Continuous and twisted $L_\infty$ morphisms

Amnon Yekutieli

Department of Mathematics, Ben Gurion University, Be’er Sheva 84105, Israel

Received 21 February 2005; received in revised form 8 September 2005
Available online 27 December 2005
Communicated by C.A. Weibel

Abstract

The purpose of this paper is to develop a suitable notion of continuous $L_\infty$ morphism between DG Lie algebras, and to study twists of such morphisms.

© 2005 Elsevier B.V. All rights reserved.

MSC: primary 53D55; secondary: 13D10, 13N10, 13J10

0. Introduction

Let $\mathbb{K}$ be a field containing $\mathbb{R}$. Consider two DG Lie algebras associated with the polynomial algebra $\mathbb{K}[t] := \mathbb{K}[t_1, \ldots, t_n]$. The first is the algebra of poly derivations $\mathcal{T}_{\text{poly}}(\mathbb{K}[t])$, and the second is the algebra of poly differential operators $\mathcal{D}_{\text{poly}}(\mathbb{K}[t])$. A very important result of Kontsevich [5], known as the Formality Theorem, gives an explicit formula for an $L_\infty$ quasi-isomorphism

$$\mathcal{U} : \mathcal{T}_{\text{poly}}(\mathbb{K}[t]) \rightarrow \mathcal{D}_{\text{poly}}(\mathbb{K}[t]).$$

Here is the main result of our paper.

Theorem 0.1. Assume $\mathbb{R} \subset \mathbb{K}$. Let $A = \bigoplus_{i \geq 0} A^i$ be a super-commutative associative unital complete DG algebra in $\text{Dir Inv Mod} \mathbb{K}$. Consider the induced continuous $A$-multilinear $L_\infty$ morphism

$$\mathcal{U}_A : A \otimes \mathcal{T}_{\text{poly}}(\mathbb{K}[t]) \rightarrow A \otimes \mathcal{D}_{\text{poly}}(\mathbb{K}[t]).$$

E-mail address: amyekut@math.bgu.ac.il.
Suppose \( \omega \in A^1 \otimes \mathcal{T}_{\text{poly}}^0(\mathbb{K}[[t]]) \) is a solution of the Maurer–Cartan equation in \( A \otimes \mathcal{T}_{\text{poly}}(\mathbb{K}[[t]]) \). Define \( \omega' := (\partial^1 U_A)(\omega) \in A^1 \otimes \mathcal{D}_{\text{poly}}^0(\mathbb{K}[[t]]) \). Then \( \omega' \) is a solution of the Maurer–Cartan equation in \( A \otimes \mathcal{D}_{\text{poly}}(\mathbb{K}[[t]]) \), and there is a continuous \( A \)-multilinear \( L_\infty \) quasi-isomorphism

\[
\mathcal{U}_{A,\omega} : (A \otimes \mathcal{T}_{\text{poly}}(\mathbb{K}[[t]]))_\omega \rightarrow (A \otimes \mathcal{D}_{\text{poly}}(\mathbb{K}[[t]]))_{\omega'},
\]

whose Taylor coefficients are

\[
(\partial^j \mathcal{U}_{A,\omega})(\alpha) := \sum_{k \geq 0} \frac{1}{(j+k)!} (\partial^{j+k} U_A)(\omega^k \wedge \alpha)
\]

for \( \alpha \in \prod^j (A \otimes \mathcal{T}_{\text{poly}}(\mathbb{K}[[t]])) \).

Below is an outline of the paper, in which we mention the various terms appearing in the theorem.

In Section 1 we develop the theory of dir-inv modules. A dir-inv structure on a \( \mathbb{K} \)-module \( M \) is a generalization of an adic topology. The category of dir-inv modules and continuous homomorphisms is denoted by \( \text{Dir Inv Mod} \). The concepts of dir-inv module, and related complete tensor product \( \hat{\otimes} \), are quite flexible, and are particularly well-suited for infinitely generated modules. Among other things we introduce the notion of DG Lie algebra in \( \text{Dir Inv Mod} \).

Section 2 concentrates on poly differential operators. The results here are mostly generalizations of material from [2].

In Section 3 we review the coalgebra approach to \( L_\infty \) morphisms. The notions of continuous, \( A \)-multilinear and twisted \( L_\infty \) morphisms are defined. The main result of this section is Theorem 3.27.

In Section 4 we recall the Kontsevich Formality Theorem. By combining it with Theorem 3.27 we deduce Theorem 0.1 (repeated as Theorem 4.15). In Theorem 0.1 the DG Lie algebras \( A \hat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[t]]) \) and \( A \hat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[[t]]) \) are the \( A \)-multilinear extensions of \( \mathcal{T}_{\text{poly}}(\mathbb{K}[[t]]) \) and \( \mathcal{D}_{\text{poly}}(\mathbb{K}[[t]]) \) respectively, and \( (A \hat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[t]]))_\omega \) and \( (A \hat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[[t]]))_{\omega'} \) are their twists. The \( L_\infty \) morphism \( \mathcal{U}_A \) is the continuous \( A \)-multilinear extension of \( \mathcal{U} \), and \( \mathcal{U}_{A,\omega} \) is its twist.

Theorem 0.1 is used in [9], in which we study deformation quantization of algebraic varieties.

1. Dir-inv modules

We begin the paper with a generalization of the notion of adic topology. In this section \( \mathbb{K} \) is a commutative base ring, and \( C \) is a commutative \( \mathbb{K} \)-algebra. The category \( \text{Mod} \ C \) is abelian and has direct and inverse limits. Unless specified otherwise, all limits are taken in \( \text{Mod} \ C \).

**Definition 1.1.** (1) Let \( M \in \text{Mod} \ C \). An *inv module structure* on \( M \) is an inverse system \( \{F^i M\}_{i \in \mathbb{N}} \) of \( C \)-submodules of \( M \). The pair \( (M, \{F^i M\}_{i \in \mathbb{N}}) \) is called an *inv \( C \)-module*.

(2) Let \( (M, \{F^i M\}_{i \in \mathbb{N}}) \) and \( (N, \{F^i N\}_{i \in \mathbb{N}}) \) be two inv \( C \)-modules. A function \( \phi : M \rightarrow N \) (\( C \)-linear or not) is said to be *continuous* if for every \( i \in \mathbb{N} \) there exists \( i' \in \mathbb{N} \) such that \( \phi(F^i M) \subset F^{i'} N \).
(3) Define $\text{Inv Mod } C$ to be the category whose objects are the inv $C$-modules, and whose morphisms are the continuous $C$-linear homomorphisms.

We do not assume that the canonical homomorphism $M \to \lim_{\to j} M/F_j M$ is surjective nor injective. There is a full embedding $\text{Mod } C \hookrightarrow \text{Inv Mod } C$, $M \mapsto (M, \{\ldots, 0, 0\})$. If $(M, \{F^j M\}_{j \in \mathbb{N}})$ and $(N, \{F^j N\}_{j \in \mathbb{N}})$ are two inv $C$-modules then $M \oplus N$ is an inv module, with inverse system of submodules $F^j (M \oplus N) := F^j M \oplus F^j N$. Thus $\text{Inv Mod } C$ is a $C$-linear additive category.

Let $(M, \{F^j M\}_{j \in \mathbb{N}})$ be an inv $C$-module, let $M', M''$ be two $C$-modules, and suppose $\phi : M' \to M$ and $\psi : M \to M''$ are $C$-linear homomorphisms. We get induced inv module structures on $M'$ and $M''$ by defining $F^j M' := \phi^{-1}(F^j M)$ and $F^j M'' := \psi(F^j M)$.

Recall that a directed set is a partially ordered set $J$ with the property that for any $j_1, j_2 \in J$ there exists $j_3 \in J$ such that $j_1, j_2 \leq j_3$.

**Definition 1.2.** (1) Let $M \in \text{Mod } C$. A dir-inv module structure on $M$ is a direct system $\{F_j M\}_{j \in J}$ of $C$-submodules of $M$, indexed by a nonempty directed set $J$, together with an inv module structure on each $F_j M$, such that for every $j_1 \leq j_2$ the inclusion $F_j M \hookrightarrow F_{j_2} M$ is continuous. The pair $(M, \{F_j M\}_{j \in J})$ is called a dir-inv $C$-module.

(2) Let $(M, \{F_j M\}_{j \in J})$ and $(N, \{F_k N\}_{k \in K})$ be two dir-inv $C$-modules. A function $\phi : M \to N$ ($C$-linear or not) is said to be continuous if for every $j \in J$ there exists $k \in K$ such that $\phi(F_j M) \subseteq F_k N$, and $\phi : F_j M \to F_k N$ is a continuous function between these two inv $C$-modules.

(3) Define $\text{Dir Inv Mod } C$ to be the category whose objects are the dir-inv $C$-modules, and whose morphisms are the continuous $C$-linear homomorphisms.

There is no requirement that the canonical homomorphism $\lim_{\to j} F_j M \to M$ will be surjective. An inv $C$-module $M$ is endowed with the dir-inv module structure $\{F_j M\}_{j \in J}$, where $J := \{0\}$ and $F_0 M := M$. Thus we get a full embedding $\text{Inv Mod } C \hookrightarrow \text{Dir Inv Mod } C$. Given two dir-inv $C$-modules $(M, \{F_j M\})_{j \in J}$ and $(N, \{F_k N\})_{k \in K}$, we make $M \oplus N$ into a dir-inv module as follows. The directed set is $J \times K$, with the component-wise partial order, and the direct system of inv modules is $F_{(j,k)} (M \oplus N) := F_j M \oplus F_k N$. The condition $J \neq \emptyset$ in part (1) of the definition ensures that the zero module $0 \in \text{Mod } C$ is an initial object in $\text{Dir Inv Mod } C$. So $\text{Dir Inv Mod } C$ is a $C$-linear additive category.

Let $(M, \{F_j M\}_{j \in J})$ be a dir-inv $C$-module, let $M', M''$ be two $C$-modules, and suppose $\phi : M' \to M$ and $\psi : M \to M''$ are $C$-linear homomorphisms. We get induced dir-inv module structures $\{F_j M'\}_{j \in J}$ and $\{F_j M''\}_{j \in J}$ on $M'$ and $M''$ as follows. Define $F_j (M') := \phi^{-1}(F_j M)$ and $F_j M'' := \psi(F_j M)$, which have induced inv module structures via the homomorphisms $\phi : F_j M' \to F_j M$ and $\psi : F_j M \to F_j M''$.

**Definition 1.3.** (1) An inv $C$-module $(M, \{F^i M\}_{i \in \mathbb{N}})$ is called discrete if $F^i M = 0$ for $i \gg 0$.

(2) An inv $C$-module $(M, \{F^i M\}_{i \in \mathbb{N}})$ is called complete if the canonical homomorphism $M \to \lim_{\to i} M/F^i M$ is bijective.

(3) A dir-inv $C$-module $M$ is called complete (resp. discrete) if it isomorphic, in $\text{Dir Inv Mod } C$, to a dir-inv module $(N, \{F_j N\}_{j \in J})$, where all the inv modules $F_j N$
are complete (resp. discrete) as defined above, and the canonical homomorphism \( \lim_{j \to} F_j N \to N \) is bijective.

(4) A dir-inv \( C \)-module \( M \) is called trivial if it is isomorphic, in \( \text{Dir Inv Mod } C \), to an object of \( \text{Mod } C \), via the embedding \( \text{Mod } C \hookrightarrow \text{Dir Inv Mod } C \).

Note that \( M \) is a trivial dir-inv module iff it is isomorphic, in \( \text{Dir Inv Mod } C \), to a discrete inv module. There do exist discrete dir-inv modules that are not trivial dir-inv modules; see Example 1.10. It is easy to see that if \( M \) is a discrete dir-inv module then it is also complete.

The base ring \( \mathbb{K} \) is endowed with the inv structure \( \{ \ldots, 0, 0 \} \), so it is a trivial dir-inv \( \mathbb{K} \)-module. But the \( \mathbb{K} \)-algebra \( C \) could have more interesting dir-inv structures (cf. Example 1.8).

If \( f^* : C \to C' \) is a homomorphism of \( \mathbb{K} \)-algebras, then there is a functor \( f_* : \text{Dir Inv Mod } C' \to \text{Dir Inv Mod } C \). In particular any dir-inv \( C \)-module is a dir-inv \( \mathbb{K} \)-module.

**Definition 1.4.** (1) Given an inv \( C \)-module \( (M, \{F_j M\}_{j \in \mathbb{N}}) \) its completion is the inv \( C \)-module \( (\hat{M}, \{\hat{F}_j M\}_{j \in \mathbb{N}}) \), defined as follows: \( \hat{M} := \lim_{j \to} M / F_j M \) and \( \hat{F}_j M := \text{Ker}(\hat{M} \to M / F_j M) \). Thus we obtain an additive endofunctor \( M \mapsto \hat{M} \) of \( \text{Inv Mod } C \).

(2) Given a dir-inv \( C \)-module \( (M, \{F_j M\}_{j \in J}) \) its completion is the dir-inv \( C \)-module \( (\hat{M}, \{\hat{F}_j M\}_{j \in J}) \) defined as follows. For any \( j \in J \) let \( \hat{F}_j M \) be the completion of the inv \( C \)-module \( F_j M \), as defined above. Then let \( \hat{M} := \lim_{j \to} \hat{F}_j M \) and \( \hat{F}_j \hat{M} := \text{Im}(\hat{F}_j M \to \hat{M}) \). Thus we obtain an additive endofunctor \( M \mapsto \hat{M} \) of \( \text{Dir Inv Mod } C \).

An inv \( C \)-module \( M \) is complete iff the functorial homomorphism \( M \to \hat{M} \) is an isomorphism; and of course \( \hat{M} \) is complete. For a dir-inv \( C \)-module \( M \) there is in general no functorial homomorphism between \( M \) and \( \hat{M} \), and we do not know if \( \hat{M} \) is complete. Nonetheless:

**Proposition 1.5.** Suppose \( M \in \text{Dir Inv Mod } C \) is complete. Then there is an isomorphism \( M \cong \hat{M} \) in \( \text{Dir Inv Mod } C \). This isomorphism is functorial.

**Proof.** For any dir-inv module \( (M, \{F_j M\}_{j \in J}) \) let us define \( M' := \lim_{j \to} F_j M \). So \( (M', \{F_j M\}_{j \in J}) \) is a dir-inv module, and there are functorial morphisms \( M' \to M \) and \( M' \to \hat{M} \). If \( M \) is complete then both these morphisms are isomorphisms. \( \square \)

Suppose \( \{M_k\}_{k \in K} \) is a collection of dir-inv modules, indexed by a set \( K \). There is an induced dir-inv module structure on \( M := \bigoplus_{k \in K} M_k \), constructed as follows. For any \( k \) let us denote by \( \{F_j M_k\}_{j \in J_k} \) the dir-inv structure of \( M_k \); so that each \( F_j M_k \) is an inv module. For each finite subset \( K_0 \subset K \) let \( J_{K_0} := \prod_{k \in K_0} J_k \), made into a directed set by component-wise partial order. Define \( J := \bigsqcup_{K_0} J_{K_0} \), where \( K_0 \) runs over the finite subsets of \( K \). For two finite subsets \( K_0 \subset K_1 \), and two elements \( j_0 = \{j_{0,k}\}_{k \in K_0} \in J_{K_0} \) and \( j_1 = \{j_{1,k}\}_{k \in K_1} \in J_{K_1} \), we declare that \( j_0 \leq j_1 \) if \( j_{0,k} \leq j_{1,k} \) for all \( k \in K_0 \). This makes \( J \) into a directed set. Now for any \( j = \{j_k\}_{k \in K_0} \in J_{K_0} \subset J \) let \( F_j M := \bigoplus_{k \in K_0} F_{j_k} M_k \), which is an inv module. The dir-inv structure on \( M \) is \( \{F_j M\}_{j \in J} \).

**Proposition 1.6.** Let \( \{M_k\}_{k \in K} \) be a collection of dir-inv \( C \)-modules, and let \( M := \bigoplus_{k \in K} M_k \), endowed with the induced dir-inv structure.
(1) \( M \) is a coproduct of \( \{M_k\}_{k \in K} \) in the category \( \text{Dir Inv Mod} \ C \).

(2) There is a functorial isomorphism \( \tilde{M} \cong \bigoplus_{k \in K} \tilde{M}_k \).

**Proof.** (1) is obvious. For (2) we note that both \( \tilde{M} \) and \( \bigoplus_{k \in K} \tilde{M}_k \) are direct limits for the direct system \( \{M_j\}_{j \in J} \). \( \square \)

Suppose \( \{M_k\}_{k \in \mathbb{N}} \) is a collection of inv \( C \)-modules. For each \( k \) let \( \{F^i M_k\}_{i \in \mathbb{N}} \) be the inv structure of \( M_k \). Then \( M := \prod_{k \in \mathbb{N}} M_k \) is an inv module, with inv structure \( F^i M := (\prod_{k \geq i} M_k) \times (\prod_{k < i} F^j M_k) \). Next let \( \{M_k\}_{k \in \mathbb{N}} \) be a collection of dir-inv \( C \)-modules, and for each \( k \) let \( \{F_j M_k\}_{j \in J_k} \) be the dir-inv structure of \( M_k \). Then there is an induced dir-inv structure on \( M := \prod_{k \in \mathbb{N}} M_k \). Define a directed set \( J := \prod_{k \in \mathbb{N}} J_k \), with component-wise partial order. For any \( j = \{j_k\}_{k \in \mathbb{N}} \in J \) define \( F_j M := \prod_{k \in \mathbb{N}} F_{j_k} M_k \), which is an inv \( C \)-module as explained above. The dir-inv structure on \( M \) is \( \{F_j M\}_{j \in J} \).

**Proposition 1.7.** Let \( \{M_k\}_{k \in \mathbb{N}} \) be a collection of dir-inv \( C \)-modules, and let \( M := \prod_{k \in \mathbb{N}} M_k \), endowed with the induced dir-inv structure. Then \( M \) is a product of \( \{M_k\}_{k \in \mathbb{N}} \) in \( \text{Dir Inv Mod} \ C \).

**Proof.** All we need to consider is continuity. First assume that all the \( M_k \) are inv \( C \)-modules. Let us denote by \( \pi_k : M \rightarrow M_k \) the projection. For each \( k, i \in \mathbb{N} \) and \( i' \geq \max(i, k) \) we have \( \pi_k(F^{i'} M) = F^{i'} M_k \). This shows that the \( \pi_k \) are continuous. Suppose \( L \) is an inv \( C \)-module and \( \phi_k : L \rightarrow M_k \) are morphisms in \( \text{Inv Mod} \ C \). For any \( i \in \mathbb{N} \) there exists \( i' \in \mathbb{N} \) such that \( \phi_k(F^{i'} L) \subset F^i M_k \) for all \( k \leq i \). Therefore the homomorphism \( \phi : L \rightarrow M \) with components \( \phi_k \) is continuous.

Now let \( M_k \) be dir-inv \( C \)-modules, with dir-inv structures \( \{F_j M_k\}_{j \in J_k} \). For any \( j = \{j_k\}_{k \in \mathbb{N}} \in J \) one has \( \pi_k(F_j M) = F_{j_k} M_k \), and as above \( \pi_k : F_j M \rightarrow F_{j_k} M_k \) is continuous. Given a dir-inv module \( L \) and morphisms \( \phi_k : L \rightarrow M_k \) in \( \text{Dir Inv Mod} C \), we have to prove that \( \phi : L \rightarrow M \) is continuous. Let \( \{F_j L\}_{j \in J_L} \) be the dir-inv structure of \( L \). Take any \( j \in J_L \). Since \( \phi_k \) is continuous, there exists some \( j_k \in J_k \) such that \( \phi_k(F_j L) \subset F_{j_k} M_k \). But then \( \phi(F_j L) \subset F_{j_k} M_k \) for \( j := \{j_k\}_{k \in \mathbb{N}} \), and by the previous paragraph \( \phi : F_j L \rightarrow F_{j_k} M_k \) is continuous. \( \square \)

The following examples should help to clarify the notion of dir-inv module.

**Example 1.8.** Let \( \mathfrak{c} \) be an ideal in \( C \). Then each finitely generated \( C \)-module \( M \) has an inv structure \( \{F^i M\}_{i \in \mathbb{N}} \), where we define the submodules \( F^i M := \mathfrak{c}^{i+1} M \). This is called the \( \mathfrak{c} \)-adic inv structure. Any \( C \)-module \( M \) has a dir-inv structure \( \{F_j M\}_{j \in J} \), which is the collection of finitely generated \( C \)-submodules of \( M \), directed by inclusion, and each \( F_j M \) is given the \( \mathfrak{c} \)-adic inv structure. We get a fully faithful functor \( \text{Mod} \ C \rightarrow \text{Dir Inv Mod} C \). This dir-inv module structure on \( M \) is called the \( \mathfrak{c} \)-adic dir-inv structure.

If \( C \) is noetherian and \( \mathfrak{c} \)-adically complete, then the finitely generated modules are complete as inv \( C \)-modules, and hence all modules are complete as dir-inv modules.

**Example 1.9.** Suppose \( \{M, \{F^i M\}_{i \in \mathbb{N}}\} \) is an inv \( C \)-module, and \( \{i_k\}_{k \in \mathbb{N}} \) is a nondecreasing sequence in \( \mathbb{N} \) with \( \lim_{k \rightarrow \infty} i_k = \infty \). Then \( \{F^{i_k} M\}_{k \in \mathbb{N}} \) is a new inv structure on \( M \), yet the identity map \( (M, \{F^i M\}_{i \in \mathbb{N}}) \rightarrow (M, \{F^{i_k} M\}_{k \in \mathbb{N}}) \) is an isomorphism in \( \text{Inv Mod} C \).

A similar modification can be done for dir-inv modules. Suppose \( \{M, \{F_j M\}_{j \in J}\} \) is a dir-inv \( C \)-module, and \( J' \subset J \) is a subset that is cofinal in \( J \). Then \( \{F_j M\}_{j \in J'} \) is a new
dir-inv structure on \( M \), yet the identity map \( (M, \{F_j M\}_{j \in J}) \to (M, \{F_j M\}_{j \in J}) \) is an isomorphism in \( \text{Dir Inv Mod } C \).

**Example 1.10.** Let \( M \) be the free \( \mathbb{K} \)-module with basis \( \{e_p\}_{p \in \mathbb{N}} \); so \( M = \bigoplus_{p \in \mathbb{N}} \mathbb{K} e_p \) in \( \text{Mod } \mathbb{K} \). We put on \( M \) the inv module structure \( \{F^i M\}_{i \in \mathbb{N}} \) with \( F^i M := 0 \) for all \( i \). Let \( N \) be the same \( \mathbb{K} \)-module as \( M \), but put on it the inv module structure \( \{F^i N\}_{i \in \mathbb{N}} \) with \( F^i N := \bigoplus_{p=i}^{\infty} \mathbb{K} e_p \). Also let \( L \) be the \( \mathbb{K} \)-module \( M \), but put on it the dir-inv module structure \( \{F_j L\}_{j \in \mathbb{N}} \), with \( F_j L := \bigoplus_{p=0}^{j} \mathbb{K} e_p \) the discrete inv module whose inv structure is \( \{\ldots, 0\} \). Both \( L \) and \( M \) are discrete and complete as dir-inv \( \mathbb{K} \)-modules, and \( \hat{N} \cong \prod_{p \in \mathbb{N}} \mathbb{K} e_p \). The dir-inv module \( M \) is trivial. \( L \) is not a trivial dir-inv \( \mathbb{K} \)-module, because it is not isomorphic in \( \text{Dir Inv Mod } \mathbb{K} \) to any inv module. The identity maps \( L \to M \to N \) are continuous. The only continuous \( \mathbb{K} \)-linear homomorphisms \( M \to L \) are those with finitely generated images.

**Remark 1.11.** In the situation of the previous example, suppose we put on the three modules \( L, M, N \) genuine \( \mathbb{K} \)-linear topologies, using the limiting processes and starting from the discrete topology. Namely \( M, N/F^2 N \) and \( F_j L \) get the discrete topologies; \( L \cong \lim_{\to} F_j L \) gets the \( \lim_{\to} \) topology; and \( N \subset \lim_{\leftarrow} N/F^2 N \) gets the \( \lim_{\leftarrow} \) topology (as in [8, Section 1.1]). Then \( L \) and \( M \) become the same discrete topological module, and \( \hat{N} \) is the topological completion of \( N \). We see that the notion of a dir-inv structure is more subtle than that of a topology, even though a similar language is used.

**Example 1.12.** Suppose \( \mathbb{K} \) is a field, and let \( M := \mathbb{K} \), the free module of rank 1. Up to isomorphism in \( \text{Dir Inv Mod } \mathbb{K} \), \( M \) has three distinct dir-inv module structures. We can denote them by \( M_1, M_2, M_3 \) in such a way that the identity maps \( M_1 \to M_2 \to M_3 \) are continuous. The only continuous \( \mathbb{K} \)-linear homomorphisms \( M_i \to M_j \) with \( i > j \) are the zero homomorphisms. \( M_2 \) is the trivial dir-inv structure, and it is the only interesting one (the others are “pathological”).

**Example 1.13.** Suppose \( M = \bigoplus_{p \in \mathbb{Z}} M^p \) is a graded \( C \)-module. The grading induces a dir-inv structure on \( M \), with \( J := \mathbb{N}, F_j M := \bigoplus_{p=-j}^{\infty} M^p \), and \( F^i F_j M := \bigoplus_{p=-j+i}^{\infty} M^p \). The completion satisfies \( \hat{M} \cong \left( \prod_{p \geq 0} M^p \right) \oplus \left( \bigoplus_{p < 0} M^p \right) \) in \( \text{Dir Inv Mod } C \), where each \( M^p \) has the trivial dir-inv module structure.

It makes sense to talk about convergence of sequences in a dir-inv module. Suppose \( (M, \{F^i M\}_{i \in \mathbb{N}}) \) is an inv \( C \)-module and \( \{m_i\}_{i \in \mathbb{N}} \) is a sequence in \( M \). We say that \( \lim_{i \to \infty} m_i = 0 \) if for every \( i_0 \) there is some \( i_1 \) such that \( \{m_i\}_{i \geq i_1} \subset F_{i_0} M \). If \( (M, \{F_j M\}_{j \in J}) \) is a dir-inv module and \( \{m_i\}_{i \in \mathbb{N}} \) is a sequence in \( M \), then we say that \( \lim_{i \to \infty} m_i = 0 \) if there exist some \( j \) and \( i_1 \) such that \( \{m_i\}_{i \geq i_1} \subset F_j M \), and \( \lim_{i \to \infty} m_i = 0 \) in the inv module \( F_j M \). Having defined \( \lim_{i \to \infty} m_i = 0 \), it is clear how to define \( \lim_{i \to \infty} m_i = m \) and \( \sum_{i=0}^{\infty} m_i = m \). Also the notion of Cauchy sequence is clear.

**Proposition 1.14.** Assume \( M \) is a complete dir-inv \( C \)-module. Then any Cauchy sequence in \( M \) has a unique limit.
Proof. Consider a Cauchy sequence \( \{m_i\}_{i \in \mathbb{N}} \) in \( M \). Convergence is an invariant of isomorphisms in \( \text{Dir Inv Mod} C \). By Definition 1.3 we may assume that in the dir-inv structure \( \{F_j M\}_{j \in J} \) of \( M \) each inv module \( F_j M \) is complete. By passing to the sequence \( \{m_i - m_{i_1}\}_{i \in \mathbb{N}} \) for suitable \( i_1 \), we can also assume the sequence is contained in one of the inv modules \( F_j M \). Thus we reduce to the case of convergence in a complete inv module, which is standard. □

Let \((M, \{F^j M\}_{j \in J})\) and \((N, \{F^j N\}_{j \in J})\) be two inv \( C \)-modules. We make \( M \otimes_C N \) into an inv module by defining

\[
F^j (M \otimes_C N) := \text{Im} \left( (M \otimes_C F^j N) \oplus (F^j M \otimes_C N) \to M \otimes_C N \right).
\]

For two dir-inv \( C \)-modules \((M, \{F_j M\}_{j \in J})\) and \((N, \{F_k N\}_{k \in K})\), we put on \( M \otimes_C N \) the dir-inv module structure \( \{F(j, k)(M \otimes_C N)\}_{(j, k) \in J \times K} \), where

\[
F(j, k)(M \otimes_C N) := \text{Im}(F_j M \otimes_C F_k N \to M \otimes_C N).
\]

Definition 1.15. Given \( M, N \in \text{Dir Inv Mod} C \) we define \( N \hat{\otimes}_C M \) to be the completion of the dir-inv \( C \)-module \( N \otimes_C M \).

Example 1.16. Let us examine the behavior of the dir-inv modules \( L, M, N \) from Example 1.10 with respect to the complete tensor product. There is an isomorphism \( L \otimes_K N \cong \bigoplus_{\rho \in \mathbb{N}} N \) in \( \text{Dir Inv Mod} \mathbb{K} \), so according to Proposition 1.6(2) there is also an isomorphism \( L \hat{\otimes}_K N \cong \bigoplus_{\rho \in \mathbb{N}} \hat{N} \) in \( \text{Dir Inv Mod} \mathbb{K} \). On the other hand \( M \otimes_K N \) is an inv \( \mathbb{K} \)-module with inv structure \( F^j (M \otimes_K N) = M \otimes_K F^j N \), so \( M \hat{\otimes}_K N \cong \prod_{\rho \in \mathbb{N}} M \) in \( \text{Dir Inv Mod} \mathbb{K} \). The series \( \sum_{p=0}^\infty e_p \otimes e_p \) converges in \( M \hat{\otimes}_K N \), but not in \( L \hat{\otimes}_K N \).

A graded object in \( \text{Dir Inv Mod} C \), or a graded dir-inv \( C \)-module, is an object \( M \in \text{Dir Inv Mod} C \) of the form \( M = \bigoplus_{i \in \mathbb{Z}} M^i \), with \( M^i \in \text{Dir Inv Mod} C \). According to Proposition 1.6 we have \( \hat{M} \cong \bigoplus_{i \in \mathbb{Z}} \hat{M}^i \). Given two graded objects \( M = \bigoplus_{i \in \mathbb{Z}} M^i \) and \( N = \bigoplus_{i \in \mathbb{Z}} N^i \) in \( \text{Dir Inv Mod} C \), the tensor product is also a graded object in \( \text{Dir Inv Mod} C \), with

\[
(M \otimes_C N)^i := \bigoplus_{p+q=i} M^p \otimes_C N^q.
\]

In this paper “algebra” is taken in the weakest possible sense: by \( C \)-algebra we mean a \( C \)-module \( A \) together with a \( C \)-bilinear function \( \mu_A : A \times A \to A \). If \( A \) is associative, or a Lie algebra, then we will specify that. However, “commutative algebra” will mean, by default, a commutative associative unital \( C \)-algebra. Another convention is that a homomorphism between unital algebras is a unital homomorphism, and a module over a unital algebra is a unital module.

Definition 1.17. (1) An algebra in \( \text{Dir Inv Mod} C \) is an object \( A \in \text{Dir Inv Mod} C \), together with a continuous \( C \)-bilinear function \( \mu_A : A \times A \to A \).

(2) A differential graded algebra in \( \text{Dir Inv Mod} C \) is a graded object \( A = \bigoplus_{i \in \mathbb{Z}} A^i \) in \( \text{Dir Inv Mod} C \), together with continuous \( C \)-bilinear functions \( \mu_A : A \times A \to A \) and
Let $d_A : A \to A$, such that $A$ is a differential graded algebra, in the usual sense, with respect to the differential $d_A$ and the multiplication $\mu_A$.

(3) Let $A$ be an algebra in $\text{Dir Inv Mod} C$, with dir-inv structure $\{F_j A\}_{j \in J}$. We say that $A$ is a unital algebra in $\text{Dir Inv Mod} C$ if it has a unit element $1_A$ (in the usual sense), such that $1_A \in \bigcup_{j \in J} F_j A$.

The base ring $K$, with its trivial dir-inv structure, is a unital algebra in $\text{Dir Inv Mod} K$. In item (3) above, the condition $1_A \in \bigcup_{j \in J} F_j A$ is equivalent to the ring homomorphism $K \to A$ being continuous.

We will use the common abbreviation “DG” for “differential graded”. An algebra in $\text{Dir Inv Mod} C$ can have further attributes, such as “Lie” or “associative”, which have their usual meanings. If $A \in \text{Inv Mod} C$ then we also say it is an algebra in $\text{Inv Mod} C$.

**Example 1.18.** In the situation of Example 1.8, the $c$-adic inv structure makes $C$ and $\hat{C}$ into unital algebras in $\text{Inv Mod} C$.

Recall that a graded algebra $A$ is called super-commutative if $ba = (-1)^{ij}ab$ and $c^2 = 0$ for all $a \in A^i, b \in A^j, c \in A^k$ and $k$ odd. There is no essential difference between left and right DG $A$-modules.

**Proposition 1.19.** Let $A$ and $g$ be DG algebras in $\text{Dir Inv Mod} C$.

1. The completion $\hat{A}$ is a DG algebra in $\text{Dir Inv Mod} C$.
2. If $A$ is complete, then the canonical isomorphism $A \cong \hat{A}$ of Proposition 1.5 is an isomorphism of DG algebras.
3. The complete tensor product $A \widehat{\otimes}_C g$ is a DG algebra in $\text{Dir Inv Mod} C$.
4. If $A$ is a super-commutative associative unital algebra, then so is $\hat{A}$.
5. If $g$ is a DG Lie algebra and $A$ is a super-commutative associative unital algebra, then $A \widehat{\otimes}_C g$ is a DG Lie algebra.

**Proof.** (1) This is a consequence of a slightly more general fact. Consider modules $M_1, \ldots, M_r, N \in \text{Dir Inv Mod} C$ and a continuous $C$-multilinear linear function $\phi : M_1 \times \cdots \times M_r \to N$. We claim that there is an induced continuous $C$-multilinear linear function $\hat{\phi} : \prod_k \hat{M}_k \to \hat{N}$. This operation is functorial (w.r.t. morphisms $M_k \to M'_k$ and $N \to N'$), and monoidal (i.e. it respects composition in the $k$th argument with a continuous multilinear function $\psi : L_1 \times \cdots \times L_s \to M_k$).

First assume $M_1, \ldots, M_r, N \in \text{Inv Mod} C$, with inv structures $\{F^i M_1\}_{i \in \mathbb{N}}$ etc. For any $i \in \mathbb{N}$ there exists $i' \in \mathbb{N}$ such that $\phi(\prod_k F^{i'} M_k) \subseteq F^i N$. Therefore there is an induced continuous $C$-multilinear function $\hat{\phi} : \prod_k \hat{M}_k \to \hat{N}$. It is easy to verify that $\phi \mapsto \hat{\phi}$ is functorial and monoidal.

Next consider the general case, i.e. $M_1, \ldots, M_r, N \in \text{Dir Inv Mod} C$. Let $\{F_j M_k\}_{j \in J_k}$ be the dir-inv structure of $M_k$, and let $\{F_j N\}_{j \in J_N}$ be the dir-inv structure of $N$. By continuity of $\phi$, given $(j_1, \ldots, j_r) \in \prod_k J_k$ there exists $j' \in J_N$ such that $\phi(\prod_k F_j M_k) \subseteq F_{j'} N$, and $\phi : \prod_k F_{j'} M_k \to F_{j'} N$ is continuous. By the previous paragraph this extends to $\hat{\phi} : \prod_k \hat{M}_k \to \hat{F}_{j'} N$. Passing to the direct limit in $(j_1, \ldots, j_r)$ we obtain $\hat{\phi} : \prod_k \hat{M}_k \to \hat{N}$. Again this operation is functorial and monoidal.
(2) Let \( A' \subset A \) be as in the proof of Proposition 1.5. This is a subalgebra. The arguments used in the proof of part (1) above show that \( A' \to A \) and \( A' \to \hat{A} \) are algebra homomorphisms.

(3) Let us write \( \cdot_A \) and \( \cdot_g \) for the two multiplications, and \( d_A \) and \( d_g \) for the differentials. Then \( A \otimes_C g \) is a DG algebra with multiplication
\[
(a_1 \otimes \gamma_1) \cdot (a_2 \otimes \gamma_2) := (-1)^{ij_1} (a_1 \cdot_A a_2) \otimes (\gamma_1 \cdot_g \gamma_2)
\]
and differential
\[
d(a_1 \otimes \gamma_1) := d_A(a_1) \otimes \gamma_1 + (-1)^{ij_1} a_1 \otimes d_g(\gamma_1)
\]
for \( a_k \in A^{ik} \) and \( \gamma_k \in g^{jh} \). These operations are continuous, so \( A \otimes_C g \) is a DG algebra in \( \text{Dir Inv Mod} C \). Now use part (1).

(4, 5) The various identities (Lie etc.) are preserved by \( \hat{\otimes} \). Definition 1.17(3) ensures that \( \hat{A} \) has a unit element. □

Definition 1.20. Suppose \( A \) is a DG super-commutative associative unital algebra in \( \text{Dir Inv Mod} C \).

(1) A DG \( A \)-module in \( \text{Dir Inv Mod} C \) is a graded object \( M \in \text{Dir Inv Mod} C \), together with continuous \( C \)-(bi)linear functions \( \mu_M : A \times M \to M \) and \( d_M : M \to M \), which make \( M \) into a DG \( A \)-module in the usual sense.

(2) A DG \( A \)-module Lie algebra in \( \text{Dir Inv Mod} C \) is a DG Lie algebra \( g \in \text{Dir Inv Mod} C \), together with a continuous \( C \)-bilinear homomorphism \( A \times g \to g \), such that \( g \) is a DG \( A \)-module, and
\[
[a_1 \gamma_1, a_2 \gamma_2] = (-1)^{ij_1} a_1 a_2 [\gamma_1, \gamma_2]
\]
for all \( a_k \in A^{ik} \) and \( \gamma_k \in g^{jh} \).

Example 1.21. If \( A \) is a DG super-commutative associative unital algebra in \( \text{Dir Inv Mod} C \), and \( g \) is a DG Lie algebra in \( \text{Dir Inv Mod} C \), then \( A \hat{\otimes}_C g \) is a DG \( \hat{A} \)-module Lie algebra in \( \text{Dir Inv Mod} C \).

Let \( A \) be a DG super-commutative associative unital algebra in \( \text{Dir Inv Mod} C \), and let \( M, N \) be two DG \( A \)-modules in \( \text{Dir Inv Mod} C \). The tensor product \( M \otimes_A N \) is a quotient of \( M \otimes_C N \), and as such it has a dir-inv structure. Moreover, \( M \otimes_A N \) is a DG \( A \)-module in \( \text{Dir Inv Mod} C \), and we define \( M \hat{\otimes}_A N \) to be its completion, which is a DG \( \hat{A} \)-module in \( \text{Dir Inv Mod} C \).

Proposition 1.22. Let \( A \) and \( B \) be DG super-commutative associative unital algebras in \( \text{Dir Inv Mod} C \), and let \( A \to B \) be a continuous homomorphism of DG \( C \)-algebras.

(1) Suppose \( M \) is a DG \( A \)-module in \( \text{Dir Inv Mod} C \). Then \( B \hat{\otimes}_A M \) is a DG \( \hat{B} \)-module in \( \text{Dir Inv Mod} C \).

(2) Suppose \( g \) is a DG \( A \)-module Lie algebra in \( \text{Dir Inv Mod} C \). Then \( B \hat{\otimes}_A g \) is a DG \( \hat{B} \)-module Lie algebra in \( \text{Dir Inv Mod} C \).

Proof. Like Proposition 1.19. □
Suppose $C, C'$ are commutative algebras in $\text{Dir Inv Mod } K$, and $f^* : C \to C'$ is a continuous $K$-algebra homomorphism. There are functors $f^* : \text{Dir Inv Mod } C \to \text{Dir Inv Mod } C'$ and $f^2 : \text{Dir Inv Mod } C \to \text{Dir Inv Mod } C'$, namely $f^* M := C' \otimes_C M$ and $f^2 M := C' \otimes C M$.

Let $M$ and $N$ be two dir-inv $C$-modules. We define

$$\text{Hom}_{\text{cont}}^C(M, N) := \text{Hom}_{\text{Dir Inv Mod } C}(M, N),$$

i.e. the $C$-module of continuous $C$-linear homomorphisms. In general this module has no obvious structure. However, if $M$ is an inv $C$-module with inv structure $\{F^i M\}_{i \in \mathbb{N}}$, and $N$ is a discrete inv $C$-module, then

$$\text{Hom}_{\text{cont}}^C(M, N) \cong \lim_{\to} \text{Hom}_C(M/F^i M, N).$$

In this case we consider each

$$F_i \text{Hom}_{\text{cont}}^C(M, N) := \text{Hom}_C(M/F^i M, N)$$

as a discrete inv module, and this endows $\text{Hom}_{\text{cont}}^C(M, N)$ with a dir-inv structure.

**Example 1.23.** In the situation of Example 1.10 one has

$$\text{Hom}_{\text{cont}}^C(N, M) \cong L \otimes_C M$$

as dir-inv $C$-modules.

**Example 1.24.** This example is taken from [8]. Assume $K$ is noetherian and $C$ is a finitely generated commutative $K$-algebra. For $q \in \mathbb{N}$ define $B_q(C) = B^{-q}(C) := C \otimes_K \cdots \otimes_K C$. Define $\hat{B}_q(C) = \hat{B}^{-q}(C)$ to be the adic completion of $B_q(C)$ with respect to the ideal $\text{Ker}(B_q(C) \to C)$.

There is a $K$-algebra homomorphism $\hat{B}^0(C) \to \hat{B}^{-q}(C)$, corresponding to the two extreme tensor factors, and in this way we view $\hat{B}^{-q}(C)$ as a complete inv $\hat{B}^0(C)$-module. There is a continuous coboundary operator that makes $\hat{B}(C) := \bigoplus_{q \in \mathbb{N}} \hat{B}^{-q}(C)$ into a complex of $\hat{B}^0(C)$-modules, and there is a quasi-isomorphism $\hat{B}(C) \to C$. We call $\hat{B}(C)$ the complete un-normalized bar complex of $C$.

Next define $\hat{C}_q(C) = \hat{C}^{-q}(C) := C \otimes_{\hat{B}^0(C)} \hat{B}^{-q}(C)$. This is a complete inv $C$-module. The complex $\hat{C}(C)$ is called the complete Hochschild chain complex of $C$. Finally let $C_{cd}^q(C) := \text{Hom}_{\text{cont}}^C(\hat{C}^{-q}(C), C)$. The complex $C_{cd}(C) := \bigoplus_{q \in \mathbb{N}} C_{cd}^q(C)$ is called the continuous Hochschild cochain complex of $C$.

2. Poly differential operators

In this section $K$ is a commutative base ring, and $C$ is a commutative $K$-algebra. The symbol $\otimes$ means $\otimes_K$. We discuss some basic properties of poly differential operators, expanding results from [9].

**Definition 2.1.** Let $M_1, \ldots, M_p, N$ be $C$-modules. A $K$-multilinear function $\phi : M_1 \times \cdots \times M_p \to N$ is called a poly differential operator (over $C$ relative to $K$) if there exists
some $d \in \mathbb{N}$ such that for any $(m_1, \ldots, m_p) \in \prod M_i$ and any $i \in \{1, \ldots, p\}$ the function $M_i \to N, m \mapsto \phi(m_1, \ldots, m_{i-1}, m, m_{i+1}, \ldots, m_p)$ is a differential operator of order $\leq d$, in the sense of [2, Section 16.8]. In this case we say that $\phi$ has order $\leq d$ in each argument.

We shall denote the set of poly differential operators $\prod M_i \to N$ over $C$ relative to $K$, of order $\leq d$ in all arguments, by

$$F_d \text{Diff}_{\text{poly}}(C; M_1, \ldots, M_p; N).$$

And we define

$$\text{Diff}_{\text{poly}}(C; M_1, \ldots, M_p; N) := \bigcup_{d \geq 0} F_d \text{Diff}_{\text{poly}}(C; M_1, \ldots, M_p; N),$$

the union being inside the set of all $K$-multilinear functions $\prod M_i \to N$. By default we only consider poly differential operators relative to $K$.

For a natural number $p$ the $p$-th un-normalized bar module $B_p(C)$ was defined in Example 1.24. Let $I_p(C)$ be the kernel of the ring homomorphism $B_p(C) \to C$. Define

$$C_p(C) := C \otimes B_0(C) B_p(C),$$

the $p$-th Hochschild chain module of $C$ (relative to $K$). For any $d \in \mathbb{N}$ define

$$B_{p,d}(C) := B_p(C) / I_p(C)^{d+1},$$

$$C_{p,d}(C) := C \otimes B_0(C) B_{p,d}(C)$$

and

$$C_{p,d}(C; M_1, \ldots, M_p) := C_{p,d}(C) \otimes B_{p-2}(C)(M_1 \otimes \cdots \otimes M_p).$$

Let

$$\phi_{\text{uni}} : \prod_{i=1}^p M_i \to C_{p,d}(C; M_1, \ldots, M_p)$$

be the $K$-multilinear function

$$\phi_{\text{uni}}(m_1, \ldots, m_p) := 1 \otimes (m_1 \otimes \cdots \otimes m_p).$$

Observe that for $p = 1$ we get $C_{1,d}(C) = \mathcal{P}^d(C)$, the module of principal parts of order $d$ (see [2]). In the same way that $\mathcal{P}^d(C)$ parametrizes differential operators, $C_{p,d}(C)$ parametrizes poly differential operators:

**Lemma 2.2.** The assignment $\psi \mapsto \psi \circ \phi_{\text{uni}}$ is a bijection

$$\text{Hom}_C(C_{p,d}(C; M_1, \ldots, M_p), N) \xrightarrow{\sim} F_d \text{Diff}_{\text{poly}}(C; M_1, \ldots, M_p; N).$$

**Proof.** The same arguments used in [2, Section 16.8] also apply here. Cf. [8, Section 1.4]. $\Box$
In case \( M_1 = \cdots = M_p = N = C \) we see that
\[
\mathcal{D}\text{iff}_{\text{poly}}(C; C, \ldots, C; C) \cong \lim_{\to} \text{Hom}_C(C_{p,d}(C), C)
\]
\[
\cong \text{Hom}_C^\text{cont}(\hat{C}_{p}(C), C) = C_{p,d}(C), \tag{2.3}
\]
with notation of Example 1.24.

**Proposition 2.4.** Suppose \( C \) is a finitely generated \( \mathbb{K} \)-algebra, with ideal \( \mathfrak{c} \subset C \). Let \( M_1, \ldots, M_p, N \) be \( C \)-modules, and let \( \phi : \prod M_i \to N \) be a multi differential operator over \( C \) relative to \( \mathbb{K} \). Then \( \phi \) is continuous for the \( \mathfrak{c} \)-adic dir-inv structures on \( M_1, \ldots, M_p, N \).

**Proof.** Suppose \( \phi \) has order \( \leq d \) in each of its arguments, and let
\[
\psi : C_{p,d}(C; M_1, \ldots, M_p) \to N
\]
be the corresponding \( C \)-linear homomorphism. As in \[8, \text{Proposition 1.4.3}\], since \( C \) is a finitely generated \( \mathbb{K} \)-algebra, it follows that \( B_{p,d}(C) \) is a finitely generated module over \( B_0(C) \); and hence \( C_{p,d}(C) \) is a finitely generated \( C \)-module. Let us denote by \( \{F_j M_i\}_{j \in J} \) and \( \{F_k N\}_{k \in K} \) the \( \mathfrak{c} \)-adic dir-inv structures on \( M_i \) and \( N \). For any \( j_1, \ldots, j_p \) the \( B_{p-2}(C) \)-module \( F_{j_1} M_{i_1} \otimes \cdots \otimes F_{j_p} M_{i_p} \) is finitely generated, and hence the \( C \)-module \( C_{p,d}(C; F_{j_1} M_{i_1}, \ldots, F_{j_p} M_{i_p}) \) is finitely generated. Therefore
\[
\psi(C_{p,d}(C; F_{j_1} M_{i_1}, \ldots, F_{j_p} M_{i_p})) = F_k N
\]
for some \( k \in K \).

It remains to prove that \( \phi : \prod_{j=1}^p F_j M_i \to F_k N \) is continuous for the \( \mathfrak{c} \)-adic inv structures. But just like \[8, \text{Proposition 1.4.6}\], for any \( i \) and \( l \) one has
\[
\phi(F_{j_1} M_{i_1}, \ldots, \mathfrak{c}^{i+l} F_{j_l} M_{i_l}, \ldots, F_{j_p} M_{i_p}) \subset \mathfrak{c}^l F_k N. \tag{2.5}
\]

Suppose \( C' \) is a commutative \( C \)-algebra with ideal \( \mathfrak{c}' \subset C' \). One says that \( C' \) is \( \mathfrak{c}' \)-adically formally étale over \( C \) if the following condition holds. Let \( D \) be a commutative \( C \)-algebra with nilpotent ideal \( \mathfrak{d} \), and let \( f : C' \to D/\mathfrak{d} \) be a \( C \)-algebra homomorphism such that \( f(\mathfrak{c}'^i) = 0 \) for \( i \gg 0 \). Then \( f \) lifts uniquely to a \( C \)-algebra homomorphism \( \tilde{f} : C' \to D \). The important instances are when \( C \to C' \) is étale (and then \( \mathfrak{c}' = 0 \)); and when \( C' \) is the \( \mathfrak{c} \)-adic completion of \( C \) for some ideal \( \mathfrak{c} \subset A \) (and \( \mathfrak{c}' = C'\mathfrak{c} \)). In both these instances \( C' \) is \( \mathfrak{c} \)-adically complete; and if \( C \) is noetherian, then \( C \to C' \) is also flat.

**Lemma 2.6.** Let \( C' \) be a \( \mathfrak{c}' \)-adically formally étale \( C \)-algebra. Define \( C'_j := C'/\mathfrak{c}^{j+1} \).

Consider \( C' \) and \( C_{p,d}(C) \) as inv \( C \)-modules, with the \( \mathfrak{c}' \)-adic and discrete inv structures respectively. Then the canonical homomorphism
\[
C' \otimes_C C_{p,d}(C) \to \lim_{\to} j C_{p,d}(C_j)
\]
is bijective.
Proof. Define ideals

\[ \mathcal{C}_p' := \text{Ker} \left( \mathcal{C}_p(C') \to \mathcal{C}_p(C'_{0}) \right) \]

and

\[ J := \text{Ker}(C'_j \otimes_C C_{p,d}(C) \to C'_j). \]

By the transitivity and the base change properties of formally étale homomorphisms, the ring homomorphism

\[ \mathcal{C}_p(C) \cong C \otimes \cdots \otimes C \to C' \otimes \cdots \otimes C' \cong \mathcal{C}_p(C') \]

is \( \mathcal{C}_p' \)-adically formally étale. Consider the commutative diagram of ring homomorphisms (with solid arrows)

The ideal \( J \) satisfies \( J^{d+1} = 0 \), and the ideal \( \text{Ker}(C'_{p,d}(C'_j) \to C'_j) \) is nilpotent too. Due to the unique lifting property the dashed arrows exist and are unique, making the whole diagram commutative. Moreover \( g : \mathcal{C}_p(C') \to \mathcal{C}_p(C'_j) \) has to be the canonical surjection, and \( \tilde{f} \) is surjective.

A little calculation shows that \( \tilde{f}(I_p(C')^{d+1}) = 0 \), and hence \( \tilde{f} \) induces a homomorphism

\[ \tilde{f} : \mathcal{C}_{p,d}(C') \to \mathcal{C}_j' \otimes_C \mathcal{C}_{p,d}(C). \]

Let

\[ \mathcal{C}_p' := \text{Ker} \left( \mathcal{C}_{p,d}(C') \to \mathcal{C}_{p,d}(C'_{0}) \right). \]

Another calculation shows that \( \tilde{f}(\mathcal{C}_{p,d}(j+1)^{(d+1)}) = 0 \). The conclusion is that there are surjections

\[ \mathcal{C}_{p,d}(C'_{j+d}) \xrightarrow{\tilde{f}} C'_j \otimes_C \mathcal{C}_{p,d}(C) \xrightarrow{e} \mathcal{C}_{p,d}(C'_j), \]

such that \( e \circ \tilde{f} \) is the canonical surjection. Passing to the inverse limit we deduce that

\[ C' \otimes_C \mathcal{C}_{p,d}(C) \to \varprojlim_j \mathcal{C}_{p,d}(C'_j) \]

is bijective. \( \square \)

Proposition 2.7. Assume \( C \) is a noetherian finitely generated \( \mathbb{K} \)-algebra, and \( C' \) is a noetherian, \( \varepsilon' \)-adically complete, flat, \( \varepsilon' \)-adically formally étale \( C \)-algebra. Let \( M_1, \ldots, M_p, N \) be \( C \)-modules, and define \( M'_i := C' \otimes_C M_i \) and \( N' := C' \otimes_C N \).
Lemma 2.6

(2.8)

we get

(2.5)

Proposition 2.4

Lemma 2.2

we have

Proposition 2.4

Suppose

The homomorphism

By

Now for any

This is because of formula

But

Combining this with (2.8) we get

But

The conclusion is that

(2.9)
Given $\phi : \prod M_i \to N$ of order $\leq d$, let $\phi' := 1 \otimes \phi$ under the isomorphism (2.9). Backtracking, we see that $\phi'$ is the unique poly differential operator extending $\phi$. □

3. $L_\infty$ morphisms and their twists

In this section we expand some results on $L_\infty$ algebras and morphisms from [5] Section 4. Much of the material presented here is based on discussions with Vladimir Hinich. There is some overlap with Section 2.2 of [3], with Section 6.1 of [6], and possibly with other accounts.

Let $K$ be a field of characteristic 0. Given a graded $K$-module $g = \bigoplus_{i \in \mathbb{Z}} g_i$ and a natural number $j$ let $T^j g := g \otimes \cdots \otimes g$ for $j$ times. The direct sum $T^g := \bigoplus_{j \in \mathbb{N}} T^j g$ is the tensor algebra. Let us denote the multiplication in $T^g$ by $\ast$. (This is just another way of writing $\otimes$, but it will be convenient to do so.)

The permutation group $S_j$ acts on $T^j g$ as follows. For any sequence of integers $d = (d_1, \ldots, d_j)$ there is a group homomorphism $\text{sgn}_d : S_j \to \{ \pm 1 \}$ such that on a transposition $\sigma = (p, p + 1)$ the value is $\text{sgn}_d(\sigma) = (-1)^{d_p d_{p+1}}$. The action of a permutation $\sigma \in S_j$ on $T^j g$ is then

$$\sigma(\gamma_1 \ast \cdots \ast \gamma_j) := \text{sgn}_d(\sigma) \gamma_{\sigma(1)} \ast \cdots \ast \gamma_{\sigma(j)}$$

for $\gamma_1 \in g^{d_1}, \ldots, \gamma_j \in g^{d_j}$. Define $\tilde{S}^j g$ to be the set of $S_j$-invariants inside $T^j g$, and $\tilde{S} g := \bigoplus_{j \geq 0} \tilde{S}^j g$.

The $K$-module $T^g$ is also a coalgebra, with coproduct $\tilde{\Delta} : T^g \to T^g \otimes T^g$ given by the formula

$$\tilde{\Delta}(\gamma_1 \ast \cdots \ast \gamma_j) := \sum_{p=0}^j (\gamma_1 \ast \cdots \ast \gamma_p) \otimes (\gamma_{p+1} \ast \cdots \ast \gamma_j).$$

The submodule $\tilde{S} g \subset T^g$ is a sub-coalgebra (but not a subalgebra!).

The super-symmetric algebra $S^g = \bigoplus_{j \geq 0} S^j g$ is defined to be the quotient of $T^g$ by the ideal generated by the elements $\gamma_1 \otimes \gamma_2 - (-1)^{d_1 d_2} \gamma_2 \otimes \gamma_1$, for all $\gamma_1 \in g^{d_1}$ and $\gamma_2 \in g^{d_2}$. In other words, $S^j g$ is the set of coinvariants of $T^j g$ under the action of the group $S_j$. The product in the algebra $S^g$ is denoted by $\cdot$. The canonical projection is $\pi : T^g \to S^g$ is an algebra homomorphism: $\pi(\gamma_1 \otimes \gamma_2) = \gamma_1 \cdot \gamma_2$.

In fact $S^g$ is a commutative cocommutative Hopf algebra. The comultiplication

$$\Delta : S^g \to S^g \otimes S^g$$

is the unique $K$-algebra homomorphism such that

$$\Delta(\gamma) = \gamma \otimes 1 + 1 \otimes \gamma$$

for all $\gamma \in g$. The antipode is $\gamma \mapsto -\gamma$. The projection $\pi : T^g \to S^g$ is not a coalgebra homomorphism. However:
Lemma 3.1. Let \( \tau : S_\mathfrak{g} \to T_\mathfrak{g} \) be the \( \mathbb{K} \)-module homomorphism defined by
\[
\tau(\gamma_1 \cdots \gamma_j) := \sum_{\sigma \in S_j} \text{sgn}(d_1,\ldots,d_j)(\sigma)\gamma_{\sigma(1)} \otimes \cdots \otimes \gamma_{\sigma(j)}
\]
for \( \gamma_1 \in \mathfrak{g}^{d_1}, \ldots, \gamma_j \in \mathfrak{g}^{d_j} \). Then \( \tau : S_\mathfrak{g} \to \tilde{S}_\mathfrak{g} \) is a coalgebra isomorphism, where \( S_\mathfrak{g} \) has the comultiplication \( \Delta \) and \( \tilde{S}_\mathfrak{g} \) has the comultiplication \( \tilde{\Delta} \).

**Proof.** Define \( \tilde{\tau} : T_\mathfrak{g} \to S_\mathfrak{g} \) to be the \( \mathbb{K} \)-module homomorphism
\[
\tilde{\tau}(\gamma_1 \cdots \gamma_j) := \frac{1}{j!} \tau(\gamma_1 \cdots \gamma_j) = \frac{1}{j!} \gamma_1 \cdots \gamma_j
\]
for \( \gamma_1, \ldots, \gamma_j \in \mathfrak{g} \). So \( \tilde{\tau} \circ \tau \) is the identity map of \( S_\mathfrak{g} \), and \( \tilde{\tau} : \tilde{S}_\mathfrak{g} \to S_\mathfrak{g} \) is bijective. It suffices to prove that
\[
(\tilde{\tau} \otimes \tilde{\tau}) \circ (\tau \otimes \tau) \circ \Delta = (\tilde{\tau} \otimes \tilde{\tau}) \circ \tilde{\Delta} \circ \tau.
\]
Take any \( \gamma_1 \in \mathfrak{g}^{d_1}, \ldots, \gamma_j \in \mathfrak{g}^{d_j} \) and write \( d := (d_1, \ldots, d_j) \). Then
\[
((\tilde{\tau} \otimes \tilde{\tau}) \circ \tilde{\Delta} \circ \tau)(\gamma_1 \cdots \gamma_j)
\]
\[
= \sum_{p=0}^{j} \sum_{\sigma \in S_p} \frac{1}{p!(j-p)!} \text{sgn}_d(\sigma)(\gamma_{\sigma(1)} \cdots \gamma_{\sigma(p)} \otimes (\gamma_{\sigma(p+1)} \cdots \gamma_{\sigma(j)}))
\]
On the other hand
\[
((\tilde{\tau} \otimes \tilde{\tau}) \circ (\tau \otimes \tau) \circ \Delta)(\gamma_1 \cdots \gamma_j)
\]
\[
= \Delta(\gamma_1 \cdots \gamma_j) = (1 \otimes \gamma_1 + \gamma_1 \otimes 1) \cdots (1 \otimes \gamma_j + \gamma_j \otimes 1)
\]
\[
\times \sum_{p=0}^{j} \sum_{\sigma \in S_{p,j-p}} \text{sgn}_d(\sigma)(\gamma_{\sigma(1)} \cdots \gamma_{\sigma(p)} \otimes (\gamma_{\sigma(p+1)} \cdots \gamma_{\sigma(j)})),
\]
where \( S_{p,j-p} \) is the set of \( (p, j-p) \)-shuffles inside the group \( S_j \). Since the algebra \( S_\mathfrak{g} \) is super-commutative the two sums are equal. \( \square \)

The grading on \( \mathfrak{g} \) induces a grading on \( S_\mathfrak{g} \), which we call the *degree*. Thus for \( \gamma_j \in \mathfrak{g}^{d_j} \) the degree of \( \gamma_1 \cdots \gamma_j \in S^j \mathfrak{g} \) is \( d_1 + \cdots + d_j \) (unless \( \gamma_1 \cdots \gamma_j = 0 \)). We consider \( S_\mathfrak{g} \) as a graded algebra for this grading. Actually there is another grading on \( S_\mathfrak{g} \), by *order*, where we define the order of \( \gamma_1 \cdots \gamma_j \) to be \( j \) (again, unless this element is zero). But this grading will have a different role.

By definition the \( j \)-th super-exterior power of \( \mathfrak{g} \) is
\[
\bigwedge^j \mathfrak{g} := S^j(\mathfrak{g}[1])[-j].
\]
(3.2)
where \( \mathfrak{g}[1] \) is the shifted graded module whose degree \( i \) component is \( \mathfrak{g}[1]^i = \mathfrak{g}^{i+1} \). When \( \mathfrak{g} \) is concentrated in degree 0 then these are the usual constructions of symmetric and exterior algebras, respectively.

We denote by \( \ln : S_\mathfrak{g} \to S^1 \mathfrak{g} = \mathfrak{g} \) the projection. So \( \ln(\gamma) \) is the first order term of \( \gamma \in S_\mathfrak{g} \). (The expression “\( \ln \)” might stand for “linear” or “logarithm”.)
Definition 3.3. Let $g$ and $g'$ be two graded $\mathbb{K}$-modules, and let $\Psi : S_{\mathbb{K}}g \to S_{\mathbb{K}}g'$ be a $\mathbb{K}$-linear homomorphism. For any $j \geq 1$ the $j$-th Taylor coefficient of $\Psi$ is defined to be

$$\partial^j \Psi := \ln \circ \Psi : S^j g \to g'.$$

We say $\Psi$ is colocal if $\Psi(S_{\mathbb{K}}^{\geq 1} g) \subset S_{\mathbb{K}}^{\geq 1} g'$ and $\Psi(S_{\mathbb{K}}^0 g) \subset S_{\mathbb{K}}^0 g'$.

Lemma 3.4. Suppose we are given a sequence of $\mathbb{K}$-linear homomorphisms $\psi_j : S^j g \to g'$, $j \geq 1$, each of degree 0. Then there is a unique colocal coalgebra homomorphism $\tilde{\psi} : \tilde{S}_g \to \tilde{S}_g'$, homogeneous of degree 0 and satisfying $\tilde{\psi}(1) = 1$, whose Taylor coefficients are $\partial^j \tilde{\psi} = \psi_j$.

Proof. Let $\tilde{S}_g' \to \tilde{S}_g g' = g'$ be the projection for this coalgebra. Consider the exact sequence of coalgebras

$$0 \to \mathbb{K} \to \tilde{S}_g \to \tilde{S}_{\mathbb{K}}^{\geq 1} g \to 0. \tag{3.5}$$

According to Kontsevich [5, Section 4.1] (see also [3, Lemma 2.1.5]) the sequence $\{\psi_j\}_{j \geq 1}$ uniquely determines a coalgebra homomorphism $\tilde{\psi} : \tilde{S}_{\mathbb{K}}^{\geq 1} g \to \tilde{S}_{\mathbb{K}}^{\geq 1} g'$ such that

$$\tilde{\ln} \circ \tilde{\psi}|_{\tilde{S}_{\mathbb{K}}^{\geq 1} g} = \psi_j \circ \tau^{-1}|_{\tilde{S}_{\mathbb{K}}^{\geq 1} g}$$

for all $j \geq 1$. Here $\tau : S_{\mathbb{K}} g \cong \tilde{S}_g$ is the coalgebra isomorphism of Lemma 3.1. Using (3.5) we can lift $\tilde{\psi}$ uniquely to a colocal coalgebra homomorphism $\tilde{\psi} : \tilde{S}_g \to \tilde{S}_g'$ by setting $\tilde{\psi}(1) := 1$. Now define the coalgebra homomorphism $\psi : S_{\mathbb{K}} g \to S_{\mathbb{K}} g'$ to be $\psi := \tau^{-1} \circ \tilde{\psi} \circ \tau$. \hfill \Box

A $\mathbb{K}$-linear map $Q : S_{\mathbb{K}} g \to S_{\mathbb{K}} g$ is a coderivation if

$$\Delta \circ Q = (Q \otimes 1 + 1 \otimes Q) \circ \Delta,$$

where $1 := 1_{S_{\mathbb{K}} g}$, the identity map.

Lemma 3.6. Given a sequence of $\mathbb{K}$-linear homomorphisms $\psi_j : S^j g \to g$, $j \geq 1$, each of degree 1, there is a unique colocal coderivation $Q$ of degree 1, such that $Q(1) = 0$ and $\partial^1 Q = \psi_j$.

Proof. According to Kontsevich [5, Section 4.3] (see also [3, Lemma 2.1.2]) the sequence $\{\psi_j\}_{j \geq 1}$ uniquely determines a coderivation $\tilde{Q} : \tilde{S}_{\mathbb{K}}^{\geq 1} g \to \tilde{S}_{\mathbb{K}}^{\geq 1} g$ such that

$$\tilde{\ln} \circ \tilde{Q}|_{\tilde{S}_{\mathbb{K}}^{\geq 1} g} = \psi_j \circ \tau^{-1}|_{\tilde{S}_{\mathbb{K}}^{\geq 1} g}$$

for all $j \geq 1$. Using (3.5) this can be lifted uniquely to a colocal coderivation $\tilde{Q} : \tilde{S}_g \to \tilde{S}_g'$ by setting $\tilde{Q}(1) := 0$. Now define the coderivation $Q : S_{\mathbb{K}} g \to S_{\mathbb{K}} g$ to be $Q := \tau^{-1} \circ \tilde{Q} \circ \tau$. \hfill \Box

We will be mostly interested in the coalgebras $S(g[1])$ and $S(g'[1])$. Observe that if $\psi : S(g[1]) \to S(g'[1])$ is a homogeneous $\mathbb{K}$-linear homomorphism of degree $i$, then, using formula (3.2), each Taylor coefficient $\partial^i \psi$ may be viewed as a homogeneous $\mathbb{K}$-linear homomorphism $\partial^i \psi : \bigwedge^i g \to g$ of degree $i + 1 - j$. 
Definition 3.7. Let $\mathfrak{g}$ be a graded $\mathbb{K}$-module. An $L_\infty$ algebra structure on $\mathfrak{g}$ is a colocal coderivation $Q : S(\mathfrak{g}[1]) \rightarrow S(\mathfrak{g}[1])$ of degree 1, satisfying $Q(1) = 0$ and $Q \circ Q = 0$. We call the pair $(\mathfrak{g}, Q)$ an $L_\infty$ algebra.

The notion of $L_\infty$ algebra generalizes that of DG Lie algebra in the following sense:

Proposition 3.8 ([5, Section 4.3]). Let $Q : S(\mathfrak{g}[1]) \rightarrow S(\mathfrak{g}[1])$ be a colocal coderivation of degree 1 with $Q(1) = 0$. Then the following conditions are equivalent.

(i) $\partial^i Q = 0$ for all $j \geq 3$, and $Q \circ Q = 0$.

(ii) $\partial^i Q = 0$ for all $j \geq 3$, and $\mathfrak{g}$ is a DG Lie algebra with respect to the differential $d := \partial^1 Q$ and the bracket $[-, -] := \partial^2 Q$.

In view of this, we shall say that $(\mathfrak{g}, Q)$ is a DG Lie algebra if the equivalent conditions of the proposition hold. An easy calculation shows that given an $L_\infty$ algebra $(\mathfrak{g}, Q)$, the function $\partial^1 Q : \mathfrak{g} \rightarrow \mathfrak{g}$ is a differential, and $\partial^2 Q$ induces a graded Lie bracket on $H(\mathfrak{g}, \partial^1 Q)$. We shall denote this graded Lie algebra by $H(\mathfrak{g}, Q)$.

Definition 3.9. Let $(\mathfrak{g}, Q)$ and $(\mathfrak{g}', Q')$ be $L_\infty$ algebras. An $L_\infty$ morphism $\Psi : (\mathfrak{g}, Q) \rightarrow (\mathfrak{g}', Q')$ is a colocal coalgebra homomorphism $\Psi : S(\mathfrak{g}[1]) \rightarrow S(\mathfrak{g}'[1])$ of degree 0, satisfying $\Psi(1) = 1$ and $\Psi \circ Q = Q' \circ \Psi$.

Proposition 3.10 ([5, Section 4.3]). Let $(\mathfrak{g}, Q)$ and $(\mathfrak{g}', Q')$ be DG Lie algebras, and let $\Psi : S(\mathfrak{g}[1]) \rightarrow S(\mathfrak{g}'[1])$ be a colocal coalgebra homomorphism of degree 0 such that $\Psi(1) = 1$. Then $\Psi$ is an $L_\infty$ morphism (i.e. $\Psi \circ Q = Q' \circ \Psi$) iff the Taylor coefficients $\psi_i := \partial^i \Psi : \bigwedge^i \mathfrak{g} \rightarrow \mathfrak{g}'$ satisfy the following identity:

\[
\begin{align*}
&d(\psi_i(\gamma_1 \wedge \cdots \wedge \gamma_i)) - \sum_{k=1}^{i} \pm \psi_i(\gamma_1 \wedge \cdots \wedge d(\gamma_k) \wedge \cdots \wedge \gamma_i) \\
&= \frac{1}{2} \sum_{\substack{k,l \geq 1 \setminus k \neq l}} \frac{1}{k! l!} \sum_{\sigma \in \mathfrak{S}_i} \pm \left[ \psi_k(\gamma_{\sigma(1)} \wedge \cdots \wedge \gamma_{\sigma(k)}), \psi_l(\gamma_{\sigma(k+1)} \wedge \cdots \wedge \gamma_{\sigma(i)}) \right] \\
&\quad + \sum_{k < l} \pm \psi_{i-1}(\{\gamma_k, \gamma_l\} \wedge \gamma_1 \wedge \cdots \wedge \gamma_{k-1} \wedge \gamma_{k+1} \wedge \cdots \wedge \gamma_l).
\end{align*}
\]

Here $\gamma_k \in \mathfrak{g}$ are homogeneous elements, $\mathfrak{S}_i$ is the permutation group of $\{1, \ldots, i\}$, and the signs depend only on the indices, the permutations and the degrees of the elements $\gamma_k$.

(See [4, Section 6] or [1, Theorem 3.1] for the explicit signs.)

The proposition shows that when $(\mathfrak{g}, Q)$ and $(\mathfrak{g}', Q')$ are DG Lie algebras and $\partial^i \Psi = 0$ for all $j \geq 2$, then $\partial^i \Psi : \mathfrak{g} \rightarrow \mathfrak{g}'$ is a homomorphism of DG Lie algebras; and conversely. It also implies that for any $L_\infty$ morphism $\psi : (\mathfrak{g}, Q) \rightarrow (\mathfrak{g}', Q')$, the map $H(\psi) : H(\mathfrak{g}, Q) \rightarrow H(\mathfrak{g}', Q')$ is a homomorphism of graded Lie algebras.

Given DG Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ we consider them as $L_\infty$ algebras $(\mathfrak{g}, Q)$ and $(\mathfrak{g}', Q')$, as explained in Proposition 3.8. If $\Psi : (\mathfrak{g}, Q) \rightarrow (\mathfrak{g}', Q')$ is an $L_\infty$ morphism, then we shall say (by slight abuse of notation) that $\Psi : \mathfrak{g} \rightarrow \mathfrak{g}'$ is an $L_\infty$ morphism.

From here until Theorem 3.21 (inclusive) $C$ is a commutative $\mathbb{K}$-algebra, and $\mathfrak{g}, \mathfrak{g}'$ are graded $\mathfrak{g}$-modules. Suppose $(\mathfrak{g}, Q)$ is an $L_\infty$ algebra structure on $\mathfrak{g}$ such that the Taylor
coefficients \( \partial^j Q : \bigwedge^j \mathfrak{g} \to \mathfrak{g} \) are all \( C \)-multilinear. Then we say \((\mathfrak{g}, Q)\) is a \textit{\( C \)-multilinear} \( \mathbb{L}_\infty \) \textit{algebra}. Similarly one defines the notion of \( C \)-multilinear \( \mathbb{L}_\infty \) \textit{morphism} \( \Psi : (g, Q) \to (g', Q') \).

With \( C \) and \( \mathfrak{g} \) as above let \( S_C \mathfrak{g} \) be the super-symmetric associative unital free algebra over \( C \). Namely \( S_C \mathfrak{g} \) is the quotient of the tensor algebra \( T_C \mathfrak{g} = C \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes_C \mathfrak{g}) \oplus \cdots \) by the ideal generated by the super-commutativity relations. The algebra \( S_C \mathfrak{g} \) is the quotient of the tensor algebra \( T_C \mathfrak{g} \) by the ideal generated by the super-commutativity relations.

\[ \Delta_C : S_C \mathfrak{g} \to S_C \mathfrak{g} \otimes_C S_C \mathfrak{g}. \]

The formulas are just as in the case \( \mathbb{K} \). It will be useful to note that \( \Delta_C \) preserves the grading by order, namely

\[ \Delta_C(S_C^i \mathfrak{g}) \subset \bigoplus_{j+k=i} S_C^j \mathfrak{g} \otimes_C S_C^k \mathfrak{g}. \]

**Lemma 3.11.** (1) Let \( \mathfrak{g} \) be a graded \( C \)-module. There is a canonical bijection \( Q \mapsto Q_C \) between the set of \( C \)-multilinear \( \mathbb{L}_\infty \) algebra structures \( Q \) on \( \mathfrak{g} \), and the set of colocal coderivations \( Q_C : S_C(\mathfrak{g}[1]) \to S_C(\mathfrak{g}[1]) \) over \( C \) of degree 1, such that \( Q_C(1) = 0 \) and \( Q_C \circ Q_C = 0 \).

(2) Let \( (\mathfrak{g}, Q) \) and \( (\mathfrak{g}', Q') \) be two \( C \)-multilinear \( \mathbb{L}_\infty \) algebras. The set of \( C \)-multilinear \( \mathbb{L}_\infty \) morphisms \( \Psi : (\mathfrak{g}, Q) \to (\mathfrak{g}', Q') \) is canonically bijective to the set of colocal coalgebra homomorphisms \( \Psi_C : S_C(\mathfrak{g}[1]) \to S_C(\mathfrak{g}'[1]) \) over \( C \) of degree 0, such that \( \Psi_C(1) = 1 \) and \( \Psi_C \circ Q_C = Q'_C \circ \Psi_C \).

**Proof.** The data for a coderivation \( Q_C : S_C(\mathfrak{g}[1]) \to S_C(\mathfrak{g}[1]) \) over \( C \) is its sequence of \( C \)-linear Taylor coefficients \( \partial^j Q_C : \bigwedge^j \mathfrak{g} \to \mathfrak{g} \). But giving such a homomorphism \( \partial^j Q_C \) is the same as giving a \( C \)-multilinear homomorphism \( \partial^j Q : \bigwedge^j \mathfrak{g} \to \mathfrak{g} \), so there is a corresponding \( C \)-multilinear coderivation \( Q : S(\mathfrak{g}[1]) \to S(\mathfrak{g}[1]) \). One checks that \( Q \circ Q = 0 \) iff \( Q_C \circ Q_C = 0 \).

Similarly for coalgebra homomorphisms. \( \square \)

An element \( \gamma \in S_C(\mathfrak{g}[1]) \) is called \textit{primitive} if \( \Delta_C(\gamma) = \gamma \otimes 1 + 1 \otimes \gamma \).

**Lemma 3.12.** The set of primitive elements of \( S_C(\mathfrak{g}[1]) \) is precisely \( S_C^1(\mathfrak{g}[1]) = \mathfrak{g}[1] \).

**Proof.** By definition of the comultiplication any \( \gamma \in \mathfrak{g}[1] \) is primitive. For the converse, let us denote by \( \mu \) the multiplication in \( S_C(\mathfrak{g}[1]) \). One checks that \( (\mu \circ \Delta_C)(\gamma) = 2^l \gamma \) for \( \gamma \in S_C^l(\mathfrak{g}[1]) \). If \( \gamma \) is primitive then \( (\mu \circ \Delta_C)(\gamma) = 2 \gamma \), so indeed \( \gamma \in S_C^1(\mathfrak{g}[1]) \). \( \square \)

Now let us assume that \( C \) is a local ring, with nilpotent maximal ideal \( m \). Suppose we are given two \( C \)-multilinear \( \mathbb{L}_\infty \) algebras \((\mathfrak{g}, Q)\) and \((\mathfrak{g}', Q')\), and a \( C \)-multilinear \( \mathbb{L}_\infty \) morphism \( \Psi : (\mathfrak{g}, Q) \to (\mathfrak{g}', Q') \). Because the coderivation \( Q \) is \( C \)-multilinear, the \( C \)-submodule \( m \mathfrak{g} \subset \mathfrak{g} \) becomes a \( C \)-multilinear \( \mathbb{L}_\infty \) algebra \((m \mathfrak{g}, Q)\). Likewise for \( m \mathfrak{g}' \), and \( \Psi : (m \mathfrak{g}, Q) \to (m \mathfrak{g}', Q') \) is a \( C \)-multilinear \( \mathbb{L}_\infty \) morphism.

The fact that \( m \) is nilpotent is essential for the next definition.
**Definition 3.13.** The **Maurer–Cartan equation** in \((\mathfrak{mg}, Q)\) is

\[
\sum_{i=1}^{\infty} \frac{1}{i!} (\partial^i Q)(\omega^i) = 0
\]

for \(\omega \in (\mathfrak{mg})^1 = (\mathfrak{mg}[1])^0\).

An element \(e \in S_C(\mathfrak{g}[1])\) is called **group-like** if \(\Delta_C(e) = e \otimes e\). For \(\omega \in \mathfrak{mg}^1\) we define

\[
\exp(\omega) := \sum_{i \geq 0} \frac{1}{i!} \omega^i \in S_C(\mathfrak{g}[1]).
\]

**Lemma 3.14.** The function \(\exp\) is a bijection from \(\mathfrak{mg}[1]\) to the set of invertible group-like elements \(e \in S_C(\mathfrak{g}[1])\) such that \(\ln(e) \in \mathfrak{mg}[1]\). The inverse of \(\exp\) is \(\ln\).

**Proof.** Let \(\omega \in \mathfrak{mg}[1]\) and \(e := \exp(\omega)\). The element \(e\) is invertible, with inverse \(\exp(-\omega)\). Using the fact that \(\Delta_C(\omega) = \omega \otimes 1 + 1 \otimes \omega\) it easily follows that \(\Delta_C(e) = e \otimes e\). And trivially \(\ln(e) = \omega\).

For the opposite direction, let \(e\) be invertible and group-like, and assume \(\ln(e) \in \mathfrak{mg}[1]\). Write it as \(e = \sum_i \gamma_i\), with \(\gamma_i \in S_C(\mathfrak{g}[1])\). The equation \(\Delta_C(e) = e \otimes e\) implies that

\[
\Delta_C(\gamma_i) = \sum_{j+k=i} \gamma_j \otimes \gamma_k
\]

for all \(i\). Hence

\[
2^i \gamma_i = \mu(\Delta_C(\gamma_i)) = \sum_{j+k=i} \gamma_j \gamma_k. \quad (3.15)
\]

For \(i = 0\) we get \(\gamma_0 = \gamma_0^2\), and since \(\gamma_0\) is invertible, it follows that \(\gamma_0 = 1\). Let \(\omega := \gamma_1 = \ln(e) \in \mathfrak{ms}_C^1(\mathfrak{g}[1]) = \mathfrak{mg}[1]\). Using induction and Eq. (3.15) we see that \(\gamma_i = \frac{1}{i!} \omega^i\) for all \(i\). Thus \(e = \exp(\omega)\). \(\square\)

**Lemma 3.16.** Let \(\omega \in (\mathfrak{mg}[1])^0 = \mathfrak{mg}^1\) and \(e := \exp(\omega)\). Then \(\omega\) is a solution of the MC equation iff \(Q(e) = 0\).

**Proof.** Since \(e\) is group-like and invertible (by Lemma 3.14) we have

\[
\Delta_C(Q(e)) = Q(e) \otimes e + e \otimes Q(e)
\]

and

\[
\Delta_C(e^{-1}Q(e)) = \Delta_C(e)^{-1} \Delta_C(Q(e)) = e^{-1}Q(e) \otimes 1 + 1 \otimes e^{-1}Q(e).
\]

So the element \(e^{-1}Q(e)\) is primitive, and by Lemma 3.12 we get \(e^{-1}Q(e) \in \mathfrak{g}[1]\). On the other hand hence \(Q(e)\) has no 0-order term, and \(Q(1) = 0\). Thus in the first order term we
get
\[ e^{-1}Q(e) = \ln \left( e^{-1}Q(e) \right) \]
\[ = \ln \left( \left( 1 - \omega + \frac{1}{2} \omega^2 \pm \cdots \right) Q(e) \right) \]
\[ = \ln (Q(e)) \]
\[ = \sum_{i=0}^{\infty} \frac{1}{i!} \ln \left( Q(\omega^i) \right) \]
\[ = \sum_{i=1}^{\infty} \frac{1}{i!} (\partial^i \Psi)(\omega^i). \]

(3.17)

Since \( e \) is invertible we are done. \( \square \)

**Lemma 3.18.** Given an element \( \omega \in \mathfrak{m}g[1] \), define \( \omega' := \sum_{i=1}^{\infty} \frac{1}{i!} (\partial^i \Psi)(\omega^i) \in \mathfrak{m}g'[1] \), \( e := \exp(\omega) \) and \( e' := \exp(\omega') \). Then \( e' = \Psi(e) \).

**Proof.** From Lemma 3.14 we see that \( \Delta_C(e) = e \otimes e \), and therefore also \( \Delta_C(\Psi(e)) = \Psi(e) \otimes \Psi(e) \in S_C(\mathfrak{g}'[1]) \). Since \( \Psi \) is \( C \)-linear and \( \Psi(1) = 1 \) we get \( \Psi(e) \in 1 + mS(\mathfrak{g}'[1]) \). Thus \( \Psi(e) \) is group-like and invertible. According to Lemma 3.14 it suffices to prove that \( \ln(e') = \ln(\Psi(e)) \). Now \( \ln(e') = \omega' \) by definition. Since \( \Psi(1) = 1 \) and \( \ln(1) = 0 \) it follows that
\[ \ln(\Psi(e)) = \ln \left( \Psi \left( \sum_{i=0}^{\infty} \frac{1}{i!} \omega^i \right) \right) = \sum_{i=0}^{\infty} \frac{1}{i!} \ln(\omega^i) = \sum_{i=1}^{\infty} \frac{1}{i!} (\partial^i \Psi)(\omega^i) = \omega'. \] \( \square \)

**Proposition 3.19.** Suppose \( \omega \in \mathfrak{m}g^1 \) is a solution of the MC equation in \((\mathfrak{m}g, Q)\). Define \( \omega' := \sum_{i=1}^{\infty} \frac{1}{i!} (\partial^i \Psi)(\omega^i) \in \mathfrak{m}g^1 \). Then \( \omega' \) is a solution of the MC equation in \((\mathfrak{m}g', Q')\).

**Proof.** Let \( e := \exp(\omega) \) and \( e' := \exp(\omega') \). By Lemma 3.16 we get \( Q(e) = 0 \). Hence \( Q'(\Psi(e)) = \Psi(Q(e)) = 0 \). According to Lemma 3.18 we have \( \Psi(e) = e' \), so \( Q'(e') = 0 \). Again by Lemma 3.16 we deduce that \( \omega' \) solves the MC equation. \( \square \)

**Definition 3.20.** Let \( \omega \in \mathfrak{m}g^1 \).

1. The colocal coderivation \( Q_\omega \) of \( S_C(\mathfrak{g}[1]) \) over \( C \), with \( Q_\omega(1) := 0 \) and with Taylor coefficients
\[ (\partial^i Q_\omega)(\gamma) := \sum_{j \geq 0} \frac{1}{j!} (\partial^{i+j} Q)(\omega^j \gamma) \]
for \( i \geq 1 \) and \( \gamma \in S_C^*(\mathfrak{g}[1]) \), is called the twist of \( Q \) by \( \omega \).

2. The colocal coalgebra homomorphism \( \Psi_\omega : S_C(\mathfrak{g}[1]) \to S_C(\mathfrak{g}'[1]) \) over \( C \), with \( \Psi_\omega(1) := 1 \) and Taylor coefficients
\[ (\partial^i \Psi_\omega)(\gamma) := \sum_{j \geq 0} \frac{1}{j!} (\partial^{i+j} \Psi)(\omega^j \gamma) \]
for \( i \geq 1 \) and \( \gamma \in S_C^*(\mathfrak{g}[1]) \), is called the twist of \( \Psi \) by \( \omega \).
Theorem 3.21. Let $C$ be a commutative local $\mathbb{K}$-algebra with nilpotent maximal ideal $m$. Let $(g, Q)$ and $(g', Q')$ be $C$-multilinear $L_\infty$ algebras and $\Psi : (g, Q) \to (g', Q')$ a $C$-multilinear $L_\infty$ morphism. Suppose $\omega \in mg^1$ a solution of the MC equation in $(mg, Q)$. Define

$$\omega' := \sum_{i=1}^{\infty} \frac{1}{j!} (\partial^j \Psi)(\omega^j) \in mg^1.$$ 

Then $(g, Q_\omega)$ and $(g', Q'_\omega)$ are $L_\infty$ algebras, and

$$\Psi_\omega : (g, Q_\omega) \to (g', Q'_\omega)$$

is an $L_\infty$ morphism.

Proof. Let $e := \exp(\omega)$. Define $\Phi_e : S_C(g[1]) \to S_C(g[1])$ to be $\Phi_e(\gamma) := e\gamma$. Since $e$ is group-like and invertible it follows that $\Phi_e$ is a coalgebra automorphism. Therefore $Q_\omega := \Phi_e^{-1} \circ Q \circ \Phi_e$ is a degree 1 colocal coderivation of $S_C(g[1])$, satisfying $Q_\omega \circ Q_\omega = 0$ and $Q_\omega(1) = e^{-1}Q(e) = 0$; cf. Lemma 3.16. So $(g, Q_\omega)$ is an $L_\infty$ algebra. Likewise we have a coalgebra automorphism $\Phi_e'$ and a coderivation $Q'_\omega := \Phi_e'^{-1} \circ Q' \circ \Phi_e'$ of $S_C(g'[1])$, where $e' := \exp(\omega')$. The degree 0 colocal coalgebra homomorphism $\tilde{\Psi}_\omega := \Phi_e^{-1} \circ \Psi \circ \Phi_e$ satisfies $\tilde{\Psi}_\omega \circ \tilde{\Psi}_\omega = \tilde{Q}'_\omega \circ \tilde{Q}'_\omega$, and also $\tilde{\Psi}_\omega(1) = e'^{-1} \Psi(e) = e'^{-1}e'^{-1} = 1$, by Lemma 3.18. Hence we have an $L_\infty$ morphism $\tilde{\Psi}_\omega : (g, Q_\omega) \to (g', Q'_\omega)$.

Let us calculate the Taylor coefficients of $Q_\omega$. For $\gamma \in S^1_C(g[1])$ one has

$$(\partial^j Q_\omega)(\gamma) = \ln(\tilde{Q}_\omega(\gamma)) = \ln(e^{-1}Q(e\gamma)).$$

But just as in (3.17), since $Q(e\gamma)$ has no zero order term, we obtain

$$\ln(e^{-1}Q(e\gamma)) = \ln(Q(e\gamma)).$$

And

$$\ln(Q(e\gamma)) = \ln\left(Q\left(\sum_{j \geq 0} \frac{1}{j!} \omega^j \gamma\right)\right)$$

$$= \sum_{j \geq 0} \frac{1}{j!} \ln(Q(\omega^j \gamma))$$

$$= \sum_{j \geq 0} \frac{1}{j!} (\partial^{j'} Q)(\omega^j \gamma)$$

$$= (\partial^j Q_\omega)(\gamma).$$

Therefore $Q_\omega = Q_\omega$. Similarly we see that $Q'_\omega = Q'_\omega$ and $\tilde{\Psi}_\omega = \Psi_\omega$. $\square$

Remark 3.23. The formulation of Theorem 3.21, as well as the idea for the proof, were suggested by Vladimir Hinich. An analogous result, for $A_\infty$ algebras, is in [6, Section 6.1].

If $(g, Q)$ is a DG Lie algebra then the sum occurring in Definition 3.20(1) is finite, so the coderivation $Q_\omega$ can be defined without a nilpotence assumption on the coefficients.
Lemma 3.24. Let \((\mathfrak{g}, Q)\) be a DG Lie algebra, and let \(\omega \in \mathfrak{g}^1\) be a solution of the MC equation. Then the \(L_\infty\) algebra \((\mathfrak{g}, Q_\omega)\) is also a DG Lie algebra. In fact, for \(\gamma_i \in \mathfrak{g}\) one has
\[
(\partial^1 Q_\omega)(\gamma_1) = (\partial^1 Q)(\gamma_1) + (\partial^2 Q)(\omega \gamma_1) = d(\gamma_1) + [\omega, \gamma_1] = (d + \mathrm{ad}(\omega))(\gamma_1),
\]
\[
(\partial^2 Q_\omega)(\gamma_1 \gamma_2) = (\partial^2 Q)(\gamma_1 \gamma_2) = [\gamma_1, \gamma_2],
\]
and \(\partial^j Q_\omega = 0\) for \(j \geq 3\).

Proof. Like Eq. (3.22), with \(C := K\) and \(e := 1\). □

In the situation of the lemma, the twisted DG Lie algebra \((\mathfrak{g}, Q_\omega)\) will usually be denoted by \(\mathfrak{g}_\omega\).

Let \(A\) be a super-commutative associative unital DG algebra in \(\text{Dir Inv Mod} K\). The notion of DG \(A\)-module Lie algebra in \(\text{Dir Inv Mod} K\) was introduced in Definition 1.20.

Definition 3.25. Let \(A\) be a super-commutative associative unital DG algebra in \(\text{Dir Inv Mod} K\), let \(g\) and \(g'\) be DG \(A\)-module Lie algebras in \(\text{Dir Inv Mod} K\), and let \(\Psi : g \to g'\) be an \(L_\infty\) morphism.

(1) If each Taylor coefficient \(\partial^j \Psi : \prod^j g \to g'\) is continuous then we say that \(\Psi\) is a continuous \(L_\infty\) morphism.

(2) Assume each Taylor coefficient \(\partial^j \Psi : \prod^j g \to g'\) is \(A\)-multilinear, i.e.
\[
(\partial^j \Psi)(a_1 \gamma_1, \ldots, a_j \gamma_j) = \pm a_1 \cdots a_j \cdot (\partial^j \Psi)(\gamma_1, \ldots, \gamma_j)
\]
for all homogeneous elements \(a_k \in A\) and \(\gamma_k \in g\), with sign according to the Koszul rule, then we say that \(\Psi\) is an \(A\)-multilinear \(L_\infty\) morphism.

Proposition 3.26. Let \(A\) and \(B\) be super-commutative associative unital DG algebras in \(\text{Dir Inv Mod} K\), and let \(g\) and \(g'\) be DG \(A\)-module Lie algebras in \(\text{Dir Inv Mod} K\). Suppose \(A \to B\) is a continuous DG algebra homomorphism, and \(\Psi : g \to g'\) is a continuous \(A\)-multilinear \(L_\infty\) morphism. Let \(\partial^j \Psi_B : \prod^j (B \otimes_A g) \to B \otimes_A g'\) be the unique continuous \(B\)-multilinear homomorphism extending \(\partial^j \Psi\). Then the degree 0 colocal coalgebra homomorphism
\[
\psi_B : S(B \otimes_A g[1]) \to S(B \otimes_A g'[1]),
\]
with \(\psi_B(1) := 1\) and with Taylor coefficients \(\partial^j \psi_B\), is an \(L_\infty\) morphism
\[
\psi_B : B \otimes_A g \to B \otimes_A g'.
\]

Proof. First consider the continuous \(B\)-multilinear homomorphisms \(\partial^j \psi_B : \prod^j (B \otimes_A g) \to B \otimes_A g'\) extending \(\partial^j \Psi\). It is a straightforward calculation to verify that the \(L_\infty\) morphism identities of Proposition 3.10 hold for the sequence of operators \(\{\partial^j \psi_B\}_{j \geq 1}\). The completion process respects these identities (cf. proof of Proposition 1.19). □

Theorem 3.27. Let \(g\) and \(g'\) be DG Lie algebras in \(\text{Dir Inv Mod} K\), and let \(\Psi : g \to g'\) be a continuous \(L_\infty\) morphism. Let \(A = \bigoplus_{i \in \mathbb{N}} A_i\) be a complete associative unital super-commutative DG algebra in \(\text{Dir Inv Mod} K\). By Proposition 3.26 there is an induced continuous \(A\)-multilinear \(L_\infty\) morphism \(\Psi_A : A \otimes g \to A \otimes g'\). Let \(\omega \in A^1 \otimes g^0\) be a
solution of the MC equation in $A \otimes \mathfrak{g}$. Assume $d_{\mathfrak{g}} = 0$, $(\partial^j \Psi_A)(\omega^j) = 0$ for all $j \geq 2$, and also that $\mathfrak{g}'$ is bounded below. Define $\omega' := (\partial^1 \Psi_A)(\omega) \in A^1 \otimes \mathfrak{g}'$. Then:

1. The element $\omega'$ is a solution of the MC equation in $A \otimes \mathfrak{g}'$.
2. Given $c \in S^j (A \otimes \mathfrak{g}[1])$ there exists a natural number $k_0$ such that $(\partial^{j+k} \Psi_A)(\omega^k c) = 0$ for all $k > k_0$.
3. The degree 0 colocal coalgebra homomorphism

$$\Psi_{A,\omega} : S (A \otimes \mathfrak{g}[1]) \to S (A \otimes \mathfrak{g}'[1]),$$

with $\Psi_{A,\omega}(1) := 1$ and Taylor coefficients

$$(\partial^j \Psi_{A,\omega})(c) := \sum_{k \geq 0} \frac{1}{(j+k)!} (\partial^{j+k} \Psi_A)(\omega^k c)$$

for $c \in S^j (A \otimes \mathfrak{g}[1])$, is a continuous $A$-multilinear $L_\infty$ morphism

$$\Psi_{A,\omega} : (A \otimes \mathfrak{g})_\omega \to (A \otimes \mathfrak{g}')_{\omega'}.$$

**Proof.** We shall use a “deformation argument”. Consider the base field $\mathbb{K}$ as a discrete inv $\mathbb{K}$-module. The polynomial algebra $\mathbb{K}[h]$ is endowed with the dir-inv $\mathbb{K}$-module structure such that the homomorphism $\bigoplus_{i \in \mathbb{N}} \mathbb{K} \to \mathbb{K}[h]$, whose $i$-th component is multiplication by $h^i$, is an isomorphism in $\text{Dir Inv Mod} \mathbb{K}$. Note that $\mathbb{K}[h]$ is a discrete dir-inv module, but it is not trivial. We view $\mathbb{K}[h]$ as a DG algebra concentrated in degree 0 (with zero differential).

For any $i \in \mathbb{N}$ let $A[h]^i := \mathbb{K}[h] \otimes A^i$, and let $A[h] := \bigoplus_{i \in \mathbb{N}} A[h]^i$, which is a DG algebra in $\text{Dir Inv Mod} \mathbb{K}$, with differential $d_{A[h]} := \mathbf{1} \otimes d_A$. We will need a “twisted” version of $A[h]_-$, which we denote by $A[h]^{-}$ and its differential is $d_{A[h]^-} := h \partial d_{A[h]}$. The dir-inv structure is such that the homomorphism $\bigoplus_{i,j \in \mathbb{N}} A^i \to A^{-}$, whose $(i, j)$-th component is multiplication by $h^{i+j}$, is an isomorphism in $\text{Dir Inv Mod} \mathbb{K}$. The specialization $h \mapsto 1$ is a continuous DG algebra homomorphism $A[h]^- \to A$. There is an induced continuous $A[h]^{-}$-multilinear $L_\infty$ morphism $\Psi_{A[h]^-} : A[h]^- \otimes \mathfrak{g} \to A[h]^- \otimes \mathfrak{g}'$.

We proceed in several steps.

Step 1. Say $r_0$ bounds $\mathfrak{g}'$ from below, i.e. $\mathfrak{g}'^r = 0$ for all $r < r_0$. Take some $j \geq 1$. For any $l \in \{1, \ldots, j\}$ choose $p_l, q_l \in \mathbb{Z}$, $y_l \in \mathfrak{g}^{p_l}$ and $a_l \in A[h]^{-q_l}$. Also choose $\gamma_0 \in \mathfrak{g}^0$ and $a_0 \in A[h]^{-1}$. Let $p := \sum_{l=1}^{j} p_l$ and $q := \sum_{l=1}^{j} q_l$. Because $\partial^{j+k} \Psi_{A[h]^-}$ is induced from $\partial^{j+k} \Psi$, and this is a homogeneous map of degree $1 - j - k$, we have

$$(\partial^{j+k} \Psi_{A[h]^-}) \left((a_0 \otimes \gamma_0)^k (a_1 \otimes y_1) \cdots (a_j \otimes y_j) \right) = \pm a_0^{q} a_1 \cdots a_j \otimes (\partial^{j+k} \Psi)(\gamma_0^k y_1 \cdots y_j) \in A[h]^{-k+q} \otimes \mathfrak{g}^{p+1-j-k}.$$

But $\mathfrak{g}^{p+1-j-k} = 0$ for all $k > p + 1 - j - r_0$.

Using multilinearity and continuity we conclude that given any $c \in S^j (A[h]^- \otimes \mathfrak{g}[1])$ there exists a natural number $k_0$ such that $(\partial^{j+k} \Psi_{A[h]^-})( (h \omega)^k c) = 0$ for all $k > k_0$.

Step 2. We are going to prove that $h \omega$ is a solution of the MC equation in $A[h]^- \otimes \mathfrak{g}$. It is given that $\omega$ is a solution of the MC equation in $A \otimes \mathfrak{g}$. Because $d_{\mathfrak{g}} = 0$, this means that

$$(d_A \otimes \mathbf{1})(\omega) + \frac{1}{2} [\omega, \omega] = 0.$$
Proposition 3.19

For Proposition 3.10 there is a twisted \( L \) coefficients satisfy the identities of \( \Psi \).

Theorem 3.21

\[
\nu \in \text{Der}_K(C) = T(C) / \text{Der}_K(C),
\]

which is a subalgebra of \( A[h]_m \), but its differential is \( d_{A[h]_m} := h d_{A[h]_m} \).

Now \( C := \mathbb{K}[h]_m \) is an artinian local ring with maximal ideal \( m := (h) \).

Step 4. Pick a natural number \( m \). Let \( h, h', \Phi, v \), and \( v' \) be as in step 3. According to Theorem 3.21 there is a twisted \( L_\infty \) morphism \( \Phi_v : \hat{h} \to \hat{h}' \).

Step 5. Specialization \( h \mapsto 1 \) induces surjective DG Lie algebra homomorphisms \( A[h]^- \otimes g \to A \otimes g \) and \( A[h]^- \otimes g' \to A \otimes g' \), sending \( h \omega \mapsto \omega \), \( h \omega' \mapsto \omega' \) and \( \hat{\psi}_{A[h^-], h \omega} \mapsto \hat{\psi}_{A[\omega]^-} \).

4. The universal \( L_\infty \) morphism of Kontsevich

In this section \( K \) is a field of characteristic 0 and \( C \) is a commutative \( K \)-algebra. Recall that we denote by \( T_C = T(C / K) := \text{Der}_K(C) \), the module of derivations of \( C \) relative to \( K \). This is a Lie algebra over \( K \). Following [5] we make the next definitions.

Definition 4.1. For \( p \geq -1 \) let

\[
\mathcal{T}^p_{\text{poly}}(C) := \bigwedge^{p+1} \mathcal{T}_C.
\]
the module of poly derivations (or poly tangents) of degree $p$ of $C$ relative to $k$. Let

$$T_{\text{poly}}(C) := \bigoplus_p T^p_{\text{poly}}(C).$$

This is a DG Lie algebra, with zero differential, and with the Schouten–Nijenhuis bracket, which is determined by the formulas

$$[\alpha_1 \wedge \alpha_2, \alpha_3] = \alpha_1 \wedge [\alpha_2, \alpha_3] + (-1)^{(p_2+1)p_3}[\alpha_1, \alpha_3] \wedge \alpha_2$$

and

$$[\alpha_1, \alpha_2] = (-1)^{1+p_1p_2}[\alpha_2, \alpha_1]$$

for elements $\alpha_i \in T^p_{\text{poly}}(C)$.

**Definition 4.2.** For any $p \geq -1$ let $D^p_{\text{poly}}(C)$ be the set of $k$-multilinear multi differential operators $\phi : C^{p+1} \to C$ (see Definition 2.1). The direct sum

$$D_{\text{poly}}(C) := \bigoplus_p D^p_{\text{poly}}(C)$$

is a DG Lie algebra. The differential $d_D$ is the shifted Hochschild differential, and the Lie bracket is the Gerstenhaber bracket (see [5, Section 3.4.2]). The elements of $D_{\text{poly}}(C)$ are called poly differential operators relative to $k$.

In the notation of Section 2 and Example 1.24 one has

$$D^p_{\text{poly}}(C) = \text{Diff}_{\text{poly}}(C; C, \ldots, C; C) = C_{\text{cd}}^{p+1}(C);$$

see formula (2.3).

Observe that $D^p_{\text{poly}}(C) \subset \text{Hom}_k(C^{\otimes(p+1)}, C)$, and $D_{\text{poly}}(C)$ is a sub DG Lie algebra of the shifted Hochschild cochain complex of $C$ relative to $k$. For $p = -1, 0$ we have $D^{-1}_{\text{poly}}(C) = C$ and $D^0_{\text{poly}}(C) = D(C)$, the ring of differential operators. Note that $D_{\text{poly}}(C)$ is a left module over $D(C)$, by the formula $D \cdot \phi := D \circ \phi$; and in this way it is also a left $C$-module.

When $C := k[t] = k[t_1, \ldots, t_n]$, the polynomial algebra in $n \geq 1$ variables, and $p \geq 1$, the following is true. The $k[t]$-module $T^{p-1}_{\text{poly}}(k[t])$ is free with finite basis $\{ \frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_p} \}$, indexed by the sequences $0 \leq i_1 < \cdots < i_p \leq n$. The $k[t]$-module $D^{p-1}_{\text{poly}}(k[t])$ is also free, with countable basis

$$\left\{ \left( \frac{\partial}{\partial t} \right)^{j_1} \otimes \cdots \otimes \left( \frac{\partial}{\partial t} \right)^{j_p} \right\}_{j_1, \ldots, j_p \in \mathbb{N}^n}, \quad (4.3)$$

where for $j_k = (j_{k,1}, \ldots, j_{k,n}) \in \mathbb{N}^n$ we write $\left( \frac{\partial}{\partial t} \right)^{j_k} := \left( \frac{\partial}{\partial t_1} \right)^{j_{k,1}} \cdots \left( \frac{\partial}{\partial t_n} \right)^{j_{k,n}}$.

For any $p \geq -1$ let $F^p_{\text{poly}}(C)$ be the set of poly differential operators of order $\leq m$ in each argument. This is a $C$-submodule of $D^p_{\text{poly}}(C)$.
Lemma 4.4. (1) For any $m$, $p$ one has

$$d_D\left(F^p_m D^\text{poly}_p(C)\right) \subset F^p_m D^\text{poly}_{p+1}(C).$$

(2) For any $m, m', p, p'$ one has

$$\left[F^p_m D^\text{poly}_p(C), F^p_{m'} D^\text{poly}_{p'}(C)\right] \subset F^p_{m+m'} D^\text{poly}_{p+p'}(C);$$

and

$$[-, -] : F^p_m D^\text{poly}_p(C) \times F^p_{m'} D^\text{poly}_{p'}(C) \to D^\text{poly}_{p+p'}(C)$$

is a poly differential operator of order $\leq m + m'$ in each of its two arguments.

Proof. These assertions follow easily from the definitions of the Hochschild differential and the Gerstenhaber bracket; cf. [5, Section 3.4.2]. \qed

Lemma 4.5. Assume $C$ is a finitely generated $\mathbb{K}$-algebra. Then $T^p_{\text{poly}}(C)$ and $F^p_m D^\text{poly}_p(C)$ are finitely generated $C$-modules.

Proof. One has

$$T^p_{\text{poly}}(C) \cong \text{Hom}_A(\Omega^{p+1}_C, A)$$

and

$$F^p_m D^\text{poly}_p(C) \cong \text{Hom}_C(C_{p+1,m}(C), C);$$

see Lemma 2.2. The $C$-modules $\Omega^{p+1}_C$ and $C_{p+1,m}(C)$ are finitely generated. \qed

Proposition 4.6. Assume $C$ is a finitely generated $\mathbb{K}$-algebra, and $C'$ is a noetherian, $c'$-adically complete, flat, $c'$-adically formally étale $C$-algebra. Let us write $G$ for either $T^p_{\text{poly}}$ or $D^\text{poly}$. Then:

1. There is a DG Lie algebra homomorphism $G(C) \to G(C')$, which is functorial in $C \to C'$.

2. The induced $C'$-linear homomorphism $C' \otimes_C G^p(C) \to G^p(C')$ is bijective.

3. For any $m$ the isomorphisms in (2), for $G = D^\text{poly}$, restrict to isomorphisms

$$C' \otimes_C F^p_m D^\text{poly}_p(C) \xrightarrow{\sim} F^p_m D^\text{poly}_p(C').$$

Proof. Consider $G = D^\text{poly}$. Let $\phi \in D^\text{poly}_p(C)$. According to Proposition 2.7, applied to the case $M_1, \ldots, M_{p+1}, N := A$, there is a unique $\phi' \in D^\text{poly}_p(C')$ extending $\phi$. From the definitions of the Gerstenhaber bracket and the Hochschild differential, it immediately follows that the function $D^\text{poly}_p(C) \to D^\text{poly}_p(C')$, $\phi \mapsto \phi'$, is a DG Lie algebra homomorphism. Parts (2,3) are also consequences of Proposition 2.7.

The case $G = T^p_{\text{poly}}$ is done similarly (and is well-known). \qed

Consider $C := \mathbb{K}[t]$ and $C' := \mathbb{K}[[t]] = \mathbb{K}[t_1, \ldots, t_n]$, the power series algebra. Since $T^p_{\text{poly}}(\mathbb{K}[t]) \cong \mathbb{K}[t] \otimes_{\mathbb{K}[t]} T^p_{\text{poly}}(\mathbb{K}[t])$, the power series algebra, Since $T^p_{\text{poly}}(\mathbb{K}[t])$ is a finitely generated left $\mathbb{K}[t]$-module, it is an
inv \mathbb{K}[[t]]\text{-module with the } (t)\text{-adic inv structure; cf. Example 1.8. Likewise } \mathcal{D}^p_{\text{poly}}(\mathbb{K}[[t]]) \text{ is a dir-inv } \mathbb{K}[[t]]\text{-module. By Proposition 4.6, }
\text{F}_m \mathcal{D}^p_{\text{poly}}(\mathbb{K}[[t]]) \cong \mathbb{K}[[t]] \otimes_{\mathbb{K}[[t]]} \text{F}_m \mathcal{D}^p_{\text{poly}}(\mathbb{K}[t]),

which is a finitely generated \mathbb{K}[[t]]\text{-module. So according to Example 1.9 we may take } (\text{F}_m \mathcal{D}^p_{\text{poly}}(\mathbb{K}[[t]]))_{m \in \mathbb{N}} \text{ as the dir-inv structure of } \mathcal{D}^p_{\text{poly}}(\mathbb{K}[[t]]). \text{Now forgetting the } \mathbb{K}[[t]]\text{-module structure, } \mathcal{T}^p_{\text{poly}}(\mathbb{K}[[t]]) \text{ becomes an inv } \mathbb{K}\text{-module, and } \mathcal{D}^p_{\text{poly}}(\mathbb{K}[[t]]) \text{ becomes a dir-inv } \mathbb{K}\text{-module.}

**Proposition 4.7.** Let \mathcal{G} stand either for \mathcal{T}_{\text{poly}} \text{ or } \mathcal{D}_{\text{poly}}. \text{Then } \mathcal{G}(\mathbb{K}[[t]]) \text{ is a complete DG Lie algebra in } \text{Dir Inv Mod } \mathbb{K}.

**Proof.** Use Proposition 2.4, and, for the case } \mathcal{G} = \mathcal{D}_{\text{poly}}, \text{ also Lemma 4.4.} \quad \square

**Remark 4.8.** One might prefer to view \mathcal{T}_{\text{poly}}(\mathbb{K}[[t]]) \text{ and } \mathcal{D}_{\text{poly}}(\mathbb{K}[[t]]) \text{ as topological DG Lie algebras. This can certainly be done: put on } \mathcal{T}^p_{\text{poly}}(\mathbb{K}[[t]]) \text{ and } \text{F}_m \mathcal{D}^p_{\text{poly}}(\mathbb{K}[[t]]) \text{ the } t\text{-adic topology, and put on } \mathcal{D}^p_{\text{poly}}(\mathbb{K}[[t]]) = \lim_{m \to \infty} \text{F}_m \mathcal{D}^p_{\text{poly}}(\mathbb{K}[[t]]) \text{ the direct limit topology (see [8, Section 1.1]). However the dir-inv structure is better suited for our work.}

**Definition 4.9.** For } p \geq 0 \text{ let } \mathcal{D}^{\text{nor},p}_{\text{poly}}(C) \text{ be the submodule of } \mathcal{D}^p_{\text{poly}}(C) \text{ consisting of poly differential operators } \phi \text{ such that } \phi(c_1, \ldots, c_{p+1}) = 0 \text{ if } c_i = \bar{1} \text{ for some } i. \text{ For } p = -1 \text{ we let } \mathcal{D}^{\text{nor},-1}_{\text{poly}}(C) := C. \text{ Define } \mathcal{D}^{\text{nor}}_{\text{poly}}(C) := \bigoplus_{p \geq -1} \mathcal{D}^{\text{nor},p}_{\text{poly}}(C). \text{ We call } \mathcal{D}^{\text{nor}}_{\text{poly}}(C) \text{ the algebra of normalized poly differential operators.}

From the formulas for the Gerstenhaber bracket and the Hochschild differential (see [5, Section 3.4.2]) it immediately follows that \mathcal{D}^{\text{nor}}_{\text{poly}}(C) \text{ is a sub DG Lie algebra of } \mathcal{D}_{\text{poly}}(C).

For any integer } p \geq 1 \text{ there is a } C\text{-linear homomorphism } \mathcal{U}_1 : \mathcal{T}^{p-1}_{\text{poly}}(C) \to \mathcal{D}^{\text{nor},p-1}_{\text{poly}}(C)

with formula

\begin{equation}
\mathcal{U}_1(\xi_1 \wedge \cdots \wedge \xi_p)(c_1, \ldots, c_p) := \frac{1}{p!} \sum_{\sigma \in S_p} \text{sgn}(\sigma)\xi_{\sigma(1)}(c_1) \cdots \xi_{\sigma(p)}(c_p)
\end{equation}

for elements } \xi_1, \ldots, \xi_p \in \mathcal{T}_C \text{ and } c_1, \ldots, c_p \in C. \text{ For } p = 0 \text{ the map } \mathcal{U}_1 : \mathcal{T}^{-1}_{\text{poly}}(C) \to \mathcal{D}^{\text{nor},-1}_{\text{poly}}(C) \text{ is the identity of } (C).

Suppose } M \text{ and } N \text{ are complexes in } \text{Dir Inv Mod } C \text{ and } \phi, \phi' : M \to N \text{ are morphisms of complexes in } \text{Dir Inv Mod } C \text{ (i.e. all maps are continuous for the dir-inv structures). We say } \phi \text{ and } \phi' \text{ are homotopic if there is a degree } \bar{-1} \text{ homomorphism of graded dir-inv modules } \eta : M \to N \text{ such that } d_N \circ \eta + \eta \circ d_M = \phi - \phi'. \text{ We say that } \phi : M \to N \text{ is a homotopy equivalence in } \text{Dir Inv Mod } C \text{ if there is a morphism of complexes } \psi : N \to M \text{ in } \text{Dir Inv Mod } C \text{ such that } \psi \circ \phi \text{ is homotopic to } 1_M \text{ and } \phi \circ \psi \text{ is homotopic to } 1_N.

**Theorem 4.11.** Let } C \text{ be a commutative } \mathbb{K}\text{-algebra with ideal } c. \text{ Assume } C \text{ is noetherian and } c\text{-adically complete. Also assume there is a } \mathbb{K}\text{-algebra homomorphism }
\( \mathbb{K}[t_1, \ldots, t_n] \to C \) which is flat and \( c \)-adically formally étale. Then the homomorphism \( U_1 : T_{\text{poly}}(C) \to T^{\text{nor}}_{\text{poly}}(C) \) and the inclusion \( T^{\text{nor}}_{\text{poly}}(C) \to T_{\text{poly}}(C) \) are both homotopy equivalences in \( \text{Dir Inv Mod} \ C \).

**Proof.** Recall that \( B_q(C) = B^{-q}(C) := C^\otimes(q+2) \), and this is a \( B_0(C) \)-algebra via the extreme factors. So \( B_q(C) \cong B_0(C) \otimes C^\otimes q \) as \( B_0(C) \)-modules. Let \( \overline{C} := C/\mathbb{K} \), the quotient \( \mathbb{K} \)-module, and define \( B^{\text{nor}}_q(C) = B^{\text{nor}, -q}(C) := B_0(C) \otimes \overline{C}^\otimes q \), the \( q \)-th normalized bar module of \( C \). According to MacLane [7, Section X.2], \( B^{\text{nor}}_q(C) := \bigoplus_q B^{\text{nor}, -q}(C) \) has a coboundary operator such that the obvious surjection \( \phi : B(C) \to B^{\text{nor}}_0(C) \) is a quasi-isomorphism of complexes of \( B^0(C) \)-modules. Define

\[
C^{\text{nor}}_q(C) = C^{\text{nor}, -q}(C) := C \otimes B_0(C) B^{\text{nor}}_q(C) \cong C \otimes \overline{C}^\otimes q.
\]

Because the complexes \( B(C) \) and \( B^{\text{nor}}(C) \) are bounded above and consist of free \( B_0(C) \)-modules, it follows that \( \phi : C(C) \to C^{\text{nor}}(C) \) is a quasi-isomorphism of \( C \)-modules. Let \( \hat{\Omega}^{q^\delta}_C \) be the \( c \)-adic completion of \( \Omega^q_C \), so that \( \hat{\Omega}^{q^\delta}_C \cong C \otimes_{\mathbb{K}[t]} \Omega^q_C \). There is a \( C \)-linear homomorphism \( \psi : C^{\text{nor}}_q(C) \to \Omega^q_C \) with formula

\[
\psi (1 \otimes (c_1 \otimes \cdots \otimes c_q)) = d(c_1) \wedge \cdots \wedge d(c_q).
\]

Consider the polynomial algebra \( \mathbb{K}[t] = \mathbb{K}[t_1, \ldots, t_n] \). For \( i \in \{1, \ldots, n\} \) and \( j \in \{1, \ldots, q\} \) let

\[
\hat{d}_j(t_i) := 1 \otimes \cdots \otimes 1 \otimes (t_i \otimes 1 - 1 \otimes t_i) \otimes 1 \otimes \cdots \otimes 1 \in B_q(\mathbb{K}[t]),
\]

and use the same expression to denote the image of this element in \( C_q(\mathbb{K}[t]) \). It is easy to verify that \( C_q(\mathbb{K}[t]) \) is a polynomial algebra over \( \mathbb{K}[t] \) in the set of generators \( \{\hat{d}_j(t_i)\} \). Another easy calculation shows that \( \text{Ker} \left( \phi : C_q(\mathbb{K}[t]) \to C^{\text{nor}}_q(\mathbb{K}[t]) \right) \) is generated as \( \mathbb{K}[t] \)-module by monomials in elements of the set \( \{\hat{d}_j(t_i)\} \).

Let us introduce a grading on \( C_q(\mathbb{K}[t]) \) by \( \text{deg}(\hat{d}_j(t_i)) := 1 \) and \( \text{deg}(t_i) := 0 \). The coboundary operator of \( C(\mathbb{K}[t]) \) has degree 0 in this grading. The grading is inherited by \( C^{\text{nor}}_q(\mathbb{K}[t]) \), and hence \( \phi : C(\mathbb{K}[t]) \to C^{\text{nor}}(\mathbb{K}[t]) \) is a quasi-isomorphism of complexes in \( \text{GrMod} \mathbb{K}[t] \), the category of graded \( \mathbb{K}[t] \)-modules. Also let us put a grading on \( \Omega^q_{\mathbb{K}[t]} \) with \( \text{deg}(d(t_i)) := 1 \). By [8, Lemma 4.3], \( \psi \circ \phi : C(\mathbb{K}[t]) \to \bigoplus_q \Omega^q_{\mathbb{K}[t]}[q] \) is a quasi-isomorphism in \( \text{GrMod} \mathbb{K}[t] \). Because we are dealing with bounded above complexes of free graded \( \mathbb{K}[t] \)-modules it follows that both \( \phi \) and \( \psi \) are homotopy equivalences in \( \text{GrMod} \mathbb{K}[t] \).

Now let us go back to the formally étale homomorphism \( \mathbb{K}[t] \to C \). We get homotopy equivalences

\[
C \otimes_{\mathbb{K}[t]} C(\mathbb{K}[t]) \xrightarrow{\phi} C \otimes_{\mathbb{K}[t]} C^{\text{nor}}(\mathbb{K}[t]) \xrightarrow{\psi} \bigoplus_q \hat{\Omega}^{q^\delta}_C[q]
\]

in \( \text{GrMod} C \). We know that \( \hat{C}_q(C) \) is a power series algebra in the set of generators \( \{\hat{d}_j(t_i)\} \); see [8, Lemma 2.6]. Therefore \( \hat{C}_q(C) \) is isomorphic to the completion of \( C \otimes_{\mathbb{K}[t]} C_q(\mathbb{K}[t]) \).
with respect to the grading (see Example 1.13). Define \( \widehat{C}^{\text{nor}}_q(C) \) to be the completion of \( C \otimes_{K[t]} C^{\text{nor}}_q(K[t]) \) with respect to the grading. We then have a homotopy equivalence of complexes in \( \text{Inv} \text{Mod} C \)

\[
\widehat{C}(C) \to \widehat{C}^{\text{nor}}(C) \to \bigoplus_q \widehat{T}^q_C[q].
\]

Applying \( \text{Hom}_C^{\text{cont}}(-, C) \) we arrive at quasi-isomorphisms

\[
\bigoplus_q \left( \bigwedge^q C \right) \to C^{\text{nor}}_q(C) \to C_\text{cd}(C),
\]

where by definition \( C^{\text{nor}}_q(C) \) is the continuous dual of \( \widehat{C}^{\text{nor}}(C) \). An easy calculation shows that \( C^{\text{nor},q}_\text{cd}(C) = T_\text{poly}^{\text{nor},q-1}(C) \).

One instance to which this theorem applies is \( C := K[[t_1, \ldots, t_n]] \). Here is another:

**Corollary 4.12.** Suppose \( C \) is a smooth \( K \)-algebra. Then the homomorphism \( U_1 : T_\text{poly}(C) \to D_\text{nor}(C) \) and the inclusion \( D_\text{nor}(C) \to D_\text{poly}(C) \) are both quasi-isomorphisms.

**Proof.** There is an open covering \( \text{Spec} C = \bigcup \text{Spec} C_i \) such that for every \( i \) there is an étale homomorphism \( K[t_1, \ldots, t_n] \to C_i \). Now use Theorem 4.11, Proposition 2.7 and faithful flatness. □

Here is a slight variation of the celebrated result of Kontsevich, known as the Formality Theorem [5, Theorem 6.4].

**Theorem 4.13.** Let \( K[t] = K[t_1, \ldots, t_n] \) be the polynomial algebra in \( n \) variables, and assume that \( R \subset K \). There is a collection of \( K \)-linear homomorphisms

\[
U_j : \bigwedge^j T_\text{poly}(K[t]) \to D_\text{poly}(K[t]),
\]

indexed by \( j \in \{1, 2, \ldots\} \), satisfying the following conditions.

(i) The sequence \( U = \{U_j\} \) is an \( L_\infty \)-morphism \( T_\text{poly}(K[t]) \to D_\text{poly}(K[t]) \).

(ii) Each \( U_j \) is a poly differential operator of \( K[t] \)-modules.

(iii) Each \( U_j \) is equivariant for the standard action of \( \text{GL}_n(K) \) on \( K[t] \).

(iv) The homomorphism \( U_1 \) is given by Eq. (4.10).

(v) For any \( j \geq 2 \) and \( \alpha_1, \ldots, \alpha_j \in T^0_\text{poly}(K[t]) \) one has \( U_j(\alpha_1 \wedge \cdots \wedge \alpha_j) = 0 \).

(vi) For any \( j \geq 2, \alpha_1 \in \text{gl}_n(K) \subset T^0_\text{poly}(K[t]) \) and \( \alpha_2, \ldots, \alpha_j \in T^0_\text{poly}(K[t]) \) one has \( U_j(\alpha_1 \wedge \cdots \wedge \alpha_j) = 0 \).

**Proof.** First let us assume that \( K = R \). Theorem 6.4 in [5] talks about the differentiable manifold \( \mathbb{R}^n \), and considers \( C^\infty \) functions on it, rather than polynomial functions. However, by construction the operators \( U_j \) are multi differential operators with polynomial coefficients (see [5, Section 6.3]). Therefore they descend to operators

\[
U_j : \bigwedge^j T_\text{poly}(\mathbb{R}[t]) \to D_\text{poly}(\mathbb{R}[t]),
\]
and conditions (i) and (ii) hold. Conditions (iii), (v) and (vi) are properties P3, P4 and P5 respectively in [5, Section 7]. For condition (iv) see [5, Sections 4.6.1–2].

For a field extension \( \mathbb{R} \subset \mathbb{K} \) use base change. \( \square \)

**Remark 4.14.** It is likely that the operator \( \mathcal{U}_j \) sends \( \bigwedge^j \mathcal{T}_{\text{poly}}(\mathbb{K}[t]) \) into \( \mathcal{D} \) \( \text{nor} \)_\( \text{poly} \)(\( \mathbb{K}[t] \)). This is clear for \( j = 1 \), where \( \mathcal{U}_1(\mathcal{T}_{\text{poly}}(\mathbb{K}[t])) = F_1 \mathcal{D} \) \( \text{nor} \)_\( \text{poly} \)(\( \mathbb{K}[t] \)); but this requires checking for \( j \geq 2 \).

In the next theorem \( \mathcal{T}_{\text{poly}}(\mathbb{K}[t]) \) and \( \mathcal{D} \) \( \text{poly} \)(\( \mathbb{K}[t] \)) are considered as DG Lie algebras in \( \text{Dir Inv Mod} \) \( \mathbb{K} \), with their \( t \)-adic dir-inv structures. Recall the notions of twisted DG Lie algebra (Lemma 3.24) and multilinear extensions of \( L_\infty \) morphisms (Proposition 3.26).

**Theorem 4.15.** Assume \( \mathbb{R} \subset \mathbb{K} \). Let \( A = \bigoplus_{i \geq 0} A^i \) be a complete super-commutative associative unital DG algebra in \( \text{Dir Inv Mod} \) \( \mathbb{K} \). Consider the induced continuous \( A \)-multilinear \( L_\infty \) morphism

\[
\mathcal{U}_A : A \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[t]) \to A \widehat{\otimes} \mathcal{D} \text{poly}(\mathbb{K}[t]).
\]

Suppose \( \omega \in A^1 \widehat{\otimes} \mathcal{T}_{\text{poly}}^0(\mathbb{K}[t]) \) is a solution of the Maurer–Cartan equation in \( A \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[t]) \). Define \( \omega' := (\partial^1 \mathcal{U}_A)(\omega) \in A^1 \widehat{\otimes} \mathcal{D}_{\text{poly}}^0(\mathbb{K}[t]) \). Then \( \omega' \) is a solution of the Maurer–Cartan equation in \( A \widehat{\otimes} \mathcal{D} \text{poly}(\mathbb{K}[t]) \), and there is continuous \( A \)-multilinear \( L_\infty \) quasi-isomorphism

\[
\mathcal{U}_{A,\omega} : (A \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[t]))_\omega \to (A \widehat{\otimes} \mathcal{D} \text{poly}(\mathbb{K}[t]))_{\omega'},
\]

whose Taylor coefficients are

\[
(\partial^j \mathcal{U}_{A,\omega})(\omega)(\alpha) := \sum_{k \geq 0} \frac{1}{(j + k)!} (\partial^{j+k} \mathcal{U}_A)(\omega^k \wedge \alpha)
\]

for \( \alpha \in \prod^j (A \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[t]))_\omega \).

**Proof.** By condition (ii) of Theorem 4.13, and by Proposition 2.4, each operator \( \partial^j \mathcal{U} := \mathcal{U}_j \) is continuous for the \( t \)-adic dir-inv structures on \( \mathcal{T}_{\text{poly}}(\mathbb{K}[t]) \) and \( \mathcal{D} \text{poly}(\mathbb{K}[t]) \). Therefore there is a unique continuous \( A \)-multilinear extension \( \partial^j \mathcal{U}_A \). Condition (v) of Theorem 4.13 implies that \( \partial^j \mathcal{U}_A(\omega^j) = 0 \) for \( j \geq 2 \). By Theorem 3.27 we get an \( L_\infty \) morphism \( \mathcal{U}_{A,\omega} \).

It remains to prove that \( \partial^j \mathcal{U}_{A,\omega} \) is a quasi-isomorphism. According to Theorem 4.11 for every \( i \) the \( K \)-linear homomorphism

\[
\partial^j \mathcal{U}_A : A^i \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[t]) \to A^i \widehat{\otimes} \mathcal{D} \text{poly}(\mathbb{K}[t])
\]

is a quasi-isomorphism. Since we are looking at bounded below complexes, a spectral sequence argument implies that

\[
\partial^j \mathcal{U}_{A,\omega} : A \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[t]) \to A \widehat{\otimes} \mathcal{D} \text{poly}(\mathbb{K}[t])
\]

is a quasi-isomorphism. \( \square \)
Acknowledgments

The author wishes to thank Vladimir Hinich, Bernhard Keller and James Stasheff for their assistance.

References