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Continuous and twisted L_∞ morphisms

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Abstract

The purpose of this paper is to develop a suitable notion of continuous L_∞ morphism between DG Lie algebras, and to study twists of such morphisms.

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0. Introduction

Let $\mathbb{K}$ be a field containing $\mathbb{R}$. Consider two DG Lie algebras associated with the polynomial algebra $\mathbb{K}[\boldsymbol{t}] := \mathbb{K}[t_1, \ldots, t_n]$. The first is the algebra of *poly derivations* $\mathcal{T}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$, and the second is the algebra of *poly differential operators* $\mathcal{D}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$. A very important result of Kontsevich [5], known as the Formality Theorem, gives an explicit formula for an L_∞ quasi-isomorphism

$$\mathcal{U} : \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}]) \to \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}]).$$

Here is the main result of our paper.

Theorem 0.1. *Assume $\mathbb{R} \subset \mathbb{K}$. Let $A = \bigoplus_{i \geq 0} A^i$ be a super-commutative associative unital complete DG algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$. Consider the induced continuous A-multilinear L_∞ morphism*

$$\mathcal{U}_A : A \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]) \to A \mathbin{\widehat{\otimes}} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]).$$

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Suppose $\omega \in A^1 \widehat{\otimes} \mathcal{T}^0_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ *is a solution of the Maurer–Cartan equation in* $A \widehat{\otimes} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$. *Define* $\omega' := (\partial^1 \mathcal{U}_A)(\omega) \in A^1 \widehat{\otimes} \mathcal{D}^0_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$. *Then* ω' *is a solution of the Maurer–Cartan equation in* $A \widehat{\otimes} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$, *and there is a continuous* A*-multilinear* L_∞ *quasi-isomorphism*

$$\mathcal{U}_{A,\omega} : \big(A \widehat{\otimes} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])\big)_\omega \to \big(A \widehat{\otimes} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])\big)_{\omega'}$$

whose Taylor coefficients are

$$(\partial^j \mathcal{U}_{A,\omega})(\alpha) := \sum_{k \geq 0} \frac{1}{(j+k)!} (\partial^{j+k} \mathcal{U}_A)(\omega^k \wedge \alpha)$$

for $\alpha \in \prod^j \big(A \widehat{\otimes} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])\big)$.

Below is an outline of the paper, in which we mention the various terms appearing in the theorem.

In Section 1 we develop the theory of dir-inv modules. A dir-inv structure on a $\mathbb{K}$-module M is a generalization of an adic topology. The category of dir-inv modules and continuous homomorphisms is denoted by $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$. The concepts of dir-inv module, and related complete tensor product $\widehat{\otimes}$, are quite flexible, and are particularly well-suited for infinitely generated modules. Among other things we introduce the notion of DG Lie algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$.

Section 2 concentrates on poly differential operators. The results here are mostly generalizations of material from [2].

In Section 3 we review the coalgebra approach to L_∞ morphisms. The notions of continuous, A-multilinear and twisted L_∞ morphisms are defined. The main result of this section is Theorem 3.27.

In Section 4 we recall the Kontsevich Formality Theorem. By combining it with Theorem 3.27 we deduce Theorem 0.1 (repeated as Theorem 4.15). In Theorem 0.1 the DG Lie algebras $A \widehat{\otimes} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ and $A \widehat{\otimes} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ are the A-multilinear extensions of $\mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ and $\mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ respectively, and $\big(A \widehat{\otimes} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])\big)_\omega$ and $\big(A \widehat{\otimes} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])\big)_{\omega'}$ are their twists. The L_∞ morphism $\mathcal{U}_A$ is the continuous A-multilinear extension of $\mathcal{U}$, and $\mathcal{U}_{A,\omega}$ is its twist.

Theorem 0.1 is used in [9], in which we study deformation quantization of algebraic varieties.

1. Dir-inv modules

We begin the paper with a generalization of the notion of adic topology. In this section $\mathbb{K}$ is a commutative base ring, and C is a commutative $\mathbb{K}$-algebra. The category $\mathsf{Mod}\,C$ is abelian and has direct and inverse limits. Unless specified otherwise, all limits are taken in $\mathsf{Mod}\,C$.

Definition 1.1. (1) Let $M \in \mathsf{Mod}\,C$. An *inv module structure* on M is an inverse system $\{\mathrm{F}^i M\}_{i \in \mathbb{N}}$ of C-submodules of M. The pair $(M, \{\mathrm{F}^i M\}_{i \in \mathbb{N}})$ is called an *inv C-module*.

(2) Let $(M, \{\mathrm{F}^i M\}_{i \in \mathbb{N}})$ and $(N, \{\mathrm{F}^i N\}_{i \in \mathbb{N}})$ be two inv C-modules. A function $\phi : M \to N$ (C-linear or not) is said to be *continuous* if for every $i \in \mathbb{N}$ there exists $i' \in \mathbb{N}$ such that $\phi(\mathrm{F}^{i'} M) \subset \mathrm{F}^i N$.

(3) Define $\mathsf{Inv}\,\mathsf{Mod}\,C$ to be the category whose objects are the inv C-modules, and whose morphisms are the continuous C-linear homomorphisms.

We do not assume that the canonical homomorphism $M \to \lim_{\leftarrow i} M/\mathrm{F}^i M$ is surjective nor injective. There is a full embedding $\mathsf{Mod}\,C \hookrightarrow \mathsf{Inv}\,\mathsf{Mod}\,C$, $M \mapsto (M, \{\ldots, 0, 0\})$. If $(M, \{\mathrm{F}^i M\}_{i\in\mathbb{N}})$ and $(N, \{\mathrm{F}^i N\}_{i\in\mathbb{N}})$ are two inv C-modules then $M \oplus N$ is an inv module, with inverse system of submodules $\mathrm{F}^i(M \oplus N) := \mathrm{F}^i M \oplus \mathrm{F}^i N$. Thus $\mathsf{Inv}\,\mathsf{Mod}\,C$ is a C-linear additive category.

Let $(M, \{\mathrm{F}^i M\}_{i\in\mathbb{N}})$ be an inv C-module, let M', M'' be two C-modules, and suppose $\phi : M' \to M$ and $\psi : M \to M''$ are C-linear homomorphisms. We get induced inv module structures on M' and M'' by defining $\mathrm{F}^i M' := \phi^{-1}(\mathrm{F}^i M)$ and $\mathrm{F}^i M'' := \psi(\mathrm{F}^i M)$.

Recall that a *directed set* is a partially ordered set J with the property that for any $j_1, j_2 \in J$ there exists $j_3 \in J$ such that $j_1, j_2 \leq j_3$.

Definition 1.2. (1) Let $M \in \mathsf{Mod}\,C$. A *dir-inv module structure* on M is a direct system $\{\mathrm{F}_j M\}_{j\in J}$ of C-submodules of M, indexed by a nonempty directed set J, together with an inv module structure on each $\mathrm{F}_j M$, such that for every $j_1 \leq j_2$ the inclusion $\mathrm{F}_{j_1} M \hookrightarrow \mathrm{F}_{j_2} M$ is continuous. The pair $(M, \{\mathrm{F}_j M\}_{j\in J})$ is called a *dir-inv C-module*.

(2) Let $(M, \{\mathrm{F}_j M\})_{j\in J}$ and $(N, \{\mathrm{F}_k N\}_{k\in K})$ be two dir-inv C-modules. A function $\phi : M \to N$ (C-linear or not) is said to be *continuous* if for every $j \in J$ there exists $k \in K$ such that $\phi(\mathrm{F}_j M) \subset \mathrm{F}_k N$, and $\phi : \mathrm{F}_j M \to \mathrm{F}_k N$ is a continuous function between these two inv C-modules.

(3) Define $\mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\,C$ to be the category whose objects are the dir-inv C-modules, and whose morphisms are the continuous C-linear homomorphisms.

There is no requirement that the canonical homomorphism $\lim_{j\to} \mathrm{F}_j M \to M$ will be surjective. An inv C-module M is endowed with the dir-inv module structure $\{\mathrm{F}_j M\}_{j\in J}$, where $J := \{0\}$ and $\mathrm{F}_0 M := M$. Thus we get a full embedding $\mathsf{Inv}\,\mathsf{Mod}\,C \hookrightarrow \mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\,C$. Given two dir-inv C-modules $(M, \{\mathrm{F}_j M\})_{j\in J}$ and $(N, \{\mathrm{F}_k N\}_{k\in K})$, we make $M \oplus N$ into a dir-inv module as follows. The directed set is $J \times K$, with the component-wise partial order, and the direct system of inv modules is $\mathrm{F}_{(j,k)}(M \oplus N) := \mathrm{F}_j M \oplus \mathrm{F}_k N$. The condition $J \neq \emptyset$ in part (1) of the definition ensures that the zero module $0 \in \mathsf{Mod}\,C$ is an initial object in $\mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\,C$. So $\mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\,C$ is a C-linear additive category.

Let $(M, \{\mathrm{F}_j M\}_{j\in J})$ be a dir-inv C-module, let M', M'' be two C-modules, and suppose $\phi : M' \to M$ and $\psi : M \to M''$ are C-linear homomorphisms. We get induced dir-inv module structures $\{\mathrm{F}_j M'\}_{j\in J}$ and $\{\mathrm{F}_j M''\}_{j\in J}$ on M' and M'' as follows. Define $\mathrm{F}_j(M') := \phi^{-1}(\mathrm{F}_j M)$ and $\mathrm{F}_j M'' := \psi(\mathrm{F}_j M)$, which have induced inv module structures via the homomorphisms $\phi : \mathrm{F}_j M' \to \mathrm{F}_j M$ and $\psi : \mathrm{F}_j M \to \mathrm{F}_j M''$.

Definition 1.3. (1) An inv C-module $(M, \{\mathrm{F}^i M\}_{i\in\mathbb{N}})$ is called *discrete* if $\mathrm{F}^i M = 0$ for $i \gg 0$.

(2) An inv C-module $(M, \{\mathrm{F}^i M\}_{i\in\mathbb{N}})$ is called *complete* if the canonical homomorphism $M \to \lim_{\leftarrow i} M/\mathrm{F}^i M$ is bijective.

(3) A dir-inv C-module M is called *complete* (resp. *discrete*) if it isomorphic, in $\mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\,C$, to a dir-inv module $(N, \{\mathrm{F}_j N\}_{j\in J})$, where all the inv modules $\mathrm{F}_j N$

are complete (resp. discrete) as defined above, and the canonical homomorphism $\lim_{j\to} \mathrm{F}_j N \to N$ is bijective.

(4) A dir-inv C-module M is called *trivial* if it is isomorphic, in $\mathsf{Dir\,Inv\,Mod}\, C$, to an object of $\mathsf{Mod}\, C$, via the embedding $\mathsf{Mod}\, C \hookrightarrow \mathsf{Dir\,Inv\,Mod}\, C$.

Note that M is a trivial dir-inv module iff it is isomorphic, in $\mathsf{Dir\,Inv\,Mod}\, C$, to a discrete inv module. There do exist discrete dir-inv modules that are not trivial dir-inv modules; see Example 1.10. It is easy to see that if M is a discrete dir-inv module then it is also complete.

The base ring $\mathbb{K}$ is endowed with the inv structure $\{\dots, 0, 0\}$, so it is a trivial dir-inv $\mathbb{K}$-module. But the $\mathbb{K}$-algebra C could have more interesting dir-inv structures (cf. Example 1.8).

If $f^* : C \to C'$ is a homomorphism of $\mathbb{K}$-algebras, then there is a functor $f_* : \mathsf{Dir\,Inv\,Mod}\, C' \to \mathsf{Dir\,Inv\,Mod}\, C$. In particular any dir-inv C-module is a dir-inv $\mathbb{K}$-module.

Definition 1.4. (1) Given an inv C-module $(M, \{\mathrm{F}^i M\}_{i\in\mathbb{N}})$ its completion is the inv C-module $(\widehat{M}, \{\mathrm{F}^i \widehat{M}\}_{i\in\mathbb{N}})$, defined as follows: $\widehat{M} := \lim_{\leftarrow i} M/\mathrm{F}^i M$ and $\mathrm{F}^i \widehat{M} := \mathrm{Ker}(\widehat{M} \to M/\mathrm{F}^i M)$. Thus we obtain an additive endofunctor $M \mapsto \widehat{M}$ of $\mathsf{Inv\,Mod}\, C$.

(2) Given a dir-inv C-module $(M, \{\mathrm{F}_j M\}_{j\in J})$ its completion is the dir-inv C-module $(\widehat{M}, \{\mathrm{F}_j \widehat{M}\}_{j\in J})$ defined as follows. For any $j \in J$ let $\widehat{\mathrm{F}_j M}$ be the completion of the inv C-module $\mathrm{F}_j M$, as defined above. Then let $\widehat{M} := \lim_{j\to} \widehat{\mathrm{F}_j M}$ and $\mathrm{F}_j \widehat{M} := \mathrm{Im}(\widehat{\mathrm{F}_j M} \to \widehat{M})$. Thus we obtain an additive endofunctor $M \mapsto \widehat{M}$ of $\mathsf{Dir\,Inv\,Mod}\, C$.

An inv C-module M is complete iff the functorial homomorphism $M \to \widehat{M}$ is an isomorphism; and of course $\widehat{M}$ is complete. For a dir-inv C-module M there is in general no functorial homomorphism between M and $\widehat{M}$, and we do not know if $\widehat{M}$ is complete. Nonetheless:

Proposition 1.5. *Suppose $M \in \mathsf{Dir\,Inv\,Mod}\, C$ is complete. Then there is an isomorphism $M \cong \widehat{M}$ in $\mathsf{Dir\,Inv\,Mod}\, C$. This isomorphism is functorial.*

Proof. For any dir-inv module $(M, \{\mathrm{F}_j M\}_{j\in J})$ let us define $M' := \lim_{j\to} \mathrm{F}_j M$. So $(M', \{\mathrm{F}_j M\}_{j\in J})$ is a dir-inv module, and there are functorial morphisms $M' \to M$ and $M' \to \widehat{M}$. If M is complete then both these morphisms are isomorphisms. □

Suppose $\{M_k\}_{k\in K}$ is a collection of dir-inv modules, indexed by a set K. There is an induced dir-inv module structure on $M := \bigoplus_{k\in K} M_k$, constructed as follows. For any k let us denote by $\{\mathrm{F}_j M_k\}_{j\in J_k}$ the dir-inv structure of M_k; so that each $\mathrm{F}_j M_k$ is an inv module. For each finite subset $K_0 \subset K$ let $J_{K_0} := \prod_{k\in K_0} J_k$, made into a directed set by component-wise partial order. Define $J := \coprod_{K_0} J_{K_0}$, where K_0 runs over the finite subsets of K. For two finite subsets $K_0 \subset K_1$, and two elements $\boldsymbol{j}_0 = \{j_{0,k}\}_{k\in K_0} \in J_{K_0}$ and $\boldsymbol{j}_1 = \{j_{1,k}\}_{k\in K_1} \in J_{K_1}$ we declare that $\boldsymbol{j}_0 \le \boldsymbol{j}_1$ if $j_{0,k} \le j_{1,k}$ for all $k \in K_0$. This makes J into a directed set. Now for any $\boldsymbol{j} = \{j_k\}_{k\in K_0} \in J_{K_0} \subset J$ let $\mathrm{F}_{\boldsymbol{j}} M := \bigoplus_{k\in K_0} \mathrm{F}_{j_k} M_k$, which is an inv module. The dir-inv structure on M is $\{F_{\boldsymbol{j}} M\}_{\boldsymbol{j}\in J}$.

Proposition 1.6. *Let $\{M_k\}_{k\in K}$ be a collection of dir-inv C-modules, and let $M := \bigoplus_{k\in K} M_k$, endowed with the induced dir-inv structure.*

(1) *M is a coproduct of $\{M_k\}_{k\in K}$ in the category* $\mathsf{Dir\,Inv\,Mod}\,C$.
(2) *There is a functorial isomorphism $\widehat{M} \cong \bigoplus_{k\in K} \widehat{M}_k$.*

Proof. (1) is obvious. For (2) we note that both $\widehat{M}$ and $\bigoplus_{k\in K} \widehat{M}_k$ are direct limits for the direct system $\{\widehat{M}_{\boldsymbol{j}}\}_{\boldsymbol{j}\in J}$. □

Suppose $\{M_k\}_{k\in\mathbb{N}}$ is a collection of inv C-modules. For each k let $\{\mathrm{F}^i M_k\}_{i\in\mathbb{N}}$ be the inv structure of M_k. Then $M := \prod_{k\in\mathbb{N}} M_k$ is an inv module, with inv structure $\mathrm{F}^i M := \big(\prod_{k>i} M_k\big) \times \big(\prod_{k\le i} \mathrm{F}^i M_k\big)$. Next let $\{M_k\}_{k\in\mathbb{N}}$ be a collection of dir-inv C-modules, and for each k let $\{\mathrm{F}_j M_k\}_{j\in J_k}$ be the dir-inv structure of M_k. Then there is an induced dir-inv structure on $M := \prod_{k\in\mathbb{N}} M_k$. Define a directed set $J := \prod_{k\in\mathbb{N}} J_k$, with component-wise partial order. For any $\boldsymbol{j} = \{j_k\}_{k\in\mathbb{N}} \in J$ define $\mathrm{F}_{\boldsymbol{j}} M := \prod_{k\in\mathbb{N}} \mathrm{F}_{j_k} M_k$, which is an inv C-module as explained above. The dir-inv structure on M is $\{\mathrm{F}_{\boldsymbol{j}} M\}_{\boldsymbol{j}\in J}$.

Proposition 1.7. *Let $\{M_k\}_{k\in\mathbb{N}}$ be a collection of dir-inv C-modules, and let $M := \prod_{k\in\mathbb{N}} M_k$, endowed with the induced dir-inv structure. Then M is a product of $\{M_k\}_{k\in\mathbb{N}}$ in* $\mathsf{Dir\,Inv\,Mod}\,C$.

Proof. All we need to consider is continuity. First assume that all the M_k are inv C-modules. Let us denote by $\pi_k : M \to M_k$ the projection. For each $k, i \in \mathbb{N}$ and $i' \ge \max(i,k)$ we have $\pi_k(\mathrm{F}^{i'} M) = \mathrm{F}^i M_k$. This shows that the π_k are continuous. Suppose L is an inv C-module and $\phi_k : L \to M_k$ are morphisms in $\mathsf{Inv\,Mod}\,C$. For any $i \in \mathbb{N}$ there exists $i' \in \mathbb{N}$ such that $\phi_k(\mathrm{F}^{i'} L) \subset \mathrm{F}^i M_k$ for all $k \le i$. Therefore the homomorphism $\phi : L \to M$ with components ϕ_k is continuous.

Now let M_k be dir-inv C-modules, with dir-inv structures $\{\mathrm{F}_j M_k\}_{j\in J_k}$. For any $\boldsymbol{j} = \{j_k\} \in J$ one has $\pi_k(\mathrm{F}_{\boldsymbol{j}} M) = \mathrm{F}_{j_k} M_k$, and as shown above $\pi_k : \mathrm{F}_{\boldsymbol{j}} M \to \mathrm{F}_{j_k} M_k$ is continuous. Given a dir-inv module L and morphisms $\phi_k : L \to M_k$ in $\mathsf{Dir\,Inv\,Mod}\,C$, we have to prove that $\phi : L \to M$ is continuous. Let $\{\mathrm{F}_j L\}_{j\in J_L}$ be the dir-inv structure of L. Take any $j \in J_L$. Since ϕ_k is continuous, there exists some $j_k \in J_k$ such that $\phi_k(\mathrm{F}_j L) \subset \mathrm{F}_{j_k} M_k$. But then $\phi(\mathrm{F}_j L) \subset \mathrm{F}_{\boldsymbol{j}} M$ for $\boldsymbol{j} := \{j_k\}_{k\in\mathbb{N}}$, and by the previous paragraph $\phi : \mathrm{F}_j L \to \mathrm{F}_{\boldsymbol{j}} M$ is continuous. □

The following examples should help to clarify the notion of dir-inv module.

Example 1.8. Let $\mathfrak{c}$ be an ideal in C. Then each finitely generated C-module M has an inv structure $\{\mathrm{F}^i M\}_{i\in\mathbb{N}}$, where we define the submodules $\mathrm{F}^i M := \mathfrak{c}^{i+1} M$. This is called the $\mathfrak{c}$-*adic inv structure*. Any C-module M has a dir-inv structure $\{\mathrm{F}_j M\}_{j\in J}$, which is the collection of finitely generated C-submodules of M, directed by inclusion, and each $\mathrm{F}_j M$ is given the $\mathfrak{c}$-adic inv structure. We get a fully faithful functor $\mathsf{Mod}\,C \to \mathsf{Dir\,Inv\,Mod}\,C$. This dir-inv module structure on M is called the $\mathfrak{c}$-*adic dir-inv structure*.

If C is noetherian and $\mathfrak{c}$-adically complete, then the finitely generated modules are complete as inv C-modules, and hence all modules are complete as dir-inv modules.

Example 1.9. Suppose $(M, \{\mathrm{F}^i M\}_{i\in\mathbb{N}})$ is an inv C-module, and $\{i_k\}_{k\in\mathbb{N}}$ is a nondecreasing sequence in $\mathbb{N}$ with $\lim_{k\to\infty} i_k = \infty$. Then $\{\mathrm{F}^{i_k} M\}_{k\in\mathbb{N}}$ is a new inv structure on M, yet the identity map $(M, \{\mathrm{F}^i M\}_{i\in\mathbb{N}}) \to (M, \{\mathrm{F}^{i_k} M\}_{k\in\mathbb{N}})$ is an isomorphism in $\mathsf{Inv\,Mod}\,C$.

A similar modification can be done for dir-inv modules. Suppose $(M, \{\mathrm{F}_j M\}_{j\in J})$ is a dir-inv C-module, and $J' \subset J$ is a subset that is cofinal in J. Then $\{\mathrm{F}_j M\}_{j\in J'}$ is a new dir-inv structure on M, yet the identity map $(M, \{\mathrm{F}_j M\}_{j\in J}) \to (M, \{\mathrm{F}_j M\}_{j\in J'})$ is an isomorphism in $\mathsf{Dir\,Inv\,Mod}\, C$.

Example 1.10. Let M be the free $\mathbb{K}$-module with basis $\{e_p\}_{p\in\mathbb{N}}$; so $M = \bigoplus_{p\in\mathbb{N}} \mathbb{K}e_p$ in $\mathsf{Mod}\,\mathbb{K}$. We put on M the inv module structure $\{\mathrm{F}^i M\}_{i\in\mathbb{N}}$ with $\mathrm{F}^i M := 0$ for all i. Let N be the same $\mathbb{K}$-module as M, but put on it the inv module structure $\{\mathrm{F}^i N\}_{i\in\mathbb{N}}$ with $\mathrm{F}^i N := \bigoplus_{p=i}^{\infty} \mathbb{K}e_p$. Also let L be the $\mathbb{K}$-module M, but put on it the dir-inv module structure $\{\mathrm{F}_j L\}_{j\in\mathbb{N}}$, with $\mathrm{F}_j L := \bigoplus_{p=0}^{j} \mathbb{K}e_p$ the discrete inv module whose inv structure is $\{\ldots, 0, 0\}$. Both L and M are discrete and complete as dir-inv $\mathbb{K}$-modules, and $\widehat{N} \cong \prod_{p\in\mathbb{N}} \mathbb{K}e_p$. The dir-inv module M is trivial. L is not a trivial dir-inv $\mathbb{K}$-module, because it is not isomorphic in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$ to any inv module. The identity maps $L \to M \to N$ are continuous. The only continuous $\mathbb{K}$-linear homomorphisms $M \to L$ are those with finitely generated images.

Remark 1.11. In the situation of the previous example, suppose we put on the three modules L, M, N genuine $\mathbb{K}$-linear topologies, using the limiting processes and starting from the discrete topology. Namely M, $N/\mathrm{F}^i N$ and $\mathrm{F}_j L$ get the discrete topologies; $L \cong \lim_{j\to} \mathrm{F}_j L$ gets the $\lim_{\to}$ topology; and $N \subset \lim_{\leftarrow i} N/\mathrm{F}^i N$ gets the $\lim_{\leftarrow}$ topology (as in [8, Section 1.1]). Then L and M become the same discrete topological module, and $\widehat{N}$ is the topological completion of N. We see that the notion of a dir-inv structure is more subtle than that of a topology, even though a similar language is used.

Example 1.12. Suppose $\mathbb{K}$ is a field, and let $M := \mathbb{K}$, the free module of rank 1. Up to isomorphism in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$, M has three distinct dir-inv module structures. We can denote them by M_1, M_2, M_3 in such a way that the identity maps $M_1 \to M_2 \to M_3$ are continuous. The only continuous $\mathbb{K}$-linear homomorphisms $M_i \to M_j$ with $i > j$ are the zero homomorphisms. M_2 is the trivial dir-inv structure, and it is the only interesting one (the others are "pathological").

Example 1.13. Suppose $M = \bigoplus_{p\in\mathbb{Z}} M^p$ is a graded C-module. The grading induces a dir-inv structure on M, with $J := \mathbb{N}$, $\mathrm{F}_j M := \bigoplus_{p=-j}^{\infty} M^p$, and $\mathrm{F}^i\mathrm{F}_j M := \bigoplus_{p=-j+i}^{\infty} M^p$. The completion satisfies $\widehat{M} \cong \left(\prod_{p\geq 0} M^p\right) \oplus \left(\bigoplus_{p<0} M^p\right)$ in $\mathsf{Dir\,Inv\,Mod}\, C$, where each M^p has the trivial dir-inv module structure.

It makes sense to talk about convergence of sequences in a dir-inv module. Suppose $(M, \{\mathrm{F}^i M\}_{i\in\mathbb{N}})$ is an inv C-module and $\{m_i\}_{i\in\mathbb{N}}$ is a sequence in M. We say that $\lim_{i\to\infty} m_i = 0$ if for every i_0 there is some i_1 such that $\{m_i\}_{i\geq i_1} \subset \mathrm{F}_{i_0} M$. If $(M, \{\mathrm{F}_j M\}_{j\in J})$ is a dir-inv module and $\{m_i\}_{i\in\mathbb{N}}$ is a sequence in M, then we say that $\lim_{i\to\infty} m_i = 0$ if there exist some j and i_1 such that $\{m_i\}_{i\geq i_1} \subset \mathrm{F}_j M$, and $\lim_{i\to\infty} m_i = 0$ in the inv module $\mathrm{F}_j M$. Having defined $\lim_{i\to\infty} m_i = 0$, it is clear how to define $\lim_{i\to\infty} m_i = m$ and $\sum_{i=0}^{\infty} m_i = m$. Also the notion of Cauchy sequence is clear.

Proposition 1.14. *Assume M is a complete dir-inv C-module. Then any Cauchy sequence in M has a unique limit.*

Proof. Consider a Cauchy sequence $\{m_i\}_{i\in\mathbb{N}}$ in M. Convergence is an invariant of isomorphisms in $\mathsf{Dir\,Inv\,Mod}\,C$. By Definition 1.3 we may assume that in the dir-inv structure $\{\mathrm{F}_j M\}_{j\in J}$ of M each inv module $\mathrm{F}_j M$ is complete. By passing to the sequence $\{m_i - m_{i_1}\}_{i\in\mathbb{N}}$ for suitable i_1, we can also assume the sequence is contained in one of the inv modules $\mathrm{F}_j M$. Thus we reduce to the case of convergence in a complete inv module, which is standard. □

Let $(M, \{\mathrm{F}^i M\}_{i\in\mathbb{N}})$ and $(N, \{\mathrm{F}^i N\}_{i\in\mathbb{N}})$ be two inv C-modules. We make $M \otimes_C N$ into an inv module by defining

$$\mathrm{F}^i(M\otimes_C N) := \mathrm{Im}\Big((M\otimes_C \mathrm{F}^i N)\oplus(\mathrm{F}^i M\otimes_C N)\to M\otimes_C N\Big).$$

For two dir-inv C-modules $(M, \{\mathrm{F}_j M\}_{j\in J})$ and $(N, \{\mathrm{F}_k N\}_{k\in K})$, we put on $M\otimes_C N$ the dir-inv module structure $\{\mathrm{F}_{(j,k)}(M\otimes_C N)\}_{(j,k)\in J\times K}$, where

$$\mathrm{F}_{(j,k)}(M\otimes_C N) := \mathrm{Im}(\mathrm{F}_j M\otimes_C \mathrm{F}_k N\to M\otimes_C N).$$

Definition 1.15. Given $M, N \in \mathsf{Dir\,Inv\,Mod}\,C$ we define $N\,\widehat{\otimes}_C\,M$ to be the completion of the dir-inv C-module $N\otimes_C M$.

Example 1.16. Let us examine the behavior of the dir-inv modules L, M, N from Example 1.10 with respect to the complete tensor product. There is an isomorphism $L\otimes_{\mathbb{K}} N \cong \bigoplus_{p\in\mathbb{N}} N$ in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$, so according to Proposition 1.6(2) there is also an isomorphism $L\,\widehat{\otimes}_{\mathbb{K}}\,N \cong \bigoplus_{p\in\mathbb{N}} \widehat{N}$ in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$. On the other hand $M\otimes_{\mathbb{K}} N$ is an inv $\mathbb{K}$-module with inv structure $\mathrm{F}^i(M\otimes_{\mathbb{K}} N) = M\otimes_{\mathbb{K}} \mathrm{F}^i N$, so $M\,\widehat{\otimes}_{\mathbb{K}}\,N \cong \prod_{p\in\mathbb{N}} M$ in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$. The series $\sum_{p=0}^{\infty} e_p\otimes e_p$ converges in $M\,\widehat{\otimes}_{\mathbb{K}}\,N$, but not in $L\,\widehat{\otimes}_{\mathbb{K}}\,N$.

A *graded object in* $\mathsf{Dir\,Inv\,Mod}\,C$, or a *graded dir-inv C-module*, is an object $M \in \mathsf{Dir\,Inv\,Mod}\,C$ of the form $M = \bigoplus_{i\in\mathbb{Z}} M^i$, with $M^i \in \mathsf{Dir\,Inv\,Mod}\,C$. According to Proposition 1.6 we have $\widehat{M} \cong \bigoplus_{i\in\mathbb{Z}} \widehat{M}^i$. Given two graded objects $M = \bigoplus_{i\in\mathbb{Z}} M^i$ and $N = \bigoplus_{i\in\mathbb{Z}} N^i$ in $\mathsf{Dir\,Inv\,Mod}\,C$, the tensor product is also a graded object in $\mathsf{Dir\,Inv\,Mod}\,C$, with

$$(M\otimes_C N)^i := \bigoplus_{p+q=i} M^p\otimes_C N^q.$$

In this paper "algebra" is taken in the weakest possible sense: by C-algebra we mean a C-module A together with a C-bilinear function $\mu_A : A\times A\to A$. If A is associative, or a Lie algebra, then we will specify that. However, "commutative algebra" will mean, by default, a commutative associative unital C-algebra. Another convention is that a homomorphism between unital algebras is a unital homomorphism, and a module over a unital algebra is a unital module.

Definition 1.17. (1) An *algebra in* $\mathsf{Dir\,Inv\,Mod}\,C$ is an object $A \in \mathsf{Dir\,Inv\,Mod}\,C$, together with a continuous C-bilinear function $\mu_A : A\times A\to A$.

(2) A *differential graded algebra in* $\mathsf{Dir\,Inv\,Mod}\,C$ is a graded object $A = \bigoplus_{i\in\mathbb{Z}} A^i$ in $\mathsf{Dir\,Inv\,Mod}\,C$, together with continuous C-(bi)linear functions $\mu_A : A\times A\to A$ and

$\mathrm{d}_A : A \to A$, such that A is a differential graded algebra, in the usual sense, with respect to the differential d_A and the multiplication μ_A.

(3) Let A be an algebra in $\mathsf{Dir\,Inv\,Mod}\,C$, with dir-inv structure $\{\mathrm{F}_j A\}_{j\in J}$. We say that A is a *unital algebra in* $\mathsf{Dir\,Inv\,Mod}\,C$ if it has a unit element 1_A (in the usual sense), such that $1_A \in \bigcup_{j\in J} \mathrm{F}_j A$.

The base ring $\mathbb{K}$, with its trivial dir-inv structure, is a unital algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$. In item (3) above, the condition $1_A \in \bigcup_{j\in J} \mathrm{F}_j A$ is equivalent to the ring homomorphism $\mathbb{K} \to A$ being continuous.

We will use the common abbreviation "DG" for "differential graded". An algebra in $\mathsf{Dir\,Inv\,Mod}\,C$ can have further attributes, such as "Lie" or "associative", which have their usual meanings. If $A \in \mathsf{Inv\,Mod}\,C$ then we also say it is an algebra in $\mathsf{Inv\,Mod}\,C$.

Example 1.18. In the situation of Example 1.8, the $\mathfrak{c}$-adic inv structure makes C and $\widehat{C}$ into unital algebras in $\mathsf{Inv\,Mod}\,C$.

Recall that a graded algebra A is called *super-commutative* if $ba = (-1)^{ij}ab$ and $c^2 = 0$ for all $a \in A^i$, $b \in A^j$, $c \in A^k$ and k odd. There is no essential difference between left and right DG A-modules.

Proposition 1.19. *Let A and $\mathfrak{g}$ be DG algebras in* $\mathsf{Dir\,Inv\,Mod}\,C$.

(1) *The completion $\widehat{A}$ is a DG algebra in* $\mathsf{Dir\,Inv\,Mod}\,C$.
(2) *If A is complete, then the canonical isomorphism $A \cong \widehat{A}$ of Proposition 1.5 is an isomorphism of DG algebras.*
(3) *The complete tensor product $A \widehat{\otimes}_C \mathfrak{g}$ is a DG algebra in* $\mathsf{Dir\,Inv\,Mod}\,C$.
(4) *If A is a super-commutative associative unital algebra, then so is $\widehat{A}$.*
(5) *If $\mathfrak{g}$ is a DG Lie algebra and A is a super-commutative associative unital algebra, then $A \widehat{\otimes}_C \mathfrak{g}$ is a DG Lie algebra.*

Proof. (1) This is a consequence of a slightly more general fact. Consider modules $M_1, \ldots, M_r, N \in \mathsf{Dir\,Inv\,Mod}\,C$ and a continuous C-multilinear linear function $\phi : M_1 \times \cdots \times M_r \to N$. We claim that there is an induced continuous C-multilinear linear function $\widehat{\phi} : \prod_k \widehat{M}_k \to \widehat{N}$. This operation is functorial (w.r.t. morphisms $M_k \to M'_k$ and $N \to N'$), and monoidal (i.e. it respects composition in the kth argument with a continuous multilinear function $\psi : L_1 \times \cdots \times L_s \to M_k$).

First assume $M_1, \ldots, M_r, N \in \mathsf{Inv\,Mod}\,C$, with inv structures $\{\mathrm{F}^i M_1\}_{i\in\mathbb{N}}$ etc. For any $i \in \mathbb{N}$ there exists $i' \in \mathbb{N}$ such that $\phi(\prod_k \mathrm{F}^{i'} M_k) \subset \mathrm{F}^i N$. Therefore there is an induced continuous C-multilinear function $\widehat{\phi} : \prod_k \widehat{M}_k \to \widehat{N}$. It is easy to verify that $\phi \mapsto \widehat{\phi}$ is functorial and monoidal.

Next consider the general case, i.e. $M_1, \ldots, M_r, N \in \mathsf{Dir\,Inv\,Mod}\,C$. Let $\{\mathrm{F}_j M_k\}_{j\in J_k}$ be the dir-inv structure of M_k, and let $\{\mathrm{F}_j N\}_{j\in J_N}$ be the dir-inv structure of N. By continuity of ϕ, given $(j_1, \ldots, j_r) \in \prod_k J_k$ there exists $j' \in J_N$ such that $\phi(\prod_k \mathrm{F}_{j_k} M_k) \subset \mathrm{F}_{j'} N$, and $\phi : \prod_k \mathrm{F}_{j_k} M_k \to \mathrm{F}_{j'} N$ is continuous. By the previous paragraph this extends to $\widehat{\phi} : \prod_k \widehat{\mathrm{F}_{j_k} M_k} \to \widehat{\mathrm{F}_{j'} N}$. Passing to the direct limit in $(j_1, \ldots, j_r)$ we obtain $\widehat{\phi} : \prod_k \widehat{M}_k \to \widehat{N}$. Again this operation is functorial and monoidal.

(2) Let $A' \subset A$ be as in the proof of Proposition 1.5. This is a subalgebra. The arguments used in the proof of part (1) above show that $A' \to A$ and $A' \to \widehat{A}$ are algebra homomorphisms.
(3) Let us write $\cdot_A$ and $\cdot_{\mathfrak{g}}$ for the two multiplications, and d_A and $\mathrm{d}_{\mathfrak{g}}$ for the differentials. Then $A \otimes_C \mathfrak{g}$ is a DG algebra with multiplication

$$(a_1 \otimes \gamma_1) \cdot (a_2 \otimes \gamma_2) := (-1)^{i_2 j_1} (a_1 \cdot_A a_2) \otimes (\gamma_1 \cdot_{\mathfrak{g}} \gamma_2)$$

and differential

$$\mathrm{d}(a_1 \otimes \gamma_1) := \mathrm{d}_A(a_1) \otimes \gamma_1 + (-1)^{i_1} a_1 \otimes \mathrm{d}_{\mathfrak{g}}(\gamma_1)$$

for $a_k \in A^{i_k}$ and $\gamma_k \in \mathfrak{g}^{j_k}$. These operations are continuous, so $A \otimes_C \mathfrak{g}$ is a DG algebra in $\mathsf{Dir\,Inv\,Mod}\, C$. Now use part (1).
(4, 5) The various identities (Lie etc.) are preserved by $\widehat{\otimes}$. Definition 1.17(3) ensures that $\widehat{A}$ has a unit element. □

Definition 1.20. Suppose A is a DG super-commutative associative unital algebra in $\mathsf{Dir\,Inv\,Mod}\, C$.

(1) A *DG A-module in* $\mathsf{Dir\,Inv\,Mod}\, C$ is a graded object $M \in \mathsf{Dir\,Inv\,Mod}\, C$, together with continuous C-(bi)linear functions $\mu_M : A \times M \to M$ and $\mathrm{d}_M : M \to M$, which make M into a DG A-module in the usual sense.
(2) A *DG A-module Lie algebra in* $\mathsf{Dir\,Inv\,Mod}\, C$ is a DG Lie algebra $\mathfrak{g} \in \mathsf{Dir\,Inv\,Mod}\, C$, together with a continuous C-bilinear homomorphism $A \times \mathfrak{g} \to \mathfrak{g}$, such that $\mathfrak{g}$ is a DG A-module, and

$$[a_1\gamma_1, a_2\gamma_2] = (-1)^{i_2 j_1} a_1 a_2 [\gamma_1, \gamma_2]$$

for all $a_k \in A^{i_k}$ and $\gamma_k \in \mathfrak{g}^{j_k}$.

Example 1.21. If A is a DG super-commutative associative unital algebra in $\mathsf{Dir\,Inv\,Mod}\, C$, and $\mathfrak{g}$ is a DG Lie algebra in $\mathsf{Dir\,Inv\,Mod}\, C$, then $A \widehat{\otimes}_C \mathfrak{g}$ is a DG $\widehat{A}$-module Lie algebra in $\mathsf{Dir\,Inv\,Mod}\, C$.

Let A be a DG super-commutative associative unital algebra in $\mathsf{Dir\,Inv\,Mod}\, C$, and let M, N be two DG A-modules in $\mathsf{Dir\,Inv\,Mod}\, C$. The tensor product $M \otimes_A N$ is a quotient of $M \otimes_C N$, and as such it has a dir-inv structure. Moreover, $M \otimes_A N$ is a DG A-module in $\mathsf{Dir\,Inv\,Mod}\, C$, and we define $M \widehat{\otimes}_A N$ to be its completion, which is a DG $\widehat{A}$-module in $\mathsf{Dir\,Inv\,Mod}\, C$.

Proposition 1.22. *Let A and B be DG super-commutative associative unital algebras in* $\mathsf{Dir\,Inv\,Mod}\, C$*, and let $A \to B$ be a continuous homomorphism of DG C-algebras.*

(1) *Suppose M is a DG A-module in* $\mathsf{Dir\,Inv\,Mod}\, C$*. Then $B \widehat{\otimes}_A M$ is a DG $\widehat{B}$-module in* $\mathsf{Dir\,Inv\,Mod}\, C$*.*
(2) *Suppose $\mathfrak{g}$ is a DG A-module Lie algebra in* $\mathsf{Dir\,Inv\,Mod}\, C$*. Then $B \widehat{\otimes}_A \mathfrak{g}$ is a DG $\widehat{B}$-module Lie algebra in* $\mathsf{Dir\,Inv\,Mod}\, C$*.*

Proof. Like Proposition 1.19. □

Suppose C, C' are commutative algebras in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$, and $f^* : C \to C'$ is a continuous $\mathbb{K}$-algebra homomorphism. There are functors $f^* : \mathsf{Dir\,Inv\,Mod}\,C \to \mathsf{Dir\,Inv\,Mod}\,C'$ and $\widehat{f^*} : \mathsf{Dir\,Inv\,Mod}\,C \to \mathsf{Dir\,Inv\,Mod}\,\widehat{C}'$, namely $f^* M := C' \otimes_C M$ and $\widehat{f^*} M := C' \widehat{\otimes}_C M$.

Let M and N be two dir-inv C-modules. We define

$$\mathrm{Hom}_C^{\mathrm{cont}}(M, N) := \mathrm{Hom}_{\mathsf{Dir\,Inv\,Mod}\,C}(M, N),$$

i.e. the C-module of continuous C-linear homomorphisms. In general this module has no obvious structure. However, if M is an inv C-module with inv structure $\{\mathrm{F}^i M\}_{i\in\mathbb{N}}$, and N is a discrete inv C-module, then

$$\mathrm{Hom}_C^{\mathrm{cont}}(M, N) \cong \lim_{i\to} \mathrm{Hom}_C(M/\mathrm{F}^i M, N).$$

In this case we consider each

$$\mathrm{F}_i \mathrm{Hom}_C^{\mathrm{cont}}(M, N) := \mathrm{Hom}_C(M/\mathrm{F}^i M, N)$$

as a discrete inv module, and this endows $\mathrm{Hom}_C^{\mathrm{cont}}(M, N)$ with a dir-inv structure.

Example 1.23. In the situation of Example 1.10 one has

$$\mathrm{Hom}_C^{\mathrm{cont}}(N, M) \cong L \otimes_C M$$

as dir-inv C-modules.

Example 1.24. This example is taken from [8]. Assume $\mathbb{K}$ is noetherian and C is a finitely generated commutative $\mathbb{K}$-algebra. For $q \in \mathbb{N}$ define $\mathcal{B}_q(C) = \mathcal{B}^{-q}(C) := C^{\otimes(q+2)} = C \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} C$. Define $\widehat{\mathcal{B}}_q(C) = \widehat{\mathcal{B}}^{-q}(C)$ to be the adic completion of $\mathcal{B}_q(C)$ with respect to the ideal $\mathrm{Ker}(\mathcal{B}_q(C) \to C)$.

There is a $\mathbb{K}$-algebra homomorphism $\widehat{\mathcal{B}}^0(C) \to \widehat{\mathcal{B}}^{-q}(C)$, corresponding to the two extreme tensor factors, and in this way we view $\widehat{\mathcal{B}}^{-q}(C)$ as a complete inv $\widehat{\mathcal{B}}^0(C)$-module. There is a continuous coboundary operator that makes $\widehat{\mathcal{B}}(C) := \bigoplus_{q\in\mathbb{N}} \widehat{\mathcal{B}}^{-q}(C)$ into a complex of $\widehat{\mathcal{B}}^0(C)$-modules, and there is a quasi-isomorphism $\widehat{\mathcal{B}}(C) \to C$. We call $\widehat{\mathcal{B}}(C)$ the *complete un-normalized bar complex* of C.

Next define $\widehat{\mathcal{C}}_q(C) = \widehat{\mathcal{C}}^{-q}(C) := C \otimes_{\widehat{\mathcal{B}}^0(C)} \widehat{\mathcal{B}}^{-q}(C)$. This is a complete inv C-module. The complex $\widehat{\mathcal{C}}(C)$ is called the *complete Hochschild chain complex* of C. Finally let $\mathcal{C}^q_{\mathrm{cd}}(C) := \mathrm{Hom}_C^{\mathrm{cont}}(\widehat{\mathcal{C}}^{-q}(C), C)$. The complex $\mathcal{C}_{\mathrm{cd}}(C) := \bigoplus_{q\in\mathbb{N}} \mathcal{C}^q_{\mathrm{cd}}(C)$ is called the *continuous Hochschild cochain complex* of C.

2. Poly differential operators

In this section $\mathbb{K}$ is a commutative base ring, and C is a commutative $\mathbb{K}$-algebra. The symbol $\otimes$ means $\otimes_{\mathbb{K}}$. We discuss some basic properties of poly differential operators, expanding results from [9].

Definition 2.1. Let $M_1, \ldots, M_p, N$ be C-modules. A $\mathbb{K}$-multilinear function $\phi : M_1 \times \cdots \times M_p \to N$ is called a *poly differential operator* (over C relative to $\mathbb{K}$) if there exists some $d \in \mathbb{N}$ such that for any $(m_1, \ldots, m_p) \in \prod M_i$ and any $i \in \{1, \ldots, p\}$ the function $M_i \to N, m \mapsto \phi(m_1, \ldots, m_{i-1}, m, m_{i+1}, \ldots, m_p)$ is a differential operator of order $\leq d$, in the sense of [2, Section 16.8]. In this case we say that ϕ has order $\leq d$ in each argument.

We shall denote the set of poly differential operators $\prod M_i \to N$ over C relative to $\mathbb{K}$, of order $\leq d$ in all arguments, by

$$\mathrm{F}_d \mathcal{D}\mathit{iff}_{\mathrm{poly}}(C; M_1, \ldots, M_p; N).$$

And we define

$$\mathcal{D}\mathit{iff}_{\mathrm{poly}}(C; M_1, \ldots, M_p; N) := \bigcup_{d \geq 0} \mathrm{F}_d \mathcal{D}\mathit{iff}_{\mathrm{poly}}(C; M_1, \ldots, M_p; N),$$

the union being inside the set of all $\mathbb{K}$-multilinear functions $\prod M_i \to N$. By default we only consider poly differential operators relative to $\mathbb{K}$.

For a natural number p the p-th un-normalized bar module $\mathcal{B}_p(C)$ was defined in Example 1.24. Let $I_p(C)$ be the kernel of the ring homomorphism $\mathcal{B}_p(C) \to C$. Define

$$\mathcal{C}_p(C) := C \otimes_{\mathcal{B}_0(C)} \mathcal{B}_p(C),$$

the *p-th Hochschild chain module* of C (relative to $\mathbb{K}$). For any $d \in \mathbb{N}$ define

$$\mathcal{B}_{p,d}(C) := \mathcal{B}_p(C) \,/\, I_p(C)^{d+1},$$
$$\mathcal{C}_{p,d}(C) := C \otimes_{\mathcal{B}_0(C)} \mathcal{B}_{p,d}(C)$$

and

$$\mathcal{C}_{p,d}(C; M_1, \ldots, M_p) := \mathcal{C}_{p,d}(C) \otimes_{\mathcal{B}_{p-2}(C)} (M_1 \otimes \cdots \otimes M_p).$$

Let

$$\phi_{\mathrm{uni}} : \prod_{i=1}^{p} M_i \to \mathcal{C}_{p,d}(C; M_1, \ldots, M_p)$$

be the $\mathbb{K}$-multilinear function

$$\phi_{\mathrm{uni}}(m_1, \ldots, m_p) := 1 \otimes (m_1 \otimes \cdots \otimes m_p).$$

Observe that for $p = 1$ we get $\mathcal{C}_{1,d}(C) = \mathcal{P}^d(C)$, the module of principal parts of order d (see [2]). In the same way that $\mathcal{P}^d(C)$ parametrizes differential operators, $\mathcal{C}_{p,d}(C)$ parametrizes poly differential operators:

Lemma 2.2. *The assignment $\psi \mapsto \psi \circ \phi_{\mathrm{uni}}$ is a bijection*

$$\mathrm{Hom}_C\big(\mathcal{C}_{p,d}(C; M_1, \ldots, M_p), N\big) \xrightarrow{\simeq} \mathrm{F}_d \mathcal{D}\mathit{iff}_{\mathrm{poly}}(C; M_1, \ldots, M_p; N).$$

Proof. The same arguments used in [2, Section 16.8] also apply here. Cf. [8, Section 1.4]. □

In case $M_1 = \cdots = M_p = N = C$ we see that

$$\begin{aligned}\mathcal{D}\mathit{iff}_{\mathrm{poly}}(C; \underbrace{C, \ldots, C}_{p}; C) &\cong \lim_{d\to} \mathrm{Hom}_C \left(\mathcal{C}_{p,d}(C), C\right) \\ &\cong \mathrm{Hom}_C^{\mathrm{cont}} \left(\widehat{\mathcal{C}}_p(C), C\right) = \mathcal{C}^p_{\mathrm{cd}}(C),\end{aligned} \tag{2.3}$$

with notation of Example 1.24.

Proposition 2.4. *Suppose C is a finitely generated $\mathbb{K}$-algebra, with ideal $\mathfrak{c} \subset C$. Let $M_1, \ldots, M_p, N$ be C-modules, and let $\phi : \prod M_i \to N$ be a multi differential operator over C relative to $\mathbb{K}$. Then ϕ is continuous for the $\mathfrak{c}$-adic dir-inv structures on $M_1, \ldots, M_p, N$.*

Proof. Suppose ϕ has order $\leq d$ in each of its arguments, and let

$$\psi : \mathcal{C}_{p,d}(C; M_1, \ldots, M_p) \to N$$

be the corresponding C-linear homomorphism. As in [8, Proposition 1.4.3], since C is a finitely generated $\mathbb{K}$-algebra, it follows that $\mathcal{B}_{p,d}(C)$ is a finitely generated module over $\mathcal{B}_0(C)$; and hence $\mathcal{C}_{p,d}(C)$ is a finitely generated C-module. Let us denote by $\{\mathrm{F}_j M_i\}_{j \in J_i}$ and $\{\mathrm{F}_k N\}_{k \in K}$ the $\mathfrak{c}$-adic dir-inv structures on M_i and N. For any $j_1, \ldots, j_p$ the $\mathcal{B}_{p-2}(C)$-module $\mathrm{F}_{j_1} M_1 \otimes \cdots \otimes \mathrm{F}_{j_p} M_p$ is finitely generated, and hence the C-module $\mathcal{C}_{p,d}(C; \mathrm{F}_{j_1} M_1, \ldots, \mathrm{F}_{j_p} M_p)$ is finitely generated. Therefore

$$\psi \left(\mathcal{C}_{p,d}(C; \mathrm{F}_{j_1} M_1, \ldots, \mathrm{F}_{j_p} M_p)\right) = \mathrm{F}_k N$$

for some $k \in K$.

It remains to prove that $\phi : \prod_{i=1}^{p} \mathrm{F}_{j_i} M_i \to \mathrm{F}_k N$ is continuous for the $\mathfrak{c}$-adic inv structures. But just like [8, Proposition 1.4.6], for any i and l one has

$$\phi(\mathrm{F}_{j_1} M_1, \ldots, \mathfrak{c}^{i+d} \mathrm{F}_{j_l} M_l, \ldots, \mathrm{F}_{j_p} M_p) \subset \mathfrak{c}^i \mathrm{F}_k N. \quad \square \tag{2.5}$$

Suppose C' is a commutative C-algebra with ideal $\mathfrak{c}' \subset C'$. One says that C' is $\mathfrak{c}'$*-adically formally étale* over C if the following condition holds. Let D be a commutative C-algebra with nilpotent ideal $\mathfrak{d}$, and let $f : C' \to D/\mathfrak{d}$ be a C-algebra homomorphism such that $f(\mathfrak{c}'^i) = 0$ for $i \gg 0$. Then f lifts uniquely to a C-algebra homomorphism $\tilde{f} : C' \to D$. The important instances are when $C \to C'$ is étale (and then $\mathfrak{c}' = 0$); and when C' is the $\mathfrak{c}$-adic completion of C for some ideal $\mathfrak{c} \subset A$ (and $\mathfrak{c}' = C'\mathfrak{c}$). In both these instances C' is $\mathfrak{c}$-adically complete; and if C is noetherian, then $C \to C'$ is also flat.

Lemma 2.6. *Let C' be a $\mathfrak{c}'$-adically formally étale C-algebra. Define $C'_j := C'/\mathfrak{c}'^{j+1}$. Consider C' and $\mathcal{C}_{p,d}(C)$ as inv C-modules, with the $\mathfrak{c}'$-adic and discrete inv structures respectively. Then the canonical homomorphism*

$$C' \mathbin{\widehat{\otimes}}_C \mathcal{C}_{p,d}(C) \to \lim_{\leftarrow j} \mathcal{C}_{p,d}(C'_j)$$

is bijective.

Proof. Define ideals

$$\mathfrak{c}'_p := \operatorname{Ker}\left(\mathcal{C}_p(C') \to \mathcal{C}_p(C'_0)\right)$$

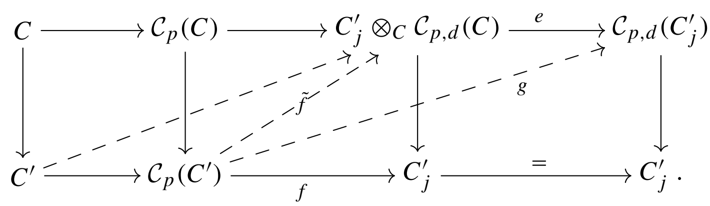
and

$$J := \operatorname{Ker}(C'_j \otimes_C \mathcal{C}_{p,d}(C) \to C'_j).$$

By the transitivity and the base change properties of formally étale homomorphisms, the ring homomorphism

$$\mathcal{C}_p(C) \cong C \otimes \cdots \otimes C \to C' \otimes \cdots \otimes C' \cong \mathcal{C}_p(C')$$

is $\mathfrak{c}'_p$-adically formally étale. Consider the commutative diagram of ring homomorphisms (with solid arrows)



The ideal J satisfies $J^{d+1} = 0$, and the ideal $\operatorname{Ker}(\mathcal{C}_{p,d}(C'_j) \to C'_j)$ is nilpotent too. Due to the unique lifting property the dashed arrows exist and are unique, making the whole diagram commutative. Moreover $g : \mathcal{C}_p(C') \to \mathcal{C}_p(C'_j)$ has to be the canonical surjection, and $\tilde{f}$ is surjective.

A little calculation shows that $\tilde{f}(I_p(C')^{d+1}) = 0$, and hence $\tilde{f}$ induces a homomorphism

$$\bar{f} : \mathcal{C}_{p,d}(C') \to C'_j \otimes_C \mathcal{C}_{p,d}(C).$$

Let

$$\mathfrak{c}'_{p,d} := \operatorname{Ker}\left(\mathcal{C}_{p,d}(C') \to \mathcal{C}_{p,d}(C'_0)\right).$$

Another calculation shows that $\bar{f}({\mathfrak{c}'_{p,d}}^{(j+1)(d+1)}) = 0$. The conclusion is that there are surjections

$$\mathcal{C}_{p,d}(C'_{jd+j+d}) \xrightarrow{\bar{f}} C'_j \otimes_C \mathcal{C}_{p,d}(C) \xrightarrow{e} \mathcal{C}_{p,d}(C'_j),$$

such that $e \circ \bar{f}$ is the canonical surjection. Passing to the inverse limit we deduce that

$$C' \widehat{\otimes}_C \mathcal{C}_{p,d}(C) \to \lim_{\leftarrow j} \mathcal{C}_{p,d}(C'_j)$$

is bijective. □

Proposition 2.7. *Assume C is a noetherian finitely generated $\mathbb{K}$-algebra, and C' is a noetherian, $\mathfrak{c}'$-adically complete, flat, $\mathfrak{c}'$-adically formally étale C-algebra. Let $M_1, \ldots, M_p, N$ be C-modules, and define $M'_i := C' \otimes_C M_i$ and $N' := C' \otimes_C N$.*

(1) *Suppose* $\phi : \prod_{i=1}^{p} M_i \to N$ *is a poly differential operator over* C. *Then* ϕ *extends uniquely to a poly differential operator* $\phi' : \prod_{i=1}^{p} M'_i \to N'$ *over* C'. *If* ϕ *has order* $\leq d$ *then so does* ϕ'.

(2) *The homomorphism*

$$C' \otimes_C \mathrm{F}_d \mathcal{D}\mathit{iff}_{\mathrm{poly}}(C; M_1, \ldots, M_p; N) \to \mathrm{F}_d \mathcal{D}\mathit{iff}_{\mathrm{poly}}(C'; M'_1, \ldots, M'_p; N'),$$

$c' \otimes \phi \mapsto c'\phi$, *is bijective.*

Proof. By Proposition 2.4, applied to C with the 0-adic inv structure, we may assume that the C-modules $M_1, \ldots, M_p, N$ are finitely generated.

Fix $d \in \mathbb{N}$. Define $C'_j := C'/\mathfrak{c}'^{j+1}$ and $N'_j := C'_j \otimes_C N$. So $C' \cong \lim_{\leftarrow j} C'_j$ and $N' \cong \lim_{\leftarrow j} N'_j$.

By Lemma 2.2 and Proposition 2.4 we have

$$\begin{aligned} &\mathrm{F}_d \mathcal{D}\mathit{iff}_{\mathrm{poly}}(C'; M'_1, \ldots, M'_p; N') \\ &\quad \cong \mathrm{Hom}_{C'}(\mathcal{C}_{p,d}(C'; M'_1, \ldots, M'_p), N') \\ &\quad \cong \lim_{\leftarrow j} \mathrm{Hom}_{C'}(\mathcal{C}_{p,d}(C'; M'_1, \ldots, M'_p), N'_j). \end{aligned} \tag{2.8}$$

Now for any $k \geq j + d$ one has

$$\mathrm{Hom}_{C'}(\mathcal{C}_{p,d}(C'; M'_1, \ldots, M'_p), N'_j) \cong \mathrm{Hom}_{C'}(\mathcal{C}_{p,d}(C'_k; M'_1, \ldots, M'_p), N'_j).$$

This is because of formula (2.5). Thus, using Lemma 2.6, we obtain

$$\begin{aligned} &\mathrm{Hom}_{C'}(\mathcal{C}_{p,d}(C'; M'_1, \ldots, M'_p), N'_j) \\ &\quad \cong \mathrm{Hom}_{C'}(\lim_{\leftarrow k} \mathcal{C}_{p,d}(C'_k; M'_1, \ldots, M'_p), N'_j) \\ &\quad \cong \mathrm{Hom}_{C'}(C' \otimes_C \mathcal{C}_{p,d}(C; M_1, \ldots, M_p), N'_j) \\ &\quad \cong \mathrm{Hom}_{C}(\mathcal{C}_{p,d}(C; M_1, \ldots, M_p), N'_j). \end{aligned}$$

Combining this with (2.8) we get

$$\begin{aligned} &\mathrm{F}_d \mathcal{D}\mathit{iff}_{\mathrm{poly}}(C'; M'_1, \ldots, M'_p; N') \\ &\quad \cong \lim_{\leftarrow j} \mathrm{Hom}_{C}(\mathcal{C}_{p,d}(C; M_1, \ldots, M_p), N'_j) \\ &\quad \cong \mathrm{Hom}_{C}\left(\mathcal{C}_{p,d}(C; M_1, \ldots, M_p), N'\right). \end{aligned}$$

But $C \to C'$ is flat, C is noetherian, and $\mathcal{C}_{p,d}(C; M_1, \ldots, M_p)$ is a finitely generated C-module. Therefore

$$\begin{aligned} &\mathrm{Hom}_{C}\left(\mathcal{C}_{p,d}(C; M_1, \ldots, M_p), N'\right) \\ &\quad \cong C' \otimes_C \mathrm{Hom}_{C}\left(\mathcal{C}_{p,d}(C; M_1, \ldots, M_p), N\right). \end{aligned}$$

The conclusion is that

$$\begin{aligned} &\mathrm{F}_m \mathcal{D}^{p+1}_{\mathrm{poly}}(C'; M'_1, \ldots, M'_p; N') \\ &\quad \cong C' \otimes_C \mathrm{F}_m \mathcal{D}^{p+1}_{\mathrm{poly}}(C; M_1, \ldots, M_p; N). \end{aligned} \tag{2.9}$$

Given $\phi : \prod M_i \to N$ of order $\leq d$, let $\phi' := 1 \otimes \phi$ under the isomorphism (2.9). Backtracking, we see that ϕ' is the unique poly differential operator extending ϕ. □

3. L_∞ morphisms and their twists

In this section we expand some results on L_∞ algebras and morphisms from [5] Section 4. Much of the material presented here is based on discussions with Vladimir Hinich. There is some overlap with Section 2.2 of [3], with Section 6.1 of [6], and possibly with other accounts.

Let $\mathbb{K}$ be a field of characteristic 0. Given a graded $\mathbb{K}$-module $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}^i$ and a natural number j let $\mathrm{T}^j \mathfrak{g} := \underbrace{\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{j}$. The direct sum $\mathrm{T}\mathfrak{g} := \bigoplus_{j \in \mathbb{N}} \mathrm{T}^j \mathfrak{g}$ is the tensor algebra. Let us denote the multiplication in $\mathrm{T}\mathfrak{g}$ by $\circledast$. (This is just another way of writing $\otimes$, but it will be convenient to do so.)

The permutation group $\mathfrak{S}_j$ acts on $\mathrm{T}^j \mathfrak{g}$ as follows. For any sequence of integers $\boldsymbol{d} = (d_1, \ldots, d_j)$ there is a group homomorphism $\mathrm{sgn}_{\boldsymbol{d}} : \mathfrak{S}_j \to \{\pm 1\}$ such that on a transposition $\sigma = (p, p+1)$ the value is $\mathrm{sgn}_{\boldsymbol{d}}(\sigma) = (-1)^{d_p d_{p+1}}$. The action of a permutation $\sigma \in \mathfrak{S}_j$ on $\mathrm{T}^j \mathfrak{g}$ is then

$$\sigma(\gamma_1 \circledast \cdots \circledast \gamma_j) := \mathrm{sgn}_{\boldsymbol{d}}(\sigma) \gamma_{\sigma(1)} \circledast \cdots \circledast \gamma_{\sigma(j)}$$

for $\gamma_1 \in \mathfrak{g}^{d_1}, \ldots, \gamma_j \in \mathfrak{g}^{d_j}$. Define $\tilde{\mathrm{S}}^j \mathfrak{g}$ to be the set of $\mathfrak{S}_j$-invariants inside $\mathrm{T}^j \mathfrak{g}$, and $\tilde{\mathrm{S}}\mathfrak{g} := \bigoplus_{j \geq 0} \tilde{\mathrm{S}}^j \mathfrak{g}$.

The $\mathbb{K}$-module $\mathrm{T}\mathfrak{g}$ is also a coalgebra, with coproduct $\tilde{\Delta} : \mathrm{T}\mathfrak{g} \to \mathrm{T}\mathfrak{g} \otimes \mathrm{T}\mathfrak{g}$ given by the formula

$$\tilde{\Delta}(\gamma_1 \circledast \cdots \circledast \gamma_j) := \sum_{p=0}^{j} (\gamma_1 \circledast \cdots \circledast \gamma_p) \otimes (\gamma_{p+1} \circledast \cdots \circledast \gamma_j).$$

The submodule $\tilde{\mathrm{S}}\mathfrak{g} \subset \mathrm{T}\mathfrak{g}$ is a sub-coalgebra (but not a subalgebra!).

The super-symmetric algebra $\mathrm{S}\mathfrak{g} = \bigoplus_{j \geq 0} \mathrm{S}^j \mathfrak{g}$ is defined to be the quotient of $\mathrm{T}\mathfrak{g}$ by the ideal generated by the elements $\gamma_1 \circledast \gamma_2 - (-1)^{d_1 d_2} \gamma_2 \circledast \gamma_1$, for all $\gamma_1 \in \mathfrak{g}^{d_1}$ and $\gamma_2 \in \mathfrak{g}^{d_2}$. In other words, $\mathrm{S}^j \mathfrak{g}$ is the set of coinvariants of $\mathrm{T}^j \mathfrak{g}$ under the action of the group $\mathfrak{S}_j$. The product in the algebra $\mathrm{S}\mathfrak{g}$ is denoted by $\cdot$. The canonical projection is $\pi : \mathrm{T}\mathfrak{g} \to \mathrm{S}\mathfrak{g}$ is an algebra homomorphism: $\pi(\gamma_1 \circledast \gamma_2) = \gamma_1 \cdot \gamma_2$.

In fact $\mathrm{S}\mathfrak{g}$ is a commutative cocommutative Hopf algebra. The comultiplication

$$\Delta : \mathrm{S}\mathfrak{g} \to \mathrm{S}\mathfrak{g} \otimes \mathrm{S}\mathfrak{g}$$

is the unique $\mathbb{K}$-algebra homomorphism such that

$$\Delta(\gamma) = \gamma \otimes 1 + 1 \otimes \gamma$$

for all $\gamma \in \mathfrak{g}$. The antipode is $\gamma \mapsto -\gamma$. The projection $\pi : \mathrm{T}\mathfrak{g} \to \mathrm{S}\mathfrak{g}$ is not a coalgebra homomorphism. However:

Lemma 3.1. *Let* $\tau : \mathrm{S}\mathfrak{g} \to \mathrm{T}\mathfrak{g}$ *be the* $\mathbb{K}$*-module homomorphism defined by*

$$\tau(\gamma_1 \cdots \gamma_j) := \sum_{\sigma \in \mathfrak{S}_j} \mathrm{sgn}_{(d_1,\ldots,d_j)}(\sigma)\gamma_{\sigma(1)} \circledast \cdots \circledast \gamma_{\sigma(j)}$$

for $\gamma_1 \in \mathfrak{g}^{d_1}, \ldots, \gamma_j \in \mathfrak{g}^{d_j}$. *Then* $\tau : \mathrm{S}\mathfrak{g} \to \tilde{\mathrm{S}}\mathfrak{g}$ *is a coalgebra isomorphism, where* $\mathrm{S}\mathfrak{g}$ *has the comultiplication* Δ *and* $\tilde{\mathrm{S}}\mathfrak{g}$ *has the comultiplication* $\tilde{\Delta}$.

Proof. Define $\tilde{\pi} : \mathrm{T}\mathfrak{g} \to \mathrm{S}\mathfrak{g}$ to be the $\mathbb{K}$-module homomorphism

$$\tilde{\pi}(\gamma_1 \circledast \cdots \circledast \gamma_j) := \frac{1}{j!}\pi(\gamma_1 \circledast \cdots \circledast \gamma_j) = \frac{1}{j!}\gamma_1 \cdots \gamma_j$$

for $\gamma_1, \ldots, \gamma_j \in \mathfrak{g}$. So $\tilde{\pi} \circ \tau$ is the identity map of $\mathrm{S}\mathfrak{g}$, and $\tilde{\pi} : \tilde{\mathrm{S}}\mathfrak{g} \to \mathrm{S}\mathfrak{g}$ is bijective. It suffices to prove that

$$(\tilde{\pi} \otimes \tilde{\pi}) \circ (\tau \otimes \tau) \circ \Delta = (\tilde{\pi} \otimes \tilde{\pi}) \circ \tilde{\Delta} \circ \tau.$$

Take any $\gamma_1 \in \mathfrak{g}^{d_1}, \ldots, \gamma_j \in \mathfrak{g}^{d_j}$ and write $\boldsymbol{d} := (d_1, \ldots, d_j)$. Then

$$\begin{aligned}&((\tilde{\pi} \otimes \tilde{\pi}) \circ \tilde{\Delta} \circ \tau)(\gamma_1 \cdots \gamma_j)\\ &\quad = \sum_{p=0}^{j} \sum_{\sigma \in \mathfrak{S}_j} \frac{1}{p!}\frac{1}{(j-p)!}\mathrm{sgn}_{\boldsymbol{d}}(\sigma)(\gamma_{\sigma(1)} \cdots \gamma_{\sigma(p)}) \otimes (\gamma_{\sigma(p+1)} \cdots \gamma_{\sigma(j)}).\end{aligned}$$

On the other hand

$$\begin{aligned}&((\tilde{\pi} \otimes \tilde{\pi}) \circ (\tau \otimes \tau) \circ \Delta)\,(\gamma_1 \cdots \gamma_j)\\ &\quad = \Delta(\gamma_1 \cdots \gamma_j) = (1 \otimes \gamma_1 + \gamma_1 \otimes 1) \cdots (1 \otimes \gamma_j + \gamma_j \otimes 1)\\ &\quad \times \sum_{p=0}^{j} \sum_{\sigma \in \mathfrak{S}_{p,j-p}} \mathrm{sgn}_{\boldsymbol{d}}(\sigma)(\gamma_{\sigma(1)} \cdots \gamma_{\sigma(p)}) \otimes (\gamma_{\sigma(p+1)} \cdots \gamma_{\sigma(j)}),\end{aligned}$$

where $\mathfrak{S}_{p,j-p}$ is the set of $(p, j-p)$-shuffles inside the group $\mathfrak{S}_j$. Since the algebra $\mathrm{S}\mathfrak{g}$ is super-commutative the two sums are equal. □

The grading on $\mathfrak{g}$ induces a grading on $\mathrm{S}\mathfrak{g}$, which we call the *degree*. Thus for $\gamma_i \in \mathfrak{g}^{d_i}$ the degree of $\gamma_1 \cdots \gamma_j \in \mathrm{S}^j\mathfrak{g}$ is $d_1 + \cdots + d_j$ (unless $\gamma_1 \cdots \gamma_j = 0$). We consider $\mathrm{S}\mathfrak{g}$ as a graded algebra for this grading. Actually there is another grading on $\mathrm{S}\mathfrak{g}$, by *order*, where we define the order of $\gamma_1 \cdots \gamma_j$ to be j (again, unless this element is zero). But this grading will have a different role.

By definition the j-th super-exterior power of $\mathfrak{g}$ is

$$\textstyle\bigwedge^j \mathfrak{g} := \mathrm{S}^j(\mathfrak{g}[1])[-j], \tag{3.2}$$

where $\mathfrak{g}[1]$ is the shifted graded module whose degree i component is $\mathfrak{g}[1]^i = \mathfrak{g}^{i+1}$. When $\mathfrak{g}$ is concentrated in degree 0 then these are the usual constructions of symmetric and exterior algebras, respectively.

We denote by $\ln : \mathrm{S}\mathfrak{g} \to \mathrm{S}^1\mathfrak{g} = \mathfrak{g}$ the projection. So $\ln(\gamma)$ is the first order term of $\gamma \in \mathrm{S}\mathfrak{g}$. (The expression "ln" might stand for "linear" or "logarithm".)

Definition 3.3. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two graded $\mathbb{K}$-modules, and let $\Psi : \mathrm{S}\mathfrak{g} \to \mathrm{S}\mathfrak{g}'$ be a $\mathbb{K}$-linear homomorphism. For any $j \geq 1$ the *j-th Taylor coefficient* of Ψ is defined to be

$$\partial^j \Psi := \ln \circ \Psi : \mathrm{S}^j \mathfrak{g} \to \mathfrak{g}'.$$

We say Ψ is colocal if $\Psi(\mathrm{S}^{\geq 1}\mathfrak{g}) \subset \mathrm{S}^{\geq 1}\mathfrak{g}'$ and $\Psi(\mathrm{S}^0\mathfrak{g}) \subset \mathrm{S}^0\mathfrak{g}'$.

Lemma 3.4. *Suppose we are given a sequence of $\mathbb{K}$-linear homomorphisms $\psi_j : \mathrm{S}^j\mathfrak{g} \to \mathfrak{g}'$, $j \geq 1$, each of degree 0. Then there is a unique colocal coalgebra homomorphism $\Psi : \mathrm{S}\mathfrak{g} \to \mathrm{S}\mathfrak{g}'$, homogeneous of degree 0 and satisfying $\Psi(1) = 1$, whose Taylor coefficients are $\partial^j \Psi = \psi_j$.*

Proof. Let $\tilde{\ln} : \tilde{\mathrm{S}}\mathfrak{g}' \to \tilde{\mathrm{S}}^1\mathfrak{g}' = \mathfrak{g}'$ be the projection for this coalgebra. Consider the exact sequence of coalgebras

$$0 \to \mathbb{K} \to \tilde{\mathrm{S}}\mathfrak{g} \to \tilde{\mathrm{S}}^{\geq 1}\mathfrak{g} \to 0. \tag{3.5}$$

According to Kontsevich [5, Section 4.1] (see also [3, Lemma 2.1.5]) the sequence $\{\psi_j\}_{j\geq 1}$ uniquely determines a coalgebra homomorphism $\tilde{\Psi} : \tilde{\mathrm{S}}^{\geq 1}\mathfrak{g} \to \tilde{\mathrm{S}}^{\geq 1}\mathfrak{g}'$ such that

$$\tilde{\ln} \circ \tilde{\Psi}|_{\tilde{\mathrm{S}}^j\mathfrak{g}} = \psi_j \circ \tau^{-1}|_{\tilde{\mathrm{S}}^j\mathfrak{g}}$$

for all $j \geq 1$. Here $\tau : \mathrm{S}\mathfrak{g} \xrightarrow{\simeq} \tilde{\mathrm{S}}\mathfrak{g}$ is the coalgebra isomorphism of Lemma 3.1. Using (3.5) we can lift $\tilde{\Psi}$ uniquely to a colocal coalgebra homomorphism $\tilde{\Psi} : \tilde{\mathrm{S}}\mathfrak{g} \to \tilde{\mathrm{S}}\mathfrak{g}'$ by setting $\tilde{\Psi}(1) := 1$. Now define the coalgebra homomorphism $\Psi : \mathrm{S}\mathfrak{g} \to \mathrm{S}\mathfrak{g}'$ to be $\Psi := \tau^{-1} \circ \tilde{\Psi} \circ \tau$. □

A $\mathbb{K}$-linear map $Q : \mathrm{S}\mathfrak{g} \to \mathrm{S}\mathfrak{g}$ is a *coderivation* if

$$\Delta \circ Q = (Q \otimes \mathbf{1} + \mathbf{1} \otimes Q) \circ \Delta,$$

where $\mathbf{1} := \mathbf{1}_{\mathrm{S}\mathfrak{g}}$, the identity map.

Lemma 3.6. *Given a sequence of $\mathbb{K}$-linear homomorphisms $\psi_j : \mathrm{S}^j\mathfrak{g} \to \mathfrak{g}$, $j \geq 1$, each of degree 1, there is a unique colocal coderivation Q of degree 1, such that $Q(1) = 0$ and $\partial^j Q = \psi_j$.*

Proof. According to Kontsevich [5, Section 4.3] (see also [3, Lemma 2.1.2]) the sequence $\{\psi_j\}_{j\geq 1}$ uniquely determines a coderivation $\tilde{Q} : \tilde{\mathrm{S}}^{\geq 1}\mathfrak{g} \to \tilde{\mathrm{S}}^{\geq 1}\mathfrak{g}$ such that

$$\tilde{\ln} \circ \tilde{Q}|_{\tilde{\mathrm{S}}^j\mathfrak{g}} = \psi_j \circ \tau^{-1}|_{\tilde{\mathrm{S}}^j\mathfrak{g}}$$

for all $j \geq 1$. Using (3.5) this can be lifted uniquely to a colocal coderivation $\tilde{Q} : \tilde{\mathrm{S}}\mathfrak{g} \to \tilde{\mathrm{S}}\mathfrak{g}$ by setting $\tilde{Q}(1) := 0$. Now define the coderivation $Q : \mathrm{S}\mathfrak{g} \to \mathrm{S}\mathfrak{g}$ to be $Q := \tau^{-1} \circ \tilde{Q} \circ \tau$. □

We will be mostly interested in the coalgebras $\mathrm{S}(\mathfrak{g}[1])$ and $\mathrm{S}(\mathfrak{g}'[1])$. Observe that if $\Psi : \mathrm{S}(\mathfrak{g}[1]) \to \mathrm{S}(\mathfrak{g}'[1])$ is a homogeneous $\mathbb{K}$-linear homomorphism of degree i, then, using formula (3.2), each Taylor coefficient $\partial^j \Psi$ may be viewed as a homogeneous $\mathbb{K}$-linear homomorphism $\partial^j \Psi : \bigwedge^j \mathfrak{g} \to \mathfrak{g}$ of degree $i + 1 - j$.

Definition 3.7. Let $\mathfrak{g}$ be a graded $\mathbb{K}$-module. An L_∞ algebra structure on $\mathfrak{g}$ is a colocal coderivation $Q : \mathrm{S}(\mathfrak{g}[1]) \to \mathrm{S}(\mathfrak{g}[1])$ of degree 1, satisfying $Q(1) = 0$ and $Q \circ Q = 0$. We call the pair $(\mathfrak{g}, Q)$ an L_∞ *algebra*.

The notion of L_∞ algebra generalizes that of DG Lie algebra in the following sense:

Proposition 3.8 (*[5, Section 4.3]*). *Let $Q : \mathrm{S}(\mathfrak{g}[1]) \to \mathrm{S}(\mathfrak{g}[1])$ be a colocal coderivation of degree* 1 *with $Q(1) = 0$. Then the following conditions are equivalent.*

(i) $\partial^j Q = 0$ *for all $j \geq 3$, and $Q \circ Q = 0$.*
(ii) $\partial^j Q = 0$ *for all $j \geq 3$, and $\mathfrak{g}$ is a DG Lie algebra with respect to the differential* $\mathrm{d} := \partial^1 Q$ *and the bracket* $[-,-] := \partial^2 Q$.

In view of this, we shall say that $(\mathfrak{g}, Q)$ is a DG Lie algebra if the equivalent conditions of the proposition hold. An easy calculation shows that given an L_∞ algebra $(\mathfrak{g}, Q)$, the function $\partial^1 Q : \mathfrak{g} \to \mathfrak{g}$ is a differential, and $\partial^2 Q$ induces a graded Lie bracket on $\mathrm{H}(\mathfrak{g}, \partial^1 Q)$. We shall denote this graded Lie algebra by $\mathrm{H}(\mathfrak{g}, Q)$.

Definition 3.9. Let $(\mathfrak{g}, Q)$ and $(\mathfrak{g}', Q')$ be L_∞ algebras. An L_∞ *morphism* $\Psi : (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ is a colocal coalgebra homomorphism $\Psi : \mathrm{S}(\mathfrak{g}[1]) \to \mathrm{S}(\mathfrak{g}'[1])$ of degree 0, satisfying $\Psi(1) = 1$ and $\Psi \circ Q = Q' \circ \Psi$.

Proposition 3.10 (*[5, Section 4.3]*). *Let $(\mathfrak{g}, Q)$ and $(\mathfrak{g}', Q')$ be DG Lie algebras, and let $\Psi : \mathrm{S}(\mathfrak{g}[1]) \to \mathrm{S}(\mathfrak{g}'[1])$ be a colocal coalgebra homomorphism of degree 0 such that $\Psi(1) = 1$. Then Ψ is an L_∞ morphism (i.e. $\Psi \circ Q = Q' \circ \Psi$) iff the Taylor coefficients $\psi_i := \partial^i \Psi : \bigwedge^i \mathfrak{g} \to \mathfrak{g}'$ satisfy the following identity:*

$$\begin{aligned}
&\mathrm{d}\,(\psi_i(\gamma_1 \wedge \cdots \wedge \gamma_i)) - \sum_{k=1}^{i} \pm \psi_i(\gamma_1 \wedge \cdots \wedge \mathrm{d}(\gamma_k) \wedge \cdots \wedge \gamma_i) \\
&\quad = \frac{1}{2} \sum_{\substack{k,l \geq 1 \\ k+l=i}} \frac{1}{k!\,l!} \sum_{\sigma \in \mathfrak{S}_i} \pm \left[\psi_k(\gamma_{\sigma(1)} \wedge \cdots \wedge \gamma_{\sigma(k)}), \psi_l(\gamma_{\sigma(k+1)} \wedge \cdots \wedge \gamma_{\sigma(i)})\right] \\
&\qquad + \sum_{k<l} \pm \psi_{i-1}([\gamma_k, \gamma_l] \wedge \gamma_1 \wedge \cdots \not\gamma_k \cdots \not\gamma_l \cdots \wedge \gamma_i).
\end{aligned}$$

Here $\gamma_k \in \mathfrak{g}$ are homogeneous elements, $\mathfrak{S}_i$ is the permutation group of $\{1, \ldots, i\}$, and the signs depend only on the indices, the permutations and the degrees of the elements γ_k. (See [4, *Section 6*] *or* [1, *Theorem 3.1*] *for the explicit signs.)*

The proposition shows that when $(\mathfrak{g}, Q)$ and $(\mathfrak{g}', Q')$ are DG Lie algebras and $\partial^j \Psi = 0$ for all $j \geq 2$, then $\partial^1 \Psi : \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism of DG Lie algebras; and conversely. It also implies that for any L_∞ morphism $\Psi : (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$, the map $\mathrm{H}(\Psi) : \mathrm{H}(\mathfrak{g}, Q) \to \mathrm{H}(\mathfrak{g}', Q')$ is a homomorphism of graded Lie algebras.

Given DG Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ we consider them as L_∞ algebras $(\mathfrak{g}, Q)$ and $(\mathfrak{g}', Q')$, as explained in Proposition 3.8. If $\Psi : (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ is an L_∞ morphism, then we shall say (by slight abuse of notation) that $\Psi : \mathfrak{g} \to \mathfrak{g}'$ is an L_∞ morphism.

From here until Theorem 3.21 (inclusive) C is a commutative $\mathbb{K}$-algebra, and $\mathfrak{g}, \mathfrak{g}'$ are graded C-modules. Suppose $(\mathfrak{g}, Q)$ is an L_∞ algebra structure on $\mathfrak{g}$ such that the Taylor coefficients $\partial^j Q : \bigwedge^j \mathfrak{g} \to \mathfrak{g}$ are all C-multilinear. Then we say $(\mathfrak{g}, Q)$ is a *C-multilinear* L_∞ *algebra*. Similarly one defines the notion of *C-multilinear* L_∞ *morphism* $\Psi : (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$.

With C and $\mathfrak{g}$ as above let $\mathrm{S}_C\mathfrak{g}$ be the super-symmetric associative unital free algebra over C. Namely $\mathrm{S}_C\mathfrak{g}$ is the quotient of the tensor algebra $\mathrm{T}_C\mathfrak{g} = C \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes_C \mathfrak{g}) \oplus \cdots$ by the ideal generated by the super-commutativity relations. The algebra $\mathrm{S}_C\mathfrak{g}$ is a Hopf algebra over C, with comultiplication

$$\Delta_C : \mathrm{S}_C\mathfrak{g} \to \mathrm{S}_C\mathfrak{g} \otimes_C \mathrm{S}_C\mathfrak{g}.$$

The formulas are just as in the case $C = \mathbb{K}$. It will be useful to note that Δ_C preserves the grading by order, namely

$$\Delta_C(\mathrm{S}^i_C\mathfrak{g}) \subset \bigoplus_{j+k=i} \mathrm{S}^j_C\mathfrak{g} \otimes_C \mathrm{S}^k_C\mathfrak{g}.$$

Lemma 3.11. (1) *Let $\mathfrak{g}$ be a graded C-module. There is a canonical bijection $Q \mapsto Q_C$ between the set of C-multilinear L_∞ algebra structures Q on $\mathfrak{g}$, and the set of colocal coderivations $Q_C : \mathrm{S}_C(\mathfrak{g}[1]) \to \mathrm{S}_C(\mathfrak{g}[1])$ over C of degree* 1*, such that $Q_C(1) = 0$ and $Q_C \circ Q_C = 0$.*

(2) *Let $(\mathfrak{g}, Q)$ and $(\mathfrak{g}', Q')$ be two C-multilinear L_∞ algebras. The set of C-multilinear L_∞ morphisms $\Psi : (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ is canonically bijective to the set of colocal coalgebra homomorphisms $\Psi_C : \mathrm{S}_C(\mathfrak{g}[1]) \to \mathrm{S}_C(\mathfrak{g}'[1])$ over C of degree* 0*, such that $\Psi_C(1) = 1$ and $\Psi_C \circ Q_C = Q'_C \circ \Psi_C$.*

Proof. The data for a coderivation $Q_C : \mathrm{S}_C(\mathfrak{g}[1]) \to \mathrm{S}_C(\mathfrak{g}[1])$ over C is its sequence of C-linear Taylor coefficients $\partial^j Q_C : \bigwedge^j_C \mathfrak{g} \to \mathfrak{g}$. But giving such a homomorphism $\partial^j Q_C$ is the same as giving a C-multilinear homomorphism $\partial^j Q : \bigwedge^j \mathfrak{g} \to \mathfrak{g}$, so there is a corresponding C-multilinear coderivation $Q : \mathrm{S}(\mathfrak{g}[1]) \to \mathrm{S}(\mathfrak{g}[1])$. One checks that $Q \circ Q = 0$ iff $Q_C \circ Q_C = 0$.

Similarly for coalgebra homomorphisms. □

An element $\gamma \in \mathrm{S}_C(\mathfrak{g}[1])$ is called *primitive* if $\Delta_C(\gamma) = \gamma \otimes 1 + 1 \otimes \gamma$.

Lemma 3.12. *The set of primitive elements of $\mathrm{S}_C(\mathfrak{g}[1])$ is precisely $\mathrm{S}^1_C(\mathfrak{g}[1]) = \mathfrak{g}[1]$.*

Proof. By definition of the comultiplication any $\gamma \in \mathfrak{g}[1]$ is primitive. For the converse, let us denote by μ the multiplication in $\mathrm{S}_C(\mathfrak{g}[1])$. One checks that $(\mu \circ \Delta_C)(\gamma) = 2^i\gamma$ for $\gamma \in \mathrm{S}^i_C(\mathfrak{g}[1])$. If γ is primitive then $(\mu \circ \Delta_C)(\gamma) = 2\gamma$, so indeed $\gamma \in \mathrm{S}^1_C(\mathfrak{g}[1])$. □

Now let us assume that C is a local ring, with nilpotent maximal ideal $\mathfrak{m}$. Suppose we are given two C-multilinear L_∞ algebras $(\mathfrak{g}, Q)$ and $(\mathfrak{g}', Q')$, and a C-multilinear L_∞ morphism $\Psi : (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$. Because the coderivation Q is C-multilinear, the C-submodule $\mathfrak{m}\mathfrak{g} \subset \mathfrak{g}$ becomes a C-multilinear L_∞ algebra $(\mathfrak{m}\mathfrak{g}, Q)$. Likewise for $\mathfrak{m}\mathfrak{g}'$, and $\Psi : (\mathfrak{m}\mathfrak{g}, Q) \to (\mathfrak{m}\mathfrak{g}', Q')$ is a C-multilinear L_∞ morphism.

The fact that $\mathfrak{m}$ is nilpotent is essential for the next definition.

Definition 3.13. The *Maurer–Cartan equation* in $(\mathfrak{mg}, Q)$ is

$$\sum_{i=1}^{\infty} \frac{1}{i!} (\partial^i Q)(\omega^i) = 0$$

for $\omega \in (\mathfrak{mg})^1 = (\mathfrak{mg}[1])^0$.

An element $e \in \mathrm{S}_C(\mathfrak{g}[1])$ is called *group-like* if $\Delta_C(e) = e \otimes e$. For $\omega \in \mathfrak{mg}^1$ we define

$$\exp(\omega) := \sum_{i \geq 0} \frac{1}{i!} \omega^i \in \mathrm{S}_C(\mathfrak{g}[1]).$$

Lemma 3.14. *The function* exp *is a bijection from* $\mathfrak{mg}[1]$ *to the set of invertible group-like elements* $e \in \mathrm{S}_C(\mathfrak{g}[1])$ *such that* $\ln(e) \in \mathfrak{mg}[1]$. *The inverse of* exp *is* ln.

Proof. Let $\omega \in \mathfrak{mg}[1]$ and $e := \exp(\omega)$. The element e is invertible, with inverse $\exp(-\omega)$. Using the fact that $\Delta_C(\omega) = \omega \otimes 1 + 1 \otimes \omega$ it easily follows that $\Delta_C(e) = e \otimes e$. And trivially $\ln(e) = \omega$.

For the opposite direction, let e be invertible and group-like, and assume $\ln(e) \in \mathfrak{mg}[1]$. Write it as $e = \sum_i \gamma_i$, with $\gamma_i \in \mathrm{S}^i_C(\mathfrak{g}[1])$. The equation $\Delta_C(e) = e \otimes e$ implies that

$$\Delta_C(\gamma_i) = \sum_{j+k=i} \gamma_j \otimes \gamma_k$$

for all i. Hence

$$2^i \gamma_i = \mu(\Delta_C(\gamma_i)) = \sum_{j+k=i} \gamma_j \gamma_k. \tag{3.15}$$

For $i = 0$ we get $\gamma_0 = \gamma_0^2$, and since γ_0 is invertible, it follows that $\gamma_0 = 1$. Let $\omega := \gamma_1 = \ln(e) \in \mathfrak{m}\mathrm{S}^1_C(\mathfrak{g}[1]) = \mathfrak{mg}[1]$. Using induction and Eq. (3.15) we see that $\gamma_i = \frac{1}{i!}\omega^i$ for all i. Thus $e = \exp(\omega)$. □

Lemma 3.16. *Let* $\omega \in (\mathfrak{mg}[1])^0 = \mathfrak{mg}^1$ *and* $e := \exp(\omega)$. *Then* ω *is a solution of the MC equation iff* $Q(e) = 0$.

Proof. Since e is group-like and invertible (by Lemma 3.14) we have

$$\Delta_C(Q(e)) = Q(e) \otimes e + e \otimes Q(e)$$

and

$$\Delta_C(e^{-1}Q(e)) = \Delta_C(e)^{-1}\Delta_C(Q(e)) = e^{-1}Q(e) \otimes 1 + 1 \otimes e^{-1}Q(e).$$

So the element $e^{-1}Q(e)$ is primitive, and by Lemma 3.12 we get $e^{-1}Q(e) \in \mathfrak{g}[1]$. On the other hand hence $Q(e)$ has no 0-order term, and $Q(1) = 0$. Thus in the first order term we get

$$
\begin{aligned}
e^{-1}Q(e) &= \ln\left(e^{-1}Q(e)\right) \\
&= \ln\left(\left(1-\omega+\frac{1}{2}\omega^2 \pm \cdots\right)Q(e)\right) \\
&= \ln\left(Q(e)\right) \\
&= \sum_{i=0}^{\infty} \frac{1}{i!}\ln\left(Q(\omega^i)\right) \\
&= \sum_{i=1}^{\infty} \frac{1}{i!}(\partial^i Q)(\omega^i).
\end{aligned} \tag{3.17}
$$

Since e is invertible we are done. □

Lemma 3.18. *Given an element $\omega \in \mathfrak{m}\mathfrak{g}[1]$, define $\omega' := \sum_{i=1}^{\infty} \frac{1}{i!}(\partial^i \Psi)(\omega^i) \in \mathfrak{m}\mathfrak{g}'[1]$, $e := \exp(\omega)$ and $e' := \exp(\omega')$. Then $e' = \Psi(e)$.*

Proof. From Lemma 3.14 we see that $\Delta_C(e) = e \otimes e$, and therefore also $\Delta_C(\Psi(e)) = \Psi(e) \otimes \Psi(e) \in \mathrm{S}_C(\mathfrak{g}'[1])$. Since Ψ is C-linear and $\Psi(1) = 1$ we get $\Psi(e) \in 1 + \mathfrak{m}\mathrm{S}(\mathfrak{g}'[1])$. Thus $\Psi(e)$ is group-like and invertible. According to Lemma 3.14 it suffices to prove that $\ln(e') = \ln(\Psi(e))$. Now $\ln(e') = \omega'$ by definition. Since $\Psi(1) = 1$ and $\ln(1) = 0$ it follows that

$$
\ln(\Psi(e)) = \ln\left(\Psi\left(\sum_{i=0}^{\infty}\frac{1}{i!}\omega^i\right)\right) = \sum_{i=0}^{\infty}\frac{1}{i!}\ln(\Psi(\omega^i)) = \sum_{i=1}^{\infty}\frac{1}{i!}(\partial^i \Psi)(\omega^i) = \omega'. \quad \square
$$

Proposition 3.19. *Suppose $\omega \in \mathfrak{m}\mathfrak{g}^1$ is a solution of the MC equation in $(\mathfrak{m}\mathfrak{g}, Q)$. Define $\omega' := \sum_{i=1}^{\infty} \frac{1}{i!}(\partial^i \Psi)(\omega^i) \in \mathfrak{m}\mathfrak{g}'^1$. Then ω' is a solution of the MC equation in $(\mathfrak{m}\mathfrak{g}', Q')$.*

Proof. Let $e := \exp(\omega)$ and $e' := \exp(\omega')$. By Lemma 3.16 we get $Q(e) = 0$. Hence $Q'(\Psi(e)) = \Psi(Q(e)) = 0$. According to Lemma 3.18 we have $\Psi(e) = e'$, so $Q'(e') = 0$. Again by Lemma 3.16 we deduce that ω' solves the MC equation. □

Definition 3.20. Let $\omega \in \mathfrak{m}\mathfrak{g}^1$.

(1) The colocal coderivation Q_ω of $\mathrm{S}_C(\mathfrak{g}[1])$ over C, with $Q_\omega(1) := 0$ and with Taylor coefficients

$$
(\partial^i Q_\omega)(\gamma) := \sum_{j \geq 0} \frac{1}{j!}(\partial^{i+j} Q)(\omega^j \gamma)
$$

for $i \geq 1$ and $\gamma \in \mathrm{S}^i_C(\mathfrak{g}[1])$, is called the *twist of Q by ω*.

(2) The colocal coalgebra homomorphism $\Psi_\omega : \mathrm{S}_C(\mathfrak{g}[1]) \to \mathrm{S}_C(\mathfrak{g}'[1])$ over C, with $\Psi_\omega(1) := 1$ and Taylor coefficients

$$
(\partial^i \Psi_\omega)(\gamma) := \sum_{j \geq 0} \frac{1}{j!}(\partial^{i+j} \Psi)(\omega^j \gamma)
$$

for $i \geq 1$ and $\gamma \in \mathrm{S}^i_C(\mathfrak{g}[1])$, is called the *twist of Ψ by ω*.

Theorem 3.21. *Let C be a commutative local $\mathbb{K}$-algebra with nilpotent maximal ideal $\mathfrak{m}$. Let $(\mathfrak{g}, Q)$ and $(\mathfrak{g}', Q')$ be C-multilinear L_∞ algebras and $\Psi : (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ a C-multilinear L_∞ morphism. Suppose $\omega \in \mathfrak{m}\mathfrak{g}^1$ a solution of the MC equation in $(\mathfrak{m}\mathfrak{g}, Q)$. Define*

$$\omega' := \sum_{i=1}^{\infty} \frac{1}{j!} (\partial^j \Psi)(\omega^j) \in \mathfrak{m}\mathfrak{g}'^1.$$

Then $(\mathfrak{g}, Q_\omega)$ and $(\mathfrak{g}', Q'_{\omega'})$ are L_∞ algebras, and

$$\Psi_\omega : (\mathfrak{g}, Q_\omega) \to (\mathfrak{g}', Q'_{\omega'})$$

is an L_∞ morphism.

Proof. Let $e := \exp(\omega)$. Define $\Phi_e : \mathrm{S}_C(\mathfrak{g}[1]) \to \mathrm{S}_C(\mathfrak{g}[1])$ to be $\Phi_e(\gamma) := e\gamma$. Since e is group-like and invertible it follows that Φ_e is a coalgebra automorphism. Therefore $\tilde{Q}_\omega := \Phi_e^{-1} \circ Q \circ \Phi_e$ is a degree 1 colocal coderivation of $\mathrm{S}_C(\mathfrak{g}[1])$, satisfying $\tilde{Q}_\omega \circ \tilde{Q}_\omega = 0$ and $\tilde{Q}_\omega(1) = e^{-1} Q(e) = 0$; cf. Lemma 3.16. So $(\mathfrak{g}, \tilde{Q}_\omega)$ is an L_∞ algebra. Likewise we have a coalgebra automorphism $\Phi_{e'}$ and a coderivation $\tilde{Q}'_{\omega'} := \Phi_{e'}^{-1} \circ Q' \circ \Phi_{e'}$ of $\mathrm{S}_C(\mathfrak{g}'[1])$, where $e' := \exp(\omega')$. The degree 0 colocal coalgebra homomorphism $\tilde{\Psi}_\omega := \Phi_{e'}^{-1} \circ \Psi \circ \Phi_e$ satisfies $\tilde{\Psi}_\omega \circ \tilde{Q}_\omega = \tilde{Q}'_{\omega'} \circ \tilde{\Psi}_\omega$, and also $\tilde{\Psi}_\omega(1) = e'^{-1} \Psi(e) = e'^{-1} e' = 1$, by Lemma 3.18. Hence we have an L_∞ morphism $\tilde{\Psi}_\omega : (\mathfrak{g}, \tilde{Q}_\omega) \to (\mathfrak{g}', \tilde{Q}'_{\omega'})$.

Let us calculate the Taylor coefficients of $\tilde{Q}_\omega$. For $\gamma \in \mathrm{S}^i_C(\mathfrak{g}[1])$ one has

$$(\partial^i \tilde{Q}_\omega)(\gamma) = \ln(\tilde{Q}_\omega(\gamma)) = \ln(e^{-1} Q(e\gamma)).$$

But just as in (3.17), since $Q(e\gamma)$ has no zero order term, we obtain

$$\ln(e^{-1} Q(e\gamma)) = \ln(Q(e\gamma)).$$

And

$$\begin{aligned} \ln(Q(e\gamma)) &= \ln\left(Q\left(\textstyle\sum_{j\geq 0} \frac{1}{j!}\omega^j \gamma\right)\right) \\ &= \sum_{j\geq 0} \frac{1}{j!} \ln(Q(\omega^j \gamma)) \\ &= \sum_{j\geq 0} \frac{1}{j!} (\partial^{i+j} Q)(\omega^j \gamma) \\ &= (\partial^i Q_\omega)(\gamma). \end{aligned} \tag{3.22}$$

Therefore $\tilde{Q}_\omega = Q_\omega$. Similarly we see that $\tilde{Q}'_{\omega'} = Q'_{\omega'}$ and $\tilde{\Psi}_\omega = \Psi_\omega$. □

Remark 3.23. The formulation of Theorem 3.21, as well as the idea for the proof, were suggested by Vladimir Hinich. An analogous result, for A_∞ algebras, is in [6, Section 6.1].

If $(\mathfrak{g}, Q)$ is a DG Lie algebra then the sum occurring in Definition 3.20(1) is finite, so the coderivation Q_ω can be defined without a nilpotence assumption on the coefficients.

Lemma 3.24. *Let* $(\mathfrak{g}, Q)$ *be a DG Lie algebra, and let* $\omega \in \mathfrak{g}^1$ *be a solution of the MC equation. Then the* L_∞ *algebra* $(\mathfrak{g}, Q_\omega)$ *is also a DG Lie algebra. In fact, for* $\gamma_i \in \mathfrak{g}$ *one has*

$$(\partial^1 Q_\omega)(\gamma_1) = (\partial^1 Q)(\gamma_1) + (\partial^2 Q)(\omega\gamma_1) = \mathrm{d}(\gamma_1) + [\omega, \gamma_1] = (\mathrm{d} + \mathrm{ad}(\omega))(\gamma_1),$$
$$(\partial^2 Q_\omega)(\gamma_1\gamma_2) = (\partial^2 Q)(\gamma_1\gamma_2) = [\gamma_1, \gamma_2],$$

and $\partial^j Q_\omega = 0$ *for* $j \geq 3$.

Proof. Like Eq. (3.22), with $C := \mathbb{K}$ and $e := 1$. □

In the situation of the lemma, the twisted DG Lie algebra $(\mathfrak{g}, Q_\omega)$ will usually be denoted by $\mathfrak{g}_\omega$.

Let A be a super-commutative associative unital DG algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$. The notion of DG A-module Lie algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$ was introduced in Definition 1.20.

Definition 3.25. Let A be a super-commutative associative unital DG algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$, let $\mathfrak{g}$ and $\mathfrak{g}'$ be DG A-module Lie algebras in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$, and let $\Psi : \mathfrak{g} \to \mathfrak{g}'$ be an L_∞ morphism.

(1) If each Taylor coefficient $\partial^j \Psi : \prod^j \mathfrak{g} \to \mathfrak{g}'$ is continuous then we say that Ψ is a *continuous* L_∞ *morphism*.
(2) Assume each Taylor coefficient $\partial^j \Psi : \prod^j \mathfrak{g} \to \mathfrak{g}'$ is A-multilinear, i.e.

$$(\partial^j \Psi)(a_1\gamma_1, \ldots, a_j\gamma_j) = \pm a_1 \cdots a_j \cdot (\partial^j \Psi)(\gamma_1, \ldots, \gamma_j)$$

for all homogeneous elements $a_k \in A$ and $\gamma_k \in \mathfrak{g}$, with sign according to the Koszul rule, then we say that Ψ is an *A-multilinear* L_∞ *morphism*.

Proposition 3.26. *Let A and B be super-commutative associative unital DG algebras in* $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$*, and let* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *be DG A-module Lie algebras in* $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$*. Suppose* $A \to B$ *is a continuous DG algebra homomorphism, and* $\Psi : \mathfrak{g} \to \mathfrak{g}'$ *is a continuous A-multilinear* L_∞ *morphism. Let* $\partial^j \Psi_{\widehat{B}} : \prod^j (B \mathbin{\widehat{\otimes}}_A \mathfrak{g}) \to B \mathbin{\widehat{\otimes}}_A \mathfrak{g}'$ *be the unique continuous* $\widehat{B}$*-multilinear homomorphism extending* $\partial^j \Psi$*. Then the degree* 0 *colocal coalgebra homomorphism*

$$\Psi_{\widehat{B}} : \mathrm{S}(B \mathbin{\widehat{\otimes}}_A \mathfrak{g}[1]) \to \mathrm{S}(B \mathbin{\widehat{\otimes}}_A \mathfrak{g}'[1]),$$

with $\Psi_{\widehat{B}}(1) := 1$ *and with Taylor coefficients* $\partial^j \Psi_{\widehat{B}}$*, is an* L_∞ *morphism*

$$\Psi_{\widehat{B}} : B \mathbin{\widehat{\otimes}}_A \mathfrak{g} \to B \mathbin{\widehat{\otimes}}_A \mathfrak{g}'.$$

Proof. First consider the continuous B-multilinear homomorphisms $\partial^j \Psi_B : \prod^j (B \otimes_A \mathfrak{g}) \to B \otimes_A \mathfrak{g}'$ extending $\partial^j \Psi$. It is a straightforward calculation to verify that the L_∞ morphism identities of Proposition 3.10 hold for the sequence of operators $\{\partial^j \Psi_B\}_{j \geq 1}$. The completion process respects these identities (cf. proof of Proposition 1.19). □

Theorem 3.27. *Let* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *be DG Lie algebras in* $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$*, and let* $\Psi : \mathfrak{g} \to \mathfrak{g}'$ *be a continuous* L_∞ *morphism. Let* $A = \bigoplus_{i \in \mathbb{N}} A^i$ *be a complete associative unital super-commutative DG algebra in* $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$*. By Proposition* 3.26 *there is an induced continuous A-multilinear* L_∞ *morphism* $\Psi_A : A \mathbin{\widehat{\otimes}} \mathfrak{g} \to A \mathbin{\widehat{\otimes}} \mathfrak{g}'$*. Let* $\omega \in A^1 \mathbin{\widehat{\otimes}} \mathfrak{g}^0$ *be a* *solution of the MC equation in* $A \mathbin{\widehat{\otimes}} \mathfrak{g}$. *Assume* $\mathrm{d}_{\mathfrak{g}} = 0$, $(\partial^j \Psi_A)(\omega^j) = 0$ *for all* $j \geq 2$, *and also that* $\mathfrak{g}'$ *is bounded below. Define* $\omega' := (\partial^1 \Psi_A)(\omega) \in A^1 \mathbin{\widehat{\otimes}} \mathfrak{g}'^0$. *Then:*

(1) *The element* ω' *is a solution of the MC equation in* $A \mathbin{\widehat{\otimes}} \mathfrak{g}'$.

(2) *Given* $c \in \mathrm{S}^j \left(A \mathbin{\widehat{\otimes}} \mathfrak{g}[1]\right)$ *there exists a natural number* k_0 *such that* $(\partial^{j+k} \Psi_A)(\omega^k c) = 0$ *for all* $k > k_0$.

(3) *The degree* 0 *colocal coalgebra homomorphism*

$$\Psi_{A,\omega} : \mathrm{S}\left(A \mathbin{\widehat{\otimes}} \mathfrak{g}[1]\right) \to \mathrm{S}\left(A \mathbin{\widehat{\otimes}} \mathfrak{g}'[1]\right),$$

with $\Psi_{A,\omega}(1) := 1$ *and Taylor coefficients*

$$(\partial^j \Psi_{A,\omega})(c) := \sum_{k \geq 0} \frac{1}{(j+k)!} (\partial^{j+k} \Psi_A)(\omega^k c)$$

for $c \in \mathrm{S}^j \left(A \mathbin{\widehat{\otimes}} \mathfrak{g}[1]\right)$, *is a continuous* A*-multilinear* L_∞ *morphism*

$$\Psi_{A,\omega} : \left(A \mathbin{\widehat{\otimes}} \mathfrak{g}\right)_\omega \to \left(A \mathbin{\widehat{\otimes}} \mathfrak{g}'\right)_{\omega'}.$$

Proof. We shall use a "deformation argument". Consider the base field $\mathbb{K}$ as a discrete inv $\mathbb{K}$-module. The polynomial algebra $\mathbb{K}[\hbar]$ is endowed with the dir-inv $\mathbb{K}$-module structure such that the homomorphism $\bigoplus_{i \in \mathbb{N}} \mathbb{K} \to \mathbb{K}[\hbar]$, whose i-th component is multiplication by $\hbar^i$, is an isomorphism in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$. Note that $\mathbb{K}[\hbar]$ is a discrete dir-inv module, but it is not trivial. We view $\mathbb{K}[\hbar]$ as a DG algebra concentrated in degree 0 (with zero differential).

For any $i \in \mathbb{N}$ let $A[\hbar]^i := \mathbb{K}[\hbar] \otimes A^i$, and let $A[\hbar] := \bigoplus_{i \in \mathbb{N}} A[\hbar]^i$, which is a DG algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$, with differential $\mathrm{d}_{A[\hbar]} := \mathbf{1} \otimes \mathrm{d}_A$. We will need a "twisted" version of $A[\hbar]$, which we denote by $A[\hbar]^\sim$. Let $A[\hbar]^{\sim i} := \hbar^i A[\hbar]^i$, and define $A[\hbar]^\sim := \bigoplus_{i \in \mathbb{N}} A[\hbar]^{\sim i}$, which has a graded subalgebra of $A[\hbar]$. The differential is $\mathrm{d}_{A[\hbar]^\sim} := \hbar \mathrm{d}_{A[\hbar]}$. The dir-inv structure is such that the homomorphism $\bigoplus_{i,j \in \mathbb{N}} A^i \to A[\hbar]^\sim$, whose (i, j)-th component is multiplication by $\hbar^{i+j}$, is an isomorphism in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$. The specialization $\hbar \mapsto 1$ is a continuous DG algebra homomorphism $A[\hbar]^\sim \to A$. There is an induced continuous $A[\hbar]^\sim$-multilinear L_∞ morphism $\Psi_{A[\hbar]^\sim} : A[\hbar]^\sim \mathbin{\widehat{\otimes}} \mathfrak{g} \to A[\hbar]^\sim \mathbin{\widehat{\otimes}} \mathfrak{g}'$.

We proceed in several steps.

Step 1. Say r_0 bounds $\mathfrak{g}'$ from below, i.e. $\mathfrak{g}'^r = 0$ for all $r < r_0$. Take some $j \geq 1$. For any $l \in \{1, \ldots, j\}$ choose $p_l, q_l \in \mathbb{Z}$, $\gamma_l \in \mathfrak{g}^{p_l}$ and $a_l \in A[\hbar]^{\sim q_l}$. Also choose $\gamma_0 \in \mathfrak{g}^0$ and $a_0 \in A[\hbar]^{\sim 1}$. Let $p := \sum_{l=1}^j p_l$ and $q := \sum_{l=1}^j q_l$. Because $\partial^{j+k} \Psi_{A[\hbar]^\sim}$ is induced from $\partial^{j+k} \Psi$, and this is a homogeneous map of degree $1 - j - k$, we have

$$\begin{aligned}(\partial^{j+k} \Psi_{A[\hbar]^\sim}) &\Big((a_0 \otimes \gamma_0)^k (a_1 \otimes \gamma_1) \cdots (a_j \otimes \gamma_j)\Big) \\ &= \pm a_0^k a_1 \cdots a_j \otimes (\partial^{j+k} \Psi)(\gamma_0^k \gamma_1 \cdots \gamma_j) \in A[\hbar]^{\sim k+q} \mathbin{\widehat{\otimes}} \mathfrak{g}^{p+1-j-k}.\end{aligned}$$

But $\mathfrak{g}^{p+1-j-k} = 0$ for all $k > p + 1 - j - r_0$.

Using multilinearity and continuity we conclude that given any $c \in \mathrm{S}^j \left(A[\hbar]^\sim \mathbin{\widehat{\otimes}} \mathfrak{g}[1]\right)$ there exists a natural number k_0 such that $(\partial^{j+k} \Psi_{A[\hbar]^\sim}) \left((\hbar\omega)^k c\right) = 0$ for all $k > k_0$.

Step 2. We are going to prove that $\hbar\omega$ is a solution of the MC equation in $A[\hbar]^\sim \mathbin{\widehat{\otimes}} \mathfrak{g}$. It is given that ω is a solution of the MC equation in $A \mathbin{\widehat{\otimes}} \mathfrak{g}$. Because $\mathrm{d}_{\mathfrak{g}} = 0$, this means that

$$(\mathrm{d}_A \otimes \mathbf{1})(\omega) + \frac{1}{2}[\omega, \omega] = 0.$$

Hence

$$\mathrm{d}_{A[\hbar]^{\sim}\,\widehat{\otimes}\,\mathfrak{g}}(\hbar\omega) + \frac{1}{2}[\hbar\omega, \hbar\omega] = \hbar^2(\mathrm{d}_A \otimes \mathbf{1})(\omega) + \frac{1}{2}\hbar^2[\omega, \omega] = 0.$$

So $\hbar\omega$ solves the MC equation in $A[\hbar]^{\sim} \widehat{\otimes} \mathfrak{g}$.

Step 3. Now we shall prove that $\hbar\omega'$ solves the MC equation in $A[\hbar]^{\sim} \widehat{\otimes} \mathfrak{g}'$. This will require an infinitesimal argument. For any natural number m define $\mathbb{K}[\hbar]_m := \mathbb{K}[\hbar]/(\hbar^{m+1})$ and $A[\hbar]_m := \mathbb{K}[\hbar]_m \otimes A$. The latter is a DG algebra with differential $\mathrm{d}_{A[\hbar]_m} := \mathbf{1} \otimes \mathrm{d}_A$. Let $A[\hbar]_m^{\sim} := \bigoplus_{i=0}^m \hbar^i A[\hbar]_m^i$, which is a subalgebra of $A[\hbar]_m$, but its differential is $\mathrm{d}_{A[\hbar]_m^{\sim}} := \hbar\mathrm{d}_{A[\hbar]_m}$. There is a surjective DG Lie algebra homomorphism $A[\hbar]^{\sim} \widehat{\otimes} \mathfrak{g}' \to A[\hbar]_m^{\sim} \widehat{\otimes} \mathfrak{g}'$, with kernel $\big(A[\hbar]^{\sim} \cap \hbar^{m+1} A[\hbar]\big) \widehat{\otimes} \mathfrak{g}'$. Since $\bigcap_{m \geq 0} \hbar^{m+1} A[\hbar] = 0$, it suffices to prove that $\hbar\omega'$ solves the MC equation in $A[\hbar]_m^{\sim} \widehat{\otimes} \mathfrak{g}'$.

Now $C := \mathbb{K}[\hbar]_m$ is an artinian local ring with maximal ideal $\mathfrak{m} := (\hbar)$. Define the DG Lie algebra $\mathfrak{h} := A[\hbar]_m \widehat{\otimes} \mathfrak{g}$, with differential $\mathrm{d}_{\mathfrak{h}} := \hbar\mathrm{d}_{A[\hbar]_m} \otimes \mathbf{1} + \mathbf{1} \otimes \mathrm{d}_{\mathfrak{g}}$; so $A[\hbar]_m^{\sim} \widehat{\otimes} \mathfrak{g} \subset \mathfrak{h}$ as DG Lie algebras. Similarly define $\mathfrak{h}'$. There is a C-multilinear L_∞ morphism $\Phi : \mathfrak{h} \to \mathfrak{h}'$ extending $\Psi_{A[\hbar]_m^{\sim}} : A[\hbar]_m^{\sim} \widehat{\otimes} \mathfrak{g} \to A[\hbar]_m^{\sim} \widehat{\otimes} \mathfrak{g}'$. By step 2 the element $\nu := \hbar\omega \in \mathfrak{m}\mathfrak{h}$ is a solution of the MC equation. According to Proposition 3.19 the element $\nu' := \sum_{k \geq 1} (\partial^k \Phi)(\nu^k)$ is a solution of the MC equation in $\mathfrak{h}'$. But $\nu' = \hbar\omega'$.

Step 4. Pick a natural number m. Let $\mathfrak{h}, \mathfrak{h}', \Phi, \nu$ and ν' be as in step 3. According to Theorem 3.21 there is a twisted L_∞ morphism $\Phi_\nu : \mathfrak{h}_\nu \to \mathfrak{h}'_{\nu'}$. Since $(A[\hbar]_m^{\sim} \widehat{\otimes} \mathfrak{g})_{\hbar\omega} \subset \mathfrak{h}_\nu$ and $(A[\hbar]_m^{\sim} \widehat{\otimes} \mathfrak{g}')_{\hbar\omega'} \subset \mathfrak{h}'_{\nu'}$ as DG Lie algebras, and Φ_ν extends $\Psi_{A[\hbar]_m^{\sim},\hbar\omega}$, it follows that $\Psi_{A[\hbar]_m^{\sim},\hbar\omega} : A[\hbar]_m^{\sim} \widehat{\otimes} \mathfrak{g} \to A[\hbar]_m^{\sim} \widehat{\otimes} \mathfrak{g}'$ is an L_∞ morphism. This means that the Taylor coefficients

$$\partial^j \Psi_{A[\hbar]_m^{\sim},\hbar\omega} : \prod\nolimits^j (A[\hbar]_m^{\sim} \widehat{\otimes} \mathfrak{g})_{\hbar\omega} \to (A[\hbar]_m^{\sim} \widehat{\otimes} \mathfrak{g}')_{\hbar\omega'}$$

satisfy the identities of Proposition 3.10. As explained in step 3, this implies that

$$\partial^j \Psi_{A[\hbar]^{\sim},\hbar\omega} : \prod\nolimits^j (A[\hbar]^{\sim} \widehat{\otimes} \mathfrak{g})_{\hbar\omega} \to (A[\hbar]^{\sim} \widehat{\otimes} \mathfrak{g}')_{\hbar\omega'}$$

also satisfy these identities. We conclude that $\Psi_{A[\hbar]^{\sim},\hbar\omega}$ is an L_∞ morphism.

Step 5. Specialization $\hbar \mapsto 1$ induces surjective DG Lie algebra homomorphisms $A[\hbar]^{\sim} \widehat{\otimes} \mathfrak{g} \to A \widehat{\otimes} \mathfrak{g}$ and $A[\hbar]^{\sim} \widehat{\otimes} \mathfrak{g}' \to A \widehat{\otimes} \mathfrak{g}'$, sending $\hbar\omega \mapsto \omega$, $\hbar\omega' \mapsto \omega'$ and $\Psi_{A[\hbar]^{\sim},\hbar\omega} \mapsto \Psi_{A,\omega}$. Therefore assertions (1–3) of the theorem hold. □

4. The universal L_∞ morphism of Kontsevich

In this section $\mathbb{K}$ is a field of characteristic 0 and C is a commutative $\mathbb{K}$-algebra. Recall that we denote by $\mathcal{T}_C = \mathcal{T}(C/\mathbb{K}) := \mathrm{Der}_{\mathbb{K}}(C)$, the module of derivations of C relative to $\mathbb{K}$. This is a Lie algebra over $\mathbb{K}$. Following [5] we make the next definitions.

Definition 4.1. For $p \geq -1$ let

$$\mathcal{T}^p_{\mathrm{poly}}(C) := \bigwedge\nolimits_C^{p+1} \mathcal{T}_C,$$

the module of *poly derivations* (or *poly tangents*) of degree p of C relative to $\mathbb{K}$. Let

$$\mathcal{T}_{\mathrm{poly}}(C) := \bigoplus_{p} \mathcal{T}^{p}_{\mathrm{poly}}(C).$$

This is a DG Lie algebra, with zero differential, and with the Schouten–Nijenhuis bracket, which is determined by the formulas

$$[\alpha_1 \wedge \alpha_2, \alpha_3] = \alpha_1 \wedge [\alpha_2, \alpha_3] + (-1)^{(p_2+1)p_3}[\alpha_1, \alpha_3] \wedge \alpha_2$$

and

$$[\alpha_1, \alpha_2] = (-1)^{1+p_1 p_2}[\alpha_2, \alpha_1]$$

for elements $\alpha_i \in \mathcal{T}^{p_i}_{\mathrm{poly}}(C)$.

Definition 4.2. For any $p \geq -1$ let $\mathcal{D}^{p}_{\mathrm{poly}}(C)$ be the set of $\mathbb{K}$-multilinear multi differential operators $\phi : C^{p+1} \to C$ (see Definition 2.1). The direct sum

$$\mathcal{D}_{\mathrm{poly}}(C) := \bigoplus_{p} \mathcal{D}^{p}_{\mathrm{poly}}(C)$$

is a DG Lie algebra. The differential $\mathrm{d}_{\mathcal{D}}$ is the shifted Hochschild differential, and the Lie bracket is the Gerstenhaber bracket (see [5, Section 3.4.2]). The elements of $\mathcal{D}_{\mathrm{poly}}(C)$ are called *poly differential operators* relative to $\mathbb{K}$.

In the notation of Section 2 and Example 1.24 one has

$$\mathcal{D}^{p}_{\mathrm{poly}}(C) = \mathcal{D}\mathit{iff}_{\mathrm{poly}}(C; \underbrace{C, \ldots, C}_{p+1}; C) = \mathcal{C}^{p+1}_{\mathrm{cd}}(C);$$

see formula (2.3).

Observe that $\mathcal{D}^{p}_{\mathrm{poly}}(C) \subset \mathrm{Hom}_{\mathbb{K}}(C^{\otimes(p+1)}, C)$, and $\mathcal{D}_{\mathrm{poly}}(C)$ is a sub DG Lie algebra of the shifted Hochschild cochain complex of C relative to $\mathbb{K}$. For $p = -1, 0$ we have $\mathcal{D}^{-1}_{\mathrm{poly}}(C) = C$ and $\mathcal{D}^{0}_{\mathrm{poly}}(C) = \mathcal{D}(C)$, the ring of differential operators. Note that $\mathcal{D}^{p}_{\mathrm{poly}}(C)$ is a left module over $\mathcal{D}(C)$, by the formula $D \cdot \phi := D \circ \phi$; and in this way it is also a left C-module.

When $C := \mathbb{K}[\boldsymbol{t}] = \mathbb{K}[t_1, \ldots, t_n]$, the polynomial algebra in $n \geq 1$ variables, and $p \geq 1$, the following is true. The $\mathbb{K}[\boldsymbol{t}]$-module $\mathcal{T}^{p-1}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$ is free with finite basis $\{\frac{\partial}{\partial t_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial t_{i_p}}\}$, indexed by the sequences $0 \leq i_1 < \cdots < i_p \leq n$. The $\mathbb{K}[\boldsymbol{t}]$-module $\mathcal{D}^{p-1}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$ is also free, with countable basis

$$\left\{ \left(\frac{\partial}{\partial \boldsymbol{t}}\right)^{\boldsymbol{j}_1} \otimes \cdots \otimes \left(\frac{\partial}{\partial \boldsymbol{t}}\right)^{\boldsymbol{j}_p} \right\}_{\boldsymbol{j}_1, \ldots, \boldsymbol{j}_p \in \mathbb{N}^n}, \tag{4.3}$$

where for $\boldsymbol{j}_k = (j_{k,1}, \ldots, j_{k,n}) \in \mathbb{N}^n$ we write $(\frac{\partial}{\partial \boldsymbol{t}})^{\boldsymbol{j}_k} := (\frac{\partial}{\partial t_1})^{j_{k,1}} \cdots (\frac{\partial}{\partial t_n})^{j_{k,n}}$.

For any $p \geq -1$ let $\mathrm{F}_m \mathcal{D}^{p}_{\mathrm{poly}}(C)$ be the set of poly differential operators of order $\leq m$ in each argument. This is a C-submodule of $\mathcal{D}^{p}_{\mathrm{poly}}(C)$.

Lemma 4.4. (1) *For any m, p one has*

$$\mathrm{d}_{\mathcal{D}}\left(\mathrm{F}_m \mathcal{D}^{p}_{\mathrm{poly}}(C)\right) \subset \mathrm{F}_m \mathcal{D}^{p+1}_{\mathrm{poly}}(C).$$

(2) *For any m, m', p, p' one has*

$$\left[\mathrm{F}_m \mathcal{D}^{p}_{\mathrm{poly}}(C), \mathrm{F}_{m'} \mathcal{D}^{p'}_{\mathrm{poly}}(C)\right] \subset \mathrm{F}_{m+m'} \mathcal{D}^{p+p'}_{\mathrm{poly}}(C);$$

and

$$[-,-] : \mathrm{F}_m \mathcal{D}^{p}_{\mathrm{poly}}(C) \times \mathrm{F}_{m'} \mathcal{D}^{p'}_{\mathrm{poly}}(C) \to \mathcal{D}^{p+p'}_{\mathrm{poly}}(C)$$

is a poly differential operator of order $\leq m + m'$ in each of its two arguments.

Proof. These assertions follow easily from the definitions of the Hochschild differential and the Gerstenhaber bracket; cf. [5, Section 3.4.2]. □

Lemma 4.5. *Assume C is a finitely generated $\mathbb{K}$-algebra. Then $\mathcal{T}^{p}_{\mathrm{poly}}(C)$ and $\mathrm{F}_m \mathcal{D}^{p}_{\mathrm{poly}}(C)$ are finitely generated C-modules.*

Proof. One has

$$\mathcal{T}^{p}_{\mathrm{poly}}(C) \cong \mathrm{Hom}_A(\Omega^{p+1}_C, A)$$

and

$$\mathrm{F}_m \mathcal{D}^{p}_{\mathrm{poly}}(C) \cong \mathrm{Hom}_C\left(\mathcal{C}_{p+1,m}(C), C\right);$$

see Lemma 2.2. The C-modules Ω^{p+1}_C and $\mathcal{C}_{p+1,m}(C)$ are finitely generated. □

Proposition 4.6. *Assume C is a finitely generated $\mathbb{K}$-algebra, and C' is a noetherian, $\mathfrak{c}'$-adically complete, flat, $\mathfrak{c}'$-adically formally étale C-algebra. Let us write $\mathcal{G}$ for either $\mathcal{T}_{\mathrm{poly}}$ or $\mathcal{D}_{\mathrm{poly}}$. Then:*

(1) *There is a DG Lie algebra homomorphism $\mathcal{G}(C) \to \mathcal{G}(C')$, which is functorial in $C \to C'$.*
(2) *The induced C'-linear homomorphism $C' \otimes_C \mathcal{G}^p(C) \to \mathcal{G}^p(C')$ is bijective.*
(3) *For any m the isomorphisms in* (2), *for $\mathcal{G} = \mathcal{D}_{\mathrm{poly}}$, restrict to isomorphisms*

$$C' \otimes_C \mathrm{F}_m \mathcal{D}^{p}_{\mathrm{poly}}(C) \xrightarrow{\simeq} \mathrm{F}_m \mathcal{D}^{p}_{\mathrm{poly}}(C').$$

Proof. Consider $\mathcal{G} = \mathcal{D}_{\mathrm{poly}}$. Let $\phi \in \mathcal{D}^{p}_{\mathrm{poly}}(C)$. According to Proposition 2.7, applied to the case $M_1, \ldots, M_{p+1}, N := A$, there is a unique $\phi' \in \mathcal{D}^{p}_{\mathrm{poly}}(C')$ extending ϕ. From the definitions of the Gerstenhaber bracket and the Hochschild differential, it immediately follows that the function $\mathcal{D}_{\mathrm{poly}}(C) \to \mathcal{D}_{\mathrm{poly}}(C')$, $\phi \mapsto \phi'$, is a DG Lie algebra homomorphism. Parts (2,3) are also consequences of Proposition 2.7.

The case $\mathcal{G} = \mathcal{T}_{\mathrm{poly}}$ is done similarly (and is well-known). □

Consider $C := \mathbb{K}[\boldsymbol{t}]$ and $C' := \mathbb{K}[[\boldsymbol{t}]] = \mathbb{K}[[t_1, \ldots, t_n]]$, the power series algebra. Since $\mathcal{T}^{p}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]) \cong \mathbb{K}[[\boldsymbol{t}]] \otimes_{\mathbb{K}[\boldsymbol{t}]} \mathcal{T}^{p}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$ is a finitely generated left $\mathbb{K}[[\boldsymbol{t}]]$-module, it is an inv $\mathbb{K}[[\boldsymbol{t}]]$-module with the $(\boldsymbol{t})$-adic inv structure; cf. Example 1.8. Likewise $\mathcal{D}^p_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ is a dir-inv $\mathbb{K}[[\boldsymbol{t}]]$-module. By Proposition 4.6,

$$\mathrm{F}_m \mathcal{D}^p_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]) \cong \mathbb{K}[[\boldsymbol{t}]] \otimes_{\mathbb{K}[\boldsymbol{t}]} \mathrm{F}_m \mathcal{D}^p_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}]),$$

which is a finitely generated $\mathbb{K}[[\boldsymbol{t}]]$-module. So according to Example 1.9 we may take $\{\mathrm{F}_m \mathcal{D}^p_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])\}_{m\in\mathbb{N}}$ as the dir-inv structure of $\mathcal{D}^p_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$. Now forgetting the $\mathbb{K}[[\boldsymbol{t}]]$-module structure, $\mathcal{T}^p_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ becomes an inv $\mathbb{K}$-module, and $\mathcal{D}^p_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ becomes a dir-inv $\mathbb{K}$-module.

Proposition 4.7. *Let $\mathcal{G}$ stand either for $\mathcal{T}_{\mathrm{poly}}$ or $\mathcal{D}_{\mathrm{poly}}$. Then $\mathcal{G}(\mathbb{K}[[\boldsymbol{t}]])$ is a complete DG Lie algebra in* $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$.

Proof. Use Proposition 2.4, and, for the case $\mathcal{G} = \mathcal{D}_{\mathrm{poly}}$, also Lemma 4.4. □

Remark 4.8. One might prefer to view $\mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ and $\mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ as topological DG Lie algebras. This can certainly be done: put on $\mathcal{T}^p_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ and $\mathrm{F}_m\mathcal{D}^p_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ the $\boldsymbol{t}$-adic topology, and put on $\mathcal{D}^p_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]) = \lim_{m\to} \mathrm{F}_m\mathcal{D}^p_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ the direct limit topology (see [8, Section 1.1]). However the dir-inv structure is better suited for our work.

Definition 4.9. For $p \geq 0$ let $\mathcal{D}^{\mathrm{nor},p}_{\mathrm{poly}}(C)$ be the submodule of $\mathcal{D}^p_{\mathrm{poly}}(C)$ consisting of poly differential operators ϕ such that $\phi(c_1, \ldots, c_{p+1}) = 0$ if $c_i = 1$ for some i. For $p = -1$ we let $\mathcal{D}^{\mathrm{nor},-1}_{\mathrm{poly}}(C) := C$. Define $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(C) := \bigoplus_{p\geq -1} \mathcal{D}^{\mathrm{nor},p}_{\mathrm{poly}}(C)$. We call $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(C)$ the algebra of *normalized poly differential operators*.

From the formulas for the Gerstenhaber bracket and the Hochschild differential (see [5, Section 3.4.2]) it immediately follows that $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(C)$ is a sub DG Lie algebra of $\mathcal{D}_{\mathrm{poly}}(C)$.

For any integer $p \geq 1$ there is a C-linear homomorphism

$$\mathcal{U}_1 : \mathcal{T}^{p-1}_{\mathrm{poly}}(C) \to \mathcal{D}^{\mathrm{nor},p-1}_{\mathrm{poly}}(C)$$

with formula

$$\mathcal{U}_1(\xi_1 \wedge \cdots \wedge \xi_p)(c_1, \ldots, c_p) := \frac{1}{p!} \sum_{\sigma\in\mathfrak{S}_p} \mathrm{sgn}(\sigma)\xi_{\sigma(1)}(c_1)\cdots\xi_{\sigma(p)}(c_p) \tag{4.10}$$

for elements $\xi_1, \ldots, \xi_p \in \mathcal{T}_C$ and $c_1, \ldots, c_p \in C$. For $p = 0$ the map $\mathcal{U}_1 : \mathcal{T}^{-1}_{\mathrm{poly}}(C) \to \mathcal{D}^{\mathrm{nor},-1}_{\mathrm{poly}}(C)$ is the identity (of C).

Suppose M and N are complexes in $\mathsf{Dir\,Inv\,Mod}\,C$ and $\phi, \phi' : M \to N$ are morphisms of complexes in $\mathsf{Dir\,Inv\,Mod}\,C$ (i.e. all maps are continuous for the dir-inv structures). We say ϕ and ϕ' are homotopic if there is a degree -1 homomorphism of graded dir-inv modules $\eta : M \to N$ such that $\mathrm{d}_N \circ \eta + \eta \circ \mathrm{d}_M = \phi - \phi'$. We say that $\phi : M \to N$ is a homotopy equivalence in $\mathsf{Dir\,Inv\,Mod}\,C$ if there is a morphism of complexes $\psi : N \to M$ in $\mathsf{Dir\,Inv\,Mod}\,C$ such that $\psi \circ \phi$ is homotopic to $\mathbf{1}_M$ and $\phi \circ \psi$ is homotopic to $\mathbf{1}_N$.

Theorem 4.11. *Let C be a commutative $\mathbb{K}$-algebra with ideal $\mathfrak{c}$. Assume C is noetherian and $\mathfrak{c}$-adically complete. Also assume there is a $\mathbb{K}$-algebra homomorphism*

$\mathbb{K}[t_1, \ldots, t_n] \to C$ *which is flat and* $\mathfrak{c}$*-adically formally étale. Then the homomorphism* $\mathcal{U}_1 : \mathcal{T}_{\mathrm{poly}}(C) \to \mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(C)$ *and the inclusion* $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(C) \to \mathcal{D}_{\mathrm{poly}}(C)$ *are both homotopy equivalences in* $\mathsf{Dir\,Inv\,Mod}\, C$.

Proof. Recall that $\mathcal{B}_q(C) = \mathcal{B}^{-q}(C) := C^{\otimes(q+2)}$, and this is a $\mathcal{B}_0(C)$-algebra via the extreme factors. So $\mathcal{B}_q(C) \cong \mathcal{B}_0(C) \otimes C^{\otimes q}$ as $\mathcal{B}_0(C)$-modules. Let $\overline{C} := C/\mathbb{K}$, the quotient $\mathbb{K}$-module, and define $\mathcal{B}^{\mathrm{nor}}_q(C) = \mathcal{B}^{\mathrm{nor},-q}(C) := \mathcal{B}_0(C) \otimes \overline{C}^{\otimes q}$, the q-th normalized bar module of C. According to MacLane [7, Section X.2], $\mathcal{B}^{\mathrm{nor}}(C) := \bigoplus_q \mathcal{B}^{\mathrm{nor},-q}(C)$ has a coboundary operator such that the obvious surjection $\phi : \mathcal{B}(C) \to \mathcal{B}^{\mathrm{nor}}(C)$ is a quasi-isomorphism of complexes of $\mathcal{B}^0(C)$-modules.

Define

$$\mathcal{C}^{\mathrm{nor}}_q(C) = \mathcal{C}^{\mathrm{nor},-q}(C) := C \otimes_{\mathcal{B}_0(C)} \mathcal{B}^{\mathrm{nor}}_q(C) \cong C \otimes \overline{C}^{\otimes q}.$$

Because the complexes $\mathcal{B}(C)$ and $\mathcal{B}^{\mathrm{nor}}(C)$ are bounded above and consist of free $\mathcal{B}_0(C)$-modules, it follows that $\phi : \mathcal{C}(C) \to \mathcal{C}^{\mathrm{nor}}(C)$ is a quasi-isomorphism of complexes of C-modules. Let $\widehat{\Omega}^q_C$ be the $\mathfrak{c}$-adic completion of Ω^q_C, so that $\widehat{\Omega}^q_C \cong C \otimes_{\mathbb{K}[\boldsymbol{t}]} \Omega^q_{\mathbb{K}[\boldsymbol{t}]}$. There is a C-linear homomorphism $\psi : \mathcal{C}^{\mathrm{nor}}_q(C) \to \Omega^q_C$ with formula

$$\psi(1 \otimes (c_1 \otimes \cdots \otimes c_q)) := \mathrm{d}(c_1) \wedge \cdots \wedge \mathrm{d}(c_q).$$

Consider the polynomial algebra $\mathbb{K}[\boldsymbol{t}] = \mathbb{K}[t_1, \ldots, t_n]$. For $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, q\}$ let

$$\tilde{\mathrm{d}}_j(t_i) := \underbrace{1 \otimes \cdots \otimes 1}_{j} \otimes (t_i \otimes 1 - 1 \otimes t_i) \otimes 1 \otimes \cdots \otimes 1 \in \mathcal{B}_q(\mathbb{K}[\boldsymbol{t}]),$$

and use the same expression to denote the image of this element in $\mathcal{C}_q(\mathbb{K}[\boldsymbol{t}])$. It is easy to verify that $\mathcal{C}_q(\mathbb{K}[\boldsymbol{t}])$ is a polynomial algebra over $\mathbb{K}[\boldsymbol{t}]$ in the set of generators $\{\tilde{\mathrm{d}}_j(t_i)\}$. Another easy calculation shows that $\mathrm{Ker}\left(\phi : \mathcal{C}_q(\mathbb{K}[\boldsymbol{t}]) \to \mathcal{C}^{\mathrm{nor}}_q(\mathbb{K}[\boldsymbol{t}])\right)$ is generated as $\mathbb{K}[\boldsymbol{t}]$-module by monomials in elements of the set $\{\tilde{\mathrm{d}}_j(t_i)\}$.

Let us introduce a grading on $\mathcal{C}_q(\mathbb{K}[\boldsymbol{t}])$ by $\deg(\tilde{\mathrm{d}}_j(t_i)) := 1$ and $\deg(t_i) := 0$. The coboundary operator of $\mathcal{C}(\mathbb{K}[\boldsymbol{t}])$ has degree 0 in this grading. The grading is inherited by $\mathcal{C}^{\mathrm{nor}}_q(\mathbb{K}[\boldsymbol{t}])$, and hence $\phi : \mathcal{C}(\mathbb{K}[\boldsymbol{t}]) \to \mathcal{C}^{\mathrm{nor}}(\mathbb{K}[\boldsymbol{t}])$ is a quasi-isomorphism of complexes in $\mathsf{GrMod}\,\mathbb{K}[\boldsymbol{t}]$, the category of graded $\mathbb{K}[\boldsymbol{t}]$-modules. Also let us put a grading on $\Omega^q_{\mathbb{K}[\boldsymbol{t}]}$ with $\deg(\mathrm{d}(t_i)) := 1$. By [8, Lemma 4.3], $\psi \circ \phi : \mathcal{C}(\mathbb{K}[\boldsymbol{t}]) \to \bigoplus_q \Omega^q_{\mathbb{K}[\boldsymbol{t}]}[q]$ is a quasi-isomorphism in $\mathsf{GrMod}\,\mathbb{K}[\boldsymbol{t}]$. Because we are dealing with bounded above complexes of free graded $\mathbb{K}[\boldsymbol{t}]$-modules it follows that both ϕ and ψ are homotopy equivalences in $\mathsf{GrMod}\,\mathbb{K}[\boldsymbol{t}]$.

Now let us go back to the formally étale homomorphism $\mathbb{K}[\boldsymbol{t}] \to C$. We get homotopy equivalences

$$C \otimes_{\mathbb{K}[\boldsymbol{t}]} \mathcal{C}(\mathbb{K}[\boldsymbol{t}]) \xrightarrow{\phi} C \otimes_{\mathbb{K}[\boldsymbol{t}]} \mathcal{C}^{\mathrm{nor}}(\mathbb{K}[\boldsymbol{t}]) \xrightarrow{\psi} \bigoplus_q \widehat{\Omega}^q_C[q]$$

in $\mathsf{GrMod}\, C$. We know that $\widehat{\mathcal{C}}_q(C)$ is a power series algebra in the set of generators $\{\tilde{\mathrm{d}}_j(t_i)\}$; see [8, Lemma 2.6]. Therefore $\widehat{\mathcal{C}}_q(C)$ is isomorphic to the completion of $C \otimes_{\mathbb{K}[\boldsymbol{t}]} \mathcal{C}_q(\mathbb{K}[\boldsymbol{t}])$

with respect to the grading (see Example 1.13). Define $\widehat{\mathcal{C}}_q^{\mathrm{nor}}(C)$ to be the completion of $C \otimes_{\mathbb{K}[\boldsymbol{t}]} \mathcal{C}_q^{\mathrm{nor}}(\mathbb{K}[\boldsymbol{t}])$ with respect to the grading. We then have a homotopy equivalence of complexes in $\mathsf{Inv}\,\mathsf{Mod}\,C$

$$\widehat{\mathcal{C}}(C) \to \widehat{\mathcal{C}}^{\mathrm{nor}}(C) \to \bigoplus_q \widehat{\Omega}_C^q[q].$$

Applying $\mathrm{Hom}_C^{\mathrm{cont}}(-, C)$ we arrive at quasi-isomorphisms

$$\bigoplus_q \left(\bigwedge\nolimits_C^q \mathcal{T}_C\right)[-q] \to \mathcal{C}_{\mathrm{cd}}^{\mathrm{nor}}(C) \to \mathcal{C}_{\mathrm{cd}}(C),$$

where by definition $\mathcal{C}_{\mathrm{cd}}^{\mathrm{nor}}(C)$ is the continuous dual of $\widehat{\mathcal{C}}^{\mathrm{nor}}(C)$. An easy calculation shows that $\mathcal{C}_{\mathrm{cd}}^{\mathrm{nor},q}(C) = \mathcal{D}_{\mathrm{poly}}^{\mathrm{nor},q-1}(C)$. □

One instance to which this theorem applies is $C := \mathbb{K}[[t_1, \ldots, t_n]]$. Here is another:

Corollary 4.12. *Suppose C is a smooth $\mathbb{K}$-algebra. Then the homomorphism $\mathcal{U}_1 : \mathcal{T}_{\mathrm{poly}}(C) \to \mathcal{D}_{\mathrm{poly}}^{\mathrm{nor}}(C)$ and the inclusion $\mathcal{D}_{\mathrm{poly}}^{\mathrm{nor}}(C) \to \mathcal{D}_{\mathrm{poly}}(C)$ are both quasi-isomorphisms.*

Proof. There is an open covering $\operatorname{Spec} C = \bigcup \operatorname{Spec} C_i$ such that for every i there is an étale homomorphism $\mathbb{K}[t_1, \ldots, t_n] \to C_i$. Now use Theorem 4.11, Proposition 2.7 and faithful flatness. □

Here is a slight variation of the celebrated result of Kontsevich, known as the *Formality Theorem* [5, Theorem 6.4].

Theorem 4.13. *Let $\mathbb{K}[\boldsymbol{t}] = \mathbb{K}[t_1, \ldots, t_n]$ be the polynomial algebra in n variables, and assume that $\mathbb{R} \subset \mathbb{K}$. There is a collection of $\mathbb{K}$-linear homomorphisms*

$$\mathcal{U}_j : \bigwedge\nolimits^j \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}]) \to \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}]),$$

indexed by $j \in \{1, 2, \ldots\}$, satisfying the following conditions.

(i) *The sequence $\mathcal{U} = \{\mathcal{U}_j\}$ is an L_∞-morphism $\mathcal{T}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}]) \to \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$.*
(ii) *Each $\mathcal{U}_j$ is a poly differential operator of $\mathbb{K}[\boldsymbol{t}]$-modules.*
(iii) *Each $\mathcal{U}_j$ is equivariant for the standard action of $\mathrm{GL}_n(\mathbb{K})$ on $\mathbb{K}[\boldsymbol{t}]$.*
(iv) *The homomorphism $\mathcal{U}_1$ is given by Eq. (4.10).*
(v) *For any $j \geq 2$ and $\alpha_1, \ldots, \alpha_j \in \mathcal{T}_{\mathrm{poly}}^0(\mathbb{K}[\boldsymbol{t}])$ one has $\mathcal{U}_j(\alpha_1 \wedge \cdots \wedge \alpha_j) = 0$.*
(vi) *For any $j \geq 2$, $\alpha_1 \in \mathfrak{gl}_n(\mathbb{K}) \subset \mathcal{T}_{\mathrm{poly}}^0(\mathbb{K}[\boldsymbol{t}])$ and $\alpha_2, \ldots, \alpha_j \in \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$ one has $\mathcal{U}_j(\alpha_1 \wedge \cdots \wedge \alpha_j) = 0$.*

Proof. First let us assume that $\mathbb{K} = \mathbb{R}$. Theorem 6.4 in [5] talks about the differentiable manifold $\mathbb{R}^n$, and considers C^∞ functions on it, rather than polynomial functions. However, by construction the operators $\mathcal{U}_j$ are multi differential operators with polynomial coefficients (see [5, Section 6.3]). Therefore they descend to operators

$$\mathcal{U}_j : \bigwedge\nolimits^j \mathcal{T}_{\mathrm{poly}}(\mathbb{R}[\boldsymbol{t}]) \to \mathcal{D}_{\mathrm{poly}}(\mathbb{R}[\boldsymbol{t}]),$$

and conditions (i) and (ii) hold. Conditions (iii), (v) and (vi) are properties P3, P4 and P5 respectively in [5, Section 7]. For condition (iv) see [5, Sections 4.6.1–2].

For a field extension $\mathbb{R} \subset \mathbb{K}$ use base change. □

Remark 4.14. It is likely that the operator $\mathcal{U}_j$ sends $\bigwedge^j \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$ into $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$. This is clear for $j = 1$, where $\mathcal{U}_1(\mathcal{T}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])) = \mathrm{F}_1\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$; but this requires checking for $j \geq 2$.

In the next theorem $\mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ and $\mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ are considered as DG Lie algebras in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$, with their $\boldsymbol{t}$-adic dir-inv structures. Recall the notions of twisted DG Lie algebra (Lemma 3.24) and multilinear extensions of L_∞ morphisms (Proposition 3.26).

Theorem 4.15. *Assume $\mathbb{R} \subset \mathbb{K}$. Let $A = \bigoplus_{i\geq 0} A^i$ be a complete super-commutative associative unital DG algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$. Consider the induced continuous A-multilinear L_∞ morphism*

$$\mathcal{U}_A : A \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]) \to A \mathbin{\widehat{\otimes}} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]).$$

Suppose $\omega \in A^1 \mathbin{\widehat{\otimes}} \mathcal{T}^0_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ is a solution of the Maurer–Cartan equation in $A \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$. Define $\omega' := (\partial^1\mathcal{U}_A)(\omega) \in A^1 \mathbin{\widehat{\otimes}} \mathcal{D}^0_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$. Then ω' is a solution of the Maurer–Cartan equation in $A \mathbin{\widehat{\otimes}} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$, and there is continuous A-multilinear L_∞ quasi-isomorphism

$$\mathcal{U}_{A,\omega} : \big(A \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])\big)_\omega \to \big(A \mathbin{\widehat{\otimes}} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])\big)_{\omega'}$$

whose Taylor coefficients are

$$(\partial^j\mathcal{U}_{A,\omega})(\alpha) := \sum_{k\geq 0} \frac{1}{(j+k)!}(\partial^{j+k}\mathcal{U}_A)(\omega^k \wedge \alpha)$$

for $\alpha \in \prod^j \big(A \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])\big)$.

Proof. By condition (ii) of Theorem 4.13, and by Proposition 2.4, each operator $\partial^j\mathcal{U} := \mathcal{U}_j$ is continuous for the $\boldsymbol{t}$-adic dir-inv structures on $\mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ and $\mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$. Therefore there is a unique continuous A-multilinear extension $\partial^j\mathcal{U}_A$. Condition (v) of Theorem 4.13 implies that $\partial^j\mathcal{U}_A(\omega^j) = 0$ for $j \geq 2$. By Theorem 3.27 we get an L_∞ morphism $\mathcal{U}_{A,\omega}$.

It remains to prove that $\partial^1\mathcal{U}_{A,\omega}$ is a quasi-isomorphism. According to Theorem 4.11 for every i the $\mathbb{K}$-linear homomorphism

$$\partial^1\mathcal{U}_A : A^i \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]) \to A^i \mathbin{\widehat{\otimes}} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$$

is a quasi-isomorphism. Since we are looking at bounded below complexes, a spectral sequence argument implies that

$$\partial^1\mathcal{U}_{A,\omega} : A \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]) \to A \mathbin{\widehat{\otimes}} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$$

is a quasi-isomorphism. □

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