Nonabelian Multiplicative Integration on Surfaces

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0. Introduction

Nonabelian 1-dimensional multiplicative integration goes back to the work of Volterra in the 19-th century.

A rudimentary theory of 2-dimensional nonabelian multiplicative integration was introduce by Schlesinger around 1930. See [DF].

In this talk I will describe a more sophisticated nonabelian multiplicative integration on surfaces, and state a few new results. Full details can be found in the book [Ye4].

The motivation for this project came from my work on *twisted deformation quantization of algebraic varieties*.

If time permits, I will say a few words about this, and about relations to other research topics, at the end of the talk.

1. Some Preliminaries

Let G be a Lie group, with Lie algebra $\mathfrak{g}$. Everything is over the field $\mathbb{R}$.

Recall that the *exponential map* of G is an analytic map

$$\exp_G : \mathfrak{g} \to G,$$

which is a diffeomorphism near $0 \in \mathfrak{g}$.

Example 1.1. For the Lie group $G = \mathrm{GL}_n(\mathbb{R})$ the Lie algebra is $\mathfrak{g} = \mathrm{M}_n(\mathbb{R})$, the algebra of matrices.

Here the exponential map is the usual matrix power series

$$\exp_G(\alpha) = \sum_{i \geq 0} \tfrac{1}{i!} \alpha^i.$$

For $n \geq 0$ we let $\mathbf{\Delta}^n$ be the n-dimensional real simplex.

This is a polyhedron embedded in $\mathbb{R}^{n+1}$.

If we use the barycentric coordinates $t_0, \ldots, t_n$ on $\mathbb{R}^{n+1}$, then $\mathbf{\Delta}^n$ is the compact subset defined by

$$t_i \geq 0 \quad \text{and} \quad \sum_{i=0}^{n} t_i = 1.$$

The vertices of $\mathbf{\Delta}^n$ are $v_0, \ldots, v_n$, where

$$v_i := (0, \ldots, 1, \ldots, 0)$$

with 1 in the i-th position.

For $n = 1$ we can identify $\mathbf{\Delta}^1$ with the unit line segment $\mathbf{I}^1$. But then we use the coordinate $t := t_1$.

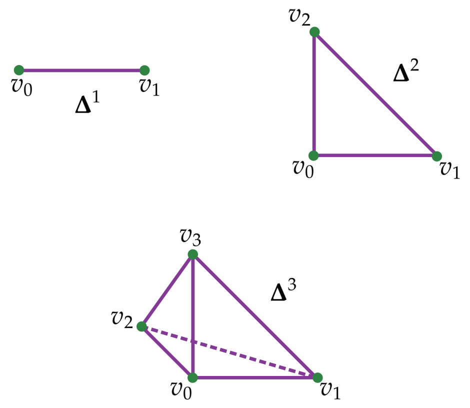


Figure : The simplices $\mathbf{\Delta}^n$ for $n = 1, 2, 3$.

Let X be an n-dimensional manifold (differentiable of type C^∞) or a convex polyhedron (such as $\mathbf{\Delta}^n$).

We denote by

$$\Omega(X) = \bigoplus_{p=0}^{n} \Omega^p(X)$$

the *de Rham algebra of smooth differential forms* on X.

In degree 0 we have $\Omega^0(X) = \mathcal{O}(X)$, the ring of smooth $\mathbb{R}$-valued functions on X.

The de Rham algebra comes with the exterior derivative

$$\mathrm{d} : \Omega^p(X) \to \Omega^{p+1}(X).$$

If Y is a manifold, and $f : X \to Y$ is a smooth map, then there is a pullback operation

$$f^* : \Omega^p(Y) \to \Omega^p(X).$$

2. MI on Curves

Let X be a manifold. By a *path* (or string) in X we mean a smooth map

$$\sigma : \mathbf{\Delta}^1 \to X.$$

Let G be a Lie group with Lie algebra $\mathfrak{g}$.

Suppose σ is a path in X, and α is a $\mathfrak{g}$-valued 1-form on X, i.e.

$$\alpha \in \Omega^1(X) \otimes \mathfrak{g}.$$

We wish to define the *nonabelian multiplicative integral* of α on σ, which is an element of the group G.

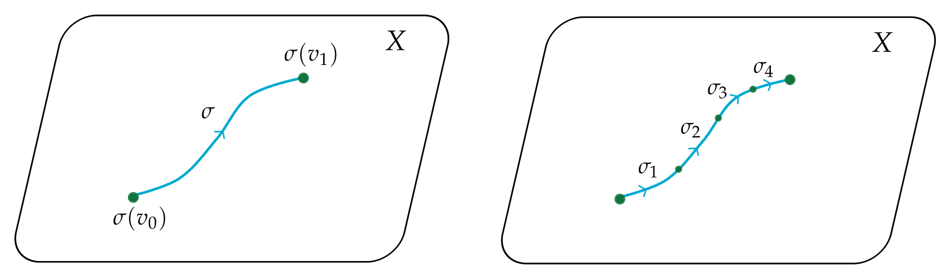
Take $k \geq 0$. We partition $\mathbf{\Delta}^1$ into 2^k equal line segments, starting from v_0.

Composing with σ we get paths

$$\sigma_1, \ldots, \sigma_{2^k} : \mathbf{\Delta}^1 \to X,$$

that we call the k-th binary subdivision of σ.

The case $k = 2$ is depicted below.



For each i there is the usual integral

$$\int_{\sigma_i} \alpha = \int_{\mathbf{\Delta}^1} \sigma_i^*(\alpha) \in \mathfrak{g}.$$

The *k-th Riemann product* is

$$\mathrm{RP}_k(\alpha \mid \sigma) := \prod_{i=1}^{2^k} \exp_G\Big(\int_{\sigma_i} \alpha\Big) \in G, \tag{2.1}$$

where the product goes from left to right.

It is not hard to prove that the limit

$$\mathrm{MI}(\alpha \mid \sigma) := \lim_{k \to \infty} \mathrm{RP}_k(\alpha \mid \sigma) \in G. \tag{2.2}$$

exists.

This is the *multiplicative integral* of α on σ.

The operation $\mathrm{MI}(\alpha \mid \sigma)$ has several nice properties.

If G is *abelian* then

$$\mathrm{MI}(\alpha \mid \sigma) = \exp_G\Big(\int_{\sigma} \alpha\Big).$$

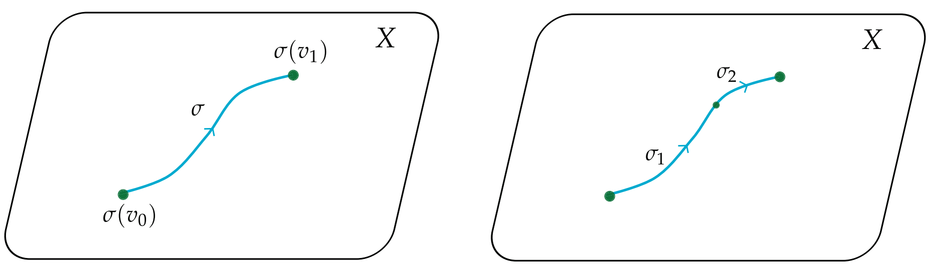
Another nice property is the *geometric multiplicativity*, which I shall now explain.

Suppose we partition $\mathbf{\Delta}^1$ into two segments or arbitrary length, starting from v_0.

This gives rise to paths

$$\sigma_1, \sigma_2 : \mathbf{\Delta}^1 \to X$$

as shown on the next slide.



Then

$$\mathrm{MI}(\alpha \mid \sigma) = \mathrm{MI}(\alpha \mid \sigma_1) \cdot \mathrm{MI}(\alpha \mid \sigma_2) \tag{2.3}$$

in the group G.

For $G = \mathrm{GL}_n(\mathbb{R})$ there is an interpretation of the 1-dimensional MI in terms of *ordinary differential equations*.

Consider a smooth function $f : \mathbf{I}^1 \to \mathrm{M}_n(\mathbb{R})$.

In other words $f = \big[f_{i,j}(t)\big]$, an $n \times n$ matrix of smooth functions of the real variable t.

Let $g : \mathbf{I}^1 \to \mathrm{M}_n(\mathbb{R})$ be the unique smooth solution of the matrix ODE

$$\tfrac{\mathrm{d}}{\mathrm{d}t} g(t) = g(t) \cdot f(t)$$

with initial condition $g(0) = 1$.

On the other hand, f defines a matrix 1-form

$$\alpha := f(t) \cdot \mathrm{d}t \in \Omega^1(\mathbf{I}^1) \otimes \mathrm{M}_n(\mathbb{R}).$$

It is not hard to show that

$$\mathrm{MI}(\alpha \mid \mathbf{I}^1) = g(1).$$

1-dimensional MI is used in various areas, such as mathematical physics and probability.

There are various names and notations for this operation. One name is *path ordered exponential integral*, with corresponding notation

$$\mathrm{Pexp} \int_{\sigma} \alpha.$$

In probability this operation is called a *time dependent continuous Markov process*.

Indeed, the geometric multiplicativity (2.3) is a manifestation of the Markov property.

In differential geometry the 1-dimensional MI has the following interpretation.

Suppose E is a vector bundle of rank n over X, with a *connection* ∇.

Assume E is trivial; so for a choice of basis the connection ∇ has a matrix

$$\alpha \in \Omega^1(X) \otimes \mathrm{M}_n(\mathbb{R}).$$

Let σ be a path in X. Then the element

$$\mathrm{MI}(\alpha \,|\, \sigma) \in \mathrm{GL}_n(\mathbb{R})$$

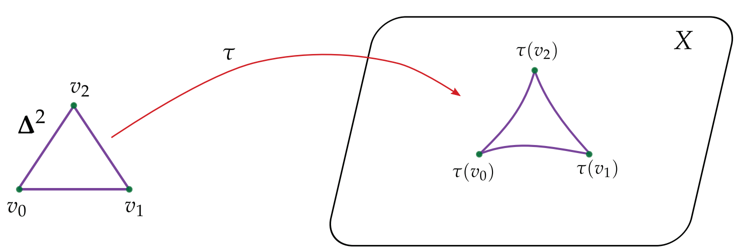
is the *holonomy* of ∇ along σ.

3. MI on Surfaces – a Naive Attempt

Consider another Lie group H, with Lie algebra $\mathfrak{h}$. As before X is a manifold.

Let β be an $\mathfrak{h}$-valued 2-form on X, i.e. $\beta \in \Omega^2(X) \otimes \mathfrak{h}$.

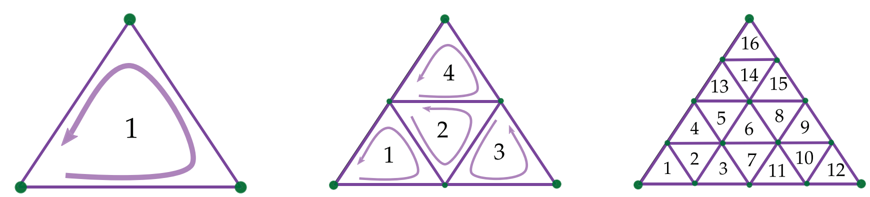
Let $\tau : \mathbf{\Delta}^2 \to X$ be a smooth map. So τ is a triangle in X:



We would like to construct a multiplicative integral

$$\mathrm{MI}(\beta \,|\, \tau) \in H.$$

For any $k \geq 0$ we partition the simplex $\mathbf{\Delta}^2$ into 4^k triangles labeled $1, \ldots, 4^k$, by the recursive rule shown below.



Composing with $\tau : \mathbf{\Delta}^2 \to X$ we obtain, for each k, a sequence of maps

$$\tau_1, \ldots, \tau_{4^k} : \mathbf{\Delta}^2 \to X.$$

We then define the k-th Riemann Product

$$\mathrm{RP}_k(\beta \,|\, \tau) := \prod_{i=1}^{4^k} \exp_H\Big(\int_{\tau_i} \beta\Big) \in H. \tag{3.1}$$

The geometry involved in these Riemann products is thus of a fractal nature.

The limit

$$\mathrm{MI}(\beta \,|\, \tau) := \lim_{k \to \infty} \mathrm{RP}_k(\beta \,|\, \tau) \in H \tag{3.2}$$

exists.

We know that when H is abelian there is equality

$$\mathrm{MI}(\beta \,|\, \tau) = \exp_H\Big(\int_{\tau} \beta\Big).$$

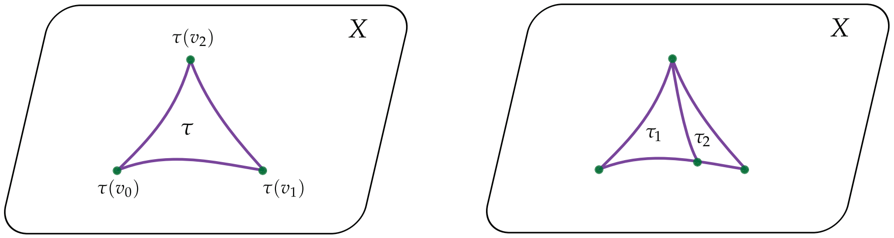
What about "geometric multiplicativity" ?

Suppose we partition $\mathbf{\Delta}^2$ into two triangles, by passing a straight line from v_2 to an arbitrary point on the opposite edge.

We get two smooth maps

$$\tau_1, \tau_2 : \mathbf{\Delta}^2 \to X$$

as shown below.



We would like $\mathrm{MI}(\beta \,|\, \tau)$ to be the product of $\mathrm{MI}(\beta \,|\, \tau_1)$ and $\mathrm{MI}(\beta \,|\, \tau_2)$.

But the product in which order? Remember that the group H is not abelian.

The answer: *in general, neither order works!*

In the next section we are going to produce a more refined MI, both in terms of the fractal geometry and in terms of the Lie theory, in an attempt to solve this problem.

4. Twisting the 2-Dimensional MI

Definition 4.1. A *Lie crossed module* is data

$$(G, H, \Psi, \Phi)$$

consisting of:

- Lie groups G and H.
- An analytic action Ψ of G on H by automorphisms of Lie groups, called the *twisting*.
- A map of Lie groups $\Phi : H \to G$, called the *feedback*.

The conditions are:

(i) The feedback Φ is G-equivariant, with respect to the twisting Ψ, and the conjugation action Ad_G of G on itself.

(ii) $\Psi \circ \Phi = \mathrm{Ad}_H$, as actions of H on itself.

Here is a commutative diagram of groups depicting the situation:

$$H \xrightarrow{\Phi} G \xrightarrow{\Psi} \mathrm{Aut}(H), \qquad \mathrm{Ad}_H : H \to \mathrm{Aut}(H) \tag{4.2}$$

The subgroup

$$H_0 := \mathrm{Ker}(\Phi) \subset H$$

is called the *inertia group*.

Note that

$$H_0 \subset \mathrm{Ker}(\mathrm{Ad}_H) = \mathrm{Z}(H),$$

where $\mathrm{Z}(H)$ is the center of the group H.

We shall denote the Lie algebras of G and H by $\mathfrak{g}$ and $\mathfrak{h}$ respectively.

Here are a few examples of Lie crossed modules (G, H, Ψ, Φ).

Example 4.3. H is any Lie group, $G = H$, $\Psi = \mathrm{Ad}_G$ and $\Phi = \mathrm{id}$. Here H_0 is the trivial group.

Example 4.4. H is an abelian Lie group, and G is the trivial group. Here $H_0 = H$.

Example 4.5. Suppose

$$1 \to N \to H \xrightarrow{\Phi} G \to 1$$

is a central extension of Lie groups.

There is an induced action Ψ of G on H (this is an easy exercise in group theory). Here $H_0 = N$ of course.

This example contains both previous examples.

Example 4.6. Consider a nonabelian *unipotent* group H, e.g.

$$H = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix} \subset \mathrm{GL}_3(\mathbb{R}).$$

Here $\exp_H : \mathfrak{h} \to H$ is a diffeomorphism.

This implies that the group

$$G := \mathrm{Aut}(H)$$

is a Lie group (isomorphic to a closed subgroup of $\mathrm{GL}(\mathfrak{h})$).

We get a Lie crossed module

$$(G, H, \Psi, \Phi)$$

with $\Phi := \mathrm{Ad}_H$.

The inertia group here is of intermediate size: $1 \subsetneqq H_0 \subsetneqq H$.

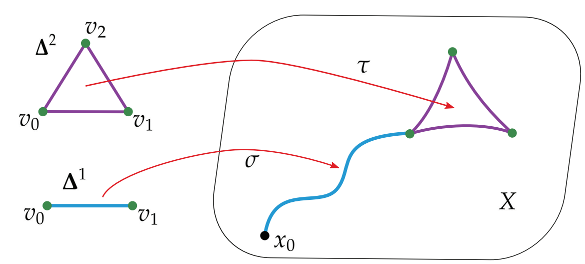
This is the sort of thing that comes up in twisted deformation quantization (the *Deligne crossed groupoid*).

By *pointed manifold* (X, x_0) we mean a manifold X, with a chosen point $x_0 \in X$ called the base point.

Definition 4.7. A *kite* in the pointed manifold (X, x_0) is a pair (σ, τ), consisting of smooth maps

$$\sigma : \mathbf{\Delta}^1 \to X \quad \text{and} \quad \tau : \mathbf{\Delta}^2 \to X,$$

satisfying $\sigma(v_0) = x_0$ and $\sigma(v_1) = \tau(v_0)$.



The integrand in our multiplicative integration is a pair of differential forms (α, β), where

$$\alpha \in \Omega^1(X) \otimes \mathfrak{g} \quad \text{and} \quad \beta \in \Omega^2(X) \otimes \mathfrak{h}. \tag{4.8}$$

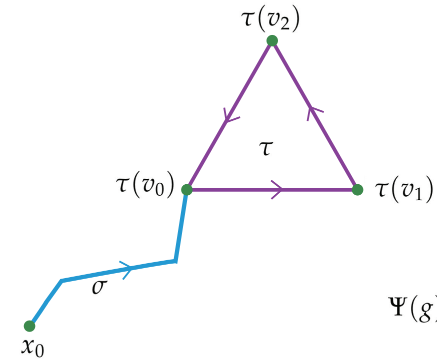
For any $k \in \mathbb{N}$ we define the k-th Riemann product

$$\mathrm{RP}_k(\alpha, \beta \,|\, \sigma, \tau) \in H.$$

For $k = 0, 1$ there are rules that will be explained on the next two slides.

For $k \geq 2$ we proceed recursively, using the rules for $k = 0, 1$.

We shall see that the fractal geometry here is very similar to what we had in the naive MI (on slide 16).



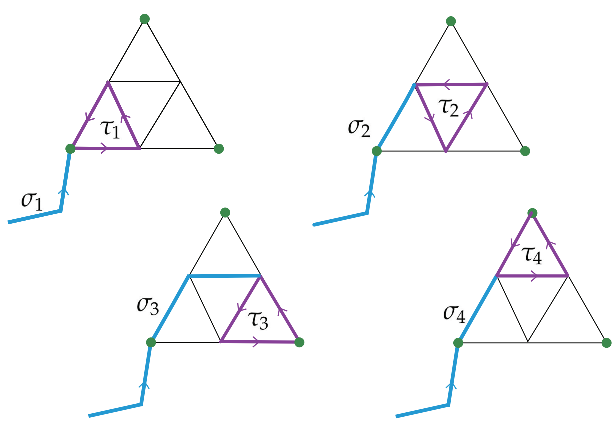
$$\int_\tau \beta \in \mathfrak{h}$$

$$g := \mathrm{MI}(\alpha \,|\, \sigma) \in G$$

$$\Psi(g)\Big(\int_\tau \beta\Big) \in \mathfrak{h} \quad \text{(with twist)}$$

$$\mathrm{RP}_0(\alpha, \beta \,|\, \sigma, \tau) := \exp_H\Big(\Psi(g)\Big(\int_\tau \beta\Big)\Big) \in H$$

Figure : The 0-th order Riemann product $\mathrm{RP}_0(\alpha, \beta \,|\, \sigma, \tau)$ of the pair (α, β) on the kite (σ, τ).



$$\mathrm{RP}_1(\alpha, \beta \,|\, \sigma, \tau) := \prod_{i=1}^{4} \mathrm{RP}_0(\alpha, \beta \,|\, \sigma_i, \tau_i)$$

Figure : The 1-st order Riemann product $\mathrm{RP}_1(\alpha, \beta \,|\, \sigma, \tau)$.

The limit

$$\mathrm{MI}(\alpha, \beta \,|\, \sigma, \tau) := \lim_{k \to \infty} \mathrm{RP}_k(\alpha, \beta \,|\, \sigma, \tau)$$

in H exists.

This is called the *multiplicative integral of* (α, β) *on* (σ, τ).

If H is abelian and G is trivial, then

$$\mathrm{MI}(\alpha, \beta \,|\, \sigma, \tau) = \exp_H\Big(\int_\tau \beta\Big),$$

as expected.

When $G = H = \mathrm{GL}_n(\mathbb{R})$, and $\Phi = \mathrm{id}$, we recover Schlesinger's old construction.

Note that if $\alpha = 0$ the twisting is trivial, so we are back with the naive MI from Section 3:

$$\mathrm{MI}(\alpha, \beta \,|\, \sigma, \tau) = \mathrm{MI}(\beta \,|\, \tau).$$

This is bad, since – as we already know – the operation $\mathrm{MI}(\beta \,|\, \tau)$ does not satisfy "geometric multiplicativity" in general.

In the next section we will see what is a sufficient (and perhaps necessary) condition on the integrand (α, β) that will make everything all right.

5. Stokes Theorem in Dimension 2

We continue with the earlier setup: (G, H, Ψ, Φ) is a Lie crossed module, (X, x_0) is a pointed manifold,

$$\alpha \in \Omega^1(X) \otimes \mathfrak{g} \quad \text{and} \quad \beta \in \Omega^2(X) \otimes \mathfrak{h}.$$

The derivative of the Lie group map $\Phi : H \to G$ is a Lie algebra homomorphism $\phi : \mathfrak{h} \to \mathfrak{g}$.

By tensoring we obtain a homomorphism

$$\phi : \Omega(X) \otimes \mathfrak{h} \to \Omega(X) \otimes \mathfrak{g}.$$

Definition 5.1. The pair (α, β) is called a *connection-curvature pair* if

$$\phi(\beta) = \mathrm{d}(\alpha) + \tfrac{1}{2}[\alpha, \alpha]$$

in $\Omega^2(X) \otimes \mathfrak{g}$.

The name should be explained. Consider the situation where

$$\alpha \in \Omega^1(X) \otimes \mathrm{M}_n(\mathbb{R})$$

is the matrix of a connection ∇ (as on page 14).

Then

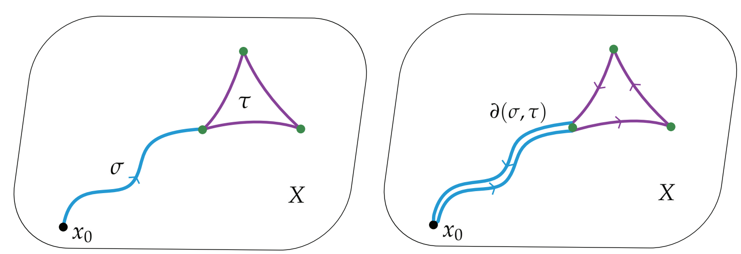
$$\mathrm{d}(\alpha) + \tfrac{1}{2}[\alpha, \alpha] \in \Omega^2(X) \otimes \mathrm{M}_n(\mathbb{R})$$

is the matrix of the *curvature* of ∇.

In [BM] the condition in the definition is called "vanishing of the fake curvature".

Definition 5.2. Let (σ, τ) be a kite in (X, x_0), shown in the figure below.

Its *boundary* is the closed path shown to the right.



Given a connection-curvature pair (α, β), we have group elements

$$\mathrm{MI}(\alpha, \beta \,|\, \sigma, \tau) \in H$$

and

$$\mathrm{MI}\big(\alpha \,|\, \partial(\sigma, \tau)\big) \in G.$$

Theorem 5.3. *[Nonabelian 2-Dimensional Stokes Theorem]*

Let (σ, τ) be a kite in (X, x_0), and let (α, β) be connection-curvature pair.

Then

$$\Phi\big(\mathrm{MI}(\alpha, \beta \,|\, \sigma, \tau)\big) = \mathrm{MI}\big(\alpha \,|\, \partial(\sigma, \tau)\big).$$

When H is abelian and G is trivial, this is an immediate consequence of the usual Stokes Theorem.

When $G = H = \mathrm{GL}_n(\mathbb{R})$, and $\Phi = \mathrm{id}$, this is Schlesinger's Theorem (proved in the 1920's). See [DF, KMR].

Schlesinger's nonabelian MI is of limited value, since (by the theorem) the quantity $\mathrm{MI}(\alpha, \beta \,|\, \sigma, \tau)$ can be computed as a 1-dimensional MI...

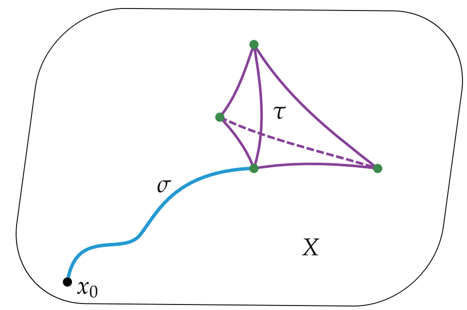
However: when the inertia group $H_0 = \mathrm{Ker}(\Phi)$ is of intermediate size, namely $1 \subsetneqq H_0 \subsetneqq H$, this is a new result.

6. Stokes Theorem in Dimension 3

Definition 6.1. A *balloon* in a pointed manifold (X, x_0) is a pair (σ, τ), consisting of smooth maps

$$\sigma : \mathbf{\Delta}^1 \to X \quad \text{and} \quad \tau : \mathbf{\Delta}^3 \to X,$$

satisfying $\sigma(v_0) = x_0$ and $\sigma(v_1) = \tau(v_0)$.



As before, (G, H, Ψ, Φ) is a Lie crossed module, (X, x_0) is a pointed manifold,

$$\alpha \in \Omega^1(X) \otimes \mathfrak{g} \quad \text{and} \quad \beta \in \Omega^2(X) \otimes \mathfrak{h}.$$

Definition 6.2. Let (σ, τ) be a balloon in (X, x_0), as shown in the previous figure.

1. The *boundary* of (σ, τ) is the sequence of kites

$$\partial(\sigma, \tau) = \big(\partial_1(\sigma, \tau),\, \partial_2(\sigma, \tau),\, \partial_3(\sigma, \tau),\, \partial_4(\sigma, \tau)\big)$$

shown on the next slide.

2. We define

$$\mathrm{MI}\big(\alpha, \beta \,|\, \partial(\sigma, \tau)\big) := \prod_{1=1}^{4} \mathrm{MI}\big(\alpha, \beta \,|\, \partial_i(\sigma, \tau)\big) \in H.$$

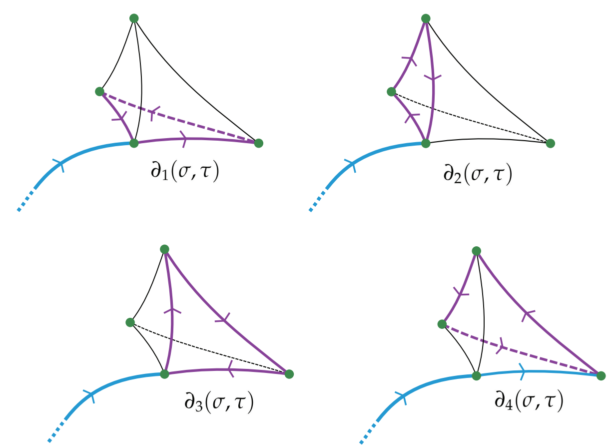


Figure : The boundary of the balloon (σ, τ) from page 34.

A 3-dimensional Stokes Theorem must involve some kind of 3-dimensional integration.

I do not know how to define a 3-dimensional nonabelian MI. Fortunately we do not need it – all we need is the *twisted abelian MI* that I will now introduce.

Recall the inertia group $H_0 = \mathrm{Ker}(\Phi)$, which is inside the center of H, so it is abelian.

Let $\mathfrak{h}_0$ denote the Lie algebra of H_0.

A differential form
$$\gamma \in \Omega^p(X) \otimes \mathfrak{h}_0$$
will be called an *inert form*.

A differential 1-form
$$\alpha \in \Omega^1(X) \otimes \mathfrak{g}$$
is called a *tame connection* if it is part of a connection-curvature pair (α, β); see Definition 5.1.

Suppose we are given a pair (α, γ) consisting of a tame connection α and an inert 3-form γ. Let (σ, τ) be a balloon in (X, x_0).

Then there is an element
$$\mathrm{MI}(\alpha, \gamma \mid \sigma, \tau) \in H_0$$
called the twisted abelian MI.

If $\alpha = 0$ then
$$\mathrm{MI}(\alpha, \gamma \mid \sigma, \tau) = \exp_{H_0}\Big(\int_\tau \gamma\Big).$$

The next definition is from [BM].

Definition 6.3. Let (α, β) be a connection-curvature pair.

There is an element
$$\gamma \in \Omega^3(X) \otimes \mathfrak{h}$$
called the 3-*curvature* of (α, β).

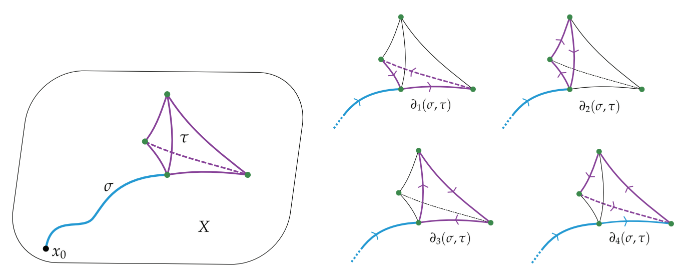
The formula for γ involves the twisting Ψ of course, but I can't give the details for lack of time.

All I will say is that when $\alpha = 0$, then $\gamma = \mathrm{d}(\beta)$.

Theorem 6.4. *[Nonabelian 3-Dimensional Stokes Theorem]*

Let (α, β) be a connection-curvature pair, with 3-curvature γ.

1. *The form γ is inert.*
2. *For any balloon (σ, τ) in (X, x_0) one has*
$$\mathrm{MI}(\alpha, \gamma \mid \sigma, \tau) = \mathrm{MI}\big(\alpha, \beta \mid \partial(\sigma, \tau)\big).$$



Theorem 6.4 is new.

Except of course when H is abelian and G is trivial. Then the result is an immediate consequence of the usual Stokes Theorem (since $\gamma = \mathrm{d}(\beta)$).

The first part of the theorem is a "generalized Bianchi identity".

Exercise 6.5. Show how Theorem 6.4 solves the problem of "geometric multiplicativity" for the 2-dimensional MI on slides 18-19.

It seems that the nonabelian 2-dimensional MI described here is the most general sort that satisfies geometric multiplicativity.

- END ? -

7. Concluding Remarks

Here are a few words on the proofs in [Ye4]. We do a lot of "hard calculus", mainly estimates for power series, like the CBH series for the nonabelian exponential.

We mostly work with cubes – not with tetrahedra. This is because the differential geometry of the binary subdivisions of the cube is much better than that of the barycentric subdivisions of the tetrahedron.

Our maps and differential forms are allowed to be *piecewise smooth*. This makes things somewhat more difficult, but is needed in several places, e.g. in the boundary of a kite (slide 31).

Our MI is related to the work of Breen and Messing [BM] on the *differential geometry of gerbes*. As mentioned above, we learned several important ideas from that paper, including the 3-curvature. The methods in [BM] are all algebro-geometric, and there is no integration.

The most important case of Theorem 6.4 is when the 3-curvature γ is 0. This case was predicted by Kontsevich [Ko]; but he gave no proof.

Baez, Schreiber and others have looked into the question of nonabelian MI on surfaces, from the point of view of mathematical physics. For them it was a question of *nonabelian gauge theory* and *higher parallel transport*. See [BS], or search on [nLab].

The paper [SW] of Schreiber and Waldorf contains a proof of the case $\gamma = 0$ of Theorem 6.4. Their methods are "soft", relying on a differential calculus of functors that they develop.

There is a relation between 1-dimensional nonabelian MI, *Chen integrals* and *noncommutative ring theory*. See Kapranov's paper [Ka1].

In the very recent paper [Ka2], Kapranov studies *nonabelian holonomy in n-dimensional crossed complexes* and *iterated Chen integrals*. For $n = 2$, crossed complexes are just the crossed modules we talked about; and there is a formal similarity to our 2-dimensional nonabelian MI.

It is likely that there should be a higher dimensional version ($n > 3$) of our nonabelian MI, and some kind of Stokes Theorem. But nothing has been done yet.

To finish, here a few words about the relation between 2-dimensional nonabelian MI and *twisted deformation quantization of algebraic varieties*.

One of the key technical problems in twisted deformation quantization is that of *gluing nonabelian gerbes*.

Kontsevich [Ko] proposed that a 2-dimensional nonabelian MI, satisfying a 3-dimensional Stokes Theorem, would provide a solution of this gluing problem.

My goal was to develop such a theory of MI. This was accomplished in the book [Ye4], and it is in fact much more intricate than what I presented in the talk.

Eventually I found another approach to the gluing problem, which is more direct, and this is the one used in [Ye2]. See the papers [Ye3] and [Ye5].

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References

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