Duality and Tilting for Commutative DG Rings

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I will discuss several results from the paper [Ye5].

Here is the plan of my lecture.

1. DG Rings
2. DG Modules
3. Resolutions and Derived Functors
4. Cohomologically Noetherian DG Rings
5. Motivation
6. Perfect DG Modules
7. Tilting DG Modules
8. Dualizing DG Modules

1. DG Rings

A *differential graded ring* (more commonly referred to as a differential graded associative unital algebra) is a graded ring

$$A = \bigoplus_{i \in \mathbb{Z}} A^i ,$$

equipped with a differential d of degree 1 satisfying the graded Leibniz rule

$$\mathrm{d}(a \cdot b) = \mathrm{d}(a) \cdot b + (-1)^i \cdot a \cdot \mathrm{d}(b)$$

for $a \in A^i$ and $b \in A^j$.

As usual "differential graded" is abbreviated to "DG".

The cohomology

$$\mathrm{H}(A) = \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^i(A)$$

is a graded ring.

A homomorphism of DG rings is a degree 0 ring homomorphism $f : A \to B$ that respects the differentials.

There is an induced graded ring homomorphism

$$\mathrm{H}(f) : \mathrm{H}(A) \to \mathrm{H}(B).$$

We call f a *quasi-isomorphism* if $\mathrm{H}(f)$ is an isomorphism.

We view rings as DG rings concentrated in degree 0.

A DG ring A is called *nonpositive* if $A^i = 0$ for all $i > 0$.

We say the DG ring A is *strictly commutative* if

$$b \cdot a = (-1)^{ij} \cdot a \cdot b$$

for all $a \in A^i$ and $b \in A^j$, and $a \cdot a = 0$ if i is odd.

For short I refer to nonpositive strictly commutative DG rings as *commutative DG rings*.

By default all DG rings in this talk are commutative. In particular all rings are commutative.

Example 1.1. Let $A := \mathbb{Z}$ and $B := \mathbb{Z}/(6)$. So B is an A-ring. For homological purposes the situation is not so nice: B is not flat over A.

We can replace B by a better "model" in the world of commutative DG rings, as follows.

Define $\tilde{B}$ to be the Koszul complex associated to the element $6 \in A$.

This is a complex concentrated in degrees -1 and 0 :

$$\tilde{B} = \left(\mathbb{Z} \cdot x \xrightarrow{\mathrm{d}} \mathbb{Z}\right) , \ \mathrm{d}(x) = 6.$$

As a graded ring we have $\tilde{B} := \mathbb{Z}[x]$, the strictly commutative polynomial ring on the variable x of degree -1. Since x is odd it satisfies $x^2 = 0$; so $\tilde{B}$ is really an exterior algebra.

There is an obvious DG ring homomorphism $f : \tilde{B} \to B$, and it is a quasi-isomorphism.

The example is a very special case of a general construction.

Suppose $A \to B$ is a homomorphism of commutative DG rings (with no finiteness assumptions at all).

Then there exists a *semi-free resolution* of $A \to B$.

This is a factorization of $A \to B$ into homomorphisms $A \to \tilde{B} \to B$, such that:

- $\tilde{B} \to B$ is a surjective quasi-isomorphism.
- $\tilde{B}$ is semi-free over A. This means that the graded ring $\tilde{B}^\natural$, gotten from $\tilde{B}$ by forgetting the differential, is a strictly commutative polynomial ring over $A^\natural$ is some graded set of variables (usually infinite).

There is a certain uniqueness of semi-free resolutions: if $\tilde{B}'$ is another semi-free resolution of $A \to B$, then there is a DG ring quasi-isomorphism $\tilde{B}' \to \tilde{B}$ that respects the homomorphisms from A and to B.

2. DG Modules

A *left DG A-module* is a graded A-module

$$M = \bigoplus_{i \in \mathbb{Z}} M^i ,$$

equipped with a differential d of degree 1 satisfying

$$\mathrm{d}(a \cdot m) = \mathrm{d}(a) \cdot m + (-1)^i \cdot a \cdot \mathrm{d}(m)$$

for $a \in A^i$ and $m \in M^j$.

If A is a ring, then a DG A-module is just a complex of A-modules.

Because A is commutative, there is no substantial difference between left and right DG A-modules.

Indeed, given a left DG A-module M, there is a right action defined by

$$m \cdot a := (-1)^{ij} \cdot a \cdot m.$$

We denote by $\mathsf{DGMod}\, A$ the category of DG A-modules. The morphisms are the degree 0 homomorphisms $\phi : M \to N$ that respect the differentials.

A *quasi-isomorphism* in $\mathsf{DGMod}\, A$ is a homomorphism $\phi : M \to N$ such that

$$\mathrm{H}(\phi) : \mathrm{H}(M) \to \mathrm{H}(N)$$

is an isomorphism.

Note that if A is a ring, then $\mathsf{DGMod}\, A$ coincides with the category $\mathsf{C}(\mathsf{Mod}\, A)$ of complexes of A-modules.

Like in the case of complexes, there is a *derived category* $\tilde{\mathsf{D}}(\mathsf{DGMod}\, A)$ gotten from $\mathsf{DGMod}\, A$ by inverting the quasi-isomorphisms. It is a triangulated category. See [Ke] for details.

There is an additive functor

$$\mathrm{Q} : \mathsf{DGMod}\, A \to \tilde{\mathsf{D}}(\mathsf{DGMod}\, A).$$

It is the identity on objects.

Any morphism ψ in $\tilde{\mathsf{D}}(\mathsf{DGMod}\, A)$ can be written as

$$\psi = \mathrm{Q}(\phi_1) \circ \mathrm{Q}(\phi_2)^{-1},$$

where ϕ_i are homomorphisms in $\mathsf{DGMod}\, A$, and ϕ_2 is a quasi-isomorphism.

We shall use the abbreviation

$$\mathsf{D}(A) := \tilde{\mathsf{D}}(\mathsf{DGMod}\, A).$$

3. Resolutions and Derived Functors

Suppose

$$F : \mathsf{DGMod}\, A \to \mathsf{DGMod}\, B$$

is a DG functor, such as the functors $M \otimes_A -$ or $\mathrm{Hom}_A(M, -)$ associated to a DG module M.

The functor F can be derived on the left and on the right.

In the world of DG modules, projective resolutions are replaced by *K-projective resolutions*. See [AFH] or [Ke].

Any DG A-module M (regardless of boundedness) admits K-projective resolutions $P \to M$.

We take any K-projective resolution $P \to M$, and define $\mathrm{L}F(M) := F(P)$.

This turns out to be a well-defined triangulated functor

$$\mathrm{L}F : \mathsf{D}(A) \to \mathsf{D}(B),$$

called the *left derived functor* of F.

For the right derived functor we use *K-injective resolutions*.

Any M has a K-injective resolution $M \to I$, and we define $\mathrm{R}F(M) := F(I)$.

This is a triangulated functor

$$\mathrm{R}F : \mathsf{D}(A) \to \mathsf{D}(B).$$

In case F is exact (i.e. it preserves quasi-isomorphisms), then it is its own left and right derived functor.

Let $f : A \to B$ be a homomorphism of DG rings.

Consider the restriction functor

$$\mathrm{rest}_f : \mathsf{DGMod}\, B \to \mathsf{DGMod}\, A.$$

It is exact, so we get

$$\mathrm{rest}_f : \mathsf{D}(B) \to \mathsf{D}(A).$$

If $f : A \to B$ is a quasi-isomorphism, then rest_f is an equivalence of triangulated categories.

This is one explanation why resolutions of DG rings are sensible.

4. Cohomologically Noetherian DG Rings

Recall that all our DG rings are commutative.

Definition 4.1. A DG ring A is called *cohomologically noetherian* if $\bar{A} := \mathrm{H}^0(A)$ is a noetherian ring, $\mathrm{H}(A)$ is bounded, and for every i the $\bar{A}$-module $\mathrm{H}^i(A)$ is finite (i.e. finitely generated).

Let us denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ the full subcategory of $\mathsf{D}(A)$ consisting of DG modules M whose cohomology $\mathrm{H}(M)$ is bounded, and the $\bar{A}$-modules $\mathrm{H}^i(M)$ are finite.

If A is cohomologically noetherian, then $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ is triangulated, and $A, \bar{A} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$.

A ring homomorphism $\mathbb{K} \to \bar{A}$ is called *essentially finite type* if $\bar{A}$ is the localization of a finitely generated $\mathbb{K}$-ring.

A ring $\mathbb{K}$ is called *regular* if it is noetherian and has finite global cohomological dimension. This is the same as requiring $\mathbb{K}$ to be noetherian of finite Krull dimension, and all its local rings $\mathbb{K}_{\mathfrak{p}}$ to be regular local rings.

Of course a field is a regular ring, and so is the ring of integers $\mathbb{Z}$.

Definition 4.2. A DG ring A is called *tractable* if it is cohomologically noetherian, and there exists some DG ring homomorphism $\mathbb{K} \to A$ from a regular ring $\mathbb{K}$, such that $\mathbb{K} \to \bar{A}$ essentially finite type.

In the applications I have in mind all DG rings are tractable.

5. Motivation

Why consider commutative DG rings?

Commutative DG rings play a central role in the *derived algebraic geometry* of Toën-Vezzosi [TV].

An *affine DG scheme* is by definition $\operatorname{Spec} A$ where A is a commutative DG ring.

A *derived stack* is a stack of groupoids on the site of affine DG schemes (with its étale topology).

It seems appropriate to initiate a thorough study of commutative DG rings and their derived module categories.

I should say that the more general theory of E_∞ *rings*, and E_∞ modules over them, was studied intensively by Lurie and others. See [Lu1], [Lu2] and [AG]. There is some overlap between these papers and our work.

Our motivation comes from another direction: commutative DG rings as resolutions of commutative rings. Let me say a few words about this.

Van den Bergh [VdB] introduced the notion of *rigid dualizing complex*.

This was in the context of *noncommutative algebraic geometry*. He considered a noncommutative algebra A over a field $\mathbb{K}$, and noncommutative dualizing complexes over A.

Later Zhang and I, in the papers [YZ1] and [YZ2], worked on a variant: the ring A is commutative, but the base ring $\mathbb{K}$ is no longer a field.

All we needed is that $\mathbb{K}$ is a *regular noetherian ring*, and A is *essentially finite type over* $\mathbb{K}$.

The first (and very difficult) step is to construct the *square* of any DG A-module M.

Let us choose a K-flat DG ring resolution $\tilde{A} \to A$ over $\mathbb{K}$.

This can be done; for instance we can take a semi-free DG ring resolution, as described in Section 1. (If A is flat over $\mathbb{K}$ we can just take $\tilde{A} = A$.)

We now define the square of M to be

$$\mathrm{Sq}_{A/\mathbb{K}}(M) := \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(A, M \otimes^{\mathrm{L}}_{\mathbb{K}} M) \in \mathsf{D}(A).$$

The hard part is to show that *this definition is independent of the choice of resolution* $\tilde{A}$. I will get back to that.

Now we can define rigidity. A *rigidifying isomorphism* for M is an isomorphism

$$\rho : M \xrightarrow{\simeq} \mathrm{Sq}_{A/\mathbb{K}}(M)$$

in $\mathsf{D}(A)$.

A *rigid complex* over A relative to $\mathbb{K}$ is a pair (M, ρ), where $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$, and ρ is a rigidifying isomorphism for M.

A *rigid dualizing complex* over A relative to $\mathbb{K}$ is a rigid complex (R_A, ρ_A), such that R_A is dualizing. (I will recall the definition of dualizing complex later.)

A rigid dualizing complex (R_A, ρ_A) exists, and it is unique up to a unique rigid isomorphism.

Rigid dualizing complexes are at the heart of a new approach to Grothendieck Duality for schemes and Deligne-Mumford stacks. See the papers [Ye3], [Ye4].

The problem is that there were errors in some proofs in the paper [YZ1], regarding the squaring operation.

The most serious error was in the proof that $\mathrm{Sq}_{A/\mathbb{K}}(M)$ is independent of the flat DG ring resolution $\tilde{A} \to A$.

A correction of this proof was provided in the paper [AILN]. A full correction of the proofs in [YZ1] (the statements there are actually true!) is now under preparation [Ye6].

One aspect of the correction requires the use of *Cohen-Macaulay DG modules over DG rings*. This was my motivation for writing [Ye5].

I will not talk about Cohen-Macaulay DG modules here (this is too technical). However I will discuss the theory leading up to Cohen-Macaulay DG modules, which I hope will be interesting for the audience.

6. Perfect DG Modules

Say A is a ring. Recall that a complex of A-modules M is called perfect if there is an isomorphism $M \cong P$ in $\mathsf{D}(A)$, where P is a bounded complex of finitely generated projective modules.

Locally on $\operatorname{Spec} A$, P is a complex of finitely generated free modules.

We will now generalize this to DG rings.

Let A be a commutative DG ring, and $\bar{A} = \mathrm{H}^0(A)$.

Given an element $s \in \bar{A}$, the localization $\bar{A}_s$ lifts to a localized DG ring A_s.

By covering sequence of $\bar{A}$ we mean a sequence $\boldsymbol{s} = (s_1, \ldots, s_m)$ such that

$$\operatorname{Spec} \bar{A} = \bigcup_i \operatorname{Spec} \bar{A}_{s_i}.$$

A DG A-module P is called *finite semi-free* if the graded $A^\natural$-module $P^\natural$ is free and finitely generated.

Definition 6.1. Let M be a DG A-module. We say that M is *perfect* if there is a covering sequence $\boldsymbol{s} = (s_1, \ldots, s_n)$ of $\bar{A}$, and for every i there is a finite semi-free DG A_{s_i}-module P_i, and an isomorphism

$$A_{s_i} \otimes_A M \cong P_i$$

in $\mathsf{D}(A_{s_i})$.

There are several notions of projective dimension of a DG A-module M. They boil down to boundedness properties of the functor $\operatorname{RHom}_A(M, -)$, when restricted to various subcategories of $\mathsf{D}(A)$.

If A is a ring then all these notions coincide; but I am not sure about DG rings.

One of these notions will appear in the next result.

Theorem 6.2. *Let A be a cohomologically noetherian DG ring, and let M be a DG A-module. Assume $\mathrm{H}(M)$ is bounded above.*

The following three conditions are equivalent:

(i) *The DG A-module M is perfect.*
(ii) *The DG $\bar{A}$-module $\bar{A} \otimes^{\mathsf{L}}_A M$ is perfect.*
(iii) *The DG A-module M is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{f}}(A)$, and it has finite projective dimension relative to $\mathsf{D}^{\mathsf{b}}(A)$.*

This theorem, and all subsequent results in the talk, are taken from [Ye5].

An object $M \in \mathsf{D}(A)$ is called *compact* if the functor $\operatorname{Hom}_{\mathsf{D}(A)}(M, -)$ commutes with infinite direct sums.

It is well known (see [Ri], [Ne]) that when A is a ring, perfect is equivalent to compact. Here is our generalization.

Theorem 6.3. *Let A be a DG ring, and let M be a DG A-module.*

The following two conditions are equivalent:

(i) *M is a perfect DG A-module.*
(ii) *M is a compact object of $\mathsf{D}(A)$.*

The proof uses the fact that being compact is local on $\operatorname{Spec} \bar{A}$, and the *Čech resolution* associated to a covering sequence as in Definition 6.1.

We would like to call a DG ring A *regular* if it is cohomologically noetherian, and it has finite global cohomological dimension.

In particular this means that any $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ is perfect.

The next surprising result says that if A is regular, then it is (quasi-isomorphic to) a ring.

Theorem 6.4. *Let A be a tractable DG ring. If $\bar{A}$ is a perfect DG A-module, then $A \to \bar{A}$ is a quasi-isomorphism.*

When $\bar{A}$ is local this was proved by Jørgensen [Jo]. The general case easily follows by our localization technique.

7. Tilting DG Modules

Definition 7.1. Let A be a DG ring. A DG A-module P is called a *tilting DG module* if there exists some DG A-module Q such that $P \otimes^{\mathrm{L}}_A Q \cong A$ in $\mathsf{D}(A)$.

Here is a derived Morita characterization of tilting DG modules, generalizing the results of Rickard for rings [Ri].

Theorem 7.2. *Let A be a cohomologically noetherian DG ring.*

The following two conditions are equivalent for a DG A-module P.

(i) *P is a tilting DG module.*
(ii) *P is perfect, and the adjunction morphism $A \to \mathrm{RHom}_A(P, P)$ in $\mathsf{D}(A)$ is an isomorphism.*

Definition 7.3. The *commutative derived Picard group* of A is the abelian group $\mathrm{DPic}(A)$ whose elements are the isomorphism classes, in $\mathsf{D}(A)$, of tilting DG A-modules.

The product is induced by the operation $- \otimes^{\mathrm{L}}_A -$, and the unit element is the class of A.

A DG ring homomorphism $A \to B$ induces a group homomorphism

$$\mathrm{DPic}(A) \to \mathrm{DPic}(B)\ ,\ P \mapsto B \otimes^{\mathrm{L}}_A P.$$

Theorem 7.4. *Let A be a DG ring, and consider the canonical DG ring homomorphism $A \to \bar{A}$. The induced group homomorphism*

$$\mathrm{DPic}(A) \to \mathrm{DPic}(\bar{A})$$

is bijective.

In earlier versions of the talk the DG ring A in the theorem was assumed to be cohomologically noetherian. But with the help of B. Antieau and J. Lurie this assumption has been was removed.

The structure of the group $\mathrm{DPic}(\bar{A})$ is known.

Let n be the number of connected components of $\operatorname{Spec} \bar{A}$.

Then

$$\mathrm{DPic}(\bar{A}) \cong \mathbb{Z}^n \times \mathrm{Pic}(\bar{A}),$$

where $\mathrm{Pic}(\bar{A})$ is the usual Picard group.

See the papers [Ye1] and [RZ].

Theorems 6.2 and 7.4 indicate that *the DG ring A behaves as though it were an infinitesimal extension, in the category of rings, of the ring $\bar{A}$.* (This observation is not new; cf. [AG], [Lu2].)

8. Dualizing DG Modules

In this section all DG rings are cohomologically noetherian.

Definition 8.1. Let A be a DG ring.

A DG A-module R is called *dualizing* if it satisfies these three conditions:

(i) $R \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$.
(ii) R has finite injective dimension.
(iii) The adjunction morphism $A \to \mathrm{RHom}_A(R, R)$ in $\mathsf{D}(A)$ is an isomorphism.

Condition (ii) says that the functor $\mathrm{RHom}_A(-, R)$ has finite cohomological dimension.

If A is a ring, then this is the original definition of Grothendieck in [RD].

Here are several results about dualizing DG modules.

Theorem 8.2. *If R is a dualizing DG module over A, then the functor*

$$\mathrm{RHom}_A(-, R) : \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$$

is an equivalence of triangulated categories.

Theorem 8.3. *Let A be a tractable DG ring. Then A has a dualizing DG module.*

Theorem 8.4. *Let R be a dualizing DG module over A.*

1. *A DG A-module R' is dualizing iff $R' \cong P \otimes^{\mathrm{L}}_A R$ for some tilting DG module P.*
2. *If P is a tilting DG module and $R \cong P \otimes^{\mathrm{L}}_A R$, then $P \cong A$ in $\mathsf{D}(A)$.*

Theorem 8.4 directly implies the next classification result.

Corollary 8.5. *Assume A has some dualizing DG module.*

The operation $R \mapsto P \otimes^{\mathrm{L}}_A R$ induces a simply transitive action of the group $\mathrm{DPic}(A)$ on the set of isomorphism classes of dualizing DG A-modules.

The results in this section up to here are generalizations of similar results of Grothendieck [RD] about rings.

After writing [Ye5] we learned about [Lu2], where Lurie considers dualizing modules over E_∞ rings. The previous results in this section can be viewed as special cases of his results.

The next corollary is totally new.

It is a combinination of Corollary 8.5 and Theorem 7.4.

Corollary 8.6. *Assume A is tractable.*

The operation $R \mapsto \mathrm{RHom}_A(\bar{A}, R)$ induces a bijection

$$\frac{\{\text{dualizing DG } A\text{-modules}\}}{\text{isomorphism}} \xrightarrow{\simeq} \frac{\{\text{dualizing DG } \bar{A}\text{-modules}\}}{\text{isomorphism}}.$$

This leads us to ask:

Question 8.7. *Is there a meaningful theory of rigid dualizing DG modules for commutative DG rings?*

If so, can it be used to establish a Grothendieck Duality for some kinds of derived schemes, and maps between them?

~ END ~

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