High Dimensional Topological Local Fields and Residues

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Notes available at
`http://www.math.bgu.ac.il/~amyekut/lectures`

(updated 12 January 2015)

I will discuss several results, mostly mostly old ones from the paper [Ye1].

Here is the plan of my lecture.

1. Background on Semi-Topological Rings
2. High Dimensional Local Fields
3. Topological Local Fields
4. The Beilinson Completion
5. The Residue Functional

If time permits I will also talk about:

6. Some Applications of the Residue Functional

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I will begin by recalling some definitions and constructions involving topologized rings.

Let us fix a nonzero commutative ring $\mathbb{k}$. By default all $\mathbb{k}$-rings in the talk are commutative. They form a category $\mathrm{Ring}_{\mathrm{c}}\,\mathbb{k}$.

Suppose A is a $\mathbb{k}$-ring. Recall that the module of differentials of A is the A-module $\Omega^1_{A/\mathbb{k}}$, with its universal derivation

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The exterior algebra of $\Omega^1_{A/\Bbbk}$ over A is the differential graded (DG) $\Bbbk$-ring

$$\Omega_{A/\Bbbk} = \bigoplus_{i \geq 0} \Omega^i_{A/\Bbbk}.$$

The multiplication is super-commutative. The differential d extends the derivation (1.1). The DG ring $\Omega_{A/\Bbbk}$ is also called the de Rham complex of A.

Example 1.2. If $A = \Bbbk[t]$, the polynomial ring in one variable, then $\Omega^1_{A/\Bbbk}$ is a free A-module of rank 1. The differential form $\mathrm{d}(t)$ is a basis.

Now take the ring of formal power series $A = \Bbbk[[t]]$. The first guess would be that $\Omega^1_{A/\Bbbk}$ is a free A-module of rank 1 with basis $\mathrm{d}(t)$.

However, if $\Bbbk$ is a field of characteristic 0, this is false! The module $\Omega^1_{A/\Bbbk}$ is not even finitely generated!

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This "problem" is well-known, as well as its solution. Cf. [Se] or [EGA-IV, Section 20.3].

The general solution is this: put a suitable $\Bbbk$-linear topology on the ring A. This will induce a $\Bbbk$-linear topology on the module $\Omega^1_{A/\Bbbk}$. The closure of 0 is an A-submodule $\overline{\{0\}}$.

The associated separated (i.e. Hausdorff) module is

$$\Omega^{1,\mathrm{sep}}_{A/\Bbbk} := \Omega^1_{A/\Bbbk} \,/\, \overline{\{0\}}.$$

If we are lucky, the A-module $\Omega^{1,\mathrm{sep}}_{A/\Bbbk}$ has the expected properties.

Example 1.3. Continuing with Example 1.2, we take the t-adic topology on $A = \Bbbk[[t]]$. Then $\Omega^{1,\mathrm{sep}}_{A/\Bbbk}$ if free with basis $\mathrm{d}(t)$.

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The rings that we will encounter (and I mean the topological local fields of dimensions ≥ 2) will have more complicated topologies.

Suppose M, N, P are linearly topologized $\Bbbk$-modules. A $\Bbbk$-bilinear function

$$\beta : M \times N \to P$$

is called semi-continuous if for any m, n the functions

$$\beta(m, -) : N \to P \quad \text{and} \quad \beta(-, n) : M \to P$$

are continuous.

Definition 1.4. A semi-topological $\Bbbk$-ring is a $\Bbbk$-ring A, endowed with a $\Bbbk$-linear topology, such that multiplication $A \times A \to A$ is a semi-continuous function.

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Definition 1.5. Let A be a semi-topological $\Bbbk$-ring. A semi-topological A-module is an A-module M, endowed with a $\Bbbk$-linear topology, such that multiplication $A \times M \to M$ is a semi-continuous function.

We write "ST" as an abbreviation for "semi-topological".

Example 1.6. Suppose A is a ST $\Bbbk$-ring.

The ring $A[[t]]$ of formal power series is isomorphic, as A-module, to $\prod_{i \in \mathbb{N}} A$, and we give it the product topology.

The ring $A((t))$ of formal Laurent series is isomorphic, as A-module, to

$$A[[t]] \oplus \left(\bigoplus\nolimits_{i \in \mathbb{N}} A \right),$$

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Both $A[[t]]$ and $A((t))$ are ST $\Bbbk$-rings.

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Let us denote by $\mathsf{STRing}_{\mathrm{c}}\, \Bbbk$ the category of commutative ST $\Bbbk$-rings. The morphisms are the continuous $\Bbbk$-ring homomorphisms.

Let A be an ST $\Bbbk$-ring. Then the DG ring of differentials $\Omega_{A/\Bbbk}$ has an induced topology, making it into an ST DG $\Bbbk$-ring.

Passing to the associated separated object we get the DG ring of separated differentials

$$\Omega^{\mathrm{sep}}_{A/\Bbbk} = \bigoplus_{i \geq 0} \Omega^{i,\mathrm{sep}}_{A/\Bbbk}.$$

If $f : A \to B$ is a homomorphism in $\mathsf{STRing}_{\mathrm{c}}\, \Bbbk$, then there is an induced homomorphism of ST DG rings

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2. High Dimensional Local Fields

This concept, introduced by Parshin [Pa1, Pa2] and Kato [Ka], was already explained in earlier talks. I will quickly recap, and introduce notation.

Definition 2.1. An *n*-dimensional local field over $\mathbb{k}$ is a field K, together with a sequence

$$\big(\mathcal{O}_1(K), \ldots, \mathcal{O}_n(K)\big)$$

of complete DVRs, such that:

- The fraction field of $\mathcal{O}_1(K)$ is K.
- The residue field $\boldsymbol{k}_i(K)$ of $\mathcal{O}_i(K)$ is the fraction field of $\mathcal{O}_{i+1}(K)$.
- All these rings and homomorphism are in the category of $\mathbb{k}$-rings, and $\mathbb{k} \to \boldsymbol{k}_n(K)$ is finite.

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Here is the picture for $n = 2$.

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K

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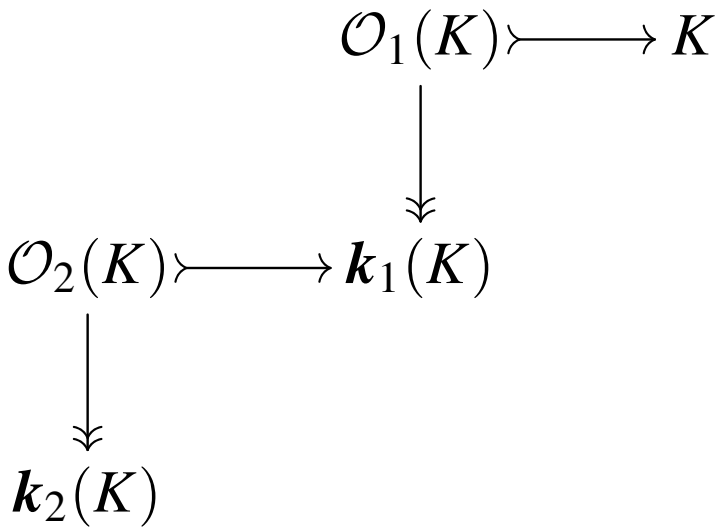

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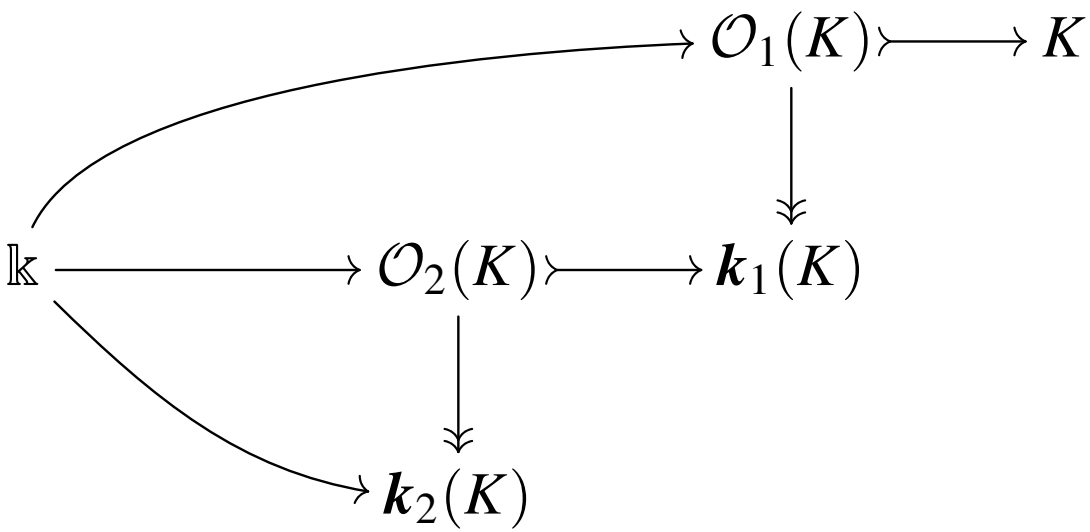

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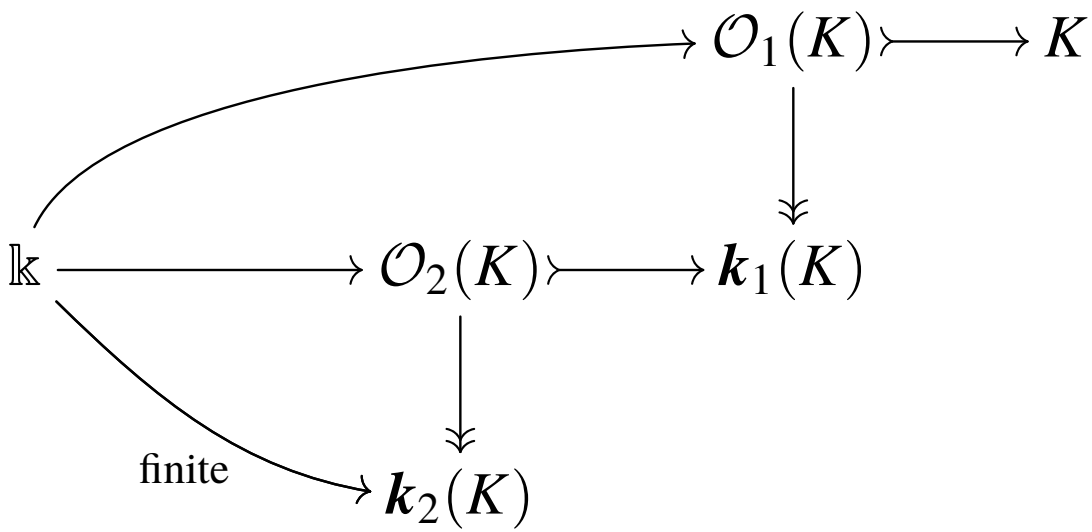

A 0-dimensional local field over $\Bbbk$ is just a field K finite over $\Bbbk$.

Example 2.2. Take $\Bbbk := \mathbb{Z}$. The fields $\widehat{\mathbb{Q}}_p$ and $\mathbb{F}_p((t))$ are 1-dimensional local fields over $\mathbb{Z}$.

Definition 2.3. Let K and L be local fields over $\Bbbk$ of dimension $n \geq 1$.

A morphism of local fields $f : K \to L$ is a $\Bbbk$-ring homomorphism such that the following conditions hold:

- $f(\mathcal{O}_1(K)) \subset \mathcal{O}_1(L)$.
- The induced $\Bbbk$-ring homomorphism $f : \mathcal{O}_1(K) \to \mathcal{O}_1(L)$ is a local homomorphism.
- The induced $\Bbbk$-ring homomorphism $\bar{f} : \boldsymbol{k}_1(K) \to \boldsymbol{k}_1(L)$ is a morphism of $(n-1)$-dimensional local fields over $\Bbbk$.

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Let us denote by $\mathsf{LF}^n\,\Bbbk$ the category of n-dimensional local fields over $\Bbbk$.

It is not hard to show that any homomorphism $K \to L$ in $\mathsf{LF}^n\,\Bbbk$ is finite.

Actually one can talk about a morphism of local fields $f : K \to L$ when $\dim(K) < \dim(L)$; but the definition is more complicated. We get a category $\mathsf{LF}\,\Bbbk$, of which $\mathsf{LF}^n\,\Bbbk$ is a full subcategory. See [Ye1] for details.

Example 2.4. If $\Bbbk$ is a field, then the field of Laurent series $K := \Bbbk((t_2))$ is a 1-dimensional local field.

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3. Topological Local Fields

From here on $\Bbbk$ is a perfect field. Therefore all our local fields are now of equal characteristics.

One of the reasons that we need this condition is as follows. Let K be an n-dimensional local field over $\Bbbk$, with last residue field $\Bbbk' := \boldsymbol{k}_n(K)$.

Since $\Bbbk$ is perfect, the finite extension $\Bbbk \to \Bbbk'$ is separable; or in other words, it is étale. An n-fold repeated application of formal lifting (also known as Hensel's Lemma) shows that there is a canonical homomorphism $\Bbbk' \to K$ in the category of $\Bbbk$-rings.

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Let $\Bbbk'$ be a finite field extension of $\Bbbk$, and let $\boldsymbol{t} = (t_1, \ldots, t_n)$ be a sequence of variables.

We denote by

$$\Bbbk'((\boldsymbol{t})) = \Bbbk'((t_1, \ldots, t_n)) := \Bbbk'((t_n)) \cdots ((t_1))$$

the iterated field of Laurent series.

The field $\Bbbk'((\boldsymbol{t}))$ has a canonical structure of n-dimensional local field over $\Bbbk$.

The DVRs are

$$\mathcal{O}_i\big(\Bbbk'((\boldsymbol{t}))\big) := \Bbbk'((t_{i+1}, \ldots, t_n))[[t_i]],$$

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Definition 3.1. ([Ye1]) An *n*-dimensional topological local field over $\Bbbk$ is a field K, together with:

(a) A structure $\{\mathcal{O}_i(K)\}_{i=1}^n$ of n-dimensional local field over $\Bbbk$.

(b) A topology, making K a semi-topological $\Bbbk$-ring.

The condition is this:

(P) There a bijection

$$f : \Bbbk'((t)) \xrightarrow{\simeq} K$$

from the standard n-dimensional topological local field with last residue field $\Bbbk' := \boldsymbol{k}_n(K)$, such that:

(i) f is an isomorphism in $\mathsf{LF}^n\,\Bbbk$ (i.e. it respects the valuations).

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The parametrization f is not part of the structure of K; it is required to exist, but (as we shall soon see) there are many distinct parametrizations.

We use the abbreviation "TLF" for "topological local field".

Here are some basic facts about TLFs.

As a ST $\Bbbk$-module, each TLF K is complete. This means that the canonical homomorphism

$$K \to \lim_{\leftarrow U} K/U,$$

where U runs over all open $\Bbbk$-submodules of K, is bijective.

In particular K is separated, so that $K^{\mathrm{sep}} = K$.

If $\dim(K) \geq 2$, then K is not a metrizable topological space, and it is not a topological ring (only ST).

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Let K be an n-dimensional TLF. A system of uniformizers in K is a sequence $(a_1, \ldots, a_n)$ of elements of $\mathcal{O}_1(K)$, such that a_1 generates the maximal ideal of $\mathcal{O}_1(K)$, and if $n \geq 2$, the sequence $(\bar{a}_2, \ldots, \bar{a}_n)$, which is the image of $(a_2, \ldots, a_n)$ under the canonical surjection $\mathcal{O}_1(K) \twoheadrightarrow \boldsymbol{k}_1(K)$, is a system of uniformizers in $\boldsymbol{k}_1(K)$.

The next theorem tells us what are all the possible parametrizations of a TLF.

Theorem 3.2. ([Ye1]) Let K be an n-dimensional TLF over $\mathbb{k}$, let $(a_1, \ldots, a_n)$ be a system of uniformizers in K, let $\mathbb{k}' := \boldsymbol{k}_n(K)$, and let $\sigma : \mathbb{k}' \to K$ be the canonical lifting.

Then σ extends uniquely to an isomorphism of TLFs

$$f : \mathbb{k}'((t_1, \ldots, t_n)) \to K$$

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Definition 3.3. Let K and L be n-dimensional TLFs.

A morphism of TLFs $f : K \to L$ is morphism of local fields which is also continuous.

We denote by $\mathsf{TLF}^n\,\Bbbk$ the category of n-dimensional TLFs over $\Bbbk$.

There is a bigger category $\mathsf{TLF}\,\Bbbk$, that allows morphisms $f : K \to L$ with $\dim(K) < \dim(L)$. See Example 2.4. $\mathsf{TLF}^n\,\Bbbk$ is a full subcategory of $\mathsf{TLF}\,\Bbbk$.

Consider the functor

$$\mathsf{TLF}^n\,\Bbbk \to \mathsf{LF}^n\,\Bbbk$$

that forgets the topology.

When $n \geq 2$ and $\mathrm{char}(\Bbbk) = 0$ this forgetful functor is far from being an equivalence.

In other words, any such local field K admits many distinct topologies, all satisfying condition (P). There is an example of this phenomenon in [Ye1].

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However:

Theorem 3.4. ([Ye1]) If $\mathrm{char}(\mathbb{k}) = p > 0$, the forgetful functor

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The rough idea of the proof is this: changing parametrizations involves Taylor series expansions (just like a change of coordinates in complex analytic geometry). The coefficients in these expansions are continuous differential operators.

It turns out that for TLFs in characteristic p, all differential operators are continuous.

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4. The Beilinson Completion

Suppose X is a finite type $\mathbb{k}$-scheme.

By a chain of points in X we mean a sequence $\xi = (x_0, \ldots, x_n)$ of points such that x_i is a specialization of x_{i-1}.

The chain ξ is saturated if each x_i is an immediate specialization of x_{i-1}.

In [Be], Beilinson defined a completion operation, which is a special case of his higher adeles.

Given a quasi-coherent sheaf $\mathcal{M}$ on X, and a chain ξ, the Beilinson completion of $\mathcal{M}$ along ξ is a $\mathbb{k}$-module $\mathcal{M}_\xi$, gotten by an n-fold zig-zag of inverse and direct limits.

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The completion $\mathcal{M}_\xi$ comes equipped with a topology, making it a ST $\Bbbk$-module.

Furthermore, the completion $\mathcal{O}_{X,\xi}$ of the structure sheaf $\mathcal{O}_X$ is a ST $\Bbbk$-ring; and for any $\mathcal{M}$, the completion $\mathcal{M}_\xi$ is a ST $\mathcal{O}_{X,\xi}$-module.

Example 4.1. If $n = 0$, so that $\xi = (x_0)$, we get $\mathcal{O}_{X,\xi} = \widehat{\mathcal{O}}_{X,x_0}$, the $\mathfrak{m}_{x_0}$-adic completion of the local ring $\mathcal{O}_{X,x_0}$, with the $\mathfrak{m}_{x_0}$-adic topology.

Given a point $x_0 \in X$, its residue field $\boldsymbol{k}(x_0)$ can be viewed as a quasi-coherent sheaf, constant on the closed set $\overline{\{x_0\}}$.

Theorem 4.2. ([Pa1], [Be], [Ye1]) Let X be a finite type $\Bbbk$-scheme, and let $\xi = (x_0, \ldots, x_n)$ be a saturated chain in X, such that x_n is a closed point.

Then the Beilinson completion $\boldsymbol{k}(x_0)_\xi$ is a finite product of n-dimensional TLFs.

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Theorem 4.2. ([Pa1], [Be], [Ye1]) Let X be a finite type $\Bbbk$-scheme, and let $\xi = (x_0, \ldots, x_n)$ be a saturated chain in X, such that x_n is a closed point.

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The completion $\mathcal{M}_\xi$ comes equipped with a topology, making it a ST $\Bbbk$-module.

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5. The Residue Functional

Let K be an n-dimensional TLF over $\Bbbk$.

Suppose $\boldsymbol{a} = (a_1, \ldots, a_n)$ is a system of uniformizers of K.

The module of separated differential 1-forms $\Omega^{1,\mathrm{sep}}_{K/\Bbbk}$ is a free K-module of rank n, with basis $\big(\mathrm{d}(a_1), \ldots, \mathrm{d}(a_n)\big)$.

In degree n the module $\Omega^{n,\mathrm{sep}}_{K/\Bbbk}$ is free of rank 1.

Any nonzero form $\alpha \in \Omega^{n,\mathrm{sep}}_{K/\Bbbk}$ determines an isomorphism of ST K-modules

$$K \to \Omega^{n,\mathrm{sep}}_{K/\Bbbk}, \quad b \mapsto b \cdot \alpha.$$

The system of uniformizers $\boldsymbol{a}$ gives a very special nonzero n-form

$$\mathrm{dlog}(\boldsymbol{a}) := a_1^{-1} \cdot \mathrm{d}(a_1) \cdots a_n^{-1} \cdot \mathrm{d}(a_n). \tag{5.1}$$

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Let $K \to L$ be a homomorphism in $\mathsf{TLF}^n\,\Bbbk$.

There is a trace homomorphism

$$\mathrm{Tr}_{L/K} : \Omega^{n,\mathrm{sep}}_{L/\Bbbk} \to \Omega^{n,\mathrm{sep}}_{K/\Bbbk} .$$

It is a nondegenerate K-linear homomorphism. By this I mean that the induced homomorphism

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is bijective.

This trace is functorial: if $L \to M$ is another homomorphism in $\mathsf{TLF}^n\,\Bbbk$, then

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If $\Bbbk'$ is a 0-dimensional TLF, then $\Omega^{0,\mathrm{sep}}_{\Bbbk'/\Bbbk} = \Bbbk'$, and $\mathrm{Tr}_{\Bbbk'/\Bbbk} = \mathrm{tr}_{\Bbbk'/\Bbbk}$, the usual trace.

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Theorem 5.2. ([Ye1]) Let K be an n-dimensional TLF over $\Bbbk$.

There is a $\Bbbk$-linear homomorphism

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called the residue functional, with these properties.

1. Continuity: the homomorphism $\mathrm{Res}^{\mathrm{TLF}}_{K/\Bbbk}$ is continuous.
2. Uniformization: let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a system of uniformizers for K, and let $\Bbbk' \to K$ be the canonical lifting of the last residue field $\Bbbk' := \boldsymbol{k}_n(K)$ into K.

   Then for any $b \in \Bbbk'$ and any $i_1, \ldots, i_n \in \mathbb{Z}$ we have

$$\mathrm{Res}^{\mathrm{TLF}}_{K/\Bbbk}\big(b \cdot a_1^{i_1} \cdots a_n^{i_n} \cdot \mathrm{dlog}(\boldsymbol{a})\big) = \begin{cases} \mathrm{tr}_{\Bbbk'/\Bbbk}(b) & \text{if } i_1 = \cdots = i_n = 0 \\ 0 & \text{otherwise .} \end{cases}$$

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3. Functoriality: let $K \to L$ be a morphism in the category $\mathsf{TLF}^n\,\Bbbk$. Then

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4. Nondegeneracy: the residue pairing

$$\langle -, - \rangle_{\mathrm{res}} : K \times \Omega^{n,\mathrm{sep}}_{K/\Bbbk} \to \Bbbk\,, \quad \langle a, \alpha \rangle_{\mathrm{res}} := \mathrm{Res}^{\mathrm{TLF}}_{K/\Bbbk}(a \cdot \alpha)$$

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However the concept of TLF was absent from their work, which resulted in ill-defined concepts.

They had erroneously asserted that there is a residue functional defined on the category $\mathsf{LF}^n\,\mathbb{k}$.

This is false when $\mathrm{char}(\mathbb{k}) = 0$ and $n \geq 2$. A counterexample to that appeared in [Ye1] (see [Ye7] for an elaborated version).

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For a TLF L of dimension n, let $\mathrm{K}_n^{\mathrm{M}}(L)$ be its n-th Milnor group. There is a group homomorphism

$$\mathrm{dlog} : \mathrm{K}_n^{\mathrm{M}}(L) \to \Omega_{L/\Bbbk}^{n,\mathrm{sep}}.$$

See (5.1)

Suppose $K \to L$ is a homomorphism of TLFs, with $\dim(K) = m$. There should be a canonical group homomorphism (I did not check the details)

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6. Some Applications of the Residue Functional

Let X be a finite type $\mathbb{k}$-scheme.

On X there is the Grothendieck residue complex $\mathcal{K}_X$. It is a dualizing complex that has very special properties.

If $\pi : X \to \operatorname{Spec} X$ is the structural map, then – in terms of [RD] – $\mathcal{K}_X$ is the Cousin complex representing the twisted inverse image $\pi^!(\mathbb{k})$.

In the more modern terminology of [Ye4], [Ye5] and [Ye6], $\mathcal{K}_X$ is called the rigid residue complex of X.

It is not hard to construct the residue complex $\mathcal{K}_X$ explicitly when X is a curve. This is classical.

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There is a very close relation between residues and differential operators.

Let A be a commutative $\mathbb{k}$-ring. We denote by $\mathcal{D}_{A/\mathbb{k}}$ the ring of differential operators of A. This is a noncommutative ring, and by definition A is a left $\mathcal{D}_A$-module.

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For a scheme X we have the sheaf of differential operators $\mathcal{D}_{X/\mathbb{k}}$, and $\mathcal{O}_X$ is a sheaf of left $\mathcal{D}_{X/\mathbb{k}}$-modules.

By the general theory of $\mathcal{D}$-modules we know that when X is smooth of dimension n, the sheaf $\Omega^n_{X/\mathbb{k}}$ is a right $\mathcal{D}_{X/\mathbb{k}}$-module.

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Since K is essentially smooth over $\Bbbk$, of relative dimension m, the general formulas give $\Omega^m_{K/k}$ a right $\mathcal{D}_{K/\Bbbk}$-module structure.

For $\alpha \in \Omega^m_{K/\Bbbk}$ and $\phi \in \mathcal{D}_{K/\Bbbk}$, this right action is denoted by

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The canonical homomorphism $K \to K_\xi$ is topologically étale in $\mathsf{STRing}_{\mathsf{c}}\, \Bbbk$.

This implies that there is a canonical ring homomorphism $\mathcal{D}_{K/\Bbbk} \to \mathcal{D}^{\mathrm{cont}}_{K_\xi/\Bbbk}$, and a canonical nondegenerate right $\mathcal{D}_{K/\Bbbk}$-module homomorphism

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Now recall the residue pairing

$$\langle -, - \rangle_{\mathrm{res}} : K_\xi \times \Omega^{n,\mathrm{sep}}_{K_\xi/\Bbbk} \to \Bbbk$$

from Theorem 5.2(4).

Since this is a topological perfect pairing, any $\phi \in \mathcal{D}^{\mathrm{cont}}_{K_\xi/\Bbbk}$, viewed as a continuous $\Bbbk$-linear homomorphism

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Theorem 6.1. ([Ye2]) In this situation, for any $\alpha \in \Omega^{m,\mathrm{sep}}_{K_\xi/\Bbbk}$ and $\phi \in \mathcal{D}^{\mathrm{cont}}_{K_\xi/\Bbbk}$ we have

$$\phi^*(\alpha) = \alpha * \phi \in \Omega^{m,\mathrm{sep}}_{K_\xi/\Bbbk}.$$

In other words, the algebraic right $\mathcal{D}$-module action on $\Omega^{m,\mathrm{sep}}_{K_\xi/\Bbbk}$, which is already defined on $\Omega^m_{K/\Bbbk}$, coincides with the analytic adjoint action.

Now recall the residue pairing

$$\langle -, - \rangle_{\mathrm{res}} : K_\xi \times \Omega^{n,\mathrm{sep}}_{K_\xi/\Bbbk} \to \Bbbk$$

from Theorem 5.2(4).

Since this is a topological perfect pairing, any $\phi \in \mathcal{D}^{\mathrm{cont}}_{K_\xi/\Bbbk}$, viewed as a continuous $\Bbbk$-linear homomorphism

$$\phi : K_\xi \to K_\xi,$$

has an adjoint operator (in the sense of functional analysis)

$$\phi^* : \Omega^{m,\mathrm{sep}}_{K_\xi/\Bbbk} \to \Omega^{m,\mathrm{sep}}_{K_\xi/\Bbbk}.$$

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Theorem 6.1 implies:

Theorem 6.2. ([Ye3]) The Grothendieck residue complex $\mathcal{K}_X$ is a complex of right $\mathcal{D}_X$-modules.

This tells us for instance that when X is an n-dimensional integral scheme, the dualizing sheaf

$$\omega_X := \mathrm{H}^{-n}(\mathcal{K}_X)$$

is a right $\mathcal{D}_X$-module.

If X is smooth, so that $\omega_X = \Omega^n_{X/\mathbb{k}}$, we recover the previous right $\mathcal{D}_X$-module structure on this sheaf.

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