Rigid Dualizing Complexes and Perverse Coherent Sheaves

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1. Background on Dualizing Complexes
2. Rigid Complexes over Rings
3. Rigid Dualizing Complexes over Rings
4. Rigid Dualizing Complexes over Schemes
5. Perverse Coherent Sheaves
6. Cohen-Macaulay Complexes
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3. Rigid Dualizing Complexes over Rings
4. Rigid Dualizing Complexes over Schemes
5. Perverse Coherent Sheaves
6. Cohen-Macaulay Complexes

This talk is about joint work with James Zhang (Seattle).
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The full subcategory of bounded complexes with coherent cohomologies is $\mathcal{D}^b_c(\text{Mod} \mathcal{O}_X)$. It is equivalent to $\mathcal{D}^b(\text{Coh} \mathcal{O}_X)$. 

Definition 1.1. (Grothendieck [RD]) A dualizing complex on $X$ is a complex $\mathcal{R} \in D^b_c(\text{Mod } \mathcal{O}_X)$ satisfying the two conditions:

(i) $\mathcal{R}$ has finite injective dimension.

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is an auto-duality of \( D^b_c(\text{Mod} \mathcal{O}_X) \).

When \( X = \text{Spec} \, A \) is affine, the complex \( R := R\Gamma(X, \mathcal{R}) \in D^b_f(\text{Mod} A) \) is called a dualizing complex over \( A \), and

\[
\mathcal{M} \mapsto R\text{Hom}_A(\mathcal{M}, R)
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is an auto-duality of \( D^b_f(\text{Mod} A) \).
Suppose \( \mathbb{K} \) is a regular noetherian ring of finite Krull dimension, and \( X \) is a finite type \( \mathbb{K} \)-scheme, with structural morphism \( \pi : X \to \text{Spec} \mathbb{K} \).
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The proof of existence of this complex, and its properties, is very difficult.

In this lecture I will explain an alternative approach to Grothendieck duality.

For other approaches see the papers in the references, mainly by Joseph Lipman and his coauthors.
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When $B$ is flat over $A$ one has

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But in general one has to use DG algebras to define $\text{Sq}_{B/A} M$. 
The functor $\text{Sq}_{B/A}$ is quadratic, in the following sense. Given a morphism $\phi : M \to N$ in $\mathbf{D} (\text{Mod} B)$, and an element $b \in B$, one has

$$\text{Sq}_{B/A} (b \phi) = b^2 \text{Sq}_{B/A} (\phi)$$

in

$$\text{Hom}_{\mathbf{D} (\text{Mod} B)} (\text{Sq}_{B/A} M, \text{Sq}_{B/A} N).$$
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**Definition 2.1.** Let \( B \) be a noetherian \( A \)-algebra, and let \( M \) be a complex in \( \mathcal{D}^b_f(\text{Mod } B) \) that has finite flat dimension over \( A \). Assume

\[
\rho : M \xrightarrow{\sim} \text{Sq}_{B/A} M
\]

is an isomorphism in \( \mathcal{D}(\text{Mod } B) \). Then the pair \((M, \rho)\) is called a rigid complex over \( B \) relative to \( A \).
Definition 2.2. Say \((M, \rho)\) and \((N, \sigma)\) are rigid complexes over \(B\) relative to \(A\). A morphism \(\phi : M \to N\) in \(\mathcal{D}(\text{Mod} B)\) is called a rigid morphism relative to \(A\) if the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\rho} & \text{Sq}_{B/A} M \\
\downarrow \phi & & \downarrow \text{Sq}_{B/A}(\phi) \\
N & \xrightarrow{\sigma} & \text{Sq}_{B/A} N
\end{array}
\]

is commutative.
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Note that the only rigid automorphism of a rigid dualizing complex $(R, \rho)$ is the identity $1_R$. Indeed, any automorphism $\phi$ of $R$ has to be of the form $\phi = a1_R$ for some invertible element $a \in A$. If $\phi$ is rigid then $a^2 = a$, and hence $a = 1$. 
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**Theorem 3.2.** Let $A$ be an essentially finite type $\mathbb{K}$-algebra. Then $A$ has a rigid dualizing complex $(R_A, \rho_A)$, which is unique up to a unique rigid isomorphism.
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Recall that a ring homomorphism $f^* : A \to B$ is called **finite** if $B$ is a finitely generated $A$-module.

**Theorem 3.3.** Let $A$ and $B$ be essentially finite type $\mathbb{K}$-algebras, and let $f^* : A \to B$ be a finite homomorphism. Then the complex $\text{RHom}_A(B, R_A)$ has an induced rigidifying isomorphism, and there is a unique rigid isomorphism

$$\text{RHom}_A(B, R_A) \cong R_B.$$
We say that $B$ is **essentially smooth** of relative dimension $n$ over $A$ if it is essentially finite type, formally smooth, and the rank of the projective $B$-module $\Omega^1_{B/A}$ is $n$. When $n = 0$ we say $B$ is **essentially étale**.
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**Example 3.4.** If $A'$ is a localization of $A$ then $A \to A'$ is essentially étale. If $B = A[t_1, \ldots, t_n]$ is a polynomial algebra then $A \to B$ is essentially smooth of relative dimension $n$. 
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**Theorem 3.5.** Let $A$ and $B$ be essentially finite type $\mathbb{K}$-algebras, and let $f^* : A \to B$ be an essentially smooth homomorphism of relative dimension $n$. Then the complex $\Omega_{B/A}^n[n] \otimes_A R_A$ has an induced rigidifying isomorphism, and there is a unique rigid isomorphism

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**Corollary 3.6.** *Given an essentially étale homomorphism $f^* : A \to B$, there is a unique rigid isomorphism*

$$B \otimes_A R_A \cong R_B.$$
4. Rigid Dualizing Complexes over Schemes
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1. $\mathcal{R} \in D^b_c(\text{Mod} \mathcal{O}_X)$ is a dualizing complex on $X$.
2. $\rho = \{\rho_U\}$ is a collection of rigidifying isomorphisms, indexed by the affine open sets of $X$. Namely, for any affine open set $U$, $\rho_U$ is a rigidifying isomorphism for the dualizing complex $R\Gamma(U, \mathcal{R})$ over the $\mathbb{K}$-algebra $A := \Gamma(U, \mathcal{O}_X)$. 
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The condition is:

(†) For any inclusion $V \subset U$ of affine open sets, with $A := \Gamma(U, \mathcal{O}_X)$ and $B := \Gamma(V, \mathcal{O}_X)$, the canonical isomorphism

$$B \otimes_A R\Gamma(U, \mathcal{R}) \cong R\Gamma(V, \mathcal{R})$$

is rigid, with respect to the rigidifying isomorphisms $\rho_U$ and $\rho_V$. 
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Dualizing complex $\mathcal{R}$ over $X$.
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$U = \text{Spec } A$

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\[ \mathcal{R} \]

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dualizing complex

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Rigid & Perverse

14 / 31
4. Rigid Dualizing Complexes over Schemes

- Rigid dualizing complex
  \( \mathcal{R} \)
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Now suppose $V \subset U$ is another affine open set, with $B := \Gamma(V, \mathcal{O}_X)$ and rigid dualizing complex $(R_B, \rho_B)$. According to Corollary 3.6 there is a unique isomorphism

$$\phi_{V/U} : \mathcal{R}_U|_V \cong \mathcal{R}_V \tag{4.2}$$

in $D(\text{Mod} \mathcal{O}_V)$ which respects rigidity.
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Therefore given an affine open set $W \subset V$, these isomorphisms satisfy

$$\phi_{W/V} \circ \phi_{V/U} = \phi_{W/U}.$$
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Grothendieck’s solution in the commutative case, in [RD], was to use Cousin complexes. This solution can be used in our setup too, but it has a disadvantage: we are forced to leave the derived category, and then return to it.
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We propose an alternative solution: perverse coherent sheaves.
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Remark 4.3. For noncommutative ringed schemes one is forced to use perverse coherent sheaves, since Cousin complexes are ill-behaved. See [YZ3].
5. Perverse Coherent Sheaves
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The notion of t-structures and perverse sheaves were introduced by Beilinson, Bernstein and Deligne [BBD] around 1980. This was in the context of intersection cohomology on singular spaces. For such a space $X$ they were interested in t-structures on subcategories of $\mathcal{D}(\text{Mod } \mathbb{K}_X)$, where $\mathbb{K}_X$ is a constant sheaf of rings on $X$. 
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Perverse coherent sheaves came into the scene only very recently, independently in the work of Bezrukavnikov (after Deligne) [Bz], Bridgeland [Br], Kashiwara [Ka] and our paper [YZ3].
Let me recall what is a t-structure on a triangulated category $\mathcal{D}$. It consists of the datum of two full subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ satisfying the axioms below, where $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n]$. 
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(i) \( \mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0} \) and \( \mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0} \).

(ii) \( \text{Hom}_\mathcal{D}(M, N) = 0 \) for \( M \in \mathcal{D}^{\leq 0} \) and \( N \in \mathcal{D}^{\geq 1} \).

(iii) For any \( M \in \mathcal{D} \) there is a distinguished triangle

\[
M' \to M \to M'' \to M'[1]
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$$M' \to M \to M'' \to M'[1]$$

in $\mathcal{D}$ with $M' \in \mathcal{D}^{\leq 0}$ and $M'' \in \mathcal{D}^{\geq 1}$.

When these conditions are satisfied one defines the heart of $\mathcal{D}$ to be the full subcategory $\mathcal{D}^0 := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. This is an abelian category.
Given a scheme $X$, the derived category $D^b_c(\text{Mod} \, \mathcal{O}_X)$ has the **standard $t$-structure**, in which

$D^b_c(\text{Mod} \, \mathcal{O}_X)_{\leq 0} := \{ \mathcal{M} \in D^b_c(\text{Mod} \, \mathcal{O}_X) \mid H^i \mathcal{M} = 0 \text{ for all } i > 0 \}$,

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\]

\[
D^b_c(\text{Mod } \mathcal{O}_X)^\geq 0 := \{ \mathcal{M} \in D^b_c(\text{Mod } \mathcal{O}_X) \mid H^i \mathcal{M} = 0 \text{ for all } i < 0 \}.
\]

The heart $D^b_c(\text{Mod } \mathcal{O}_X)^0$ is equivalent to the category $\text{Coh } \mathcal{O}_X$ of coherent sheaves.

Other t-structures will be referred to as perverse t-structures.
Here is an observation. Suppose \( A \) is an essentially finite type \( \mathbb{K} \)-algebra, where as before \( \mathbb{K} \) is a finite dimensional regular noetherian ring. Let \( R_A \) be the rigid dualizing complex of \( A \).
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Then the duality $D := \text{RHom}_A(-, R_A)$ gives rise to a perverse t-structure

\[
pD^b_f(\text{Mod} A)_{\leq 0} := \{ M \mid H^iDM = 0 \text{ for all } i < 0 \},
\]

\[
pD^b_f(\text{Mod} A)_{\geq 0} := \{ M \mid H^iDM = 0 \text{ for all } i > 0 \}.
\]

on $\text{D}^b_f(\text{Mod} A)$. 

5. Perverse Coherent Sheaves

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Then the duality $D := \text{RHom}_A(\_ , R_A)$ gives rise to a perverse t-structure

$$\mathcal{P}^b_{\leq 0} := \{M \mid H^iDM = 0 \text{ for all } i < 0\},$$

$$\mathcal{P}^b_{\geq 0} := \{M \mid H^iDM = 0 \text{ for all } i > 0\}.$$

on $\mathcal{D}^b_{\mathfrak{f}}(\text{Mod} A)$.

We call it the rigid perverse t-structure. The heart is denoted by $\mathcal{P}^b_{\mathfrak{f}}(\text{Mod} A)^0$. 
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on $D^b_f(\text{Mod} A)$.

We call it the rigid perverse t-structure. The heart is denoted by $\text{pD}^b_f(\text{Mod} A)^0$.

The next theorem was proved in [YZ3].
Theorem 5.1. Let $X$ be a finite type $\mathbb{K}$-scheme. Let $\star$ denote either $\leq 0$, $\geq 0$ or 0.
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Define $\mathcal{P} \mathcal{D}^b_{c}(\text{Mod } \mathcal{O}_X)^\star$ to be the class of complexes $\mathcal{M} \in \mathcal{D}^b_c(\text{Mod } \mathcal{O}_X)$ such that

$$R\Gamma(U, \mathcal{M}) \in \mathcal{P} \mathcal{D}^b_f(\text{Mod } A)^\star$$

for any affine open set $U$, with $A := \Gamma(U, \mathcal{O}_X)$. 

Theorem 5.1. Let \( X \) be a finite type \( \mathbb{K} \)-scheme. Let \( \star \) denote either \( \leq 0 \), \( \geq 0 \) or \( 0 \).

Define \( \mathcal{P}\mathcal{D}_c^b(\mathrm{Mod}\mathcal{O}_X)^\star \) to be the class of complexes \( \mathcal{M} \in \mathcal{D}_c^b(\mathrm{Mod}\mathcal{O}_X) \) such that

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\]

for any affine open set \( U \), with \( A := \Gamma(U, \mathcal{O}_X) \).

Then:

1. The pair

\[
(\mathcal{P}\mathcal{D}_c^b(\mathrm{Mod}\mathcal{O}_X)^{\leq 0}, \mathcal{P}\mathcal{D}_c^b(\mathrm{Mod}\mathcal{O}_X)^{\geq 0})
\]

is a t-structure on \( \mathcal{D}_c^b(\mathrm{Mod}\mathcal{O}_X) \).
Theorem 5.1. Let $X$ be a finite type $\mathbb{K}$-scheme. Let $\star$ denote either $\leq 0$, $\geq 0$ or $0$.

Define $\mathcal{D}^b_c(\text{Mod } \mathcal{O}_X)^\star$ to be the class of complexes $\mathcal{M} \in \mathcal{D}^b_c(\text{Mod } \mathcal{O}_X)$ such that

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for any affine open set $U$, with $A := \Gamma(U, \mathcal{O}_X)$.

Then:

1. The pair $$(\mathcal{D}^b_c(\text{Mod } \mathcal{O}_X)^{\leq 0}, \mathcal{D}^b_c(\text{Mod } \mathcal{O}_X)^{\geq 0})$$

is a $t$-structure on $\mathcal{D}^b_c(\text{Mod } \mathcal{O}_X)$.

2. The assignment $V \mapsto \mathcal{D}^b_c(\text{Mod } \mathcal{O}_V)^0$, for $V \subset X$ open, is a stack of abelian categories on $X$. 
Part (2) says that the objects of $\text{pD}_c^b(\text{Mod} \mathcal{O}_X)^0$, which we call \textit{perverse coherent sheaves}, can be glued. They behave like sheaves, and hence the name.
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For any affine open set $U = \text{Spec} \ A$, the dualizing complex $\mathcal{R}_U$ (the sheafification of the rigid dualizing complex $R_A$) is clearly a perverse coherent sheaf on $U$. 
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For any affine open set $U = \text{Spec} \, A$, the dualizing complex $\mathcal{R}_U$ (the sheafification of the rigid dualizing complex $R_A$) is clearly a perverse coherent sheaf on $U$.

Thus we can use the isomorphisms

$$\phi_{U,V} : \mathcal{R}_U|_{U \cap V} \xrightarrow{\sim} \mathcal{R}_V|_{U \cap V},$$

deduced from equation (4.2), to glue the affine dualizing complexes.
$U = \text{Spec } A$

$\mathcal{R}_U \in \text{pD}_c^b(\text{Mod } \mathcal{O}_U)^0$
$\mathcal{R}_V \in \mathcal{D}^b_c(\text{Mod } \mathcal{O}_V)^0$

$\mathcal{R}_U \in \mathcal{D}^b_c(\text{Mod } \mathcal{O}_U)^0$

$X$

$V = \text{Spec } B$

$U = \text{Spec } A$
5. Perverse Coherent Sheaves

\[ \mathcal{R}_V \in \mathcal{D}^b_c(\text{Mod } \mathcal{O}_V)^0 \]

\[ \phi_{U,V} : \mathcal{R}_U|_{U \cap V} \cong \mathcal{R}_V|_{U \cap V} \text{ in } \mathcal{D}^b_c(\text{Mod } \mathcal{O}_{U \cap V})^0 \]

\[ V = \text{Spec } B \]

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In this way we obtain:
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**Theorem 5.2.** Let $X$ be a finite type $\mathbb{K}$-scheme. There exists a rigid dualizing complex $(\mathcal{R}_X, \rho_X)$ over $X$ relative to $\mathbb{K}$, and it is unique up to a unique rigid isomorphism.
Along the same lines, using Theorems 3.3 and 3.5, we can also prove:
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**Theorem 5.3.** Let $X$ and $Y$ be finite type $\mathbb{K}$-schemes, and let $f : X \to Y$ be a morphism.
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**Theorem 5.3.** Let $X$ and $Y$ be finite type $\mathbb{K}$-schemes, and let $f : X \to Y$ be a morphism.

1. If $f$ is finite, then there is a unique isomorphism

$$Rf_* R^0 X \cong R\text{Hom}_{O_Y}(f_* O_X, R^0 Y)$$

which respects rigidity.
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**Theorem 5.3.** Let $X$ and $Y$ be finite type $\mathbb{K}$-schemes, and let $f : X \to Y$ be a morphism.

1. If $f$ is finite, then there is a unique isomorphism

$$Rf_* R_X \cong R\text{Hom}_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathcal{R}_Y)$$

which respects rigidity.

2. If $f$ is smooth of relative dimension $n$, then there is a unique isomorphism

$$\Omega^n_{X/Y}[n] \otimes_{\mathcal{O}_X} f^* \mathcal{R}_Y \cong R_X$$

which respects rigidity.
Proper morphisms and residues are treated in [Ye6].
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**Remark 5.4.** Recently I discovered a totally new proof of the duality theorem for proper morphisms, which uses perverse sheaves only, avoiding residue calculations. If correct, the new proof will make the paper [Ye6] much shorter.
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**Remark 5.4.** Recently I discovered a totally new proof of the duality theorem for proper morphisms, which uses perverse sheaves only, avoiding residue calculations. If correct, the new proof will make the paper [Ye6] much shorter.

**Remark 5.5.** I think all the results here work also for essentially finite type \( \mathbb{K} \)-schemes.
6. Cohen-Macaulay Complexes
As before $X$ is a finite type $\mathbb{K}$-scheme. Let $\mathcal{R}_X$ be the rigid dualizing complex of $X$. 
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As before $X$ is a finite type $\mathbb{K}$-scheme. Let $\mathcal{R}_X$ be the rigid dualizing complex of $X$.

Given a point $x \in X$ let $k(x)$ be its residue field. We denote by $\dim_{\mathbb{K}}(x)$ the unique integer $i$ such that

$$\text{Ext}^{-i}_{\mathcal{O}_{X,x}}(k(x), \mathcal{R}_{X,x}) \neq 0.$$
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As before $X$ is a finite type $\mathbb{K}$-scheme. Let $\mathcal{R}_X$ be the rigid dualizing complex of $X$.

Given a point $x \in X$ let $k(x)$ be its residue field. We denote by $\dim_K(x)$ the unique integer $i$ such that

$$\operatorname{Ext}_{\mathcal{O}_{X,x}}^{-i}(k(x), \mathcal{R}_{X,x}) \neq 0.$$ 

Then the function

$$\dim_K : X \rightarrow \mathbb{Z}$$

is a dimension function, i.e.

$$\dim_K(y) = \dim_K(x) - 1$$

when $y$ is an immediate specialization of $x$. 
Example 6.1. Take $\mathbb{K} := \mathbb{Z}$, the ring of integers, and $X := \mathbb{A}^1_\mathbb{K} = \text{Spec } \mathbb{K}[t]$, the affine line. Consider the following points in $X$: $x_0$ is the generic point; $x_1$ is the prime ideal $(t)$; and $x_2$ is the maximal ideal $(t, 2)$. Then

$$\text{dim}_{\mathbb{K}}(x_i) = 1 - i.$$
6. Cohen-Macaulay Complexes

\[ X = \mathbb{A}^1_{\mathbb{Z}} \]

\( x_0 = (0) \)
generic point
\( \dim_{\mathbb{Z}}(x_0) = 1 \)

\( x_1 = (t) \)
curve
\( \dim_{\mathbb{Z}}(x_1) = 0 \)

\( x_2 = (t, 2) \)
closed point
\( \dim_{\mathbb{Z}}(x_2) = -1 \)

\( \text{Spec } \mathbb{Z} \)
Recall from [RD] that a complex $\mathcal{M} \in D^b_c(\text{Mod} \mathcal{O}_X)$ is called \textbf{Cohen-Macaulay} if for every point $x$ the local cohomologies $H^i_x \mathcal{M}$ all vanish except for $i = -\dim_k(x)$.
Recall from [RD] that a complex $\mathcal{M} \in D^b_c(\text{Mod } \mathcal{O}_X)$ is called **Cohen-Macaulay** if for every point $x$ the local cohomologies $H^i_x \mathcal{M}$ all vanish except for $i = - \dim_K(x)$.

Here is another result from [YZ3].
Theorem 6.2. Let $X$ be a finite type scheme over $\mathbb{K}$, let $\mathcal{R}_X$ be the rigid dualizing complex of $X$, and let $D$ be the duality functor $R\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{R}_X)$. 
Theorem 6.2. Let $X$ be a finite type scheme over $\mathbb{K}$, let $\mathcal{R}_X$ be the rigid dualizing complex of $X$, and let $\mathcal{D}$ be the duality functor $\mathbb{R}\mathcal{H}\mathcal{om}_{\mathcal{O}_X}(-, \mathcal{R}_X)$. Then the following conditions are equivalent for $\mathcal{M} \in \mathcal{D}_c^b(\text{Mod } \mathcal{O}_X)$.
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(i) $\mathcal{M}$ is a perverse coherent sheaf (for the rigid perverse t-structure).
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(i) $\mathcal{M}$ is a perverse coherent sheaf (for the rigid perverse t-structure).

(ii) $\mathcal{D}\mathcal{M}$ is a coherent sheaf, i.e. $H^i\mathcal{D}\mathcal{M} = 0$ for all $i \neq 0$. 
Theorem 6.2. Let $X$ be a finite type scheme over $\mathbb{K}$, let $\mathcal{R}_X$ be the rigid dualizing complex of $X$, and let $D$ be the duality functor $\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_{\mathcal{O}_X}(-, \mathcal{R}_X)$. Then the following conditions are equivalent for $\mathcal{M} \in D^b_c(\text{Mod } \mathcal{O}_X)$.

(i) $\mathcal{M}$ is a perverse coherent sheaf (for the rigid perverse t-structure).

(ii) $D\mathcal{M}$ is a coherent sheaf, i.e. $H^iD\mathcal{M} = 0$ for all $i \neq 0$.

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(iii) $\mathcal{M}$ is a Cohen-Macaulay complex.

In particular this implies the Cohen-Macaulay complexes form an abelian subcategory of $D^b_c(\text{Mod} \, \mathcal{O}_X)$, a fact that seems to have eluded Grothendieck.
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(i) $\mathcal{M}$ is a perverse coherent sheaf (for the rigid perverse $t$-structure).
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M. Kashiwara, T-structures on the derived categories of holonomic D-modules and coherent O-modules, Moscow Mathematical Journal, **4** (2004), no. 4, 847-868.


