The Derived Category of Sheaves of Commutative DG Rings

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1. Background on Derived Intersection

The starting point of this story is *Serre's local intersection multiplicity formula*.

Let $\mathbb{K}$ be a field, and let X be a smooth $\mathbb{K}$-scheme.

Consider irreducible closed subschemes $Y_1, Y_2 \subseteq X$.

Let us assume that their intersection $Z := Y_1 \cap Y_2$ has the expected dimension; this is called a proper intersection.

Take the generic point z of one of the irreducible components of Z.

The local intersection multiplicity of Y_1 and Y_2 at z is the number

$$\text{(1.1)} \qquad \operatorname{int}(Y_1, Y_2; z) := \sum_{i \geq 0} (-1)^i \cdot \operatorname{length}_{\mathcal{O}_{X,z}} \big(\operatorname{Tor}_i^{\mathcal{O}_{X,z}} (\mathcal{O}_{Y_1,z}, \mathcal{O}_{Y_2,z}) \big).$$

We can approach formula (1.1) from another direction.

Consider the object

$$\text{(1.2)} \qquad \mathcal{B} = \mathcal{O}_{Y_1} \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathcal{O}_{Y_2}$$

in $\mathrm{D}(\operatorname{Mod} \mathcal{O}_X)$, the derived category of $\mathcal{O}_X$-modules.

Because

$$\mathrm{H}^{-i}(\mathcal{B}) = \operatorname{Tor}_i^{\mathcal{O}_X} (\mathcal{O}_{Y_1}, \mathcal{O}_{Y_2})$$

as coherent $\mathcal{O}_X$-modules,

we can express the local intersection multiplicity as follows:

$$\operatorname{int}(Y_1, Y_2; z) = \sum_{i \geq 0} (-1)^i \cdot \operatorname{length}_{\mathcal{O}_{X,z}} \mathrm{H}^{-i}(\mathcal{B})_z .$$

Some time around 1990, a few mathematicians (including Deligne, Drinfeld and Kontsevich) had the idea that the complex of $\mathcal{O}_X$-modules $\mathcal{B}$ from formula (1.2) should be upgraded, and given a ring structure.

The pair $(X, \mathcal{B})$ should be viewed as a *DG scheme*, and it should be the *derived intersection* of Y_1 and Y_2:

$$\text{(1.3)} \qquad Y_1 \times^{\mathrm{R}}_X Y_2 = (X, \mathcal{B}).$$

Here "DG" is short for "differential graded".

This was made precise in the paper [CK] of Ciocan-Fontanine and Kapranov from 2001, that dealt with quasi-projective schemes X over a field $\mathbb{K}$ of characteristic 0.

Later there appeared much more sophisticated (and difficult to comprehend) treatments of derived intersection, as part of the emerging discipline of *derived algebraic geometry.*

In derived algebraic geometry the theme is to replace ringed spaces with certain objects (e.g. derived schemes) that live in complicated homotopical settings. This makes the objects hard to understand and to manipulate geometrically.

See the preprint [Be] of Behrend for one approach, and a survey of the approaches of Toën et al. and of Lurie under "derived stack" in [nLab].

In this talk I will explain a simplified sort of derived algebraic geometry, with temporary name *flexible DG schemes.* It is a variation of the construction of [CK], and is still work in progress.

Here are some advantages of the theory of flexible DG schemes.

- The geometric intuition is retained – a flexible DG scheme $(X, \mathcal{A})$ is a sheaf of DG rings $\mathcal{A}$ on a scheme X.
- The methods are pretty elementary: not much beyond algebraic geometry and derived categories.
- The theory applies to schemes over an arbitrary base ring $\mathbb{K}$ (including $\mathbb{K} = \mathbb{Z}$), to other ringed spaces (like complex manifolds), and should even work on many ringed sites (like the étale site of a scheme).
- It is easy to present the derived intersection of closed subschemes Y_1, Y_2 of a scheme X as a flexible DG scheme.
- We expect to be able to define rigid dualizing complexes over flexible DG schemes, and to prove their uniqueness and existence.
- This theory should make it possible to geometrize the recent derived completion of DG rings of Shaul [Sh].

2. Commutative DG Rings

Let us fix some nonzero commutative base ring $\mathbb{K}$ (e.g. a field or $\mathbb{Z}$). All rings are going to be central over $\mathbb{K}$.

A *DG ring* is a graded ring

$$A = \bigoplus_{p \in \mathbb{Z}} A^p,$$

with a differential $\mathrm{d} : A \to A$ of degree $+1$ that satisfies $\mathrm{d} \circ \mathrm{d} = 0$ and

$$\mathrm{d}(a \cdot b) = \mathrm{d}(a) \cdot b + (-1)^p \cdot a \cdot \mathrm{d}(b)$$

for all $a \in A^p$ and $b \in A^q$.

A homomorphism of DG rings $\phi : A \to B$ must respect degrees and differentials.

We say that A is *nonpositive* if $A^p = 0$ for all $p > 0$.

The DG ring A is called *strongly commutative* if

$$b \cdot a = (-1)^{p \cdot q} \cdot a \cdot b \tag{2.1}$$

for all $a \in A^p$ and $b \in A^q$, and

$$a \cdot a = 0 \tag{2.2}$$

if p is odd.

Definition 2.3. A DG ring A is called *commutative* if it is both nonpositive and strongly commutative.

Commutative rings are viewed as commutative DG rings concentrated in degree 0.

Example 2.4. Let A be a commutative ring, and let $a \in A$ be an element.

Consider the *Koszul complex* $B := \mathrm{K}(A; a)$.

It is a free graded A-module

$$B = B^{-1} \oplus B^0 = (A \cdot t) \oplus A,$$

where t is a variable of degree -1.

The graded module B is made into a commutative DG ring by defining $t \cdot t := 0$ and $\mathrm{d}(t) := a \in B^0$.

The inclusion $A \to B$ is a DG ring homomorphism.

Note that de Rham complexes are excluded, because they are not nonpositive.

3. Semi-Free DG Rings

From here on we assume *all graded rings and DG rings are commutative*, in the sense of Definition 2.3.

The category of DG rings is denoted by $\mathsf{DGRng}/\mathbb{K}$.

Suppose we fix some $A \in \mathsf{DGRng}/\mathbb{K}$. A *DG ring over A* is a DG ring B, equipped with a homomorphism $\phi : A \to B$. These form the category DGRng/A.

A *nonpositive graded set* is a set I with a decomposition $I = \coprod_{p \leq 0} I^p$.

For each index $i \in I^p$ we take a variable t_i of degree p.

The *free graded ring* $\mathbb{K}[I]$ is the graded ring generated over $\mathbb{K}$ by the variables $\{t_i\}_{i \in I}$, subject to the strong commutativity relations (2.1) and (2.2).

Given a DG ring A, we can forget the differential. The resulting graded ring is $A^\natural$.

Definition 3.1. Let A be a DG ring.

A *semi-free DG A-ring* is a DG A-ring B, such that there is an isomorphism of graded $A^\natural$-rings

$$B^\natural \cong A^\natural \otimes_{\mathbb{K}} \mathbb{K}[I]$$

for some nonpositive graded set I.

A homomorphism $\phi : A \to B$ in $\mathsf{DGRng}/\mathbb{K}$ is called a quasi-isomorphism if

$$\mathrm{H}(\phi) : \mathrm{H}(A) \to \mathrm{H}(B)$$

is an isomorphism of graded rings.

Definition 3.2. Let B be a DG A-ring.

A *semi-free resolution* of B over A is a quasi-isomorphism $\tilde{B} \to B$ in DGRng/A, where $\tilde{B}$ is semi-free over A.

Every $B \in \mathsf{DGRng}/A$ admits a semi-free resolution $\tilde{B} \to B$.

Usually the indexing set I of the resolution $\tilde{B}$ is infinite.

But sometimes there is a finite semi-free resolution:

Example 3.3. Take $A := \mathbb{Z}$ and $B := \mathbb{Z}/(6)$.

Let $\tilde{B} := \mathrm{K}(A; a)$, the Koszul complex of the element $a := 6$. See Example 2.4.

Then $\tilde{B} \to B$ is a semi-free resolution in DGRng/A.

There is one index only, in degree -1.

4. DG Ringed Spaces

Definition 4.1. A *DG ringed space* is a pair $(X, \mathcal{A})$, consisting of a topological space X and a sheaf of DG $\mathbb{K}$-rings

$$\mathcal{A} = \bigoplus_{p \leq 0} \mathcal{A}^p$$

on X.

We shall only consider commutative DG ringed spaces, i.e. $\mathcal{A}$ is a sheaf of commutative DG $\mathbb{K}$-rings.

Definition 4.2. If $(Y, \mathcal{B})$ is another DG ringed space, then a map of DG ringed spaces

$$f : (Y, \mathcal{B}) \to (X, \mathcal{A})$$

is a map of topological spaces $f : Y \to X$, with a DG ring homomorphism

$$f^* : f^{-1}(\mathcal{A}) \to \mathcal{B}$$

on Y.

The category of DG $\mathcal{A}$-modules on X is denoted by $\mathrm{DGMod}\,\mathcal{A}$. It has a derived category $\mathrm{D}(\mathrm{DGMod}\,\mathcal{A})$.

Every DG $\mathcal{A}$-module $\mathcal{M}$ admits K-flat resolutions $\mathcal{P} \to \mathcal{M}$ and K-injective resolutions $\mathcal{M} \to \mathcal{I}$ (under mild finiteness conditions on X).

Hence the standard triangulated derived functors exist.

These include the functors $\mathrm{R}f_*$ and $\mathrm{L}f^*$ associated to a map of DG ringed spaces f.

5. The Derived Category of Sheaves DG Rings

Let $(X, \mathcal{A})$ be a DG ringed space.

A *DG $\mathcal{A}$-ring* is a sheaf of DG rings $\mathcal{B}$ on X equipped with a DG ring homomorphism $\mathcal{A} \to \mathcal{B}$.

Of course $(X, \mathcal{B})$ in itself is a DG ringed space.

The DG $\mathcal{A}$-rings form the category $\mathrm{DGRng}/\mathcal{A}$.

A homomorphism $\phi : \mathcal{B} \to \mathcal{C}$ in $\mathrm{DGRng}/\mathcal{A}$ is called a *quasi-isomorphism* if

$$\mathrm{H}(\phi) : \mathrm{H}(\mathcal{B}) \to \mathrm{H}(\mathcal{C})$$

is an isomorphism of sheaves of graded rings.

We can now define the notion appearing in the title of the talk:

Definition 5.1. The *derived category* of $\mathsf{DGRng}/\mathcal{A}$ is the category $\mathsf{D}(\mathsf{DGRng}/\mathcal{A})$ gotten by formally inverting the quasi-isomorphisms.

There is a functor

$$\mathrm{Q} : \mathsf{DGRng}/\mathcal{A} \to \mathsf{D}(\mathsf{DGRng}/\mathcal{A}) \tag{5.2}$$

that we call categorical localization, which is the identity on objects.

We want to understand the functor (5.2).

Before going further, let us mention that categorical localization commutes with geometric localization.

Namely, if $U \subseteq X$ is a subset, then there is a restriction functor

$$\mathrm{Rest}_{U/X} : \mathsf{DGRng}/\mathcal{A} \to \mathsf{DGRng}/\mathcal{A}|_U, \quad \mathcal{B} \mapsto \mathcal{B}|_U.$$

It is exact, and sits in this commutative diagram:

$$\begin{array}{ccc} \mathsf{DGRng}/\mathcal{A} & \xrightarrow{\mathrm{Q}} & \mathsf{D}(\mathsf{DGRng}/\mathcal{A}) \\ {\scriptstyle \mathrm{Rest}_{U/X}} \downarrow & & \downarrow {\scriptstyle \mathrm{Rest}_{U/X}} \\ \mathsf{DGRng}/\mathcal{A}|_U & \xrightarrow{\mathrm{Q}} & \mathsf{D}(\mathsf{DGRng}/\mathcal{A}|_U) \end{array} \tag{5.3}$$

6. Semi-Pseudo-Free Resolutions

We have a topological space X. The constant sheaf on X with values in $\mathbb{K}$ is $\mathbb{K}_X$.

Definition 6.1. Let $U \subseteq X$ be an open set, with inclusion morphism $g : U \to X$.

The extension by zero sheaf

$$\mathcal{T} := g_!(\mathbb{K}_U)$$

is called the *pseudo-free* $\mathbb{K}_X$*-module* with pseudo-rank 1 and pseudo-support U.

Note that there is a section

$$t \in \Gamma(U, \mathcal{T}) \tag{6.2}$$

that we call the *pseudo-basis* of $\mathcal{T}$.

Also note that the sheaf $\mathcal{T}$ is a flat $\mathbb{K}_X$-module, and its stalk at a point $x \in X$ is

$$\mathcal{T}_x = \begin{cases} \mathbb{K} \cdot t \cong \mathbb{K} & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{T}$ be a pseudo-free $\mathbb{K}_X$-module.

The $\mathcal{O}_X$-module
$$\mathcal{L} := \mathcal{O}_X \otimes_{\mathbb{K}_X} \mathcal{T}$$
is called the *pseudo-free $\mathcal{O}_X$-module induced from $\mathcal{T}$*.

Note that
$$\mathcal{L} \cong g_!(\mathcal{O}_U)$$
in the notation of Definition 6.1.

In [RD], pseudo-free $\mathcal{O}_X$-modules were used for several purposes, among them to create flat resolutions.

In [KS2] these were called "almost free $\mathcal{O}_X$-modules".

Warning: Suppose $(X, \mathcal{O}_X)$ is a scheme, and $\mathcal{L}$ is a pseudo-free $\mathcal{O}_X$-module. Usually $\mathcal{L}$ is *not quasi-coherent.*

Recall (from Section 3) that a free graded $\mathbb{K}$-ring $\mathbb{K}[I]$ has a basis $\{t_i\}_{i \in I}$, which is a collection of graded variables, indexed nonpositive graded set I.

Our first main idea is to replace the collection of variables $\{t_i\}_{i \in I}$ with a collection $\{\mathcal{T}_i\}_{i \in I}$ of pseudo-free $\mathbb{K}_X$-modules, also indexed by a nonpositive graded set I.

Each pseudo-free $\mathbb{K}_X$-module $\mathcal{T}_i$ has its own pseudo-support U_i, and its pseudo-basis
$$t_i \in \Gamma(U_i, \mathcal{T}_i).$$

The noncommutative pseudo-free $\mathbb{K}_X$-ring $\mathbb{K}_X\langle I \rangle$ is the direct sum of the "monomials"
$$\mathcal{T}_{i_1} \otimes_{\mathbb{K}_X} \cdots \otimes_{\mathbb{K}_X} \mathcal{T}_{i_n}, \quad (i_1, \ldots, i_n) \in \underbrace{I \times \cdots \times I}_{n},$$
with the obvious multiplication.

Definition 6.3. Let $\{\mathcal{T}_i\}_{i \in I}$ be a collection of pseudo-free $\mathbb{K}_X$-modules.

The *commutative pseudo-free graded $\mathbb{K}_X$-ring* $\mathbb{K}_X[I]$ is the quotient of $\mathbb{K}_X\langle I \rangle$ modulo the commutativity relations (2.1) and (2.2).

The local structure of the sheaf of rings $\mathbb{K}_X[I]$ is very nice:

At each point $x \in X$, the stalk $\mathbb{K}_X[I]_x$ is the commutative free graded $\mathbb{K}$-ring $\mathbb{K}[I_x]$, with indexing set
$$I_x := \{ i \in I \mid x \in U_i \} \subseteq I$$
and basis $\{t_i\}_{i \in I_x}$.

Definition 6.4. Let $(X, \mathcal{A})$ be a DG ringed space.

A *semi-pseudo-free DG $\mathcal{A}$-ring* is a DG $\mathcal{A}$-ring $\mathcal{B}$, such that there is a graded $\mathcal{A}^\natural$-ring isomorphism
$$\mathcal{B}^\natural \cong \mathcal{A}^\natural \otimes_{\mathbb{K}_X} \mathbb{K}_X[I]$$
for some collection $\{\mathcal{T}_i\}_{i \in I}$ of pseudo-free $\mathbb{K}_X$-modules.

This implies that $\mathcal{B}$ is K-flat as a DG $\mathcal{A}$-module.

Definition 6.5. Let $\mathcal{B} \in \mathsf{DGRng}/\mathcal{A}$.

A *semi-pseudo-free resolution* of $\mathcal{B}$ is a quasi-isomorphism $\tilde{\mathcal{B}} \to \mathcal{B}$ in $\mathsf{DGRng}/\mathcal{A}$, from some semi-pseudo-free DG $\mathcal{A}$-ring $\tilde{\mathcal{B}}$.

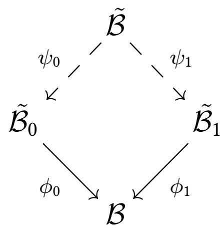
Theorem 6.6. *[Ye3] Every $\mathcal{B} \in \mathsf{DGRng}/\mathcal{A}$ admits some semi-pseudo-free resolution.*

Theorem 6.7. *[Ye3] Suppose $\phi_0 : \tilde{\mathcal{B}}_0 \to \mathcal{B}$ and $\phi_1 : \tilde{\mathcal{B}}_1 \to \mathcal{B}$ are quasi-isomorphisms in* $\mathrm{DGRng}/\mathcal{A}$.

Then there is a semi-pseudo-free DG ring $\tilde{\mathcal{B}}$, and quasi-isomorphisms $\psi_0 : \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}_0$ and $\psi_1 : \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}_1$, such that

$$\phi_0 \circ \psi_0 = \phi_1 \circ \psi_1.$$

(6.8)



Remark 6.9. In [CK] the authors only considered *quasi-coherent DG $\mathcal{O}_X$-rings* on a scheme $(X, \mathcal{O}_X)$.

Namely DG rings $\mathcal{A} \in \mathrm{DGRng}/\mathcal{O}_X$ such that each $\mathcal{A}^i$ is a quasi-coherent $\mathcal{O}_X$-module.

The resolutions in [CK] were by *quasi-coherent semi-free DG rings*. These are like in our Definition 6.4, except that instead of the pseudo-free $\mathcal{O}_X$-modules $\mathcal{O}_X \otimes_{\mathbb{K}_X} \mathcal{T}_i$ that we use, they had rank 1 locally free O_X-modules $\mathcal{L}_i$ (i.e. line bundles).

Existence of resolutions (like our Theorem 6.6) was proved only for a quasi-projective scheme X over a field $\mathbb{K}$.

Some examples of quasi-coherent semi-free resolutions are worked out in the Appendix (Section 10).

Remark 6.10. As far as I can tell, a single semi-pseudo-free DG ring $\tilde{\mathcal{B}}$ cannot, in general, lift an *infinite* collection of quasi-isomorphisms $\{\phi_i\}$, as in diagram (6.8).

The reason – deep down in the details – is that an infinite intersection of open sets is not open.

There is an analogous finiteness in the quasi-coherent semi-free resolutions of [CK], when X is a quasi-projective scheme. In this setup finiteness shows up in the poles of sections of the line bundles $\mathcal{L}_i$.

This sort of behavior is typical in algebraic geometry. It resembles the fact that a single open covering of a scheme X does not trivialize all line bundles on X.

Remark 6.11. What the last remark seems to imply is that the semi-pseudo-free DG rings are not cofibrant objects, in the sense of Quillen.

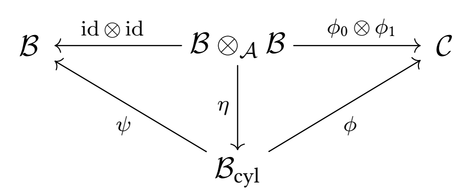
Thus it is plausible that there is no Quillen model structure on $\mathrm{DGRng}/\mathcal{A}$.

7. The Quasi-Homotopy Relation

Recall our setup: $(X, \mathcal{A})$ is a commutative DG ringed space over $\mathbb{K}$.

Definition 7.1. Suppose
$$\phi_0, \phi_1 : \mathcal{B} \to \mathcal{C}$$
are homomorphisms in $\mathrm{DGRng}/\mathcal{A}$.

A *homotopy* from ϕ_0 to ϕ_1 is a commutative diagram



in $\mathrm{DGRng}/\mathcal{A}$, in which ψ is a quasi-isomorphism.

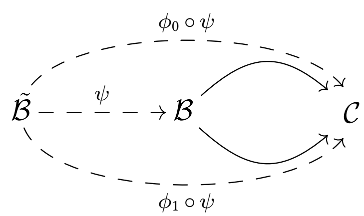
The definition above is borrowed from the theory of Quillen model structures, where it is called a "left homotopy", and $\mathcal{B}_{\mathrm{cyl}}$ is called a "cylinder object".

Here is the second main innovation in this talk.

Definition 7.2. Suppose $\phi_0, \phi_1 : \mathcal{B} \to \mathcal{C}$ are homomorphisms in $\mathsf{DGRng}/\mathcal{A}$.

A *quasi-homotopy* from ϕ_0 to ϕ_1 is:

- a quasi-isomorphism $\psi : \tilde{\mathcal{B}} \to \mathcal{B}$,
- and a homotopy from $\phi_0 \circ \psi$ to $\phi_1 \circ \psi$.



Theorem 7.3. *[Ye3] The relation of quasi-homotopy is a congruence on the category* $\mathsf{DGRng}/\mathcal{A}$.

This means that the next definition is possible.

Definition 7.4. The *homotopy category* of $\mathsf{DGRng}/\mathcal{A}$ is the category $\mathsf{K}(\mathsf{DGRng}/\mathcal{A})$, that has the same objects as $\mathsf{DGRng}/\mathcal{A}$, and its morphisms are the quasi-homotopy classes of DG ring homomorphisms.

There is a functor

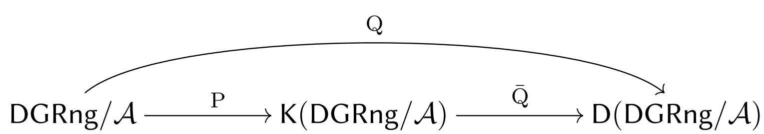
$$\mathrm{P} : \mathsf{DGRng}/\mathcal{A} \to \mathsf{K}(\mathsf{DGRng}/\mathcal{A}) \tag{7.5}$$

that is the identity on objects and surjective on morphisms.

It is quite easy to see that if $\phi_0, \phi_1 : \mathcal{B} \to \mathcal{C}$ are quasi-homotopic morphisms in $\mathsf{DGRng}/\mathcal{A}$, then $\mathrm{Q}(\phi_0) = \mathrm{Q}(\phi_1)$ in $\mathsf{D}(\mathsf{DGRng}/\mathcal{A})$.

This implies that there is a functor $\bar{\mathrm{Q}}$ that fits into this commutative diagram:

(7.6)



Theorem 7.7. *[Ye3] The functor $\bar{\mathrm{Q}}$ is a faithful right Ore localization.*

Therefore every morphism $\psi : \mathcal{B} \to \mathcal{C}$ in $\mathsf{D}(\mathsf{DGRng}/\mathcal{A})$ is a simple right fraction:

$$\psi = \mathrm{Q}(\phi_0) \circ \mathrm{Q}(\phi_1)^{-1},$$

where ϕ_i are homomorphisms in $\mathsf{DGRng}/\mathcal{A}$ and ϕ_1 is a quasi-isomorphism.

8. Derived Tensor Products

We are still in the general geometric setup: $(X, \mathcal{A})$ is a commutative DG ringed space over $\mathbb{K}$.

The tensor product is a functor

$$(- \otimes_{\mathcal{A}} -) : (\mathsf{DGRng}/\mathcal{A}) \times (\mathsf{DGRng}/\mathcal{A}) \to \mathsf{DGRng}/\mathcal{A}.$$

It turns out to have a left derived functor with respect to quasi-isomorphisms:

Theorem 8.1. *[Ye3] There is a functor*

$$(- \otimes^{\mathrm{L}}_{\mathcal{A}} -) : \mathsf{D}(\mathsf{DGRng}/\mathcal{A}) \times \mathsf{D}(\mathsf{DGRng}/\mathcal{A}) \to \mathsf{D}(\mathsf{DGRng}/\mathcal{A}) \, ,$$

together with a morphism of functors

$$\eta^{\mathrm{L}} : (- \otimes^{\mathrm{L}}_{\mathcal{A}} -) \to (- \otimes_{\mathcal{A}} -).$$

The functor $(- \otimes^{\mathrm{L}}_{\mathcal{A}} -)$ is determined by this property:

If $\mathcal{B}, \mathcal{C} \in \mathsf{DGRng}/\mathcal{A}$ are such that at least one of them is K-flat, then the morphism

$$\eta^{\mathrm{L}}_{\mathcal{B},\mathcal{C}} : \mathcal{B} \otimes^{\mathrm{L}}_{\mathcal{A}} \mathcal{C} \to \mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}$$

in $\mathsf{D}(\mathsf{DGRng}/\mathcal{A})$ is an isomorphism.

See [Ye4, Subsection 8.3] regarding nonadditive derived functors.

The proof of Theorem 8.1 uses some facts about pseudo-semi-free resolutions.

Here are two good properties of the functor $(- \otimes^{\mathrm{L}}_{\mathcal{A}} -)$:

- It commutes with the forgetful functor

$$\mathsf{D}(\mathsf{DGRng}/\mathcal{A}) \to \mathsf{D}(\mathsf{Mod}\, \mathcal{A}).$$

- It commutes with geometric localization to a subset $U \subseteq X$; see diagram (5.3).

9. Flexible DG Schemes

We need a geometric version of quasi-isomorphism.

Definition 9.1. A map of DG ringed spaces

$$f : (Y, \mathcal{B}) \to (X, \mathcal{A})$$

is called a *quasi-isomorphism* if $f : Y \to X$ is a homeomorphism, and

$$\mathrm{H}(f^*) : \mathrm{H}(f^{-1}(\mathcal{A})) \to \mathrm{H}(\mathcal{B})$$

is an isomorphism of graded $\mathbb{K}_Y$-rings.

Given a DG ring A, its 0-th cohomology $\mathrm{H}^0(A)$ is a ring. Consider the affine scheme $X := \mathrm{Spec}(\mathrm{H}^0(A))$.

As explained in [Ye5, Section 4], there is a canonical way to sheafify A on X. The resulting DG ringed space $(X, \mathcal{A})$ is denoted by $\mathrm{DGSpec}(A)$.

Here are a few tentative definitions.

Definition 9.2. A *flexible DG scheme* is a DG ringed space $(X, \mathcal{A})$ with the following two properties:

(i) The ringed space $(X, \mathrm{H}^0(\mathcal{A}))$ is a scheme.

(ii) Every point $x \in X$ has an open neighborhood U such that the DG ringed space $(U, \mathcal{A}|_U)$ is quasi-isomorphic to $\mathrm{DGSpec}(A)$, for some DG ring A.

Definition 9.3. A *map of flexible DG schemes*

$$f : (Y, \mathcal{B}) \to (X, \mathcal{A})$$

is a map of DG ringed spaces such that:

- $$\mathrm{H}^0(f) : (Y, \mathrm{H}^0(\mathcal{B})) \to (X, \mathrm{H}^0(\mathcal{A}))$$
  is a map of schemes.
- Locally, in terms of property (ii), f is represented by a homomorphism $A \to B$ of DG rings.

The resulting category is denoted by $\mathsf{DGSch}/\mathbb{K}$.

By inverting the quasi-isomorphisms in $\mathsf{DGSch}/\mathbb{K}$ we obtain the *derived category of flexible DG schemes*, with notation $\mathsf{D}(\mathsf{DGSch}/\mathbb{K})$.

We can finally say what is derived intersection in the context of flexible DG schemes.

Recall the derived tensor product of sheaves of DG rings from Theorem 8.1.

Definition 9.4. Let Y_1 and Y_2 be closed subschemes of a scheme X.

Their *derived intersection* is the flexible DG scheme

$$Y_1 \times^{\mathrm{R}}_X Y_2 := (Z, \mathcal{C}), \tag{9.5}$$

where the topological space is

$$Z := Y_1 \cap Y_1 \subseteq X, \tag{9.6}$$

and the sheaf of DG rings is

$$\mathcal{C} := (\mathcal{O}_{Y_1} \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathcal{O}_{Y_2})|_Z \in \mathsf{D}(\mathsf{DGRng}/\mathcal{O}_X). \tag{9.7}$$

Note that

$$\mathrm{H}^0(\mathcal{C}) = (\mathcal{O}_{Y_1} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_2})|_Z,$$

so $(Z, \mathrm{H}^0(\mathcal{C}))$ is indeed a scheme.

As far as we can tell, the pair $(Z, \mathcal{C})$ is a well-defined object of $\mathsf{D}(\mathsf{DGSch}/\mathbb{K})$.

There are a few concrete examples of derived intersections in the Appendix (Section 11).

$\sim$ *END* $\sim$

10. Appendix: Examples of Quasi-Coherent Semi-Free Resolutions

Let X be a scheme over a field $\mathbb{K}$.

Recall from Remark 6.9 that $\mathcal{A} \in \mathsf{DGRng}/\mathcal{O}_X$ is called quasi-coherent if every $\mathcal{A}^i$ is a quasi-coherent $\mathcal{O}_X$-module.

If $Y \subseteq X$ is a closed subscheme, then $\mathcal{O}_Y$ is a quasi-coherent DG $\mathcal{O}_X$-ring.

Sometimes closed subschemes have quasi-coherent semi-free resolutions.

Here are some examples.

Example 10.1. Suppose X is an affine scheme and $Y \subseteq X$ is a closed subscheme.

Let $A := \Gamma(X, \mathcal{O}_X)$ and $B := \Gamma(X, \mathcal{O}_Y)$.

So $A \to B$ is a surjective ring homomorphism.

We choose a semi-free resolution $\phi : \tilde{B} \to B$ in DGRng/A, as in Definition 3.2.

Passing to quasi-coherent sheaves on X we obtain a quasi-coherent semi-free resolution $\phi : \tilde{\mathcal{B}} \to \mathcal{O}_Y$ in $\mathrm{DGRng}/\mathcal{O}_X$.

Example 10.2. Now take $X = \mathbf{P}^n_{\mathbb{K}}$, the n-dimensional projective space.

Let $Y \subseteq X$ be a hypersurface defined by a homogeneous polynomial f of degree d.

There is an exact sequence of coherent sheaves

$$0 \to \mathcal{O}_X(-d) \xrightarrow{f} \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

This can be viewed as a "projective Koszul complex".

Define the graded sheaf

$$\tilde{\mathcal{B}} = \tilde{\mathcal{B}}^{-1} \oplus \tilde{\mathcal{B}}^{0} := \big(\mathcal{O}_X(-d) \cdot s\big) \,\oplus\, \mathcal{O}_X,$$

where s is a variable of degree -1.

We make $\tilde{\mathcal{B}}$ into a DG $\mathcal{O}_X$-ring by $s^2 := 0$ and $\mathrm{d}(s) := f$.

Then $\tilde{\mathcal{B}} \to \mathcal{O}_Y$ is a quasi-coherent semi-free resolution in $\mathrm{DGRng}/\mathcal{O}_X$.

11. Appendix: Examples of Derived Intersection

In this last section there are a few examples of derived intersection – all in the quasi-coherent framework, as in [CK].

Example 11.1. Consider the affine plane

$$X := \mathbf{A}^2_{\mathbb{K}} = \mathrm{Spec}(A)$$

where $A := \mathbb{K}[t_1, t_2]$ is the polynomial ring over a field $\mathbb{K}$.

We look at the curves $Y_i \subseteq X$ defined by the equations $f_i(t_1, t_2) = 0$, where

$$f_1(t_1, t_2) := t_2 \quad \text{and} \quad f_2(t_1, t_2) := t_2 - t_1^2.$$

So $Y_i = \mathrm{Spec}(B_i)$, where $B_i := A/(f_i)$.

(cont.) Here is the picture:

The set-theoretic intersection of the curves Y_1 and Y_2 is of course

$$Y_1 \cap Y_2 = Z = \{z\},$$

where z is the origin.

(cont.) This intersection is *proper*: it has the correct dimension, which is 0.

But the intersection is *not transversal*, and this is recorded by the multiplicity, which is 2.

In terms of schemes, the intersection is $(Z, \mathcal{O}_Z)$, where $\mathcal{O}_Z$ is the sheafification of the ring

(11.2) $$C' := A/(f_1, f_2) \cong \mathbb{K}[\epsilon], \quad \epsilon^2 = 0.$$

Let's calculate the derived intersection

$$Y_1 \times^{\mathrm{R}}_X Y_2 = (Z, \mathcal{C}).$$

We start on the ring level.

(cont.) Take the semi-free resolution $\tilde{B}_1 \to B_1$ in DGRng/A to be the Koszul complex

$$\tilde{B}_1 := \mathrm{K}(A; f_1) = \big(A \xrightarrow{f_1} A\big).$$

Passing to sheaves, this gives us a quasi-coherent semi-free resolution

(11.3) $$\tilde{\mathcal{B}}_1 \to \mathcal{O}_{Y_1}$$

in $\mathrm{DGRng}/\mathcal{O}_X$.

Let

$$C := \tilde{B}_1 \otimes_A B_2 = \big(A \xrightarrow{f_1} A\big) \otimes_A B_2 = \big(B_2 \xrightarrow{f_1} B_2\big).$$

So there is a quasi-isomorphism DGRng/A :

(11.4) $$C = \tilde{B}_1 \otimes_A B_2 \to C',$$

where C' is from (11.2).

(cont.) Sheafifying (11.4), we see that there is a quasi-isomorphism

$$\mathcal{C} = \tilde{\mathcal{B}}_1 \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_2} \to \mathcal{O}_Z$$

in $\mathrm{DGRng}/\mathcal{O}_X$.

We see that in this case the derived intersection $(Z, \mathcal{C})$ is isomorphic, in the derived category $\mathrm{D}(\mathrm{DGRSch}/\mathbb{K})$, to the usual intersection $(Z, \mathcal{O}_Z)$.

Example 11.5. Let us calculate the derived self-intersection of Y_1 :

$$Y_1 \times^{\mathrm{R}}_X Y_1 = (Y_1, \mathcal{D}).$$

We can use the resolution (11.3) from the previous example.

But now

$$D := \tilde{B}_1 \otimes_A B_1 = \big(A \xrightarrow{f_1} A\big) \otimes_A B_1 = \big(B_1 \xrightarrow{f_1} B_1\big) = \big(B_1 \xrightarrow{0} B_1\big).$$

Thus

$$D = D^{-1} \oplus D^0 \cong B_1[s] = (B_1 \cdot s) \oplus B_1,$$

where s is a variable of degree -1, and we define $s^2 = 0$, and $\mathrm{d}(s) = 0$.

In sheaves we have $\mathcal{D} = \mathcal{O}_{Y_1}[s]$.

We see that derived intersection, like classical intersection, is not deformation invariant.

Example 11.6. Here we replace the affine picture by its projective closure.

So $X = \mathbf{P}^2_{\mathbb{K}}$, and $Y_1, Y_2 \subseteq X$ are the closures of the curves from Example 11.1.

(cont.) Thus $Y_i \subseteq X$ is the projective curve defined by the homogeneous equation $f_i(t_0, t_1, t_2) = 0$, where

$$f_1(t_0, t_1, t_2) := t_2 \quad \text{and} \quad f_2(t_0, t_1, t_2) := t_0 \cdot t_2 - t_1^2 .$$

We can use Example 10.2 to cook up a quasi-coherent semi-free resolution of $\mathcal{O}_{Y_1}$.

I leave it as an exercise to prove:

1. The derived intersection $Y_1 \times^{\mathrm{R}}_X Y_2$ in the projective case is $(Z, \mathcal{O}_Z)$, where $\mathcal{O}_Z$ is the $\mathcal{O}_X$-ring of length 2 from Example 11.1.
2. The derived self-intersection $Y_1 \times^{\mathrm{R}}_X Y_1$ in the projective case is $(Y_1, \mathcal{D})$, where $\mathcal{D} = \mathcal{O}_{Y_1}[s]$, s has degree -1, $s^2 = 0$, and $\mathrm{d}(s) = 0$.

It is instructive to use the open embedding $\mathbf{A}^2_{\mathbb{K}} \subseteq \mathbf{P}^2_{\mathbb{K}}$ to compare these calculations to Examples 11.1 and 11.5.

For a deeper discussion of derived self-intersection see the paper [AC].

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