

Twisted Deformation Quantization of Algebraic Varieties

Lecture Notes ¹

Amnon Yekutieli
Ben Gurion University, ISRAEL
<http://www.math.bgu.ac.il/~amyekut>

Here is the plan of my lecture:

1. Some background on Deformation Quantization
2. Poisson Deformations of Algebraic Varieties
3. Associative Deformations of Algebraic Varieties
4. Deformation Quantization
5. Twisted Deformations of Algebraic Varieties
6. Twisted Deformation Quantization

Notes are available online. The notes also contain four appendices, and a bibliography.

Part of this work is joint with Frederick Leitner.

1. SOME BACKGROUND ON DEFORMATION QUANTIZATION

Let \mathbb{K} be a field of characteristic 0, and let C be a commutative \mathbb{K} -algebra.

Recall that a *Poisson bracket* on C is a \mathbb{K} -bilinear function

$$\{-, -\} : C \times C \rightarrow C$$

which makes C into a Lie algebra, and is a biderivation (i.e. a derivation in each argument).

The pair $(C, \{-, -\})$ is called a Poisson algebra.

Poisson algebras arise in several ways, e.g. classical Hamiltonian mechanics, or Lie theory.

Let $\mathbb{K}[[\hbar]]$ be the ring of formal power series in the variable \hbar .

Let $C[[\hbar]]$ be the set of formal power series with coefficients in C , which we view only as a $\mathbb{K}[[\hbar]]$ -module.

A *star product* on $C[[\hbar]]$ is a function

$$\star : C[[\hbar]] \times C[[\hbar]] \rightarrow C[[\hbar]]$$

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which makes $C[[\hbar]]$ into an associative $\mathbb{K}[[\hbar]]$ -algebra, with unit $1 \in C$, and such that

$$f \star g \equiv fg \pmod{\hbar}$$

for any $g, f \in C$.

The pair $(C[[\hbar]], \star)$ is called an *associative deformation* of C .

Example 1.1. Suppose $(C[[\hbar]], \star)$ is an associative deformation of C .

Given $f, g \in C$, we know that

$$f \star g - g \star f \equiv 0 \pmod{\hbar}.$$

Hence there is a unique element

$$\{f, g\}_\star \in C$$

such that

$$\frac{1}{\hbar}(f \star g - g \star f) \equiv \{f, g\}_\star \pmod{\hbar}.$$

It is quite easy to show that $\{-, -\}_\star$ is a Poisson bracket on C . We call it the *first order bracket* of \star .

Deformation quantization seeks to reverse Example 1.1.

Definition 1.2. Given a Poisson bracket $\{-, -\}$ on the algebra C , a *deformation quantization* of $\{-, -\}$ is an associative deformation $(C[[\hbar]], \star)$ of C whose first order bracket is $\{-, -\}$.

In physics \hbar is the *Planck constant*. For a quantum phenomenon depending on \hbar , the limit as $\hbar \rightarrow 0$ is thought of as the classical limit of this phenomenon.

The original idea by the physicists Flato et. al. ([BFFLS], 1978) was that deformation quantization should model the transition from classical Hamiltonian mechanics to quantum mechanics.

Special cases (like the Moyal product) were known. The problem arose: *does any Poisson bracket admit a deformation quantization?*

For a symplectic manifold X and $C = C^\infty(X)$ the problem was solved by De Wilde and Lecomte ([DL], 1983). A more geometric solution was discovered by Fedosov ([Fe], 1994).

The general case, i.e. $C = C^\infty(X)$ for a Poisson manifold X , was solved by Kontsevich ([Ko1], 1997). See surveys in the book [CKTB].

2. POISSON DEFORMATIONS OF ALGEBRAIC VARIETIES

In algebraic geometry we have to consider deformations as sheaves.

Let X be a smooth algebraic variety over \mathbb{K} , with structure sheaf \mathcal{O}_X .

We view \mathcal{O}_X as a Poisson \mathbb{K} -algebra with zero bracket.

Definition 2.1. A *Poisson deformation* of \mathcal{O}_X is a sheaf \mathcal{A} of flat, \hbar -adically complete, commutative Poisson $\mathbb{K}[[\hbar]]$ -algebras on X , with an isomorphism of Poisson

algebras

$$\psi : \mathcal{A}/(\hbar) \xrightarrow{\cong} \mathcal{O}_X,$$

called an augmentation.

A *gauge equivalence* $\mathcal{A} \rightarrow \mathcal{A}'$ between Poisson deformations is a $\mathbb{K}[[\hbar]]$ -linear isomorphism of sheaves of Poisson algebras, that commutes with the augmentations to \mathcal{O}_X .

Given a Poisson deformation \mathcal{A} of \mathcal{O}_X , we may define the *first order bracket*

$$\{-, -\}_{\mathcal{A}} : \mathcal{O}_X \times \mathcal{O}_X \rightarrow \mathcal{O}_X.$$

This is a Poisson bracket whose formula is

$$\{f, g\}_{\mathcal{A}} := \psi\left(\frac{1}{\hbar}\{\tilde{f}, \tilde{g}\}\right),$$

where $f, g \in \mathcal{O}_X$ are local sections, and $\tilde{f}, \tilde{g} \in \mathcal{A}$ are arbitrary local lifts.

The first order bracket is invariant under gauge equivalence.

Example 2.2. Let $\{-, -\}_1$ be some Poisson bracket on \mathcal{O}_X .

Define

$$\mathcal{A} := \mathcal{O}_X[[\hbar]].$$

This is a sheaf of $\mathbb{K}[[\hbar]]$ -algebras, with the usual commutative multiplication, and the obvious augmentation $\mathcal{A}/(\hbar) \cong \mathcal{O}_X$.

Put on \mathcal{A} the $\mathbb{K}[[\hbar]]$ -bilinear Poisson bracket $\{-, -\}$ such that

$$\{f, g\} = \hbar\{f, g\}_1$$

for $f, g \in \mathcal{O}_X$.

Then \mathcal{A} is a Poisson deformation of \mathcal{O}_X . The first order bracket in this case is just

$$\{-, -\}_{\mathcal{A}} = \{-, -\}_1.$$

Poisson deformations are controlled by a sheaf of DG (differential graded) Lie algebras $\mathcal{T}_{\text{poly}, X}$, called the *poly derivations*.

This is explained in Appendix A.

3. ASSOCIATIVE DEFORMATIONS OF ALGEBRAIC VARIETIES

Definition 3.1. An *associative deformation* of \mathcal{O}_X is a sheaf \mathcal{A} of flat, \hbar -adically complete, associative, unital $\mathbb{K}[[\hbar]]$ -algebras on X , with an isomorphism of algebras

$$\psi : \mathcal{A}/(\hbar) \xrightarrow{\cong} \mathcal{O}_X,$$

called an augmentation.

There is a suitable notion of gauge equivalence between associative deformations.

Given an associative deformation \mathcal{A} we may define the first order bracket

$$\{-, -\}_{\mathcal{A}} : \mathcal{O}_X \times \mathcal{O}_X \rightarrow \mathcal{O}_X.$$

The formula is

$$\{f, g\}_{\mathcal{A}} := \psi\left(\frac{1}{\hbar}(\tilde{f} \star \tilde{g} - \tilde{g} \star \tilde{f})\right).$$

The first order bracket is invariant under gauge equivalence.

Note that both kinds of deformations – Poisson and associative – include as special cases the classical commutative deformations of \mathcal{O}_X .

Associative deformations are controlled by a quasi-coherent sheaf of DG Lie algebras $\mathcal{D}_{\text{poly}, X}$, called the *poly differential operators*.

This is explained in Appendix A.

4. DEFORMATION QUANTIZATION

Kontsevich [K01] proved that any Poisson deformation of a real C^∞ manifold X can be canonically quantized.

In this section we present an algebraic version of this result. But first a definition.

Definition 4.1. Let \mathcal{A} be a Poisson deformation of \mathcal{O}_X . A *quantization* of \mathcal{A} is an associative deformation \mathcal{B} , such that the first order brackets satisfy

$$\{-, -\}_{\mathcal{B}} = \{-, -\}_{\mathcal{A}}.$$

Recalling Example 2.2, we see that this definition captures the essence of deformation quantization, namely quantizing a Poisson bracket on \mathcal{O}_X .

Theorem 4.2. ([Ye1]) *Let \mathbb{K} be a field containing \mathbb{R} , and let X be a smooth affine algebraic variety over \mathbb{K} .*

There is a canonical bijection

$$\begin{aligned} \text{quant} : \frac{\{\text{Poisson deformations of } \mathcal{O}_X\}}{\text{gauge equivalence}} \\ \xrightarrow{\cong} \frac{\{\text{associative deformations of } \mathcal{O}_X\}}{\text{gauge equivalence}}, \end{aligned}$$

which is a quantization as defined above.

By “canonical” I mean that this quantization map commutes with étale morphisms $X' \rightarrow X$.

Actually our result in [Ye1] is stronger – it holds for a wider class of varieties, not just affine varieties. However all these cases are subsumed in Corollary 6.2 below.

Theorem 4.2 is a consequence of the following more general result.

Theorem 4.3. ([Ye1]) *Let \mathbb{K} be a field containing \mathbb{R} , and let X be a smooth algebraic variety over \mathbb{K} .*

Then there is a diagram

$$\begin{array}{ccc} \mathcal{T}_{\text{poly}, X} & & \mathcal{D}_{\text{poly}, X} \\ \downarrow & & \downarrow \\ \text{Mix}(\mathcal{T}_{\text{poly}, X}) & \longrightarrow & \text{Mix}(\mathcal{D}_{\text{poly}, X}) \end{array}$$

where:

- $\text{Mix}(\mathcal{T}_{\text{poly},X})$ and $\text{Mix}(\mathcal{D}_{\text{poly},X})$ are sheaves of DG Lie algebras on X , called mixed resolutions;
- the vertical arrows are DG Lie algebra quasi-isomorphisms;
- and the horizontal arrow is an L_∞ quasi-isomorphism.

The mixed resolutions combine the *commutative Čech resolution* associated to an affine open covering of X , and the *Grothendieck sheaf of jets*.

An L_∞ *quasi-isomorphism* is a generalization of a DG Lie algebra quasi-isomorphism.

Theorem 4.3 is proved using the *Formality Theorem* of Kontsevich [Ko1] and *formal geometry*.

More on the proof of Theorem 4.3 in Appendices B and C.

Remark 4.4. Calaque and Van den Bergh [CV] proved (using ideas of Tamarkin and Halbout) that a global quantization map exists over \mathbb{Q} .

According to Kontsevich, changing the local formality isomorphism (i.e. changing the Drinfeld associator) can have a nontrivial effect on the global quantization process.

This phenomenon is very intriguing and should be studied.

5. TWISTED DEFORMATIONS OF ALGEBRAIC VARIETIES

What can be done in general, when the variety X is not affine? Can we still make use of Theorem 4.3?

In the paper [Ko3] Kontsevich suggests that in general the deformation quantization of a Poisson bracket might have to be a *stack of algebroids*. This is a generalization of the notion of sheaf of algebras.

Actually stacks of algebroids appeared earlier, under the name *sheaves of twisted modules*, in the work of Kashiwara [Ka]. See also [DP], [PS], [KS].

I will use the term *twisted associative deformation*, and present an approach that treats the Poisson case as well.

This approach was suggested to us by Kontsevich. A similar point of view is taken in [BGNT].

Here I will explain only a naive definition of twisted deformations. A more sophisticated definition, involving gerbes, may be found in Appendix D.

The fact that the two definitions agree follows from our work on central extensions of gerbes and obstructions classes [Ye5].

Let $U \subset X$ be an affine open set, and let $C := \Gamma(U, \mathcal{O}_X)$.

Suppose A is an associative or Poisson deformation of the \mathbb{K} -algebra C .

One may assume that $A = C[[\hbar]]$, and it is either endowed with a Poisson bracket $\{-, -\}$, or with a star product \star .

In either case A becomes a pronilpotent Lie algebra, and $\hbar A$ is a Lie subalgebra.

In the Poisson case the Lie bracket is $\{-, -\}$, and in the associative case the Lie bracket is the commutator

$$[a, b] := a \star b - b \star a.$$

Let us denote the corresponding pronilpotent group by

$$\mathrm{IG}(A) := \exp(\hbar A),$$

and call it the *group of inner gauge transformations* of A .

The group $\mathrm{IG}(A)$ acts on the deformation A by gauge equivalences. We denote this action by Ad .

In the Poisson case the gauge transformation $\mathrm{Ad}(g)$, for $g \in \mathrm{IG}(A)$, can be viewed as a formal hamiltonian flow.

In the associative case the intrinsic exponential function

$$\exp(a) = \sum_{i \geq 0} \frac{1}{i!} a^i,$$

for $a \in \hbar A$, allows us to identify the group $\mathrm{IG}(A)$ with the multiplicative subgroup

$$\{g \in A \mid g \equiv 1 \pmod{\hbar}\}.$$

Under this identification the operation $\mathrm{Ad}(g)$ is just conjugation by the invertible element g .

The above can be sheaffied: to a deformation \mathcal{A} of \mathcal{O}_X we assign the sheaf of groups $\mathrm{IG}(\mathcal{A})$.

Let us fix an affine open covering $\{U_0, \dots, U_m\}$ of X . We write

$$U_{i,j,\dots} := U_i \cap U_j \cap \dots.$$

Definition 5.1. A *twisted associative (resp. Poisson) deformation* \mathcal{A} of \mathcal{O}_X consists of the following data:

- (1) For any i , a deformation \mathcal{A}_i of \mathcal{O}_{U_i} .
- (2) For any $i < j$, a gauge equivalence

$$g_{i,j} : \mathcal{A}_i|_{U_{i,j}} \rightarrow \mathcal{A}_j|_{U_{i,j}}.$$

- (3) For any $i < j < k$, an element

$$a_{i,j,k} \in \Gamma(U_{i,j,k}, \mathrm{IG}(\mathcal{A}_i)).$$

The conditions are:

- (i) For any $i < j < k$ one has

$$g_{i,k}^{-1} \circ g_{j,k} \circ g_{i,j} = \mathrm{Ad}(a_{i,j,k}^{-1}).$$

- (ii) For any $i < j < k < l$ one has

$$a_{i,j,l}^{-1} \cdot a_{i,k,l} \cdot a_{i,j,k} = g_{i,j}^{-1}(a_{j,k,l}).$$

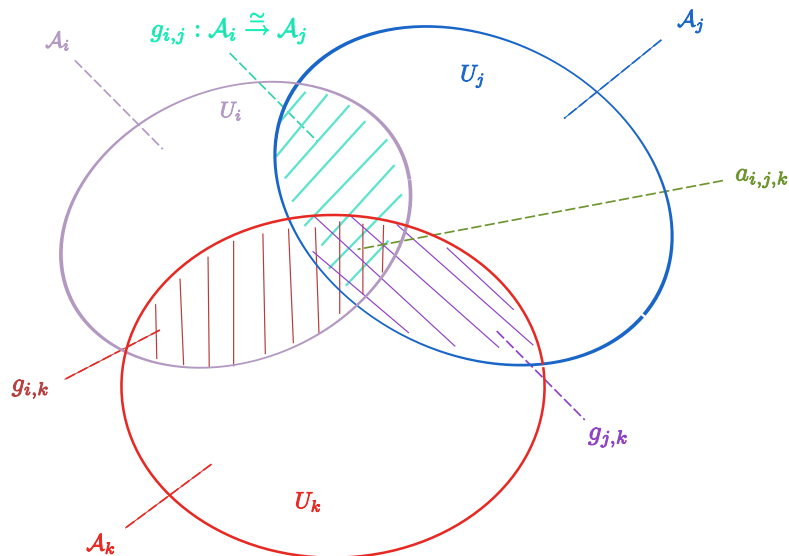


FIGURE 1

See Figure 1 for an illustration.

Condition (i) says that the 2-cochain $\{\text{Ad}(a_{i,j,k})\}$ measures the failure of the 1-cochain $\{g_{i,j}\}$ to be a cocycle.

This tells us whether the collection $\{\mathcal{A}_i\}$ of local deformations can be glued into a global deformation of \mathcal{O}_X .

Condition (ii) – usually called the tetrahedron equation – says that the 2-cochain $\{a_{i,j,k}\}$ satisfies a twisted cocycle condition.

Example 5.2. If \mathcal{A} is a usual deformation of \mathcal{O}_X , then we obtain a twisted deformation \mathcal{A} by taking $\mathcal{A}_i := \mathcal{A}|_{U_i}$, $g_{i,j} := \mathbf{1}$ and $a_{i,j,k} := 1$.

Remark 5.3. For a twisted associative deformation \mathcal{A} there is a well defined abelian category $\text{Coh } \mathcal{A}$ of “coherent left \mathcal{A} -modules”, which is a deformation of the abelian category $\text{Coh } \mathcal{O}_X$. See the work of Lowen and Van den Bergh [LV].

Indeed, there is a geometric Morita theory, which says that twisted associative deformations of \mathcal{O}_X are the same as deformations of $\text{Coh } \mathcal{O}_X$ (as stack of abelian categories). This is explained in the new book by Kashiwara and Schapira [KS].

We do not know of a similar interpretation of twisted Poisson deformations.

Remark 5.4. It appears that twisted deformations behave very similarly when X is a complex analytic manifold.

For a differentiable manifold X there is nothing interesting, since all obstruction classes vanish.

6. TWISTED DEFORMATION QUANTIZATION

There is a notion of *twisted gauge equivalence* $\mathcal{A} \rightarrow \mathcal{B}$ between twisted associative (resp. Poisson) deformations of \mathcal{O}_X .

Just as in the case of usual deformations, given a twisted (associative or Poisson) deformation \mathcal{A} of \mathcal{O}_X , we can define the first order bracket $\{-, -\}_{\mathcal{A}}$ on \mathcal{O}_X .

Let \mathcal{A} be a twisted Poisson deformation, and let \mathcal{B} be a twisted associative deformation. We say that \mathcal{B} is a *twisted quantization* of \mathcal{A} if

$$\{-, -\}_{\mathcal{B}} = \{-, -\}_{\mathcal{A}}.$$

The next theorem is influenced by ideas of Kontsevich (from [Ko3] and private communications).

Theorem 6.1. ([Ye5]) *Let \mathbb{K} be a field containing \mathbb{R} , and let X be a smooth algebraic variety over \mathbb{K} .*

Then there is a canonical bijection

$$\begin{aligned} \text{quant} : & \frac{\{\text{twisted Poisson deformations of } \mathcal{O}_X\}}{\text{twisted gauge equivalence}} \\ & \xrightarrow{\cong} \frac{\{\text{twisted associative deformations of } \mathcal{O}_X\}}{\text{twisted gauge equivalence}}, \end{aligned}$$

which is a twisted quantization in the sense above.

As before, by “canonical” we mean that this quantization map commutes with étale morphisms $X' \rightarrow X$.

The proof of Theorem 6.1 relies on a rather complicated calculation of Maurer-Cartan equations in cosimplicial DG Lie algebras, and on a new theory of *nonabelian integration on surfaces*.

The theorem, together with the results on obstruction classes for gerbes, implies:

Corollary 6.2. ([Ye5])

Assume

$$H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0.$$

Then the quantization map of the theorem gives a bijection

$$\begin{aligned} \text{quant} : & \frac{\{\text{Poisson deformations of } \mathcal{O}_X\}}{\text{gauge equivalence}} \\ & \xrightarrow{\cong} \frac{\{\text{associative deformations of } \mathcal{O}_X\}}{\text{gauge equivalence}}. \end{aligned}$$

Let me finish with a question.

Given a variety X , with Poisson bracket $\{-, -\}_1$ on \mathcal{O}_X , we can form the Poisson deformation $\mathcal{A} := \mathcal{O}_X[[\hbar]]$, with bracket $\hbar\{-, -\}_1$.

By viewing \mathcal{A} as a twisted Poisson deformation, and applying Theorem 6.1, we get a twisted associative deformation $\mathcal{B} := \text{quant}(\mathcal{A})$.

We say \mathcal{B} is *really twisted* if it is not equivalent to any usual deformation \mathcal{B} .

Question 6.3. *Does there exist a variety X , with a symplectic Poisson bracket $\{-, -\}_1$, such that the corresponding twisted associative deformation \mathcal{B} is really twisted?*

My feeling is that the answer is positive.

And moreover, an example should be when X is any Calabi-Yau surface, and $\{-, -\}_1$ is any nonzero Poisson bracket on X .

- END -

Here are the appendices:

- A. DG Lie Algebras and Deformations
- B. The Universal Quantization Map
- C. The L_∞ quasi-isomorphism of the Level of Sheaves
- D. Twisted Deformations via Stacks of Gluing Groupoids

APPENDIX A. DG LIE ALGEBRAS AND DEFORMATIONS

The idea that DG (differential graded) Lie algebras control deformation problems is attributed to Deligne. See [GM].

Recall that a DG Lie algebra is a graded \mathbb{K} -module $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}^p$, with a bracket $[-, -]$ satisfying the graded version of the Lie algebra identities, together with a graded derivation d of degree 1 and square 0.

Given a DG Lie algebra \mathfrak{g} , let us define a new DG Lie algebra

$$\mathfrak{g}[[\hbar]]^+ := \bigoplus_p \hbar \mathfrak{g}^p[[\hbar]] \subset \bigoplus_p \mathfrak{g}^p[[\hbar]],$$

in which \hbar is central.

Remark A.1. Everywhere in these notes we can replace $\mathbb{K}[[\hbar]]$ with any complete noetherian local \mathbb{K} -algebra R , with maximal ideal \mathfrak{m} , such that $R/\mathfrak{m} = \mathbb{K}$. There would have to be slight modifications of course; e.g. instead of $\mathfrak{g}[[\hbar]]^+$ we would have to take $\mathfrak{g} \widehat{\otimes} \mathfrak{m}$.

The *Maurer-Cartan equation* in $\mathfrak{g}[[\hbar]]^+$ is

$$d(\alpha) + \frac{1}{2}[\alpha, \alpha] = 0$$

for

$$\alpha = \sum_{j=1}^{\infty} \alpha_j \hbar^j \in \mathfrak{g}^1[[\hbar]]^+.$$

Let $\exp(\mathfrak{g}^0[[\hbar]]^+)$ be the pro-unipotent group associated to the pro-nilpotent Lie algebra $\mathfrak{g}^0[[\hbar]]^+$.

There is an action of the group $\exp(\mathfrak{g}^0[[\hbar]]^+)$ on $\mathfrak{g}^1[[\hbar]]^+$, and this action preserve the set of solutions of the Maurer-Cartan equation.

One defines

$$\text{MC}(\mathfrak{g}[[\hbar]]^+) := \frac{\{\text{solutions of MC equation in } \mathfrak{g}[[\hbar]]^+\}}{\exp(\mathfrak{g}^0[[\hbar]]^+)}.$$

Let us return to our deformation problem, where X is a smooth algebraic variety over \mathbb{K} . Take an affine open set $U \subset X$, and let $C := \Gamma(U, \mathcal{O}_X)$.

One can show that any Poisson (resp. associative) deformation of C is isomorphic to $C[[\hbar]]$ as $\mathbb{K}[[\hbar]]$ -algebra (resp. $\mathbb{K}[[\hbar]]$ -module). Thus it suffices to understand Poisson brackets and star products on $C[[\hbar]]$.

Let \mathcal{T}_C denote the module of derivations. For $p \geq -1$ define

$$\mathcal{T}_{\text{poly}}^p(C) := \bigwedge_C^{p+1} \mathcal{T}_C.$$

So $\mathcal{T}_{\text{poly}}^{-1}(C) = C$, $\mathcal{T}_{\text{poly}}^0(C) = \mathcal{T}_C$ and $\mathcal{T}_{\text{poly}}^1(C) = \bigwedge_C^2 \mathcal{T}_C$.

The direct sum

$$\mathcal{T}_{\text{poly}}(C) := \bigoplus_p \mathcal{T}_{\text{poly}}^p(C)$$

is a DG Lie algebra, called the algebra of *poly derivations* of C . The Lie bracket is the Schouten-Nijenhuis bracket, and the differential is 0.

A calculation shows that the solutions of the Maurer-Cartan equation in $\mathcal{T}_{\text{poly}}(C)[[\hbar]]^+$ are the $\mathbb{K}[[\hbar]]$ -bilinear Poisson brackets on $C[[\hbar]]$ that vanish modulo \hbar , and that the group $\exp(\mathcal{T}_{\text{poly}}^0(C)[[\hbar]]^+)$ is the group of gauge equivalences.

In this sense $\mathcal{T}_{\text{poly}}(C)$ controls Poisson deformations of C .

The second DG Lie algebra in this picture is that of the *poly differential operators*.

For $p \geq -1$ one defines

$$\mathcal{D}_{\text{poly}}^p(C) := \{\phi : C^{p+1} \rightarrow C \mid \phi \text{ is a differential operator in each argument}\}.$$

So $\mathcal{D}_{\text{poly}}^{-1}(C) = C$ and $\mathcal{D}_{\text{poly}}^0(C) = \mathcal{D}(C)$, the ring of differential operators.

$\mathcal{D}_{\text{poly}}(C)$ is a sub DG Lie algebra of the Hochschild cochain complex of C , with the Gerstenhaber bracket.

Solutions $\beta = \sum_{j=1}^{\infty} \beta_j \hbar^j$ of the Maurer-Cartan equation in $\mathcal{D}_{\text{poly}}(C)[[\hbar]]^+$ correspond to star products on $C[[\hbar]]$, by the formula

$$f \star g := fg + \sum_{j=1}^{\infty} \beta_j(f, g) \hbar^j.$$

And the group $\exp(\mathcal{D}_{\text{poly}}^0(C)[[\hbar]]^+)$ is the group of gauge equivalences.

Remark A.2. There is a delicate issue hidden here. One can show that any star product is gauge equivalent to a differential star product. This follows from the fact that $\mathcal{D}_{\text{poly}}(C)$ is quasi-isomorphic to the full Hochschild cochain complex of C .

Geometrically, there are sheaves of DG Lie algebras $\mathcal{T}_{\text{poly}, X}$ and $\mathcal{D}_{\text{poly}, X}$ on X , that are quasi-coherent as \mathcal{O}_X -modules. For any affine open set U as above we have

$$\Gamma(U, \mathcal{T}_{\text{poly}, X}) = \mathcal{T}_{\text{poly}}(C),$$

and likewise for $\mathcal{D}_{\text{poly}}$.

In order to control global deformations one has to resort to some kind of resolution of these sheaves of DG Lie algebras, such as the mixed resolutions mentioned in Theorem 4.3.

APPENDIX B. THE UNIVERSAL QUANTIZATION MAP

Let C be a smooth \mathbb{K} -algebra.

There is a canonical map of complexes

$$\mathcal{U}_1 : \mathcal{T}_{\text{poly}}(C) \rightarrow \mathcal{D}_{\text{poly}}(C)$$

given by

$$\mathcal{U}_1(\partial_1 \wedge \cdots \wedge \partial_k)(f_1, \dots, f_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \partial_{\sigma(1)}(f_1) \cdots \partial_{\sigma(k)}(f_k)$$

for $f_i \in C$ and $\partial_i \in \mathcal{T}_C$.

It is known that \mathcal{U}_1 is a quasi-isomorphism – see [Ko1] for the case $C = C^\infty(U)$, and [Ye1] for the case $C = \mathcal{O}(U)$ – and it induces an isomorphism of graded Lie algebras in cohomology.

But \mathcal{U}_1 is not a DG Lie algebra homomorphism!

Theorem B.1. (Kontsevich Formality Theorem)

Let $C := \mathbb{K}[[t_1, \dots, t_n]]$, the formal power series ring. Assume $\mathbb{R} \subset \mathbb{K}$.

Then \mathcal{U}_1 extends to an L_∞ quasi-isomorphism

$$\mathcal{U} = \{\mathcal{U}_j\}_{j=1}^\infty : \mathcal{T}_{\text{poly}}(C) \rightarrow \mathcal{D}_{\text{poly}}(C).$$

In other words, \mathcal{U}_1 is a DG Lie algebra quasi-isomorphism, up to specified higher homotopies $\mathcal{U}_2, \mathcal{U}_3, \dots$

Each of the maps \mathcal{U}_j is invariant under linear change of coordinates.

There is an induced L_∞ quasi-isomorphism

$$\mathcal{U} : \mathcal{T}_{\text{poly}}(C)[[\hbar]]^+ \rightarrow \mathcal{D}_{\text{poly}}(C)[[\hbar]]^+,$$

and a calculation shows that we get a bijection

$$\text{MC}(\mathcal{U}) : \text{MC}(\mathcal{T}_{\text{poly}}(C)[[\hbar]]^+) \xrightarrow{\cong} \text{MC}(\mathcal{D}_{\text{poly}}(C)[[\hbar]]^+)$$

with an explicit formula. Therefore:

Corollary B.2. Assume $\mathbb{R} \subset \mathbb{K}$ and $C = \mathbb{K}[[t_1, \dots, t_n]]$. Then there is a canonical bijection of sets

$$\text{quant} : \frac{\{\text{Poisson brackets on } C[[\hbar]]\}}{\text{gauge equivalence}} \xrightarrow{\cong} \frac{\{\text{star products on } C[[\hbar]]\}}{\text{gauge equivalence}}$$

preserving first order brackets.

APPENDIX C. THE L_∞ QUASI-ISOMORPHISM OF THE LEVEL OF SHEAVES

Here is an outline of the proof of Theorem 4.3. We assume $\mathbb{R} \subset \mathbb{K}$, and X is a smooth n -dimensional algebraic variety over \mathbb{K} .

A formal coordinate system at a closed point $x \in X$ is an isomorphism of \mathbb{K} -algebras

$$\mathbf{k}(x)[[\mathbf{t}]] = \mathbf{k}(x)[[t_1, \dots, t_n]] \xrightarrow{\cong} \widehat{\mathcal{O}}_{X,x},$$

where $\mathbf{k}(x)$ is the residue field.

There is an infinite dimensional scheme $\text{Coor } X$, with a projection $\pi : \text{Coor } X \rightarrow X$, which is a moduli space for formal coordinate systems. (In [Ko1] the notation for $\text{Coor } X$ is X^{coor} .)

In particular, for every closed point $x \in X$, the $\mathbf{k}(x)$ -rational points in the fiber $\pi^{-1}(x)$ stand in bijection to the set of formal coordinate systems at x .

To get an idea of how the scheme $\text{Coor } X$ looks, let us note that $\text{Coor } X = \lim_{\leftarrow} \text{Coor}^i X$, where each $\text{Coor}^i X$ is the variety parametrizing formal coordinate systems up to order i .

Any function f on X has a universal Taylor expansion, when we pull it up to $\text{Coor } X$ via π . Thus the pullback of the structure sheaf \mathcal{O}_X embeds inside the power series algebra $\mathcal{O}_{\text{Coor } X}[[\mathbf{t}]]$.

Likewise the pullbacks to $\text{Coor } X$ of the sheaves $\mathcal{T}_{\text{poly},X}$ and $\mathcal{D}_{\text{poly},X}$ are embedded inside $\mathcal{O}_{\text{Coor } X} \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$ and $\mathcal{O}_{\text{Coor } X} \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$ respectively.

Due to the Formality Theorem we obtain an L_∞ quasi-isomorphism

$$\mathcal{U} : \mathcal{O}_{\text{Coor } X} \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]]) \rightarrow \mathcal{O}_{\text{Coor } X} \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]]).$$

If we had a section $\sigma : X \rightarrow \text{Coor } X$ then we could pull \mathcal{U} down to an L_∞ quasi-isomorphism on X . However usually there are no global sections of $\text{Coor } X$.

The group GL_n acts on $\text{Coor } X$ by linear change of coordinates. Let us define $\text{LCC } X$ to be the quotient scheme $\text{Coor } X / \text{GL}_n$. (“LCC” stands for “linear coordinate classes”.)

Recall that the universal deformation of Kontsevich is invariant under linear change of coordinates, namely under the action of the group GL_n . This implies that the L_∞ morphism \mathcal{U} descends to $\text{LCC } X$; and hence it suffices to work with sections $\sigma : X \rightarrow \text{LCC } X$.

In the C^∞ context such global sections $\sigma : X \rightarrow \text{LCC } X$ do exist (because the fibers of the bundle $\text{LCC } X$ are contractible). But this is not the case in algebraic geometry. So we must use a trick.

Let G be the group of \mathbb{K} -algebra automorphisms of $\mathbb{K}[[\mathbf{t}]]$. So $G \cong \text{GL}_n \rtimes N$, where N is the subgroup of elements that act trivially modulo $(\mathbf{t})^2$. The group N is pro-unipotent. It turns out that $\text{Coor } X$ is a G -torsor over X .

Suppose we are given a finite number of sections

$$\sigma_0, \dots, \sigma_q : U \rightarrow \text{LCC } X$$

over some open set U .

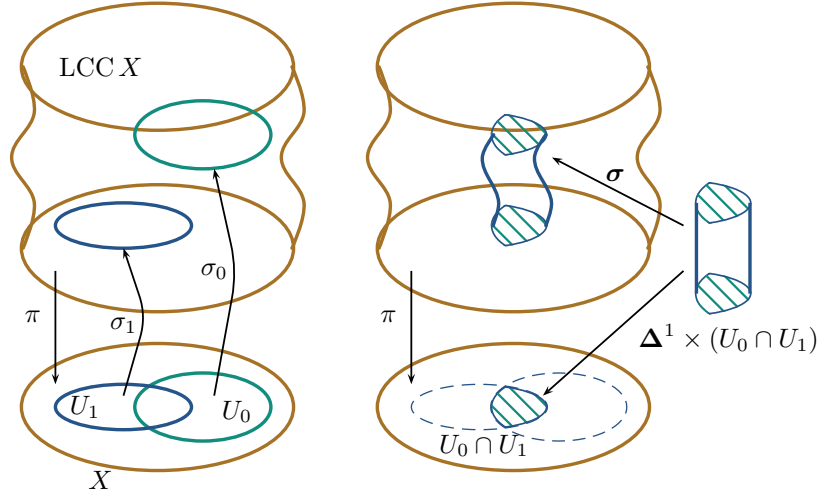


FIGURE 2. Simplicial sections, $q = 1$. We start with sections over two open sets U_0 and U_1 in the left diagram; and we pass to a simplicial section σ on the right.

Using an averaging process for unipotent group actions [Ye4], we show that there exists a canonical morphism

$$\sigma : \Delta_{\mathbb{K}}^q \times U \rightarrow \text{LCC } X$$

which restricts to σ_j on the j -th vertex of $\Delta_{\mathbb{K}}^q$. Here $\Delta_{\mathbb{K}}^q$ is the q -dimensional geometric simplex.

Since sections exist locally, we can choose an open covering $X = \bigcup U_i$ with sections $\sigma_i : U_i \rightarrow \text{LCC } X$. For any i_0, \dots, i_q we then obtain a morphism

$$\sigma : \Delta_{\mathbb{K}}^q \times (U_{i_0} \cap \dots \cap U_{i_q}) \rightarrow \text{LCC } X.$$

(See Figure 1 for an illustration of the case $q = 1$.)

As q varies we have a *simplicial section* of $\text{LCC } X \rightarrow X$. See [Ye2].

Another device we use is *mixed resolutions*. The mixed resolution $\text{Mix}(\mathcal{T}_{\text{poly}, X})$ is a sheaf of DG Lie algebras on X which is quasi-isomorphic to $\mathcal{T}_{\text{poly}, X}$. Likewise for $\mathcal{D}_{\text{poly}, X}$.

The simplicial section σ allows us to pull down \mathcal{U} , and after twisting (because of the Grothendieck differential occurring in the mixed resolution) we obtain an L_{∞} quasi-isomorphism

$$\Psi_{\sigma} : \text{Mix}(\mathcal{T}_{\text{poly}, X}) \rightarrow \text{Mix}(\mathcal{D}_{\text{poly}, X})$$

between sheaves of DG Lie algebras on X .

APPENDIX D. TWISTED DEFORMATIONS VIA STACKS OF GLUING GROUPOIDS

Recall that a *groupoid* G is a category in which all morphisms are invertible.

We denote by $G(i, j)$ the set of arrows from the object i to the object j .

Note that $G(i, i)$ is a group.

Any element $g \in G(i, j)$ defines a group isomorphism

$$\text{Ad}(g) : G(i, i) \xrightarrow{\cong} G(j, j).$$

A *stack of groupoids* \mathcal{G} on X is the geometrization of the notion of groupoid, in the same way that a sheaf of groups is the geometrization of the notion of a group.

Thus for any open set $U \subset X$ there is a groupoid $\mathcal{G}(U)$.

And there are restriction functors

$$\mathcal{G}(U) \rightarrow \mathcal{G}(V)$$

for any inclusion $V \subset U$.

These satisfy a rather complicated list of conditions. For details see [Gi, BM, KS].

In particular, given any open set $U \subset X$ and any object $i \in \text{ob } \mathcal{G}(U)$, there is a sheaf of groups $\mathcal{G}(i, i)$ on U .

A stack of groupoids \mathcal{G} is called a *gerbe* if it is locally nonempty and locally connected.

Definition D.1. Let X be a smooth algebraic variety over \mathbb{K} . A *twisted associative (resp. Poisson) deformation* \mathcal{A} of \mathcal{O}_X is the following data:

- (1) A gerbe \mathcal{G} on X , called the *gluing gerbe* of \mathcal{A} .
- (2) For any open set $U \subset X$ and $i \in \text{ob } \mathcal{G}(U)$, an associative (resp. Poisson) deformation \mathcal{A}_i of \mathcal{O}_{U_i} .

The conditions are:

- (a) For any $i \in \text{ob } \mathcal{G}(U)$, the sheaf of groups $\mathcal{G}(i, i)$ coincides with $\text{IG}(\mathcal{A}_i)$, the sheaf of inner gauge transformations of the deformation \mathcal{A}_i .
- (b) For any $i \in \text{ob } \mathcal{G}(U)$, any $j \in \text{ob } \mathcal{G}(V)$, any $W \subset U \cap V$ and any $g \in \mathcal{G}(W)(i, j)$, the isomorphism of sheaves of groups

$$\text{Ad}(g) : \mathcal{G}(i, i)|_W \xrightarrow{\cong} \mathcal{G}(j, j)|_W$$

is induced from a (necessarily unique) gauge equivalence

$$\mathcal{A}_i|_W \xrightarrow{\cong} \mathcal{A}_j|_W.$$

Theorem D.2. ([Ye5]) *Definitions 5.1 and D.1 are equivalent.*

Remark D.3. Let \mathcal{A} be a twisted deformation, with gluing groupoid \mathcal{G} .

It is important to note that the set $\text{ob } \mathcal{G}(X)$ could be empty, meaning that \mathcal{A} is really twisted; i.e. it is not equivalent to a deformation in the usual sense.

This can be detected by the non-vanishing of suitable obstruction classes in $H^2(X, \mathcal{O}_X)$.

Indeed, Theorem D.2 is a consequence of the fact that all relevant obstruction classes vanish on affine open sets.

Finally let me say a few words on the proof of Theorem 6.1 in [Ye5].

Fix an affine open covering $U = \{U_0, \dots, U_m\}$ of X , such that for each i there is an étale morphism $U_i \rightarrow \mathbf{A}_{\mathbb{K}}^n$.

Consider the cosimplicial DG Lie algebra

$$\mathfrak{t} := \Gamma(X, C(U, \mathcal{T}_{\text{poly}, X})).$$

Here $C(\mathbf{U}, -)$ denotes the cosimplicial Čech resolution based on \mathbf{U} .

Likewise there is a cosimplicial DG Lie algebra

$$\mathfrak{d} := \Gamma(X, C(\mathbf{U}, \mathcal{D}_{\text{poly}, X}^{\text{nor}})).$$

Suppose \mathfrak{g} is any cosimplicial DG Lie algebra. Let $\overline{\text{DT}}(\mathfrak{g}[[\hbar]]^+)$ be the set of equivalence classes of *descent triples* in the cosimplicial DG Lie algebra $\mathfrak{g}[[\hbar]]^+$.

Now twisted Poisson deformations of \mathcal{O}_X correspond canonically to elements of $\overline{\text{DT}}(\mathfrak{t}[[\hbar]]^+)$. And similarly, twisted associative deformations are parametrized by $\overline{\text{DT}}(\mathfrak{d}[[\hbar]]^+)$.

On the other hand, let $\tilde{\mathfrak{N}}\mathfrak{g}$ denote the Thom-Sullivan normalization of a cosimplicial DG Lie algebra \mathfrak{g} . We can look at the set $\overline{\text{MC}}(\tilde{\mathfrak{N}}\mathfrak{g}[[\hbar]]^+)$ of equivalence classes of Maurer-Cartan elements in the DG Lie algebra $\tilde{\mathfrak{N}}\mathfrak{g}[[\hbar]]^+$.

It is easy to deduce from Theorem 4.3 that there is a canonical bijection

$$\text{quant} : \overline{\text{MC}}(\tilde{\mathfrak{N}}\mathfrak{t}[[\hbar]]^+) \xrightarrow{\cong} \overline{\text{MC}}(\tilde{\mathfrak{N}}\mathfrak{g}[[\hbar]]^+).$$

To wrap it all up we prove, by a rather complicated calculation, that for any cosimplicial DG Lie algebra of *quantum type* \mathfrak{g} , namely for which $\mathfrak{g}^i = 0$ for $i < -1$, there is a functorial bijection

$$\overline{\text{dt}} : \overline{\text{MC}}(\tilde{\mathfrak{N}}\mathfrak{g}[[\hbar]]^+) \xrightarrow{\cong} \overline{\text{DT}}(\mathfrak{g}[[\hbar]]^+).$$

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