Twisted Deformation Quantization of Algebraic Varieties

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I talked about twisted deformation quantization in the past [Ye4]. However this is a new lecture, written especially for this workshop. It incorporates some recent progress on the subject.

I am assuming some knowledge of deformation quantization, algebraic geometry and stacks.

Here is the plan of my lecture:

1. Associative Deformations
2. Poisson Deformations
3. Twisted Deformations
4. The Twisted Quantization Theorem
5. About the Proof; Intermediate Results

1. Associative Deformations

Let $\mathbb{K}$ be a field of characteristic 0.

A *parameter algebra* over $\mathbb{K}$ is a complete noetherian local commutative $\mathbb{K}$-algebra R, with maximal ideal $\mathfrak{m}$, such that $R/\mathfrak{m} = \mathbb{K}$.

The important example is $R = \mathbb{K}[[\hbar]]$, formal power series in a variable $\hbar$.

For a $\mathbb{K}$-module V and an R-module M we denote by $M \widehat{\otimes} V$ the $\mathfrak{m}$-adic completion of $M \otimes V := M \otimes_{\mathbb{K}} V$.

Let X be a smooth (i.e. nonsingular) algebraic variety over $\mathbb{K}$. The structure sheaf of X (the sheaf of functions) is denoted by $\mathcal{O}_X$.

Let $\mathcal{M}$ be a sheaf of R-modules on X. We say $\mathcal{M}$ is *flat* if at each point $x \in X$, the stalk $\mathcal{M}_x$ is a flat R-module.

We say that $\mathcal{M}$ is $\mathfrak{m}$-*adically complete* if the canonical sheaf homomorphism

$$\mathcal{M} \to \lim_{\leftarrow i} (\mathcal{M}/\mathfrak{m}^i\mathcal{M})$$

is an isomorphism.

Proposition 1.1. *Suppose $\mathcal{M}$ is a flat $\mathfrak{m}$-adically complete sheaf of R-modules on X, such that $\mathbb{K} \otimes_R \mathcal{M}$ is isomorphic to a quasi-coherent $\mathcal{O}_X$-module.*

Let $U \subset X$ be an affine open set.

Then the R-module $\Gamma(U, \mathcal{M})$ is flat and $\mathfrak{m}$-adically complete,

and the canonical homomorphism $\mathcal{M} \to \mathbb{K} \otimes_R \mathcal{M}$ induces an isomorphism

$$\mathbb{K} \otimes_R \Gamma(U, \mathcal{M}) \cong \Gamma(U, \mathbb{K} \otimes_R \mathcal{M}).$$

Definition 1.2. An *associative R-deformation* of $\mathcal{O}_X$ is a sheaf $\mathcal{A}$ of associative unital R-algebras on X, which is flat and $\mathfrak{m}$-adically complete, together with an isomorphism of $\mathbb{K}$-algebras $\mathbb{K} \otimes_R \mathcal{A} \cong \mathcal{O}_X$, called an augmentation.

The multiplication of $\mathcal{A}$ is denoted by $\star$.

I will sometimes be sloppy and say "algebra" instead of "sheaf of algebras".

Definition 1.3. Suppose $\mathcal{A}$ and $\mathcal{A}'$ are associative R-deformations of $\mathcal{O}_X$. A *gauge transformation* $g : \mathcal{A} \to \mathcal{A}'$ is an isomorphism of sheaves of $\mathbb{K}$-algebras that commutes with the augmentations.

The category of associative R-deformations of $\mathcal{O}_X$, where the morphisms are the gauge transformations, is denoted by $\mathsf{AssDef}(R, X)$. This is a groupoid of course.

Take an affine open set $U \subset X$, and let $A := \Gamma(U, \mathcal{A})$ and $C := \Gamma(U, \mathcal{O}_X)$.

According to Proposition 1.1 we know that as an R-module, A is flat, $\mathfrak{m}$-adically complete, and augmented: there is a homomorphism $A \to C$ that induces an isomorphism $\mathbb{K} \otimes_R A \cong C$.

We can choose a $\mathbb{K}$-linear splitting $C \to A$ of the augmentation. This gives an isomorphism of augmented R-modules

$$R \mathbin{\widehat{\otimes}} C \cong A. \tag{1.4}$$

The multiplication $\star$ on $R \mathbin{\widehat{\otimes}} C$, coming from the isomorphism (1.4), is called a *star product*.

We can express $\star$ as follows: there is a $\mathbb{K}$-linear homomorphism

$$\omega : C \otimes C \to \mathfrak{m} \mathbin{\widehat{\otimes}} C \tag{1.5}$$

such that

$$c_1 \star c_2 = c_1 \cdot c_2 + \omega(c_1, c_2)$$

for any $c_1, c_2 \in C$.

Example 1.6. In case $R = \mathbb{K}[[\hbar]]$ we have $R \mathbin{\widehat{\otimes}} C = C[[\hbar]]$, and so we can expand ω into an $\hbar$-adic series: $\omega = \sum_{j \geq 1} \omega_j \hbar^j$, where $\omega_j : C \otimes C \to C$.

It is well-known that ω is a solution of the Maurer-Cartan equation in the Hochschild complex of C (shifted by 1 and tensored with $\mathfrak{m}$).

A less known fact is that by a suitable choice of splitting (1.4), ω can be made a bi-differential operator. We will return to this later.

Let $\mathcal{A}$ be an associative R-deformation of $\mathcal{O}_X$. The subsheaf $\mathcal{A}^\times$ of invertible elements is a sheaf of groups on X.

But there is a smaller sheaf of groups, that is more important for us.

Definition 1.7. For an associative R-deformation $\mathcal{A}$, the *inner gauge group* is the sheaf of groups

$$\mathrm{IG}(\mathcal{A}) := \mathrm{Ker}(\mathcal{A}^\times \xrightarrow{\mathrm{aug}} \mathcal{O}_X^\times).$$

We can also express $\mathrm{IG}(\mathcal{A})$ as a pronilpotent group.

The sheaf $\mathfrak{m}\mathcal{A}$ is a pronilpotent Lie algebra, with the commutator bracket, and there is a canonical isomorphism

$$\mathrm{IG}(\mathcal{A}) \cong \mathrm{Exp}(\mathfrak{m}\mathcal{A}). \tag{1.8}$$

A gauge transformation $g : \mathcal{A} \to \mathcal{A}'$ (see Definition 1.3) induces an isomorphism of groups

(1.9) $$\mathrm{IG}(g) : \mathrm{IG}(\mathcal{A}) \to \mathrm{IG}(\mathcal{A}').$$

There is an action ig of the group $\mathrm{IG}(\mathcal{A})$ on the algebra $\mathcal{A}$. A local section $g \in \mathrm{IG}(\mathcal{A}) \subset \mathcal{A}^{\times}$ acts on $\mathcal{A}$ by conjugation:

(1.10) $$\mathrm{ig}(g)(a) := \mathrm{Ad}(g)(a) = g \star a \star g^{-1}.$$

This extra structure makes $\mathrm{AssDef}(R, X)$ into a *crossed groupoid* (also known as a strict 2-groupoid). The elements of $\Gamma(X, \mathrm{IG}(\mathcal{A}))$ are the 2-morphisms. I will say more about this later.

2. Poisson Deformations

Let $\mathcal{A}$ be a sheaf of commutative R-algebras on X. An R-bilinear Poisson bracket on $\mathcal{A}$ is an R-linear sheaf homomorphism

$$\{-,-\} : \mathcal{A} \otimes_R \mathcal{A} \to \mathcal{A}$$

that is a derivation in each argument, and is also a Lie bracket.

A Poisson R-algebra is a sheaf of commutative R-algebras $\mathcal{A}$ equipped with an R-bilinear Poisson bracket.

We view $\mathcal{O}_X$ as a Poisson $\mathbb{K}$-algebra with the zero bracket.

Definition 2.1. A *Poisson R-deformation* of $\mathcal{O}_X$ is a sheaf $\mathcal{A}$ of Poisson R-algebras on X, which is flat and $\mathfrak{m}$-adically complete, together with an isomorphism of Poisson $\mathbb{K}$-algebras $\mathbb{K} \otimes_R \mathcal{A} \cong \mathcal{O}_X$, called an augmentation.

Definition 2.2. Suppose $\mathcal{A}$ and $\mathcal{A}'$ are Poisson R-deformations of $\mathcal{O}_X$. A *gauge transformation* $g : \mathcal{A} \to \mathcal{A}'$ is an isomorphism of sheaves of Poisson $\mathbb{K}$-algebras that commutes with the augmentations.

The category of Poisson R-deformations of $\mathcal{O}_X$, where the morphisms are the gauge transformations, is denoted by $\mathrm{PoisDef}(R, X)$. This is a groupoid.

Note that the sheaf $\mathfrak{m}\mathcal{A}$ is a pronilpotent Lie algebra, using the Poisson bracket.

Definition 2.3. For a Poisson R-deformation $\mathcal{A}$, the *inner gauge group* is the sheaf of groups

$$\mathrm{IG}(\mathcal{A}) := \mathrm{Exp}(\mathfrak{m}\mathcal{A}).$$

Like (1.9), these groups respect gauge transformations.

There is an action ig of the group $\mathrm{IG}(\mathcal{A})$ on the Poisson algebra $\mathcal{A}$. A local section $g = \exp(\gamma) \in \mathrm{IG}(\mathcal{A})$ acts on $\mathcal{A}$ by a *formal Hamiltonian flow*:

(2.4) $$\mathrm{ig}(g)(a) := \exp(\mathrm{ad}(\gamma))(a),$$

where $\mathrm{ad}(\gamma)(a) := \{\gamma, a\}$.

The inner gauge groups make $\mathrm{PoisDef}(R, X)$ into a crossed groupoid.

3. Twisted Deformations

Here I explain what are twisted (or stacky) R-deformations of $\mathcal{O}_X$.

A first approach is to use the notion of *stack of R-algebroids*, which was introduced by Kontsevich in [Ko2]. A similar notion of *twisted sheaf* appeared earlier in Kashiwara's paper [Ka].

By definition a stack of R-algebroids on X is a stack $\boldsymbol{\mathcal{A}}$ of R-linear categories, which is locally nonempty and locally connected by isomorphisms.

If i, j are local objects of the stack $\boldsymbol{\mathcal{A}}$, defined on some open set U, then $\boldsymbol{\mathcal{A}}(i,j)$ is a sheaf of R-modules on U. For $i = j$ we get a sheaf of R-algebras $\boldsymbol{\mathcal{A}}(i) := \boldsymbol{\mathcal{A}}(i,i)$.

In [KS] the authors define a *DQ algebroid* to be a stack of R-algebroids $\boldsymbol{\mathcal{A}}$ which is locally equivalent to an associative R-deformation of $\mathcal{O}_X$.

This means that for any local object i, the sheaf of R-algebras $\boldsymbol{\mathcal{A}}(i)$ is locally isomorphic to an associative R-deformation.

For related notions of stacky deformations see [BGNT1], [BGNT2], [DP1], [Lo], [LV], [Po], [PS].

However this is not the type of stacky deformation that we need. To explain this, let me now introduce the *gauge gerbe* $\boldsymbol{\mathcal{G}}$ of an R-algebroid $\boldsymbol{\mathcal{A}}$.

For any R-linear category A we consider the groupoid $A^{\times}$, which is the subcategory of A with all the objects, but only the invertible morphisms.

For a prestack of R-linear categories $\boldsymbol{\mathcal{A}}$ there is a prestack of groupoids $\boldsymbol{\mathcal{A}}^{\times}$, defined by

$$\boldsymbol{\mathcal{A}}^{\times}(U) := \boldsymbol{\mathcal{A}}(U)^{\times}$$

for any open set $U \subset X$.

Here is an exercise:

Proposition 3.1. *If the prestack of algebroids $\boldsymbol{\mathcal{A}}$ is a stack, then $\boldsymbol{\mathcal{A}}^{\times}$ is a gerbe.*

For this reason we call $\boldsymbol{\mathcal{G}} := \boldsymbol{\mathcal{A}}^{\times}$ the gauge gerbe of the stack of algebroids $\boldsymbol{\mathcal{A}}$.

Note that the categorical information about the algebroid $\boldsymbol{\mathcal{A}}$ (how many objects on each open set, and which objects are isomorphic) is encoded by the gauge gerbe $\boldsymbol{\mathcal{G}}$.

Also the classification of R-algebroids (in terms of descent data, i.e. cocycles) depends only on their gauge gerbes.

The type of stacky associative R-deformation that we want is slightly different.

We want a prestack of R-algebroids $\boldsymbol{\mathcal{A}}$, together with a sub-prestack $\boldsymbol{\mathcal{G}} \subset \boldsymbol{\mathcal{A}}^{\times}$ which is a gerbe, such that for every local object i the automorphism group is

$$\boldsymbol{\mathcal{G}}(i) \cong \mathrm{IG}(\boldsymbol{\mathcal{A}}(i)) = \mathrm{Exp}(\mathfrak{m}\boldsymbol{\mathcal{A}}(i)).$$

The reason is that deformation quantization can only deal with the pronilpotent inner gauge groups $\mathrm{IG}(\boldsymbol{\mathcal{A}}(i))$, and not with the bigger groups $\boldsymbol{\mathcal{A}}(i)^{\times}$.

I will make it precise in the next slides. The definition I will give might be too complicated in the associative case, but it is the only one known in the Poisson case.

We already encountered *crossed groupoids*, but without details. I will now give some details.

For a full discussion see [Br], [Bw], [Ye5], [Ye7].

Note that in [Bw] a crossed groupoid is called a crossed module over a groupoid, and also a 2-truncated crossed complex.

A crossed groupoid

$$\mathsf{G} = (\mathsf{G}_1, \mathsf{G}_2, \mathrm{Ad}_{\mathsf{G}_1 \curvearrowright \mathsf{G}_2}, \mathrm{D})$$

is made up of groupoids G_1 and G_2 that share the same set of objects $\mathrm{Ob}(\mathsf{G})$. The groupoid G_2 is totally disconnected.

There is an action $\mathrm{Ad}_{\mathsf{G}_1 \curvearrowright \mathsf{G}_2}$ of G_1 on G_2, and a functor $\mathrm{D} : \mathsf{G}_2 \to \mathsf{G}_1$ called the feedback. There are a few conditions (suppressed).

The morphisms in the groupoid G_p are called p-morphisms.

If there is one object, then G is a *crossed module*.

The similarity to a *strict* 2*-groupoid* is no coincidence. In fact these two concepts are the same; but the translation between them is a bit messy. See [Ye5, Section 5].

Example 3.2. Let $U \subset X$ be an open set, and let P be either $\mathsf{AssDef}(R, U)$ or $\mathsf{PoisDef}(R, U)$. Then P is a crossed groupoid.

For any $\mathcal{A}, \mathcal{A}' \in \mathrm{Ob}(\mathsf{P})$ the 1-morphisms, namely the elements of $\mathsf{P}_1(\mathcal{A}, \mathcal{A}')$, are the gauge transformations $g : \mathcal{A} \to \mathcal{A}'$.

In particular the group of 1-morphisms is $\mathsf{P}_1(\mathcal{A}) = \mathrm{Aut}_{\mathsf{P}}(\mathcal{A})$.

The group of 2-morphisms of $\mathcal{A}$ is $\mathsf{P}_2(\mathcal{A}) := \mathrm{IG}(\mathcal{A})$.

The action of a 1-morphism $g : \mathcal{A} \to \mathcal{A}'$ on 2-morphisms is the group isomorphism

$$\mathrm{Ad}_{G_1 \curvearrowright G_2}(g) := \mathrm{IG}(g) : \mathrm{IG}(\mathcal{A}) \to \mathrm{IG}(\mathcal{A}').$$

The feedback

$$\mathrm{D} : \mathrm{IG}(\mathcal{A}) \to \mathrm{Aut}_{\mathsf{P}}(\mathcal{A})$$

is $\mathrm{D}(a) := \mathrm{ig}(a)$. Recall that this is either conjugation by the invertible element a, or a formal hamiltonian flow.

Because deformations are sheaves, we obtain *stacks of crossed groupoids* $\mathbf{AssDef}(R, X)$ and $\mathbf{PoisDef}(R, X)$, defined by

$$\mathbf{AssDef}(R, X)(U) := \mathsf{AssDef}(R, U)$$

and

$$\mathbf{PoisDef}(R, X)(U) := \mathsf{PoisDef}(R, U)$$

for $U \subset X$ open.

Similarly there are these stacks of crossed groupoids:

- The stack $\mathbf{Grp}(X)$ of sheaves of groups on X. For a group $\mathcal{G}$ we take $\mathbf{Grp}(X)_1(\mathcal{G}) := \mathrm{Aut}(\mathcal{G})$ and $\mathbf{Grp}(X)_2(\mathcal{G}) := \mathcal{G}$.
- The stack $\mathbf{Ass}(R, X)$ of sheaves of R-algebras on X. For an algebra $\mathcal{A}$ we take $\mathbf{Ass}(R, X)_1(\mathcal{A}) := \mathrm{Aut}(\mathcal{A})$ and $\mathbf{Ass}(X)_2(\mathcal{A}) := \mathcal{A}^{\times}$.

Suppose P is a stack of crossed groupoids. If we forget the 2-morphisms, this leaves us with a stack of groupoids P_1.

There is a functor of stacks of groupoids

$$\mathrm{IG} : \mathsf{P}_1 \to \mathsf{Grp}(X)_1,$$

defined by

$$\mathrm{IG}(\mathcal{A}) := \mathsf{P}_2(\mathcal{A})$$

and

$$\mathrm{IG}(g) := \mathrm{Ad}_{G_1 \curvearrowright G_2}(g) : \mathsf{P}_2(\mathcal{A}) \to \mathsf{P}_2(\mathcal{A}').$$

Given a gerbe $\mathcal{G}$, there is a "tautological" functor of stacks

$$\mathrm{Aut}_{\mathcal{G}} : \mathcal{G} \to \mathsf{Grp}(X)_1.$$

The formula is this:

$$\mathrm{Aut}_{\mathcal{G}}(i) := \mathcal{G}(i)$$

for a local object i, and

$$\mathrm{Aut}_{\mathcal{G}}(g) := \mathrm{Ad}(g) : \mathcal{G}(i) \to \mathcal{G}(j)$$

for a local isomorphism $g : i \to j$.

Definition 3.3. Let P be a stack of crossed groupoids on X. A *twisted object* of P is a triple $(\mathcal{G}, \boldsymbol{\mathcal{A}}, \mathrm{cp})$ consisting of:

1. A gerbe $\mathcal{G}$, called the *gauge gerbe*.
2. A functor of stacks $\boldsymbol{\mathcal{A}} : \mathcal{G} \to \mathsf{P}_1$, called the *representation*.
3. An isomorphism
$$\mathrm{cp} : \mathrm{Aut}_{\mathcal{G}} \xrightarrow{\simeq} \mathrm{IG} \circ \boldsymbol{\mathcal{A}}$$
of functors of stacks $\mathcal{G} \to \mathsf{Grp}(X)_1$, called the *coupling*.

There is a compatibility condition on the coupling (supressed).

Let me explain. Say $(\mathcal{G}, \boldsymbol{\mathcal{A}}, \mathrm{cp})$ is a twisted object of the stack of crossed groupoids P.

To any local object i of $\mathcal{G}$, the representation $\boldsymbol{\mathcal{A}}$ assigns a local object $\mathcal{A}_i := \boldsymbol{\mathcal{A}}(i)$. We can think of $\mathcal{A}_i$ as a sheaf.

To a local isomorphism $g : i \to j$ in $\mathcal{G}$, the representation assigns a gauge transformation

$$\boldsymbol{\mathcal{A}}(g) : \mathcal{A}_i \to \mathcal{A}_j.$$

The coupling provides an isomorphism of groups

$$\mathrm{cp} : \mathcal{G}(i) \xrightarrow{\simeq} \mathrm{IG}(\mathcal{A}_i) = \mathsf{P}_2(\mathcal{A}_i).$$

Definition 3.4.

1. A *twisted associative R-deformation* of $\mathcal{O}_X$ is a twisted object of the stack $\mathbf{AssDef}(R, X)$.
2. A *twisted Poisson R-deformation* of $\mathcal{O}_X$ is a twisted object of the stack $\mathbf{PoisDef}(R, X)$.

Given a twisted deformation $(\mathcal{G}, \mathcal{A}, \mathrm{cp})$, it will be more convenient to talk about a twisted deformation $\mathcal{A}$, with gauge gerbe $\mathcal{G}$ (and to leave the coupling implicit).

Note that here the coupling makes the gauge gerbe $\mathcal{G}$ into a *pronilpotent gerbe*, in the sense of [Ye2].

To end this section, let me explain how other, more familar, kinds of stacks can be expressed as twisted objects.

Example 3.5. Consider the stack of crossed groupoids $\mathbf{Grp}(X)$. A twisted object of this stack is the same as a gerbe $\mathcal{G}$.

Indeed, $\mathcal{G}$ is its own gauge grebe; the representation is $\mathcal{A} := \mathrm{Aut}_{\mathcal{G}}$; and the coupling cp is the identity.

Example 3.6. An R-linear algebroid $\mathcal{A}$ is twisted object of the stack of crossed groupoids $\mathbf{Ass}(R, X)$.

Example 3.7. A DQ algebroid $\mathcal{A}$ is twisted object of the stack of crossed groupoids $\mathbf{AssDef}(R, X)$, but for the big inner gauge group $\mathrm{IG}(\mathcal{A}) := \mathcal{A}(i)^{\times}$.

Similar ideas on classification of twisted objects can be found in [Br], [Ko2], [BGNT1], [BGNT2], [DP1], [DP2], [PS].

4. The Twisted Quantization Theorem

Definition 4.1. Let $(\mathcal{A}, \mathcal{G}, \mathrm{cp})$ and $(\mathcal{A}', \mathcal{G}', \mathrm{cp}')$ be twisted R-deformations of $\mathcal{O}_X$ of the same kind (both associative or Poisson).

A *twisted gauge transformation* $F : \mathcal{A} \to \mathcal{A}'$ consists of:

- An equivalence $F_{\mathrm{gau}} : \mathcal{G} \xrightarrow{\simeq} \mathcal{G}'$ between the gauge gerbes.
- An isomorphism
$$F_{\mathrm{rep}} : \mathcal{A} \overset{\simeq}{\Longrightarrow} \mathcal{A}' \circ F_{\mathrm{gau}}$$
of functors $\mathcal{G} \to \mathbf{P}$.

There is a compatibility condition regarding the couplings.

Our main result is this.

Theorem 4.2. (Twisted Quantization, [Ye8])

Assume $\mathbb{K}$ contains $\mathbb{R}$. Let X be a smooth algebraic variety over $\mathbb{K}$, and let R be a parameter algebra.

There is a bijection

$$\mathrm{tw.quant} : \frac{\{\text{twisted Poisson } R\text{-deformations of } \mathcal{O}_X\}}{\text{twisted gauge equivalence}} \xrightarrow{\simeq} \frac{\{\text{twisted associative } R\text{-deformations of } \mathcal{O}_X\}}{\text{twisted gauge equivalence}}$$

called twisted quantization.

The bijection tw.quant *commutes with homomorphisms of parameter algebras $R \to R'$, and with étale morphisms of varieties $X' \to X$.*

I did not explain how to pull back twisted deformations along an étale map of varieties $f : X' \to X$; but it is clear what to do when f is an isomorphism, or even an open embedding.

The condition that $\mathbb{R} \subset \mathbb{K}$ can be removed, if we use another universal quantization formula (not the original one from [Ko1]). Cf. [Ta], [CV].

Theorem 4.2 was suggested, in a much weaker form and with no proof, in the paper [Ko2] of M. Kontsevich.

Results of similar flavor can be found in [Ka], [BK], [BGNT1], [BGNT2], [CH].

Here is a special case of the main theorem.

Corollary 4.3. ([Ye1], [Ye8]) *In the situation of Theorem* 4.2, *assume the cohomology groups* $\mathrm{H}^1(X, \mathcal{O}_X)$, $\mathrm{H}^2(X, \mathcal{O}_X)$, $\mathrm{H}^1(X, \mathcal{T}_X)$ *and* $\mathrm{H}^1(X, \mathcal{D}_X)$ *all vanish (e.g. X is affine).*

Then there is a canonical bijection

$$\mathrm{quant} : \frac{\{\text{Poisson brackets on } R \mathbin{\widehat{\otimes}} \mathcal{O}_X\}}{\text{gauge equivalence}} \xrightarrow{\cong} \frac{\{\text{star products on } R \mathbin{\widehat{\otimes}} \mathcal{O}_X\}}{\text{gauge equivalence}} .$$

I will finish this section with two questions.

Question 4.4. It is easy to construct an example of a *commutative* twisted associative $\mathbb{K}[[\hbar]]$-deformation of $\mathcal{O}_X$ that is *really twisted*, namely it is not twisted-equivalent to a usual deformation.

But does there exist a variety X, with a *symplectic* twisted associative $\mathbb{K}[[\hbar]]$-deformation $\boldsymbol{\mathcal{A}}$ of $\mathcal{O}_X$ which is *really twisted*?

A more concrete (but perhaps much more challenging) question is:

Question 4.5. Let X be a Calabi-Yau surface, with symplectic Poisson bracket $\{-,-\}$. Consider the corresponding Poisson $\mathbb{K}[[\hbar]]$-deformation $\mathcal{A}$ of $\mathcal{O}_X$, which we can view as a twisted Poisson deformation $\boldsymbol{\mathcal{A}}$.

Let $\boldsymbol{\mathcal{B}} := \mathrm{tw.quant}(\boldsymbol{\mathcal{A}})$, which is a twisted associative $\mathbb{K}[[\hbar]]$-deformation of $\mathcal{O}_X$, well-defined up to twisted gauge equivalence. *Is* $\boldsymbol{\mathcal{B}}$ *really twisted?*

5. About the Proof; Intermediate Results

The proof of the main theorem, Theorem 4.2, relies on several intermediate theorems, that are of independent interest.

I will give a explanation of a few notions, and then I will present these theorems (as much as time permits).

Finally I will show a diagram that tells how these intermediate results are combined to prove the main result.

Consider a *cosimplicial crossed groupoid* $G = \{G^q\}_{q \in \mathbb{N}}$.

A *descent datum* in G is a triple (i, g, a) consisting of:

- An object i in the crossed groupoid G^0.
- A 1-morphism g in the crossed groupoid G^1.
- A 2-morphism a in the crossed groupoid G^2.

These must satisfy the familiar conditions of gerbe cocycles (see [Br]). I will give an example in the next slide.

The set of all descent data is denoted by $\mathrm{Desc}(G)$.

There is an equivalence relation on this set, and we write $\overline{\mathrm{Desc}}(G)$ for the resulting quotient set.

Example 5.1. On the topological space X we have the stack of crossed groupoids $\mathbf{P} := \mathbf{Alg}(R, X)$. Recall that the local objects are the sheaves of R-algebras.

Take an open covering $\boldsymbol{U} = \{U_k\}$ of X. The *Čech construction* give rise to a cosimplicial crossed groupoid $\mathrm{C}(\boldsymbol{U}, \mathbf{P})$.

A descent datum in $\mathrm{C}(\boldsymbol{U}, \mathbf{P})$ is precisely a cocycle description of an R-algebroid. See [Ko2].

The set $\overline{\mathrm{Desc}}(\mathrm{C}(\boldsymbol{U}, \mathbf{P}))$ is an approximation of the set of equivalence classes of R-algebroids. (To capture everything we must go to the limit over refinements; and sometimes we also need hypercoverings.)

Fortunately, for twisted deformations a single affine open covering suffices! This is the significance of Theorem 5.2 below.

Let us denote by $\overline{\mathrm{TwOb}}(\mathbf{P})$ the set of twisted gauge equivalence classes of twisted objects of the stack $\mathbf{P}$.

Theorem 5.2. (Decomposition, [Ye8]) *Let $\boldsymbol{U}$ be an affine open covering of X, and let $\mathbf{P}$ be either $\mathbf{PoisDef}(R, X)$ or $\mathbf{PoisDef}(R, X)$.*

There is a canonical bijection

$$\overline{\mathrm{dec}} : \overline{\mathrm{TwOb}}(\mathbf{P}) \xrightarrow{\simeq} \overline{\mathrm{Desc}}(\mathrm{C}(\boldsymbol{U}, \mathbf{P}))$$

called decomposition.

A DG Lie algebra $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}^i$ is called *quantum type* if $\mathfrak{g}^i = 0$ for all $i < -1$.

The *Deligne crossed groupoid* $\mathrm{Del}(\mathfrak{g}, R)$ associated to such a DG Lie algebra has as objects the solutions of the MC equation in $\mathfrak{m} \mathbin{\hat{\otimes}} \mathfrak{g}$. See [De], [Ge].

If $\mathfrak{g}$ is a cosimplicial DG Lie algebra, then we obtain a cosimplicial crossed groupoid $\mathrm{Del}(\mathfrak{g}, R)$, and we can talk about descent data.

On the variety X there are two important sheaves of DG Lie algebras:

- The sheaf $\mathcal{T}_{\mathrm{poly},X}$ of poly-derivations.
- The sheaf $\mathcal{D}_{\mathrm{poly},X}$ of poly-differential operators.

An open covering $\boldsymbol{U}$ gives rise to cosimplicial DG Lie algebras $\mathrm{C}(\boldsymbol{U}, \mathcal{T}_{\mathrm{poly},X})$ and $\mathrm{C}(\boldsymbol{U}, \mathcal{D}_{\mathrm{poly},X})$.

Theorem 5.3. (Geometrization, [Ye6])

Let $\boldsymbol{U}$ a finite affine open covering of X.

There are canonical bijections

$$\overline{\text{geo}} : \overline{\text{Desc}}\big(\text{Del}\big(\text{C}\big(\boldsymbol{U}, \mathcal{D}_{\text{poly},X}\big), R\big)\big) \xrightarrow{\simeq} \overline{\text{Desc}}\big(\text{C}\big(\textbf{AssDef}(R, X), \boldsymbol{U}\big)\big)$$

and

$$\overline{\text{geo}} : \overline{\text{Desc}}\big(\text{Del}\big(\text{C}\big(\boldsymbol{U}, \mathcal{T}_{\text{poly},X}\big), R\big)\big) \xrightarrow{\simeq} \overline{\text{Desc}}\big(\text{C}\big(\textbf{PoisDef}(R, X), \boldsymbol{U}\big)\big)$$

called geometrization.

The following theorem is a far reaching generalization of the original Equivalence Theorem from [GM], which is attributed to Deligne. There are similar results in [Ge], [BGNT1].

Theorem 5.4. (Equivalence for Lie Descent, [Ye8])

Let $\Psi : \mathfrak{g} \to \mathfrak{h}$ be a cosimplicial L_∞ quasi-isomorphism between cosimplicial quantum type DG Lie algebras.

There is a bijection

$$\overline{\text{Desc}}(\text{Del})(\Psi) : \overline{\text{Desc}}(\text{Del}(\mathfrak{g}, R)) \to \overline{\text{Desc}}(\text{Del}(\mathfrak{h}, R)),$$

functorial in Ψ and R.

The last ingredient we need is a version of the Kontsevich Formality Theorem. It is based on our work in [Ye1]. See [VdB] for an alternative proof.

Let $U \subset X$ be an affine open set. An étale coordinate system on U is an étale map $U \to \mathbf{A}^n_{\mathbb{K}}$.

An *affine open covering with étale coordinates* is an affine open covering $\boldsymbol{U} = \{U_k\}$, together with an étale coordinate system on each U_k.

Say $\mathcal{M}$ is a quasi-coherent sheaf on X. The *mixed resolution* of $\mathcal{M}$ is a complex of sheaves $\text{Mix}_{\boldsymbol{U}}(\mathcal{M})$, made up of the Čech resolution coming from the covering $\boldsymbol{U}$, and the *jet resolution*. There is a canonical quasi-isomorphism

$$\text{mix} : \mathcal{M} \to \text{Mix}_{\boldsymbol{U}}(\mathcal{M}).$$

Theorem 5.5. (Cosimplicial Formality, [Ye8])

Let $\boldsymbol{U}$ be a finite affine open covering with étale coordinates of X.

There is a diagram

$$\begin{array}{ccc}
\text{C}(\boldsymbol{U}, \mathcal{T}_{\text{poly},X}) & & \text{C}(\boldsymbol{U}, \mathcal{D}_{\text{poly},X}) \\
\downarrow{\scriptstyle \text{mix}} & & \downarrow{\scriptstyle \text{mix}} \\
\text{C}(\boldsymbol{U}, \text{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\text{poly},X})) & \xrightarrow{\Psi} & \text{C}(\boldsymbol{U}, \text{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\text{poly},X}))
\end{array},$$

where the objects are cosimplicial quantum type DG Lie algebras, the vertical arrows are cosimplicial DG Lie quasi-isomorphisms, and the horizontal arrow is a cosimplicial L_∞ quasi-isomorphism.

Theorem 5.4 says that when we apply the functor $\overline{\text{Desc}}(\text{Del})$ these arrows become bijections.

Finally here is the proof of the main theorem: the *twisted quantization* operation is the unique arrow making the diagram below commutative.

$$\begin{array}{ccc}
\overline{\mathrm{TwOb}}(\mathbf{PoisDef}(R,X)) & \xrightarrow{\text{tw.quant}} & \overline{\mathrm{TwOb}}(\mathbf{AssDef}(R,X)) \\
\downarrow \overline{\mathrm{dec}} & & \downarrow \overline{\mathrm{dec}} \\
\overline{\mathrm{Desc}}(\mathrm{C}(\boldsymbol{U},\mathbf{PoisDef}(R,X))) & & \overline{\mathrm{Desc}}(\mathrm{C}(\boldsymbol{U},\mathbf{AssDef}(R,X)))\\
\uparrow \overline{\mathrm{geo}} & & \uparrow \overline{\mathrm{geo}}\\
\overline{\mathrm{Desc}}(\mathrm{Del}(\mathrm{C}(\boldsymbol{U},\mathcal{T}_{\mathrm{poly},X}),R)) & & \overline{\mathrm{Desc}}(\mathrm{Del}(\mathrm{C}(\boldsymbol{U},\mathcal{D}_{\mathrm{poly},X}),R))\\
\downarrow \overline{\mathrm{mix}} & & \downarrow \overline{\mathrm{mix}}\\
\overline{\mathrm{Desc}}(\mathrm{Del}(\mathrm{C}(\boldsymbol{U},\mathrm{Mix}(\mathcal{T}_{\mathrm{poly},X})),R)) & \xrightarrow{\overline{\Psi}} & \overline{\mathrm{Desc}}(\mathrm{Del}(\mathrm{C}(\boldsymbol{U},\mathrm{Mix}(\mathcal{D}_{\mathrm{poly},X})),R))
\end{array}$$

~ END ~

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