Duality in Noncommutative Algebra and Geometry

Lecture Notes
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Here is the plan of my lecture:

1. Recalling Duality
2. The Derived Category
3. Duality in Commutative Algebraic Geometry
4. Duality in Noncommutative Algebra
5. Applications in Ring Theory
6. Noncommutative Algebraic Geometry

1. Recalling Duality

Duality is one of the fundamental concepts in mathematics.

The most basic duality is that of linear algebra. We take a vector space V over a field $\mathbb{K}$ and assign to it

$$V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K}).$$

If V is finite dimensional then $V \cong V^{**}$.

This can be generalized in many ways. For instance we can make V infinite, and then often we impose a topology to retain reflexivity (e.g. a Banach space).

Now change the ring of coefficients: instead of the field $\mathbb{K}$ take a commutative ring A, and instead of a vector space V we consider an A-module M. Immediately we run into difficulties.

Example 1.1. Let us look at $A := \mathbb{Z}$, the ring of integers. A $\mathbb{Z}$-module is just an abelian group.

There are two distinct dualities for finitely generated abelian groups. For a finite group H the dual is

$$H^* = \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Q}/\mathbb{Z}),$$

whereas for a free group G the dual is

$$G^* = \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Z}).$$

(We want to stay within finitely generated groups, so the Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{R}/\mathbb{Z})$ is ruled out.)

We know that $G^{**} \cong G$ and $H^{**} \cong H$.

Is it possible to unite these two dualities into one? What about more complicated rings?

2. The Derived Category

This is where the *derived category* enters. The idea of Grothendieck and Verdier (see [RD]) was to work with *complexes of modules*.

A complex of A-modules consists of

$$M = \left(\cdots \to M^{i-1} \xrightarrow{\mathrm{d}^{i-1}} M^i \xrightarrow{\mathrm{d}^i} M^{i+1} \to \cdots \right)$$

where the M^i are A-modules, the d^i are A-linear maps, and $\mathrm{d}^i \circ \mathrm{d}^{i-1} = 0$.

The module M^i is called the degree i component of M.

The derived category $\mathsf{D}(\mathsf{Mod}\, A)$ has complexes as objects. Morphisms in $\mathsf{D}(\mathsf{Mod}\, A)$ are more complicated to explain.

For any i, the *i-th cohomology* of a complex M is the module

$$\mathrm{H}^i M := \mathrm{Ker}(\mathrm{d}^i) / \mathrm{Im}(\mathrm{d}^{i-1}).$$

This is a functor

$$\mathrm{H}^i : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{Mod}\, A.$$

The category of modules $\mathsf{Mod}\, A$ embeds as a full subcategory into $\mathsf{D}(\mathsf{Mod}\, A)$, as the complexes concentrated in degree 0:

$$\left(\cdots \to 0 \to M^0 \to 0 \to \cdots \right).$$

There is a *shift* operation on $\mathsf{D}(\mathsf{Mod}\, A)$. Given a complex M and an integer k, the shifted complex $M[k]$ has M^{k+i} as its degree i component.

Given two complexes M and N, we get a new complex

$$\mathrm{RHom}_A(M, N) \in \mathsf{D}(\mathsf{Mod}\, A).$$

The operation $\mathrm{RHom}_A(-,-)$ is called the *right derived Hom functor*.

The complex $\mathrm{RHom}_A(M, N)$ is related to the familiar Ext modules. Indeed,

$$\mathrm{H}^i\, \mathrm{RHom}_A(M, N) = \mathrm{Ext}^i_A(M, N).$$

Now suppose that for a certain complex $R \in \mathsf{D}(\mathsf{Mod}\, A)$, the functor

$$M \mapsto M^* := \mathrm{RHom}_A(M, R)$$

is a duality, in the sense that $M \cong M^{**}$.

Then we call R a *dualizing complex*.

Remark 2.1. There are certain *finiteness conditions* that we must impose on R and M in order to make things really work. These are analogous to the finiteness already appearing in linear algebra.

I will ignore such conditions (as well as other technicalities) in this lecture.

Example 2.2. Getting back to the ring $\mathbb{Z}$, and the setup of Example 1.1, one can show (without much difficulty) that the module $R := \mathbb{Z}$ is a dualizing complex.

The two dualities, for finite groups and free groups, are included in the duality $\mathrm{RHom}_{\mathbb{Z}}(-, \mathbb{Z})$.

Indeed, for a free group G we have

$$G^* = \mathrm{Ext}^0_{\mathbb{Z}}(G, \mathbb{Z});$$

and for a finite group H:

$$H^* = \mathrm{Ext}^1_{\mathbb{Z}}(H, \mathbb{Z}).$$

Generally speaking, most commutative rings encountered in algebra will admit a dualizing complex.

3. Duality in Commutative Algebraic Geometry

Grothendieck's duality theory has geometric aspects to it, when we pass from commutative rings to *algebraic varieties*.

It has local features (such as local duality and residues). It also has global features – indeed, it is a vast generalization of Serre duality, which talks about vector bundles on smooth projective varieties.

Moreover, Grothendieck's duality is a relative theory, in the sense that it deals with maps between varieties.

Applications of Grothendieck duality in algebraic geometry are numerous: moduli problems, resolution of singularities, enumerative geometry. . . .

I'll now try to give a taste of how Grothendieck duality relates to familiar notions from complex differential geometry.

Example 3.1. Consider the ring $A = \mathbb{C}[t]$ of polynomials over the field $\mathbb{C}$.

Like in Example 2.2, the module A is a dualizing complex. However it is not the "correct" dualizing complex.

We shall view A as the ring of algebraic $\mathbb{C}$-valued functions on the affine line $\mathbf{A}^1(\mathbb{C})$. Consider the module of algebraic differential 1-forms Ω^1_A, which is a free A-module of rank 1.

The "correct" dualizing complex is the shifted complex

$$R := \Omega^1_A[1],$$

i.e. the complex which has the module Ω^1_A in degree -1, and the zero module in all other degrees.

The reason is that we can integrate meromorphic 1-forms.

Indeed, suppose $f(t)\mathrm{d}t$ is a meromorphic differential form on the projective line

$$\mathbf{P}^1(\mathbb{C}) = \mathbf{A}^1(\mathbb{C}) \cup \{\infty\},$$

and $x \in \mathbf{P}^1(\mathbb{C})$ is a point.

By contour integration we get the residue

$$\operatorname{Res}_x f(t)\mathrm{d}t := \frac{1}{2\pi i} \oint_x f(t)\mathrm{d}t \in \mathbb{C}.$$

An indication that the residue has geometric significance is the Residue Theorem:

$$\sum_{x \in \mathbf{P}^1(\mathbb{C})} \operatorname{Res}_x f(t)\mathrm{d}t = 0.$$

The residue $\operatorname{Res}_x f(t)\mathrm{d}t$ above can be defined on any algebraic curve (over any field $\mathbb{K}$), and there is a Residue Theorem.

This residue map was used by Serre to build his duality, and to prove the Riemann-Roch Theorem for curves [Se].

Grothendieck expanded this idea enormously by in the book Residues and Duality [RD].

Indeed, Grothendieck showed how to define residues, in an abstract form, on any algebraic variety X. These residues were used to construct his global duality.

See the work of Lipman and others [Li, LS, Ye2] for more on this direction.

4. Duality in Noncommutative Algebra

Consider a noncommutative ring A. What is to prevent us from seeking a duality of the form $\operatorname{RHom}_A(-, R)$? Nothing really, as long as we take R to be a complex of *A-bimodules*.

Suppose A is a unital associative algebra over a field $\mathbb{K}$. Recall that an A-bimodule M has two commuting actions by A: left multiplication and right multiplication.

It is convenient to view the bimodule M as a left module over the algebra $A \otimes_{\mathbb{K}} A^{\mathrm{op}}$. Here A^{op} is the opposite algebra, where the order of multiplication is reversed.

Thus for $m \in M$ and

$$a_1 \otimes a_2 \in A \otimes_{\mathbb{K}} A^{\mathrm{op}}$$

we have

$$(a_1 \otimes a_2) \cdot m = a_1 \cdot m \cdot a_2 \in M.$$

Let us denote by $\mathsf{Mod}\, A$ the category of left A-modules. Then $\mathsf{Mod}\, A^{\mathrm{op}}$ is really the category of right A-modules.

Given a complex of bimodules R, we actually have two duality functors:

$$\operatorname{RHom}_A(-, R) : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{op}})$$

and

$$\operatorname{RHom}_{A^{\mathrm{op}}}(-, R) : \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{op}}) \to \mathsf{D}(\mathsf{Mod}\, A).$$

If these two functors are equivalences, then R is called a *dualizing complex over A*. This definition goes back to my paper [Ye1] from 1990.

It turns out that many interesting noncommutative rings admit dualizing complexes. A very important existence criterion was discovered by Van den Bergh [VdB1] in 1995.

It should be noted that many of the methods of commutative algebra and geometry do not apply in the noncommutative setting (e.g. localization).

However the theory of duality works extremely well over noncommutative rings, and hence it has become one of the standard homological tools of ring theory.

One of the first observations when passing to noncommutative rings is that the notion of "correct" dualizing complex becomes sharper.

By considerations of *local cohomology* (for graded rings) or the *rigidity equation* of Van den Bergh

$$R \cong \mathrm{RHom}_{A \otimes_{\mathbb{K}} A^{\mathrm{op}}}(A, R \otimes_{\mathbb{K}} R),$$

a particular dualizing complex is chosen. This is the *balanced*, respectively *rigid*, dualizing complex. See [Ye1, VdB1, YZ2].

The rigid dualizing complex enjoys several functorial properties, similar to the commutative situation. As we shall see this has nice consequences.

5. Applications in Ring Theory

Let us look at some applications.

Example 5.1. Let $\mathfrak{g}$ be an n-dimensional semisimple Lie algebra over $\mathbb{K}$, and let $A = \mathrm{U}(\mathfrak{g})$ be its universal enveloping algebra. Van den Bergh [VdB2] proved a Poincare-type duality formula between Hochschild homology and cohomology of A.

This formula relies on the properties of the rigid dualizing complex R of A, which in this case is just $R = A[n]$.

Example 5.2. Suppose $\mathfrak{b}$ is an n-dimensional Lie algebra over $\mathbb{K}$.

In [Ye3] I proved that the rigid dualizing complex of $\mathrm{U}(\mathfrak{b})$ is the complex $M[n]$, where M is the twisted bimodule

$$M := \mathrm{U}(\mathfrak{b}) \otimes_{\mathbb{K}} \bigwedge^{n} \mathfrak{b}. \tag{5.3}$$

When $\mathfrak{b}$ is not semisimple, M could be a nontrivial bimodule (cf. previous example).

Now suppose $\mathfrak{g}$ is a simple Lie algebra. Duflo [Du] and Brown-Levasseur [BL] obtained important structural results on Verma modules over $\mathrm{U}(\mathfrak{g})$ and their duals.

By looking at the Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, and using formula (5.3), together with the functorial properties of rigid dualizing complexes, I got a new proof of these results, as well as some generalizations.

Example 5.4. Suppose G is a semisimple Lie group over $\mathbb{C}$. Let $\mathcal{O}_q(G)$ be its quantized coordinate ring. Goodearl and Zhang [GZ] obtained results on the structure of $\mathcal{O}_q(G)$, using rigid dualizing complexes.

Example 5.5. The notion of Cohen-Macaulay module is important in commutative algebra, and it has several equivalent definitions. It appears that the only useful definition in the noncommutative context is the one based on the rigid dualizing complex (cf. [YZ2]). This was used, for instance, in the paper of Etingof and Ginzburg on symplectic reflection algebras [EG].

Example 5.6. Suppose D is a division algebra over a field $\mathbb{K}$. Zhang and I [YZ3] introduced the homological transcendence degree $\operatorname{Htr} D$. This is a numerical invariant of D, which, when D is commutative, is the usual transcendence degree.

The definition itself is quite simple minded. But the important properties of this invariant rely on the theory of rigid dualizing complexes.

6. Noncommutative Algebraic Geometry

By *noncommutative algebraic geometry* people usually refer to the confluence of ring theory and algebraic geometry, with an emphasis on homological methods. This is a new and developing area of research.

Initial work in this area was done by Artin, Van den Bergh, Tate, Zhang, Stafford and others (the so called "Artin school"), in the 1980's. See the papers [ATV1, AZ, SV] and their references.

Independently, the "Moscow school", consisting of Manin, Bondal, Orlov, Kapranov and others, worked on related problems. See [BK, BO1, BO2].

The main object of study of the Artin school was *noncommutative projective varieties.*

Suppose A is a noncommutative graded algebra over a field. Artin and Zhang [AZ] showed that a meaningful geometric object, namely the projective spectrum $\operatorname{Proj} A$, can be constructed by "reversing" Serre's theorem about coherent sheaves on projective varieties.

In this way A becomes the "homogeneous coordinate ring" of the "projective variety" $\operatorname{Proj} A$.

This only works well if the algebra A satisfies a condition called χ. This condition is satisfied if A is commutative, but also in many other interesting cases.

In [YZ1] Zhang and I we proved that if the graded algebra A has a balanced dualizing complex, then condition χ holds, and moreover $\operatorname{Proj} A$ has Serre duality. The converse was proved by Van den Bergh [VdB1].

The main idea of the Moscow school was the *Serre functor.* This is a very powerful abstraction of Serre's original duality.

Example 6.1. Suppose X is a smooth projective algebraic variety of dimension n.

Let Ω^n_X be the sheaf of degree n differential forms.

It is a line bundle on X, sometimes called the *dualizing sheaf* or the *canonical bundle.*

Let us denote by $\mathsf{D}(X)$ the derived category of coherent sheaves on X.

The rigid dualizing complex of X is the complex

$$\Omega^n_X[n] \in \mathsf{D}(X).$$

And the Serre functor S of $\mathsf{D}(X)$ is

$$S\mathcal{M} = \Omega^n_X[n] \otimes_{\mathcal{O}_X} \mathcal{M}$$

for $\mathcal{M} \in \mathsf{D}(X)$.

A big push to noncommutative algebraic geometry came with the realization that it is useful for some problems in mathematical physics. See Examples below.

With the progress on the Homological Mirror Symmetry Conjecture of Kontsevich (see [Ko], [KS]), it became clear that all the previously disparate ideas on noncommutative algebraic geometry are actually pretty tightly linked.

There are even suggestions now that there is a close connection to noncommutative geometry in the sense of Connes and Quillen (e.g. cyclic homology).

To finish, here are several examples that illustrate recent work.

Example 6.2. This first example shows how the distinction between commutative algebraic geometry and noncommutative ring theory can blur.

Let A be the ring of $n \times n$ upper triangular matrices over a field $\mathbb{K}$.

The rigid dualizing complex of A is the bimodule

$$A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K}).$$

A calculation by Kreines and myself [Ye4] showed that

$$\underbrace{A^* \otimes^{\mathrm{L}}_A \cdots \otimes^{\mathrm{L}}_A A^*}_{n+1} \cong A[n-1].$$

Here $\otimes^{\mathrm{L}}_A$ is the derived tensor product.

Now the Serre functor S of $\mathsf{D}(\mathsf{Mod}\, A)$ is

$$SM = A^* \otimes^{\mathrm{L}}_A M$$

for $M \in \mathsf{D}(\mathsf{Mod}\, A)$.

Let us denote by T the shift functor, so that

$$TM = M[1].$$

We see that

$$S^{n+1} = T^{n-1}.$$

According to Kontsevich, this says that "A is a Calabi-Yau manifold" of dimension $\frac{n-1}{n+1}$.

Example 6.3. Kapustin, Kuznetsov and Orlov [KKO] studied noncommutative instantons (this is mathematical physics). Their classification turned out to involve noncommutative projective surfaces. A significant part of the paper is devoted to expanding and applying the noncommutative Serre duality.

Example 6.4. Zhang and I studied rigid dualizing complexes on certain noncommutative spaces. This work involved the notion of perverse coherent sheaves. See [YZ3], [YZ5].

The idea that dualizing complexes should be related to perverse coherent sheaves was realized independently by Kashiwara [Ka].

Example 6.5. More recently, Zhang and I showed how to obtain a large part of the original Grothendieck duality theory (namely the commutative picture) very efficiently, using Van den Bergh's rigid dualizing complexes, plus the theory of perverse coherent sheaves. See [YZ6], [YZ7].

Finally:

Example 6.6. In the paper [AKO] by Auroux, Katzarkov and Orlov, the authors prove the homological mirror symmetry conjecture for weighted projective spaces and their noncommutative deformations. (Again, mathematical physics.) Their work uses the duality for noncommutative projective schemes.

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