutative diagram



Passing to enveloping DG rings, we get a diagram of quasi-isomorphisms

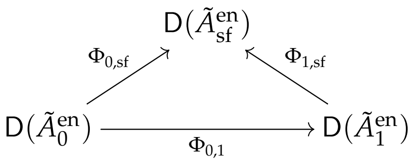


and thus a diagram of equivalences



These equivalences can be easily shown to respect the package of standard derived functors.

Finally, the diagram of equivalences



is commutative up to a canonical isomorphism.

– END –

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