Higher Descent

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Outline

1. Descent for Sheaves

Let X be a topological space, and consider a sheaf $\mathcal{F}$ of sets on X.

By a *twisted form* of $\mathcal{F}$ we mean a sheaf $\mathcal{F}'$ which is locally isomorphic to $\mathcal{F}$.

Namely for some open covering $\boldsymbol{U} = \{U_k\}_{k \in K}$ of X, there are isomorphisms of sheaves of sets

$$\mathcal{F}'|_{U_k} \cong \mathcal{F}|_{U_k}.$$

Such a covering $\boldsymbol{U}$ is called a *trivializing covering* for $\mathcal{F}$.

The idea of descent is to classify these twisted forms $\mathcal{F}'$ (up to isomorphism) in terms of *descent data*, or in other words *cocycles*.

This concept is of course not restricted to sheaves of sets.

We may consider twisted forms of a sheaf of groups, or a sheaf of rings, and so on.

Perhaps the most familiar situation is this: suppose $(X, \mathcal{O}_X)$ is an algebraic variety, or a differentiable manifold.

Fix a positive integer r, and consider the $\mathcal{O}_X$-module $\mathcal{F} = \mathcal{O}_X^r$. An $\mathcal{O}_X$-module locally isomorphic to $\mathcal{F}$ is just a rank r locally free $\mathcal{O}_X$-module, which is the same as the sheaf of sections of a rank r vector bundle on X.

Thus classifying twisted forms of $\mathcal{F} = \mathcal{O}_X^r$ amounts to classifying rank r vector bundles on X.

There is no reason to restrict attention to topological spaces – we could look at sheaves on sites (like the étale site of a scheme X).

Indeed, Galois descent, which is a special case of étale descent, is very useful in algebra, number theory and group theory. But in this talk I'll stick to spaces.

Before explaining how descent works, I need to recall some combinatorial constructions.

2. Cosimplicial Groups

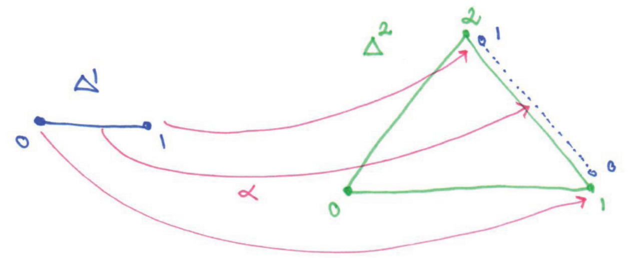
We first recall the *simplex category* $\mathbf{\Delta}$. The set of objects of $\mathbf{\Delta}$ is $\mathbb{N}$, and the morphisms $\alpha : p \to q$ are the order preserving functions

$$\alpha : \{0, \ldots, p\} \to \{0, \ldots, q\}.$$

A *cosimplicial object* in a category C is a functor $C : \mathbf{\Delta} \to \mathsf{C}$.

So in fact C is a collection $\{C^p\}_{p \in \mathbb{N}}$ of objects, together with a collection of morphisms $C(\alpha) : C^p \to C^q$, satisfying certain relations.

It is useful to picture p as the real p-dimensional simplex $\mathbf{\Delta}^p$; and to view arrows in the simplex category $\mathbf{\Delta}$ as linear maps between these real simplices.

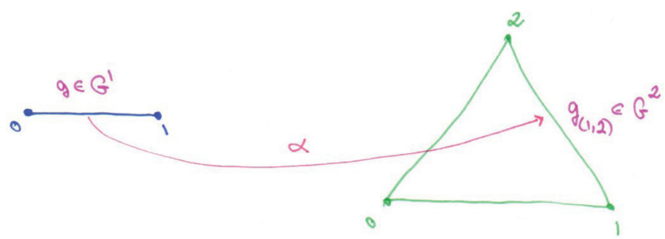


A morphism $\alpha : \Delta^1 \to \Delta^2$ in the simplex category. As a sequence we have $\alpha = (1, 2)$.

Suppose $G = \{G^p\}_{p \in \mathbb{N}}$ is a cosimplicial group. Take an element $g \in G^p$, and an arrow $\alpha : \mathbf{\Delta}^p \to \mathbf{\Delta}^q$. We then write

$$g_\alpha := G(\alpha)(g) \in G^q,$$

and think of this as "the element g pushed to the α face of $\mathbf{\Delta}^q$".

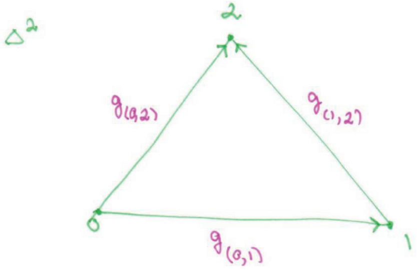


Consider an element g of the group G^1.

We say that g is a 1-*cocycle*, or a *descent datum*, if the equation

$$g_{(0,2)}^{-1} \cdot g_{(1,2)} \cdot g_{(0,1)} = 1 \tag{2.1}$$

holds in the group G^2.



The set of 1-cocycles of G is denoted by $\mathrm{Desc}(G)$.

There is an action of the group G^0 on the set $\mathrm{Desc}(G)$. The quotient set is denoted by $\mathrm{H}^1(G) = \overline{\mathrm{Desc}}(G)$, and called the 1-*st cohomology of* G.

In fact $\mathrm{H}^1(G)$ is a *pointed set*; the special point is the class of the trivial element $1 \in G^1$.

If G is an abelian cosimplicial group, then $\mathrm{H}^1(G)$ is an abelian group.

3. Back to Sheaves

We now return to the question of classifying twisted forms of a sheaf of sets $\mathcal{F}$ on a topological space X.

There is a sheaf of groups $\mathcal{G} := \mathcal{A}ut(\mathcal{F})$ on X. This is the sheaf whose sections on an open set U is the group

$$\Gamma(U, \mathcal{G}) = \mathrm{Aut}(\mathcal{F}|_U)$$

of automorphisms of the sheaf $\mathcal{F}|_U$.

Let let $\boldsymbol{U} = \{U_k\}_{k \in K}$ be an open covering of X.

For any natural number p define the group

$$\mathrm{C}^p(\boldsymbol{U}, \mathcal{G}) := \prod_{k_0, \ldots, k_p \in K} \Gamma(U_{k_0, \ldots, k_p}, \mathcal{G}).$$

The restriction homomorphisms give the collection

$$\mathrm{C}(\boldsymbol{U}, \mathcal{G}) := \big\{ \mathrm{C}^p(\boldsymbol{U}, \mathcal{G}) \big\}_{p \in \mathbb{N}}$$

a structure of a cosimplicial group. This is the *Čech cosimplicial group*.

The 1-*st Čech cohomology of* $\mathcal{G}$ is the pointed set

$$\check{\mathrm{H}}^1(\boldsymbol{U}, \mathcal{G}) := \mathrm{H}^1\big(\mathrm{C}(\boldsymbol{U}, \mathcal{G})\big) = \overline{\mathrm{Desc}}\big(\mathrm{C}(\boldsymbol{U}, \mathcal{G})\big).$$

Here is the result summarizing descent theory:

Theorem 3.1. *Let $\mathcal{F}$ be a sheaf of sets on X, with sheaf of automorphisms $\mathcal{G} = \mathcal{A}ut(\mathcal{F})$, and let $\boldsymbol{U}$ be an open covering of X. There is a canonical bijection of pointed sets*

$$\frac{\{\text{twisted forms } \mathcal{F}' \text{ of } \mathcal{F} \text{ that trivialize on } \boldsymbol{U}\}}{\text{isomorphism}} \;\cong\; \check{\mathrm{H}}^1(\boldsymbol{U}, \mathcal{G}).$$

In this theorem, "sheaf of sets" can of course be replaced by "rank r vector bundle" etc.

The corresponding statement for vector bundles is: the pointed set $\check{\mathrm{H}}^1(\boldsymbol{U}, \mathrm{GL}_r(\mathcal{O}_X))$ classifies rank r vector bundles that trivialize on $\boldsymbol{U}$.

Can we capture all twisted forms of $\mathcal{F}$ this way?

Let me give two answers.

First, it could happen that the covering $\boldsymbol{U}$ trivializes all twisted forms of $\mathcal{F}$. Then $\check{\mathrm{H}}^1(\boldsymbol{U}, \mathcal{G})$ classifies all twisted forms of $\mathcal{F}$.

Example 3.2. Assume X is an algebraic variety, $\mathcal{G} = \mathcal{A}ut(\mathcal{F})$ is abelian, and is isomorphic (as sheaf of groups) to a coherent $\mathcal{O}_X$-module.

Let $\boldsymbol{U}$ be an affine open covering of X. Then $\check{\mathrm{H}}^1(\boldsymbol{U}, \mathcal{G})$ classifies all twisted forms of $\mathcal{F}$.

Here is the second and complete answer.

We can pass to the colimit on all open coverings and define:

$$\check{\mathrm{H}}^1(X, \mathcal{G}) := \lim_{\boldsymbol{U} \to} \check{\mathrm{H}}^1(\boldsymbol{U}, \mathcal{G}).$$

Theorem 3.1 shows that:

Corollary 3.3. *The pointed set $\check{\mathrm{H}}^1(X, \mathcal{G})$ classifies all twisted forms of $\mathcal{F}$.*

4. Higher Descent: Stacks

I will quickly explain the geometric meaning of higher descent, for those who know stacks. Stacks will appear also in the examples that I will give. However the results stated in the talk do not require knowledge of stacks.

Let X be a topological space, and let $\boldsymbol{\mathcal{F}}$ be a stack of categories on X. A twisted form of $\boldsymbol{\mathcal{F}}$ is a stack $\boldsymbol{\mathcal{F}}'$ which is locally equivalent to $\boldsymbol{\mathcal{F}}$.

Higher descent (or "nonabelian second cohomology") is the way to classify twisted forms of $\boldsymbol{\mathcal{F}}$.

I should say that unlike classical descent, in higher descent usually open coverings (and their limits) are not sufficient. In many instances it is necessary to use *hypercoverings*.

5. Crossed Groupoids

The passage to higher descent, from the combinatorial point of view, is not so hard: we replace cosimplicial groups with *cosimplicial crossed groupoids*.

I have to define what is a *crossed groupoid*. This is also known as *strict 2-groupoid*, or 2-*truncated crossed complex*, or *groupoid with inner gauge groups*.

Recall that a groupoid is a category in which all morphisms are invertible.

For a groupoid $\boldsymbol{G}$ and $x \in \mathrm{Ob}(\boldsymbol{G})$ we write $\boldsymbol{G}(x) := \boldsymbol{G}(x, x)$, the set of morphisms from x to itself. This is the automorphism group of x.

Suppose $\boldsymbol{N}$ is another groupoid, such that $\operatorname{Ob}(\boldsymbol{N}) = \operatorname{Ob}(\boldsymbol{G})$.

An *action* Ψ of $\boldsymbol{G}$ on $\boldsymbol{N}$ is a collection of group isomorphisms

$$\Psi(g) : \boldsymbol{N}(x) \xrightarrow{\simeq} \boldsymbol{N}(y)$$

for all $x, y \in \operatorname{Ob}(\boldsymbol{G})$ and $g \in \boldsymbol{G}(x, y)$, such that

$$\Psi(h \circ g) = \Psi(h) \circ \Psi(g)$$

whenever g and h are composable arrows in $\boldsymbol{G}$, and $\Psi(\mathbf{1}_x) = \mathbf{1}_{\boldsymbol{N}(x)}$.

The prototypical example is the adjoint action $\operatorname{Ad}_{\boldsymbol{G}}$ of $\boldsymbol{G}$ on itself, namely

$$\operatorname{Ad}_{\boldsymbol{G}}(g)(h) := g \circ h \circ g^{-1}$$

for $g \in \boldsymbol{G}(x, y)$ and $h \in \boldsymbol{G}(x)$.

Definition 5.1. A *crossed groupoid* is a structure

$$\boldsymbol{G} = (\boldsymbol{G}_1, \boldsymbol{G}_2, \operatorname{Ad}_{\boldsymbol{G}_1 \curvearrowright \boldsymbol{G}_2}, \mathrm{D})$$

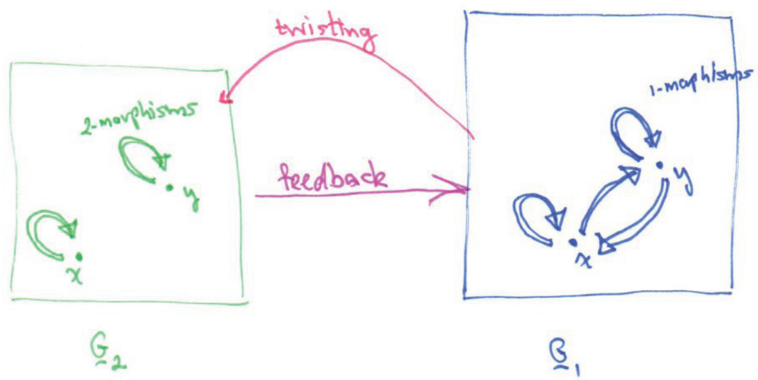
consisting of:

- Groupoids $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$, such that $\boldsymbol{G}_2$ is totally disconnected, and $\operatorname{Ob}(\boldsymbol{G}_1) = \operatorname{Ob}(\boldsymbol{G}_2)$. We write $\operatorname{Ob}(\boldsymbol{G}) := \operatorname{Ob}(\boldsymbol{G}_1)$.
- An action $\operatorname{Ad}_{\boldsymbol{G}_1 \curvearrowright \boldsymbol{G}_2}$ of $\boldsymbol{G}_1$ on $\boldsymbol{G}_2$, called the *twisting*.
- A morphism of groupoids (i.e. a functor) $\mathrm{D} : \boldsymbol{G}_2 \to \boldsymbol{G}_1$ called the *feedback*, which is the identity on objects.

There are two conditions, that I will not specify.

We sometimes refer to the morphisms in the groupoid $\boldsymbol{G}_1$ as 1-*morphisms*, and to the morphisms in $\boldsymbol{G}_2$ as 2-*morphisms*.

Crossed Groupoid $\underline{G} = (\underline{G}_1, \underline{G}_2, Ad, D)$



If the crossed groupoid $\boldsymbol{G}$ has only one object, then it is called a *crossed group*, or a *crossed module*, or a *strict* 2-*group*.

In this case often the notation is $(G_2 \to G_1)$. The arrow denotes the feedback, and the twisting is implicit.

Example 5.2. A stupid example: groups posing as crossed groups.

Take any group G. We then have a crossed group $(1 \to G)$.

Example 5.3. An interesting example of a crossed group.

Take any group G. There is a crossed group

$$\boldsymbol{G} := (G \to \operatorname{Aut}(G)).$$

The twisting is the canonical action of $\operatorname{Aut}(G)$ on G; the feedback is $\mathrm{D} := \operatorname{Ad}_G$.

Example 5.4. Let Ring be the category of rings. Consider the subcategory $\mathsf{Ring}^{\times}$ that has the same set of objects, but the morphisms are the ring *isomorphisms* $g : A \to B$.

We can upgrade the groupoid $\boldsymbol{G}_1 := \mathsf{Ring}^{\times}$ to a crossed groupoid

$$\boldsymbol{G} = (\boldsymbol{G}_1, \boldsymbol{G}_2, \mathrm{Ad}_{\boldsymbol{G}_1 \curvearrowright \boldsymbol{G}_2}, \mathrm{D})$$

as follows.

For an object A, i.e. a ring, the group $\boldsymbol{G}_2(A)$ is the multiplicative group $A^{\times}$ of invertible elements.

The feedback

$$\mathrm{D} : \boldsymbol{G}_2(A) = A^{\times} \to \boldsymbol{G}_1(A) = \mathrm{Aut}_{\mathsf{Ring}}(A)$$

is the conjugation action.

The twisting $\mathrm{Ad}_{\boldsymbol{G}_1 \curvearrowright \boldsymbol{G}_2}$ is obvious.

Example 5.5. The most fascinating crossed groupoid that I encountered is the *Deligne crossed groupoid.*

Let $\mathbb{K}$ be a field of characteristic 0. By a *parameter algebra* over $\mathbb{K}$ we mean a complete noetherian local commutative $\mathbb{K}$ algebra $(R, \mathfrak{m})$, with residue field $R/\mathfrak{m} = \mathbb{K}$. The main example is of course $R = \mathbb{K}[[\hbar]]$.

A DG Lie algebra $\mathfrak{g}$ is called *quantum type* if $\mathfrak{g} = \bigoplus_{i \geq -1} \mathfrak{g}^i$.

There is an induced pronilpotent DG Lie algebra

$$\mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g} = \bigoplus_{i \geq -1} \mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g}^i .$$

The *Deligne crossed groupoid* $\mathrm{Del}(\mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g})$ is made up of these components: the objects are the solutions of the *Maurer-Cartan equation* in $\mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g}$; the 1-morphisms and 2-morphisms are certain symmetries of these solutions. See [Ge, Ye3].

Let

$$\boldsymbol{G} = (\boldsymbol{G}_1, \boldsymbol{G}_2, \mathrm{Ad}_{\boldsymbol{G}_1 \curvearrowright \boldsymbol{G}_2}, \mathrm{D})$$

and

$$\boldsymbol{H} = (\boldsymbol{H}_1, \boldsymbol{H}_2, \mathrm{Ad}_{\boldsymbol{H}_1 \curvearrowright \boldsymbol{H}_2}, \mathrm{D})$$

be crossed groupoids. A morphism of crossed groupoids $\boldsymbol{G} \to \boldsymbol{H}$ consists of groupoid morphisms $\boldsymbol{G}_i \to \boldsymbol{H}_i$ that respect the rest of the structure. We get a category structure on the set of crossed groupoids.

This allows us to us consider *cosimplicial crossed groupoids* $\boldsymbol{G} = \{\boldsymbol{G}^p\}_{p \in \mathbb{N}}$, i.e. functors

$$\boldsymbol{G} : \boldsymbol{\Delta} \to \{\text{crossed groupoids}\}.$$

6. Cosimplicial Crossed Groupoids

Let $\boldsymbol{G} = \{\boldsymbol{G}^p\}_{p \in \mathbb{N}}$ be a cosimplicial crossed groupoid.

Fix $p \in \mathbb{N}$; so $\boldsymbol{G}^p$ is a crossed groupoid. For any $x \in \mathrm{Ob}(\boldsymbol{G}^p)$ there is a group homomorphism (the feedback)

$$\mathrm{D} : \boldsymbol{G}_2^p(x) \to \boldsymbol{G}_1^p(x).$$

And for every morphism $g : x \to y$ in $\boldsymbol{G}_1^p$ there is a group isomorphism (the twisting)

$$\mathrm{Ad}(g) : \boldsymbol{G}_2^p(x) \to \boldsymbol{G}_2^p(y).$$

Definition 6.1. Let $\boldsymbol{G} = \{\boldsymbol{G}^p\}_{p \in \mathbb{N}}$ be a cosimplicial crossed groupoid.

A *combinatorial descent datum* in $\boldsymbol{G}$ is a triple (x, g, a) of elements of the following sorts:

(0) $x \in \mathrm{Ob}(\boldsymbol{G}^0)$.
(1) $g \in \boldsymbol{G}^1_1(x_{(0)}, x_{(1)})$.
(2) $a \in \boldsymbol{G}^2_2(x_{(0)})$.

The conditions are as follows:

(i) (Failure of 1-cocycle)

$$g^{-1}_{(0,2)} \circ g_{(1,2)} \circ g_{(0,1)} = \mathrm{D}(a)$$

in the group $\boldsymbol{G}^2_1(x_{(0)})$.

(ii) (Twisted 2-cocycle)

$$a^{-1}_{(0,1,3)} \circ a_{(0,2,3)} \circ a_{(0,1,2)} = \mathrm{Ad}(g^{-1}_{(0,1)})(a_{(1,2,3)})$$

in the group $\boldsymbol{G}^3_2(x_{(0)})$.

We denote by $\mathrm{Desc}(\boldsymbol{G})$ the set of all descent data in $\boldsymbol{G}$.

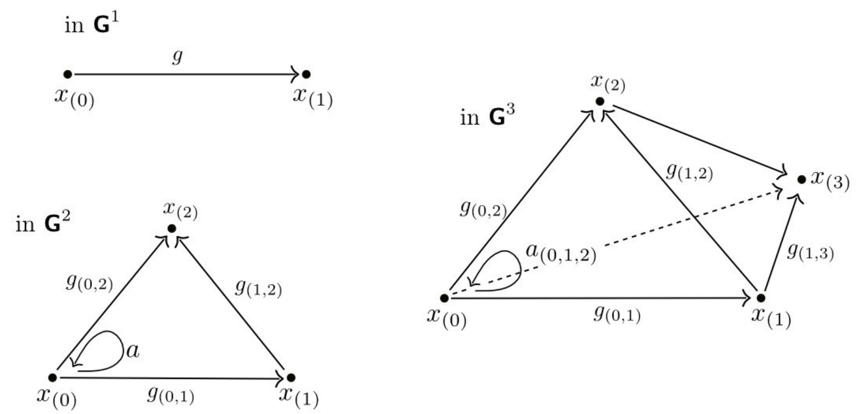


Figure: Illustration of a combinatorial descent datum (x, g, a) in the cosimplicial crossed groupoid $\mathbf{G} = \{\mathbf{G}^p\}_{p \in \mathbb{N}}$.

There is an equivalence relation on the set $\mathrm{Desc}(\boldsymbol{G})$, and we denote the quotient set by $\overline{\mathrm{Desc}}(\boldsymbol{G})$.

Observe that if $\boldsymbol{G}$ is a cosimplicial group posing as a cosimplicial crossed groupoid (cf. Example 5.2), then

$$\overline{\mathrm{Desc}}(\boldsymbol{G}) = \check{\mathrm{H}}^1(G).$$

Example 6.2. Let $\mathcal{G}$ be a sheaf of groups on a topological space X.

Consider the gerbe $\boldsymbol{\mathcal{F}}$ of left $\mathcal{G}$-torsors. A twisted form of $\boldsymbol{\mathcal{F}}$ is called a *$\mathcal{G}$-gerbe.*

Here is the higher descent classification of $\mathcal{G}$-gerbes, following Breen [Br].

Consider the sheaf of crossed groups $\boldsymbol{\mathcal{G}} := (\mathcal{G} \to \mathcal{A}ut(\mathcal{G}))$, as in Example 5.3.

Let $\boldsymbol{U}$ be an open covering of X. The the Čech construction gives rise to a cosimplicial crossed group $\boldsymbol{G} := \mathrm{C}(\boldsymbol{U}, \boldsymbol{\mathcal{G}})$.

The set $\overline{\mathrm{Desc}}(\boldsymbol{G})$ is an approximate classification of $\mathcal{G}$-gerbes; to get it right we need to use hypercoverings, and to pass to the colimit by refinement.

Sometimes a single open covering $\boldsymbol{U}$ is enough to classify $\mathcal{G}$-gerbes.

A sufficient condition is that $\boldsymbol{U}$ *totally trivializes* all $\mathcal{G}$-gerbes.

By definition the covering $\boldsymbol{U} = \{U_k\}_{k \in K}$ totally trivializes a gerbe $\boldsymbol{\mathcal{F}}$ if for every $k_0, k_1 \in K$ the groupoids $\boldsymbol{\mathcal{F}}(U_{k_0})$ and $\boldsymbol{\mathcal{F}}(U_{k_0,k_1})$ are nonempty and connected.

Example 6.3. Assume X is an algebraic variety, and $\mathcal{G}$ is a sheaf of abelian groups that is isomorphic to a quasi-coherent $\mathcal{O}_X$-module.

Then any affine open covering $\boldsymbol{U}$ totally trivializes all $\mathcal{G}$-gerbes.

Example 6.4. This example is taken from [Ye2].

Let X be a smooth algebraic variety over a field $\mathbb{K}$ of characteristic 0.

There is a sheaf $\mathcal{T}_{\mathrm{poly},X}$ of quantum type DG Lie algebras on X, called the sheaf of *polyderivations.*

Let $\boldsymbol{U} = \{U_k\}$ be a finite affine open covering of X. The Čech construction gives rise to a cosimplicial quantum type DG Lie algebra $\mathrm{C}(\boldsymbol{U}, \mathcal{T}_{\mathrm{poly},X})$.

Take a parameter algebra $(R, \mathfrak{m})$. Applying the Deligne construction we obtain a cosimplicial crossed groupoid

$$\boldsymbol{G} := \mathrm{Del}\big(\mathfrak{m} \mathbin{\widehat{\otimes}} \mathrm{C}(\boldsymbol{U}, \mathcal{T}_{\mathrm{poly},X})\big).$$

The set $\overline{\mathrm{Desc}}(\boldsymbol{G})$ classifies *twisted Poisson R-deformations* $\boldsymbol{\mathcal{A}}$ of $\mathcal{O}_X$.

(cont.) There is another important sheaf of quantum type DG Lie algebras on X: it is the sheaf $\mathcal{D}_{\mathrm{poly},X}$ of *polydifferential operators.*

We get another cosimplicial crossed groupoid

$$\boldsymbol{H} := \mathrm{Del}\big(\mathfrak{m} \mathbin{\widehat{\otimes}} \mathrm{C}(\boldsymbol{U}, \mathcal{D}_{\mathrm{poly},X})\big).$$

The set $\overline{\mathrm{Desc}}(\boldsymbol{H})$ classifies *twisted associative R-deformations* $\boldsymbol{\mathcal{A}}$ of $\mathcal{O}_X$. These are very similar to *stacks of algebroids* in the sense of [Ko2].

Let me give a partial explanation why a single affine open covering suffices for the classification of twisted R-deformations.

A twisted deformation $\boldsymbol{\mathcal{A}}$ comes equipped with a *gauge gerbe* $\boldsymbol{\mathcal{G}}$. If $\boldsymbol{\mathcal{A}}$ is a twisted associative deformation, then $\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{A}}^{\times}$, the gerbe of invertible morphisms. In the Poisson case it is more subtle.

Our work on pronilpotent gerbes and obstruction classes in [Ye1] implies that any affine open covering $\boldsymbol{U}$ totally trivializes the gauge gerbe $\boldsymbol{\mathcal{G}}$.

7. The Equivalence Theorem

A crossed groupoid $\boldsymbol{G}$ has homotopy groups. A morphism of crossed groupoids $\boldsymbol{G} \to \boldsymbol{H}$ is called a *weak equivalence* if it induced bijections on all homotopy groups.

Next consider cosimplicial crossed groupoids $\boldsymbol{G} = \{\boldsymbol{G}^p\}_{p \in \mathbb{N}}$ and $\boldsymbol{H} = \{\boldsymbol{H}^p\}_{p \in \mathbb{N}}$. A morphism of cosimplicial crossed groupoids $\boldsymbol{G} \to \boldsymbol{H}$ is called a weak equivalence if in every simplicial dimension p the morphism $\boldsymbol{G}^p \to \boldsymbol{H}^p$ is a weak equivalence.

Here is a recent result of mine from [Ye4].

Theorem 7.1. (Combinatorial Equivalence) *Let $\boldsymbol{G}$ and $\boldsymbol{H}$ be cosimplicial crossed groupoids, and let $\Phi : \boldsymbol{G} \to \boldsymbol{H}$ be a cosimplicial weak equivalence. Then the induced function*

$$\overline{\mathrm{Desc}}(\Phi) : \overline{\mathrm{Desc}}(\boldsymbol{G}) \to \overline{\mathrm{Desc}}(\boldsymbol{H})$$

is bijective.

The proof of this theorem is by a direct combinatorial calculation.

I was initially looking for a homotopical proof of the theorem, but all "obvious" approaches (like using the Reedy model structure on cosimplicial simplicial sets) failed.

Very recently Prezma [Pr] was able to come up with a proof using homotopy theory. His proof gives more information than mine.

Using Theorem 7.1 and the bar-cobar construction for DG Lie algebras, we prove the next result in [Ye2]:

Theorem 7.2. (Lie Equivalence) *Let $\Psi : \mathfrak{g} \to \mathfrak{h}$ be a cosimplicial L_∞ morphism between cosimplicial quantum type DG Lie algebras, and let $(R, \mathfrak{m})$ be a parameter algebra.*

There is a function

$$\overline{\mathrm{Desc}}(\mathrm{Del})(\Psi) : \overline{\mathrm{Desc}}(\mathrm{Del}(\mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g})) \to \overline{\mathrm{Desc}}(\mathrm{Del}(\mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{h}))$$

with explicit formula.

The function $\overline{\mathrm{Desc}}(\mathrm{Del})(\Psi)$ is functorial in Ψ and $\mathfrak{m}$.

If Ψ is a cosimplicial L_∞ quasi-isomorphism, then $\overline{\mathrm{Desc}}(\mathrm{Del})(\Psi)$ is bijective.

8. An Application: Twisted Deformation Quantization

To finish, let me explain how Theorem 7.2 is used in [Ye2]. We continue with the setup of Example 6.4.

There is a quasi-isomorphism of sheaves of DG Lie algebras

$$\mathcal{T}_{\mathrm{poly},X} \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})$$

called the *mixed resolution*. The sheaf $\mathrm{Mix}_{\boldsymbol{U}}(-)$ is a "mixture" of the Čech resolution and the jet resolution.

Similarly there is the mixed resolution

$$\mathcal{D}_{\mathrm{poly},X} \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X}).$$

The Kontsevich Formality gives rise to an L_∞ quasi-isomorphism

$$\Psi : \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X}) \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})$$

between these sheaves of DG Lie algebras.

The Čech construction gives a diagram

$$\begin{array}{ccc} \mathrm{C}(\boldsymbol{U}, \mathcal{T}_{\mathrm{poly},X}) & & \mathrm{C}(\boldsymbol{U}, \mathcal{D}_{\mathrm{poly},X}) \\ \downarrow & & \downarrow \\ \mathrm{C}\big(\boldsymbol{U}, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})\big) & \xrightarrow{\mathrm{C}(\boldsymbol{U},\Psi)} & \mathrm{C}\big(\boldsymbol{U}, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})\big) \end{array} .$$

The objects are cosimplicial quantum type DG Lie algebras, the vertical arrows are cosimplicial DG Lie quasi-isomophisms, and the horizontal arrow is a cosimplicial L_∞ quasi-isomorphism.

Theorem 7.2 tells us that when we apply the operation

$$\overline{\mathrm{Desc}}\big(\mathrm{Del}(\mathfrak{m}\,\widehat{\otimes}\,-)\big)$$

to this diagram, the three arrows become bijections.

Therefore there is an induced bijection in the top row:

$$\overline{\mathrm{Desc}}\big(\mathrm{Del}(\mathfrak{m}\,\widehat{\otimes}\,\mathrm{C}(\boldsymbol{U}, \mathcal{T}_{\mathrm{poly},X}))\big) \xrightarrow{\cong} \overline{\mathrm{Desc}}\big(\mathrm{Del}(\mathfrak{m}\,\widehat{\otimes}\,\mathrm{C}(\boldsymbol{U}, \mathcal{D}_{\mathrm{poly},X}))\big) .$$

This is the *twisted quantization* map

$$\mathrm{tw.quant} : \frac{\{\text{twisted Poisson } R\text{-deformations of } \mathcal{O}_X\}}{\text{twisted gauge equivalence}} \xrightarrow{\cong} \frac{\{\text{twisted associative } R\text{-deformations of } \mathcal{O}_X\}}{\text{twisted gauge equivalence}}$$

It is conjectured that the twisted quantization map destroys the geometry: it could take a Poisson deformation (a sheaf), and send it to a twisted associative deformation which is really twisted (i.e. not equivalent to a sheaf).

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