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Rings with Auslander Dualizing Complexes*

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A ring with an Auslander dualizing complex is a generalization of an Auslander–Gorenstein ring. We show that many results which hold for Auslander–Gorenstein rings also hold in the more general setting. On the other hand we give criteria for existence of Auslander dualizing complexes which show these occur quite frequently. The most powerful tool we use is the Local Duality Theorem for connected graded algebras over a field. Filtrations allow the transfer of results to nongraded algebras. We also prove some results of a categorical nature, most notably the functoriality of rigid dualizing complexes.

Contents.

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0. INTRODUCTION

Dualizing complexes play an essential role in the Serre–Grothendieck Duality Theory on schemes (see [RD]). The duality formalism was generalized to noncommutative rings by the first author, in order to answer some questions which arose in this context, mainly regarding local duality for noncommutative graded algebras (see [Ye1]). A version of Serre duality for noncommutative projective schemes was established using dualizing complexes (see Jørgensen [Jo2] and our [YZ1]). Dualizing complexes, and more generally derived categories, are powerful tools for proving abstract properties of noncommutative rings. For examples, consider the noncommutative graded versions of the Auslander–Buchsbaum Theorem, The Bass Theorem, and the No-Holes Theorem for Bass numbers (see [Jo1, Theorems 3.2, 4.5, 4.6, and 4.8]). Under the synonym ''cotilting complexes,'' dualizing complexes were studied by Miyachi [Mi2]. Cotilting bimodules occur often in papers on representations of finite-dimensional algebras, cf. [Ha].

In this paper we will provide further evidence that the dualizing complex (Definition 1.1) is an effective tool for studying noncommutative rings. We are especially interested in those dualizing complexes which satisfy an extra homological condition called the *Auslander property* (Definition 2.1). The basic idea here is that if a statement holds for Auslander–Gorenstein rings in the sense of [Bj], then an appropriate version of the statement should hold for rings with Auslander dualizing complexes. The Gorenstein condition, i.e., the ring itself having finite injective dimension, is considered to be very restrictive; in contrast, having an Auslander dualizing complex is considered to be a mild condition.

There are a few ways to show existence of Auslander dualizing complexes. For example, if A is a connected graded algebra (over a field k) with enough normal elements, then A has an Auslander dualizing complex. Recall that a connected graded k-algebra A has *enough normal elements* if every graded prime factor $A/\mathfrak{p} \neq k$ has a nonzero normal element of positive degree. This class of rings has been studied recently by many algebraists, because of developments in quantum groups and noncommutative algebraic geometry.

In this paper we prove a diverse collection of results, whose common thread is that their proofs are based on the existence of an Auslander dualizing complex. Throughout k denotes a fixed base field, and a k-algebra means an associative algebra with 1.

First we generalize [GL, Theorem 1.6] by dropping the Gorenstein condition. Note that the hypothesis on $\operatorname{gr} A$ in the next theorem is easy to check in practice (see Example 6.11), and we suspect it can even be weakened.

THEOREM 0.1. *Assume A is a normally separated filtered k-algebra such that* $\operatorname{gr} A$ *is a noetherian connected graded k-algebra with enough normal elements. Then* $\operatorname{Spec} A$ *is catenary.*

This is proved after Corollary 6.22.

We generalize some results in [ASZ], two of which are:

THEOREM 0.2. *Assume A is a noetherian k-algebra with an Auslander dualizing complex. Then there is a step duality between the category* $\mathsf{Mod}_f\, A$ *of finitely generated left A-modules and the category* $\mathsf{Mod}_f\, A^{\circ}$ *of finitely generated right A-modules.*

Actually we prove a more general result involving two algebras—see Theorem 2.15. It follows that if the algebra A has an Auslander dualizing complex then the left and right Krull dimensions of A are finite.

THEOREM 0.3. *Let A be an Auslander–Gorenstein noetherian k-algebra. Assume A has a filtration such that* $\operatorname{gr} A$ *is an AS-Gorenstein noetherian connected graded k-algebra* (*e.g., if A is connected graded*). *Then A has an artinian self-injective ring of fractions.*

For more details see Theorem 6.23.

The notion of characteristic variety $\operatorname{Ch} M$ of a module M was introduced in $\mathscr{D}$-module theory (cf. [Co, p. 98]). The next theorem generalizes a result of Gabber by dropping the Gorenstein condition. The possibility of making this generalization was suggested by Van den Bergh.

THEOREM 0.4 (Purity of Characteristic Variety). *Assume A is a filtered k-algebra such that* $\operatorname{gr} A$ *is a commutative connected graded affine k-algebra. If M is a finitely generated* GKdim-*pure left A-module, then the characteristic variety* $\operatorname{Ch} M$ *is pure.*

Given an algebra A there might exist non-isomorphic Auslander dualizing complexes over it (see Example 2.3(c)). For various reasons (like functoriality) it is desirable to find dualizing complexes which are canonical in some sense. In the graded case, the *balanced dualizing complex* (Definition 4.2), introduced by the first author in [Ye1], is a natural choice. In the upgraded case, one should consider the *rigid dualizing complex* (Definition 3.1) introduced by Van den Bergh [VdB]. Rigid dualizing complexes (and balanced dualizing complexes in the graded case) are uniquely determined up to a unique isomorphism. A balanced dualizing complex is always rigid (see [VdB, Proposition 8.2(2)] and our Corollary 6.7). The next theorem on the functoriality of rigid dualizing complexes is a combination of Theorem 3.2 and Corollary 3.4. A homomorphism $A \to B$ of algebras is called *finite* if B is a finitely generated left and right A-module.

THEOREM 0.5. *Let A be a noetherian k-algebra. A rigid dualizing complex R_A over A is unique up to a unique isomorphism. If $A \to B$ is a finite homomorphism and R_A, R_B are rigid dualizing complexes over A, B, respectively, then there is at most one morphism $\mathrm{Tr}_{B/A} : R_B \to R_A$ compatible with the rigidity.*

The existence of a dualizing complex is not automatic. A very effective criterion for existence of balanced complexes is given in [VdB, Theorem 6.3] (which is Theorem 4.6 here). Van den Bergh's idea was to first prove the Local Duality Theorem, and then to show this duality is represented by a balanced dualizing complex. The Rees algebra allows us to transfer results on graded algebras to non-graded algebras (see Theorem 6.2).

We prove that Auslander rigid dualizing complexes exist for a large class of rings. First, the similar submodule condition on graded A-modules (see Definition 5.12) enables induction on GKdim. Combining Theorems 5.13 and 5.14 we get:

THEOREM 0.6. *Assume A is a noetherian connected graded k-algebra which has a balanced dualizing complex R and satisfies the similar submodule condition (e.g., A is FBN or has enough normal elements). Then the balanced dualizing complex R is graded Auslander.*

Next, Theorem 6.2 says:

THEOREM 0.7. *Suppose A is a filtered k-algebra such that the associated graded algebra* $\mathrm{gr}\, A$ *is noetherian. If* $\mathrm{gr}\, A$ *has a graded Auslander balanced dualizing complex, then A has an Auslander rigid dualizing complex.*

By results of Grothendieck, a commutative affine connected graded k-algebra has a graded Auslander balanced dualizing complex (this also follows from Theorem 0.6). Since factor rings of universal enveloping algebras of finite dimensional Lie algebras are filtered, and their associated graded algebras are commutative, Theorem 0.7 tells us that these algebras have Auslander rigid dualizing complexes. In the same way we may use Theorems 0.7 and 0.6 to show that many quantum algebras and their factor algebras have Auslander rigid dualizing complexes. A key step in the proof of Theorem 0.7 is the following theorem (see Theorem 5.1 for full details and proof).

THEOREM 0.8. *Let A be a noetherian connected graded k-algebra. Suppose $t \in A$ is a nonzero homogeneous normal element of positive degree. Then A has a graded Auslander balanced dualizing complex if and only if so does $A/(t)$.*

As noted on [Zh, p. 399], the proof of [SZ, Lemma 6.1(ii)] has a gap, and an alternative proof of the result (under extra hypotheses) is given in [Zh, Theorem 3.1]. We now give a complete proof of [SZ, Lemma 6.1(ii)] using Auslander dualizing complexes (the proof is at the end of Section 5).

PROPOSITION 0.9. *Let A be a noetherian locally finite $\mathbb{N}$-graded k-algebra. If A is graded FBN, or A has enough normal elements, then* GKdim $M =$ Kdim $M \in \mathbb{N}$ *for every finitely generated left or right graded A-module M.*

Here are some other results we prove:

(1) Gabber's Maximality Principle (Theorem 2.19).

(2) Existence of double-Ext spectral sequence (Proposition 1.7).

(3) The existence of an Auslander rigid dualizing complex is transferred to related algebras (Propositions 4.18 and 4.20, Corollaries 4.17, 5.10, and 5.11).

The canonical dimension, denoted by Cdim, is defined when A has an Auslander dualizing complex (Definition 2.9). It is an exact finitely partitive dimension function (Theorem 2.10). Local duality implies that the canonical dimension is symmetric in the graded case (Proposition 4.13). Therefore if A is a connected graded algebra with an Auslander balanced dualizing complex, the canonical dimension Cdim is exact, finitely partitive, and symmetric on graded modules. Note that the Krull dimension, denoted by Kdim, is exact and finitely partitive, but it is unknown whether it is symmetric. On the other hand the Gelfand–Kirillov dimension, denoted by GKdim, is symmetric, but neither exact nor finitely partitive in general. Hence the canonical dimension is the better dimension function —at least in the graded or filtered case.

The study of dualizing complexes over noncommutative rings presents many interesting and subtle questions. We conclude the introduction by mentioning two of them:

QUESTION 0.10. *Which (noetherian, affine) k-algebras have (rigid) dualizing complexes?*

QUESTION 0.11. *Is a rigid dualizing complex always Auslander?*

1. DUALIZING COMPLEXES

Let k be a field and let A be an associative k-algebra with 1. All A-modules will be by default left modules, and we denote by $\mathsf{Mod}\, A$ the category of left A-modules. Let A° be the opposite algebra, and let $A^e := A \otimes A^\circ$ where $\otimes = \otimes_k$. Thus an A^e-module M is, in the conventional notation, an A-A-bimodule ${}_AM_A$ central over k. Most of our definitions and results have a left–right symmetry, expressible by the exchange of $A \leftrightarrow A^{\circ}$. Since these symmetries are evident we shall usually not mention them.

Let $\mathsf{D}(\mathsf{Mod}\, A)$ be the derived category of A-modules, and let $\mathsf{D}^{\star}(\mathsf{Mod}\, A)$, for $\star = b,\ +,\ -$, or blank, be the full subcategories of bounded, bounded below, bounded above, or unbounded complexes, respectively [RD].

Given another k-algebra B, the forgetful functor $\mathsf{Mod}(A \otimes B^{\circ}) \to \mathsf{Mod}\, A$ is exact, and so induces a functor $\mathsf{D}^{\star}(\mathsf{Mod}(A \otimes B^{\circ})) \to \mathsf{D}^{\star}(\mathsf{Mod}\, A)$. Now $A \otimes B^{\circ}$ is a projective A-module, so any projective (resp. flat, injective) $(A \otimes B^{\circ})$-module is projective (resp. flat, injective) over A.

Consider k-algebras A, B, C. For complexes $M \in \mathsf{D}(\mathsf{Mod}(A \otimes B^{\circ}))$ and $N \in \mathsf{D}(\mathsf{Mod}(A \otimes C^{\circ}))$, with either $M \in \mathsf{D}^{-}$ or $N \in \mathsf{D}^{+}$, there is a derived functor

$$\mathrm{R}\,\mathrm{Hom}_A(M, N) \in \mathsf{D}(\mathsf{Mod}(B \otimes C^{\circ})).$$

It is calculated by replacing M with an isomorphic complex in $\mathsf{D}^{-}(\mathsf{Mod}(A \otimes B^{\circ}))$ which consists of projective modules over A, or by replacing N with an isomorphic complex in $\mathsf{D}^{+}(\mathsf{Mod}(A \otimes C^{\circ}))$ which consists of injective modules over A. For full details see [RD, Ye1]. Note that for modules M and N, viewed as complexes concentrated in degree 0, one has

$$\mathrm{H}^{q}\,\mathrm{R}\,\mathrm{Hom}_A(M, N) = \mathrm{Ext}^{q}_A(M, N),$$

the latter being the usual Ext.

Because the forgetful functors $\mathsf{Mod}(A \otimes B^{\circ}) \to \mathsf{Mod}\, A$, etc., commute with $\mathrm{R}\,\mathrm{Hom}_A(-, -)$ there is no need to mention them explicitly.

A complex $N \in \mathsf{D}^{+}(\mathsf{Mod}\, A)$ is said to have finite injective dimension if there is an integer q_0 with $\mathrm{Ext}^{q}_A(M, N) = 0$ for all $q > q_0$ and $M \in \mathsf{Mod}\, A$.

For the rest of this section A denotes a left noetherian k-algebra and B denotes a right noetherian k-algebra. (For instance we could take $A = B$ a two-sided noetherian algebra.) Observe that the algebra $A \otimes B^{\circ}$ need not be left noetherian.

The subcategory $\mathsf{Mod}_f\, A$ of finitely generated A-modules is abelian and closed under extensions. Hence there is a full triangulated subcategory $\mathsf{D}_f(\mathsf{Mod}\, A) \subset \mathsf{D}(\mathsf{Mod}\, A)$ consisting of all complexes with finitely generated cohomologies.

Dualizing complexes over commutative rings were introduced in [RD]. The noncommutative graded version first appeared in [Ye1], and we now give a slightly more general version.

DEFINITION 1.1. Assume A and B are k-algebras, with A left noetherian and B right noetherian. A complex $R \in \mathsf{D}^b(\mathsf{Mod}(A \otimes B^\circ))$ is called a *dualizing complex* if it satisfies the three conditions below:

(i) R has finite injective dimension over A and B°.

(ii) R has finitely generated cohomology modules over A and B°.

(iii) The canonical morphisms $B \to \mathrm{R\,Hom}_A(R, R)$ in $\mathsf{D}(\mathsf{Mod}\, B^e)$, and $A \to \mathrm{R\,Hom}_{B^\circ}(R, R)$ in $\mathsf{D}(\mathsf{Mod}\, A^e)$, are both isomorphisms.

In case $A = B$, we shall say that R is a dualizing complex over A.

Condition (i) is equivalent to having an isomorphism $R \cong I \in \mathsf{D}^b(\mathsf{Mod}\, A \otimes B^\circ)$, where each I^q is injective over A and over B°.

EXAMPLE 1.2. Suppose A is commutative and R is a dualizing complex in the sense of [RD]. If we consider R as a complex of bimodules, by identifying $A = A^\circ$, then R is a dualizing complex in the sense of the definition above. According to [Ye3], if Spec A is connected, then any dualizing complex R' over A is isomorphic to $R \otimes_A P[n]$, where P is an invertible bimodule (not necessarily central!) and $n \in \mathbb{Z}$.

Some easy examples of dualizing complexes over noncommutative rings are given in Example 2.3.

The next proposition offers an explanation of the name "dualizing complex." The duality functors associated to R are the contravariant functors

$$D := \mathrm{R\,Hom}_A(-, R) : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B^\circ)$$
$$D^\circ := \mathrm{R\,Hom}_{B^\circ}(-, R) : \mathsf{D}(\mathsf{Mod}\, B^\circ) \to \mathsf{D}(\mathsf{Mod}\, A).$$

PROPOSITION 1.3. *Let $R \in \mathsf{D}(\mathsf{Mod}(A \otimes B^\circ))$ be a dualizing complex.*

(1) *For any $M \in \mathsf{D}_f(\mathsf{Mod}\, A)$ one has $DM \in \mathsf{D}_f(\mathsf{Mod}\, B^\circ)$ and $M \cong D^\circ DM$.*

(2) *The functors D and D° determine a duality, i.e., an anti-equivalence, of triangulated categories between $\mathsf{D}_f(\mathsf{Mod}\, A)$ and $\mathsf{D}_f(\mathsf{Mod}\, B^\circ)$, restricting to a duality between $\mathsf{D}^b_f(\mathsf{Mod}\, A)$ and $\mathsf{D}^b_f(\mathsf{Mod}\, B^\circ)$.*

Proof. (1) This is slightly stronger than [Ye1, Lemma 3.5]. By adjunction we get a functorial morphism $M \to D^\circ DM$. Since the functor $D^\circ D$ is way out in both directions and $D^\circ DA \cong A$ by assumption, the claim follows from the reversed forms of [RD, Propositions I.7.1 and I.7.3].

(2) This is immediate from part (1), together with the fact that $M \in \mathsf{D}^b(\mathsf{Mod}\, A)$ implies $DM \in \in \mathsf{D}^b(\mathsf{Mod}\, B^\circ)$. ∎

Remark 1.4. The noetherian hypothesis can be relaxes—dualizing complexes can be defined over any coherent algebra A (see [Ye1, Mi1]). The category of finitely generated A-modules is then replaced by the category of coherent modules. Many definitions and results in our paper hold for coherent algebras, as can be easily checked.

Perhaps one can even work over an arbitrary algebra, using the category of coherent complexes, as defined by Illusie (see [SGA6, Exposé I]).

Another direction to extend the theory is to allow k to be any commutative ring. In this case the derived category of bimodules should be $\mathsf{D}(\mathsf{DGMod}(A \otimes^L_k B^\circ))$, where $A \otimes^L_k B^\circ$ is a differential graded algebra. See [Ye3, Remark 1.12].

Remark 1.5. Miyachi proved a converse to Proposition 1.3(2): if there are contravariant triangle functors $\mathsf{D}(\mathsf{Mod}\ A) \to \mathsf{D}(\mathsf{Mod}\ B^\circ)$ and $\mathsf{D}(\mathsf{Mod}\ B^\circ) \to \mathsf{D}(\mathsf{Mod}\ A)$ which send $\oplus$ to $\prod$, preserve D^b, and induce duality on D^b_f, then there is a dualizing complex in $\mathsf{D}(\mathsf{Mod}(A \otimes B^\circ))$ (see [Mi2, Theorem 3.3]).

Remark 1.6. There are examples of algebras A and B where there is a dualizing complex $R \in \mathsf{D}(\mathsf{Mod}(A \otimes B^\circ))$, but there is no dualizing complex in $\mathsf{D}(\mathsf{Mod}\ A^e)$; cf. [WZ]. The algebras A and B are necessarily not derived Morita equivalent, since given a tilting complex $T \in \mathsf{D}(\mathsf{Mod}(B \otimes A^\circ))$, the complex $R \otimes^L_B T \in \mathsf{D}(\mathsf{Mod}\ A^e)$ would be dualizing (cf. [Ye3]).

There are Grothendieck spectral sequences for the isomorphism of functors $1_{\mathsf{D}^b_f(\mathsf{Mod}\ A)} \cong D^\circ D$ and $1_{\mathsf{D}^b_f(\mathsf{Mod}\ B^\circ)} \cong DD^\circ$. For modules they take this form:

PROPOSITION 1.7. *Let $R \in \mathsf{D}(\mathsf{Mod}(A \otimes B^\circ))$ be a dualizing complex. Then three are convergent* double-Ext *spectral sequences*

$$E_2^{p,q} := \mathrm{Ext}^p_{B^\circ}\big(\mathrm{Ext}^{-q}_A(M,R),R\big) \Rightarrow M \tag{1.8}$$

for all $M \in \mathsf{Mod}_f\ A$, and

$$E_2^{p,q} := \mathrm{Ext}^p_A\big(\mathrm{Ext}^{-q}_{B^\circ}(N,R),R\big) \Rightarrow N \tag{1.9}$$

for all $N \in \mathsf{Mod}_f\ B^\circ$.

Proof. By symmetry it suffices to consider (1.8) only. We can assume R is a bounded complex of bimodules with each R^q an injective module over A and B°. Given a nonzero finitely generated A-module M, define the complex

$$H := \mathrm{Hom}_{B^\circ}\big(\mathrm{Hom}_A(M,R),R\big).$$

Then the adjunction homomorphism $M \to H$ is a quasi-isomorphism. Pick a positive integer d large enough so that $R^q = 0$ if $|q| > d$. Consider the decreasing filtration on H given by the subcomplexes

$$F^p H := \operatorname{Hom}_{B^\circ}(\operatorname{Hom}_A(M, R), R^{\geq p}).$$

Then F is an exhaustive filtration, and it determines the convergent spectral sequence (1.8). ∎

Given $M \in \mathsf{D}^+(\mathsf{Mod}\, A)$, there is a quasi-isomorphism $M \to I$ in $\mathsf{D}^+(\mathsf{Mod}\, A)$, where each I^q is injective and $\operatorname{Ker}(I^q \to I^{q+1}) \subset I^q$ is essential. Such I is unique (up to a non-unique isomorphism), and it is called the *minimal injective resolution* of M (cf. [Ye1, Lemma 4.2]). If M has finite injective dimension then I is bounded.

The next two results are straightforward generalizations of [ASZ, Lemma 2.2 and Theorem 2.3], so the proofs are omitted.

LEMMA 1.10. *Let $R \in \mathsf{D}(\mathsf{Mod}(A \otimes B^\circ))$ be a dualizing complex, and let I be the minimal injective resolution of R in $\mathsf{D}^b(\mathsf{Mod}\, A)$. Let $Z_i := \operatorname{Ker}(I^i \to I^{i+1})$ and let M be a finitely generated left A-module. Then there exist $f_1, \dots, f_n \in \operatorname{Hom}(M, Z_i)$ such that for every $N \subset \bigcap_j \operatorname{Ker}(f_j)$ the natural map $\operatorname{Ext}^i_A(M, R) \to \operatorname{Ext}^i_A(N, R)$ is zero; or equivalently, the natural map $\operatorname{Ext}^i_A(M/N, R) \to \operatorname{Ext}^i_A(M, R)$ is surjective.*

THEOREM 1.11. *Let $R \in \mathsf{D}(\mathsf{Mod}(A \otimes B^\circ))$ be a dualizing complex, let I be the minimal injective resolution of R in $\mathsf{D}^b(\mathsf{Mod}\, A)$, and let $Z_i := \operatorname{Ker}(I^i \to I^{i+1})$. Then:*

(1) *For every nonzero A-module M there is a nonzero submodule $N \subset M$ which embeds in some Z_i.*

(2) *Every indecomposable injective A-module appears in I.*

We conclude this section with a discussion of dualizing complexes in $\mathsf{D}(\mathsf{Mod}(A \otimes B^\circ))$ when A is commutative. For a prime ideal $\mathfrak{p} \subset A$ let $J_A(\mathfrak{p})$ be an injective hull of $A/\mathfrak{p}$. Let us recall a result of Grothendieck.

PROPOSITION 1.12 [RD, Proposition V.7.3]. *Suppose A is a commutative noetherian ring and $R \in \mathsf{D}^b_f(\mathsf{Mod}\, A)$ is a (central) dualizing complex. Let I be the minimal injective resolution of R in $\mathsf{Mod}\, A$.*

(1) *There is a function $d : \operatorname{Spec} A \to \mathbb{Z}$ such that*

$$I^q \cong \bigoplus_{d(\mathfrak{p}) = q} J_A(\mathfrak{p}).$$

(2) *If $\mathfrak{p} \subset \mathfrak{q}$ are primes and $\mathfrak{q}/\mathfrak{p} \subset A/\mathfrak{p}$ has height 1, then $d(\mathfrak{p}) = d(\mathfrak{q}) - 1$.*

The function d is called a codimension function in [RD]. In our case we get:

THEOREM 1.13. *Suppose A is a commutative noetherian k-algebra, B is a right noetherian k-algebra, and $R \in \mathsf{D}(\mathsf{Mod}(A \otimes B^{\circ}))$ is a dualizing complex. Let I be the minimal injective resolution R in $\mathsf{Mod}\, A$. Then*:

(1) *There are functions $d, r : \operatorname{Spec} A \to \mathbb{Z}$, with $r \geq 1$ and constant on connected components of $\operatorname{Spec} A$, s.t.*

$$I^q \cong \bigoplus_{d(\mathfrak{p})=q} J_A(\mathfrak{p})^{r(\mathfrak{p})}.$$

(2) *If $\mathfrak{p} \subset \mathfrak{q}$ are primes of A and $\mathfrak{q}/\mathfrak{p} \subset A/\mathfrak{p}$ has height 1, then $d(\mathfrak{p}) = d(\mathfrak{q}) - 1$.*

(3) *A is catenary, and (if $A \neq 0$) its Krull dimension is*

$$\operatorname{Kdim} A \leq \max\{d(\mathfrak{p})\} - \min\{d(\mathfrak{p})\} < \infty.$$

(4) *B is an Azumaya A-algebra.*

The proof of the theorem is after the next lemma.

LEMMA 1.14. *Assume in addition that A is local. Then there is an integer d such that $\operatorname{Ext}^q_A(M, R) = 0$ and $\operatorname{Ext}^q_{B^{\circ}}(N, R) = 0$ for all $q \neq d$, all finite length A-modules M, and all finite length B°-modules N.*

Proof. According to [Ye3, Proposition 5.4] (which works even when $A \neq B$; cf. ibid. Proposition 2.5), left and right multiplications on R induce ring isomorphisms $A \cong \operatorname{End}_{\mathsf{D}(\mathsf{Mod}(A\otimes B^{\circ}))}(R) \cong Z(B)$. Moreover since A is noetherian and $B^{\circ} \cong \operatorname{Ext}^0_A(R, R)$ we see that B is a finite A-algebra. If N is an A-central $(A \otimes B^{\circ})$-module, then $\operatorname{Ext}^q_{B^{\circ}}(N, R)$ is a central A-bimodule.

Denote by K the residue field of A. Let $p_0 := \min\{p | \operatorname{Ext}^p_A(K, R) \neq 0\}$ and $p_1 := \max\{p | \operatorname{Ext}^p_A(K, R) \neq 0\}$. By induction on length we see that for every finite length A-module M, $p_0 = \min\{p | \operatorname{Ext}^p_A(M, R) \neq 0\}$ and $p_1 = \max\{p | \operatorname{Ext}^p_A(M, R) \neq 0\}$.

Now take a nonzero finite length B°-module N. Since we can view N as a central A-bimodule, it follows that $\operatorname{Ext}^q_{B^{\circ}}(N, R)$ is also a central A-bimodule for every q; and so it has finite length. Define $q_0 := \min\{q | \operatorname{Ext}^q_{B^{\circ}}(N, R) \neq 0\}$ and $q_1 := \max\{q | \operatorname{Ext}^q_{B^{\circ}}(N, R) \neq 0\}$. In the E_2-page of the spectral sequence of Proposition 1.7 we have nonzero terms $E_2^{p_1, -q_0} = \operatorname{Ext}^{p_1}_A(\operatorname{Ext}^{q_0}_{B^{\circ}}(N, R))$ and $E_2^{p_0, -q_1} = \operatorname{Ext}^{p_0}_A(\operatorname{Ext}^{q_1}_{B^{\circ}}(N, R))$, that appear in the right-top and left-bottom corners, respectively. The convergence of the spectral sequence forces $p_1 = q_0$ and $p_0 = q_1$. Therefore we get $d := p_1 = q_0 = q_1 = p_0$. ∎

Proof of Theorem 1.13. (1) As observed in the proof of the lemma, B is a finite A-algebra. Take a prime $\mathfrak{p} \subset A$, and define $B_{\mathfrak{p}} := B \otimes_A A_{\mathfrak{p}}$ and $R_{\mathfrak{p}} := A_{\mathfrak{p}} \otimes_A R \otimes_B B_{\mathfrak{p}} \in \mathsf{D}^b(\mathsf{Mod}(A_{\mathfrak{p}} \otimes B_{\mathfrak{p}}^{\circ}))$. We claim that $R_{\mathfrak{p}}$ is a dualizing complex.

First note that the cohomologies $H^q R$ are central A-bimodules. Since $A \to A_{\mathfrak{p}}$ is flat and $A_{\mathfrak{p}} \otimes_A A_{\mathfrak{p}} = A_{\mathfrak{p}}$, there are isomorphisms $H^q R_{\mathfrak{p}} \cong A_{\mathfrak{p}} \otimes_A H^q R \cong H^q R \otimes_B B_{\mathfrak{p}}$. Therefore $R_{\mathfrak{p}} \cong A_{\mathfrak{p}} \otimes_A R \cong R \otimes_B B_{\mathfrak{p}}$ in $\mathsf{D}^b(\mathsf{Mod}(A \otimes B^{\circ}))$. It is easy to see that the cohomology bimodules $H^q R$ are finitely generated on both sides, $\mathrm{R}\operatorname{Hom}_{A_{\mathfrak{p}}}(R_{\mathfrak{p}}, R_{\mathfrak{p}}) \cong B_{\mathfrak{p}}^{\circ}$ and $\mathrm{R}\operatorname{Hom}_{B_{\mathfrak{p}}^{\circ}}(R_{\mathfrak{p}}, R_{\mathfrak{p}}) \cong A_{\mathfrak{p}}$.

In order to verify that $R_{\mathfrak{p}}$ has finite injective dimension over $B_{\mathfrak{p}}^{\circ}$ it suffices to show that $\operatorname{Ext}^q_{B_{\mathfrak{p}}^{\circ}}(N, R_{\mathfrak{p}})$ vanishes for all finitely generated $B_{\mathfrak{p}}^{\circ}$-modules N, for large q. Now we can write $N \cong N' \otimes_B B_{\mathfrak{p}}$ for some finitely generated B°-module N'. Then $\operatorname{Ext}^q_{B_{\mathfrak{p}}^{\circ}}(N, R_{\mathfrak{p}}) \cong \operatorname{Ext}^q_{B^{\circ}}(N', R) \otimes_A A_{\mathfrak{p}}$. Likewise for the injective dimension over $A_{\mathfrak{p}}$. So indeed $R_{\mathfrak{p}}$ is dualizing.

Since the multiplicity of the indecomposable injective $J_A(\mathfrak{p})$ is measured by $\operatorname{Ext}^q_{A_{\mathfrak{p}}}(k(\mathfrak{p}), R_{\mathfrak{p}})$, where $k(\mathfrak{p})$ is the residue field, the lemma says that $J_A(\mathfrak{p})$ occurs in the complex I in only one degree, say $d(\mathfrak{p})$. The fact that the multiplicity $r(\mathfrak{p})$ is locally constant will be proved in part 4 below.

(2) Choose $a \in \mathfrak{q} - \mathfrak{p}$, so in the exact sequence

$$0 \to (A/\mathfrak{p})_{\mathfrak{q}} \xrightarrow{a} (A/\mathfrak{p})_{\mathfrak{q}} \to M \to 0$$

the $A_{\mathfrak{q}}$-module M has finite length. Applying $\operatorname{Ext}^q_A(-, R)$ to this sequence we obtain an exact sequence of finitely generated $A_{\mathfrak{q}}$-modules

$$\operatorname{Ext}^q_A\big((A/\mathfrak{p})_{\mathfrak{q}}, R\big) \xrightarrow{a} \operatorname{Ext}^q_A\big((A/\mathfrak{p})_{\mathfrak{q}}, R\big) \to \operatorname{Ext}^{q+1}_A(M, R)$$

for each q. By Nakayama's Lemma and part (1) we find that $\operatorname{Ext}^q_A((A/\mathfrak{p})_{\mathfrak{q}}, R) = 0$ unless $q + 1 = d(\mathfrak{q})$. Hence $d(\mathfrak{p}) = d(\mathfrak{q}) - 1$ as claimed.

(3) This follows trivially from (2).

(4) Pick a prime ideal $\mathfrak{p} \subset A$. Since $R_{\mathfrak{p}}$ is a dualizing complex in $\mathsf{D}(\mathsf{Mod}(A_{\mathfrak{p}} \otimes B_{\mathfrak{p}}^{\circ}))$, the lemma tells us that the functors $\operatorname{Ext}^{d(\mathfrak{p})}_{A_{\mathfrak{p}}}(-, R_{\mathfrak{p}})$ and $\operatorname{Ext}^{d(\mathfrak{p})}_{B_{\mathfrak{p}}^{\circ}}(-, R_{\mathfrak{p}})$ are a duality between the categories of finite length modules over $A_{\mathfrak{p}}$ and $B_{\mathfrak{p}}^{\circ}$. Furthermore since this duality is $A_{\mathfrak{p}}$-linear, it restricts to a duality between $\mathsf{Mod}_f A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^n$ and $\mathsf{Mod}_f (B_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^n B_{\mathfrak{p}})^{\circ}$ for every $n \geq 1$. Since $\mathsf{Mod}_f A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^n$ has an auto-duality, namely $\operatorname{Hom}_{A_{\mathfrak{p}}}(-, J_A(\mathfrak{p}))$, it follows that there is an equivalence between $\mathsf{Mod}_f A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^n$ and $\mathsf{Mod}_f B_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^n B_{\mathfrak{p}}$. Morita Theory says there is an isomorphism $\phi_n : B_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^n B_{\mathfrak{p}} \xrightarrow{\simeq} M_{r(n)}(A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}^n)$ for some number $r(n)$.

Since the isomorphisms ϕ_n all arise from the same equivalence, they can be made compatible, and in particular we have $r(n) = r(\mathfrak{p})$ for all n. In the inverse limit we get $\hat{A}_{\mathfrak{p}} \otimes_A B \cong M_{r(\mathfrak{p})}(\hat{A}_{\mathfrak{p}})$ as $\hat{A}_{\mathfrak{p}}$-algebras.

Since $\mathfrak{p}$ was an arbitrary prime ideal of A, we conclude that the multiplication map $B \otimes_A B^{\circ} \to \mathrm{End}_A(B)$ is bijective. Therefore B is Azumaya over A. As a projective A-module, the rank of B at a prime $\mathfrak{p}$ is precisely $r(\mathfrak{p})^2$, so the function r is locally constant. ∎

Remark 1.15. A complex such as I in Theorem 1.12 is called a residual complex. It actually depends functorially on R: $I = ER$, where E is the Cousin functor. Noncommutative variants of the Cousin functor are studied in [Ye2, YZ2]. In particular one can show that the complex I can be made into a complex of bimodules, where on the right it is the minimal injective resolution of R in $\mathsf{Mod}\, B^{\circ}$. It follows that R is an Auslander dualizing complex, as defined in Section 2. In [YZ2] we show that a more complicated version of Theorem 1.13 holds when $A = B$ is a PI algebra.

EXAMPLE 1.16. Let A be a noetherian commutative regular ring of infinite Krull dimension—see Nagata's example [Na, Appendix, p. 203, Example 1]. The complex $R = A$ is a pointwise dualizing complex (see [RD, Sect. V.8]), and $\mathrm{R\,Hom}_A(-, A) : \mathsf{D}_f(\mathsf{Mod}\, A) \to \mathsf{D}_f(\mathsf{Mod}\, A)$ is a duality. However, by Theorem 1.13, there is no dualizing complex in $\mathsf{D}^b(\mathsf{Mod}(A \otimes B^{\circ}))$ for any right noetherian k-algebra B.

EXAMPLE 1.17. Let A be the non-catenary noetherian commutative local ring of [Na, Appendix, p. 203, Example 2]. Then again there is no dualizing complex in $\mathsf{D}^b(\mathsf{Mod}(A \otimes B^{\circ}))$ for any right noetherian k-algebra B.

2. AUSLANDER DUALIZING COMPLEXES

The basic ideas in this section already appear in [Ye2, Sect. 1] (which treats graded algebras).

Assume $R \in \mathsf{D}^b(\mathsf{Mod}(A \otimes B^{\circ}))$ is a dualizing complex. Let M be a finitely generated A-module. The *grade of M with respect to R* is

$$j_{R;A}(M) := \inf\{q \mid \mathrm{Ext}^q_A(M, R) \neq 0\} \in \mathbb{Z} \cup \{\infty\}.$$

Similarly define $j_{R;B^{\circ}}$ for a B°-module.

We are ready to define the notion appearing in the title of the paper.

DEFINITION 2.1. Let A and B be k-algebras, with A left noetherian and B right noetherian, and let $R \in \mathsf{D}(\mathsf{Mod}(A \otimes B^{\circ}))$ be a dualizing complex. We say that R has the *Auslander property*, or that R is an *Auslander dualizing complex*, if

(i) for every finitely generated A-module M, integer q, and B°-submodule $N \subset \mathrm{Ext}^q_A(M, R)$ one has $j_{R;\, B^\circ}(N) \geq q$;

(ii) the same holds after exchanging A and B°.

Note that the role of the algebras A and B° is symmetric. Also note that if R is an Auslander dualizing complex, then any shift $R[n]$ is also an Auslander dualizing complex (the shift cancels out in the double dual).

EXAMPLE 2.2. Let A be a commutative k-algebra and R a central dualizing complex over it. From Proposition 1.12 it is clear that R is Auslander. For a prime ideal $\mathfrak{p}$ one has $j_{R;\, A}(A/\mathfrak{p}) = d(\mathfrak{p})$.

EXAMPLE 2.3. (a) If A is Gorenstein, i.e., the bimodule A has finite injective dimension on both sides, then $R = A$ is a dualizing complex over A. If A is an Auslander–Gorenstein ring in the sense of [Bj], then $R = A$ is an Auslander dualizing complex.

(b) If A is a finite k-algebra, i.e., $\mathrm{rank}_k\, A < \infty$, then the bimodule $A^* := \mathrm{Hom}_k(A, k)$ is injective on both sides, and $R = A^*$ is an Auslander dualizing complex over A. Clearly $j_{R;\, A}(M) = 0$ for all M.

(c) Let A be the matrix algebra $\left(\begin{smallmatrix} k & V \\ 0 & k \end{smallmatrix}\right)$ where V is a finite rank k-module. Since A is hereditary it is also Gorenstein, so both A and A^* are dualizing complexes, and if $V \neq 0$ they are non-isomorphic. According to [ASZ, Example 5.4], the dualizing complex A is Auslander iff $\mathrm{rank}_k\, V \leq 1$. See also [Ye3, Sect. 3].

More examples of algebras with Auslander dualizing complexes are in Examples 6.11–6.15.

The next definition is taken form [MR, Sect. 6.8.4] (with a slight modification—we allow negative dimensions).

DEFINITION 2.4. An *exact dimension function* is a function dim which assigns to each module $M \in \mathsf{Mod}_f\, A$ a value $\dim M$ in an ordered set containing $-\infty$, $\mathbb{R}$, and some infinite ordinals, and satisfies the following axioms:

(i) $\dim 0 = -\infty$.

(ii) For every short exact sequence $0 \to M' \to M \to M'' \to 0$ one has $\dim M = \max\{\dim M', \dim M''\}$.

(iii) If $\mathfrak{p} M = 0$ for some prime ideal $\mathfrak{p}$, and M is a torsion $A/\mathfrak{p}$-module, then $\dim M \leq \dim A/\mathfrak{p} - 1$.

Familiar examples of exact dimension functions are the Krull dimension Kdim and (sometimes) the Gelfand–Kirillov dimension GKdim.

LEMMA 2.5. *A function* dim *defined on* $\mathsf{Mod}_f\ A$ *and satisfying axioms* (i)–(ii) (*resp. axioms* (i)–(iii)) *extends uniquely to a function on* $\mathsf{Mod}\ A$, *satisfying axioms* (i)–(ii) (*resp. axioms* (i)–(iii)) *and the axiom*

(iv) $\dim M = \sup\{\dim M' \mid M' \subset M$ *finitely generated*$\}$.

The proof of the lemma is standard.

Usually we will have a pair of dimension functions, one on $\mathsf{Mod}\ A$ and the other on $\mathsf{Mod}\ B^\circ$; when necessary we shall distinguish between them by writing $\dim_A$ and $\dim_{B^\circ}$, respectively.

DEFINITION 2.6. An exact dimension function dim is called *finitely partitive* if given a finitely generated module M there is a number l_0, such that for every chain of submodules $M = M_0 \supsetneqq M_1 \cdots \supsetneqq M_l$ with $\dim M_i/M_{i+1} = \dim M$ one has $l \leq l_0$.

In the next two definitions (taken from [ASZ, Ye2]) dim denotes a function on $\mathsf{Mod}\ A$ satisfying axioms (i), (ii), and (iv).

DEFINITION 2.7. (1) A module M is called dim-*pure* if $\dim M' = \dim M$ for every nonzero submodule $M' \subset M$.

(2) A module M is called dim-*essentially pure* if M contains an essential submodule which is pure.

(3) A module M is called dim-*critical* if $M \neq 0$, and $\dim M/M' < \dim M$ for every $0 \neq M' \subsetneqq M$.

DEFINITION 2.8. (1) Let $\mathsf{M}_q(\dim)$ be the full subcategory of $\mathsf{Mod}\ A$ consisting of modules M with $\dim M \leq q$, and let $\mathsf{M}_{q,f}(\dim) := \mathsf{M}_q(\dim) \cap \mathsf{Mod}_f\ A$.

(2) Given a module M let $\Gamma_{\mathsf{M}_q(\dim)} M \subset M$ be the largest submodule $M' \subset M$ such that $\dim M' \leq q$.

Since $\mathsf{M}_q(\dim)$ is a localizing subcategory of $\mathsf{Mod}\ A$ the submodule $\Gamma_{\mathsf{M}_q(\dim)} M$ is well defined (and in fact $\Gamma_{M_q(\dim)}$ is a left exact idempotent functor). The corresponding subcategories of $\mathsf{Mod}\ B^\circ$ shall be denoted by $\mathsf{M}_q^\circ(\dim)$ and $\mathsf{M}_{q,f}^\circ(\dim)$.

DEFINITION 2.9. Let M be a finitely generated A-module. The *canonical dimension of M with respect to R* is

$$\operatorname{Cdim}_{R;A} M := -j_{R;A}(M) \in \mathbb{Z} \cup \{-\infty\}.$$

Likewise define $\operatorname{Cdim}_{R;B^\circ}$.

The canonical dimension will not be an exact dimension function in general. However, we have the following theorem, which generalizes results of Björk and Levasseur (the graded case was proved in [Ye2]).

THEOREM 2.10. *If R is an Auslander dualizing complex then* $\mathrm{Cdim}_{R;A}$ *is a finitely partitive exact dimension function.*

The proof of this theorem appears later in the section. The key step is:

LEMMA 2.11. *Let* $0 \to M' \to M' \to M'' \to 0$ *be a short exact sequence of finitely generated A-modules. Then*

$$j_{R;A}(M) = \inf\{j_{R;A}(M'), j_{R;A}(M'')\}.$$

Proof. The proof goes along the lines of the proofs in [Bj, Lev]. By Proposition 1.7 we have a convergent spectral sequence

$$E_2^{p,q} := \mathrm{Ext}^p_{B^\circ}(\mathrm{Ext}^{-q}_A(M,R),R) \Rightarrow M, \tag{2.12}$$

so there is a corresponding filtration (called the b-filtration in [Lev, Theorem 2.2])

$$M = F^{-d}M \supset F^{-d+1}M \supset \cdots \supset F^{d+1}M = 0.$$

The Auslander condition tells us that $E_2^{p,q} = 0$ if $p < -q$. So the spectral sequence lives in a bounded region of the (p,q) plane: $p \geq -q, q \leq -j_{A;R}(M)$ and $p \leq d$ (see Fig. 1). The coboundary operator of E_r has bidegree $(r, 1-r)$ and $r \geq 2$. We conclude that for every $|p| \leq d$ there is an exact sequence of A-modules

$$0 \to \frac{F^p m}{F^{p+1}M} \to E_2^{p,-p} \to Q^p \to 0 \tag{2.13}$$

with Q^p a subquotient of $\bigoplus_{r \geq 2} E_2^{p+r,-p+(1-r)}$ (cf. [Bj, Theorem 1.3; Lev, Theorem 2.2]).

By the Auslander property it then follows that $j_{R;A}(F^pM/F^{p+1}M) \geq p$ for all M and p. Just as in [Bj, Proposition 1.6], one uses descending induction on p, starting at $p = d+1$, to prove that $j_{R;A}(F^pM) \geq p$ for all p. This implies that

$$j_{R;B^\circ}(\mathrm{Ext}^{j_{R;A}(M)}(M,R)) = j_{R;A}(M).$$

Now continue exactly like in [Bj, Proposition 1.8]. ∎

We conclude that $\mathrm{Cdim}_{R;A}$ verifies axiom (ii). Axiom (i) holds trivially. By symmetry $\mathrm{Cdim}_{R;B^\circ}$ also verifies axioms (i)–(ii).

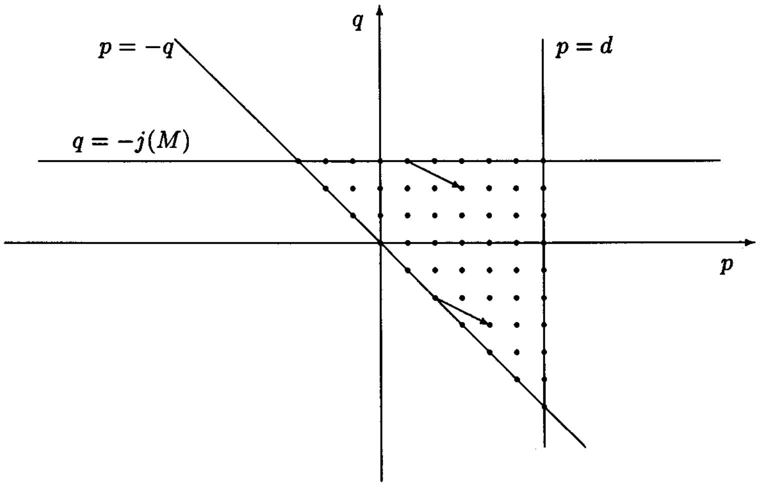

FIG. 1. The E_2 term of the spectral sequence (2.12) in the (p, q) plane.

THEOREM 2.14. *Suppose $R \in \mathsf{D}(\mathsf{Mod}(A \otimes B^{\circ}))$ is an Auslander dualizing complex. Let $M \in \mathsf{Mod}_f\, A$ be nonzero and $\operatorname{Cdim}_R M = n$. Then*:

(1) $\operatorname{Ext}_A^{-n}(M, R)$ *is* Cdim_R*-pure of dimension* n.

(2) *For each* p, $\operatorname{Ext}_{B^{\circ}}^{-p}(\operatorname{Ext}_A^{-p}(M, R), R)$ *is* Cdim_R*-pure of dimension* p *or is* 0.

(3) *For each* p *there is an exact sequence*

$$0 \to \Gamma_{\mathsf{M}_{p-1}} M \to \Gamma_{\mathsf{M}_p} M \to \operatorname{Ext}_{B^{\circ}}^{-p}\big(\operatorname{Ext}_A^{-p}(M, R), R\big) \to Q^p \to 0,$$

functorial in M, *where* $\operatorname{Cdim}_R Q^p \leq p - 2$ *and* $\mathsf{M}_p = \mathsf{M}_p(\operatorname{Cdim}_R)$.

Proof. (1) Because the line $q = -j_{R;A}(M)$ is on the boundary of the region of support of the spectral sequence (2.12), and the coboundary operator of E_r has bidegree $(r, 1 - r)$, $r \geq 2$, we see that for this value of q there is a bounded filtration $E_2^{p,q} \supset E_3^{p,q} \supset \dots$, with $E_r^{p,q}/E_{r+1}^{p,q}$ a subquotient of $E_r^{p+r,q+(1-r)}$. Now the abutment of the spectral sequence is concentrated on the line $p = -q$ of the (p, q)-plane, so $E_r^{p,q} = 0$ for $p > -q$ and $r \gg 0$. By Lemma 2.11 we conclude that for $q = -j_{R;A}(M)$ and $p > -q$,

$$j_{R;A}\big(\operatorname{Ext}_{B^{\circ}}^{p}\big(\operatorname{Ext}_A^{-q}(M, R), R\big)\big) = j_{R;A}\big(E_2^{p,q}\big) \geq p + 2$$

(cf. [Bj, formula (1.10)]. Just like in [Bj, Proposition 1.11] it follows that $\operatorname{Ext}_A^{p}(\operatorname{Ext}_{B^{\circ}}^{p}(N, R), R) = 0$ for $p > j_{R;A}(M)$ and $N = \operatorname{Ext}_A^{j_{R;A}(M)}(M, R)$. So by [Bj, Proposition 1.9] we conclude that N is pure.

(2) Take $N := \mathrm{Ext}_A^{-p}(M, R)$. Then $\mathrm{Cdim}_{R;\,B^\circ} N \le p$ and part (1) applies.

(3) By part (2), the sequence (2.13) and induction on p we see that $\Gamma_{\mathsf{M}_p} M = F^{-p} M$. ▮

For an integer q let $\mathsf{M}_q := \mathsf{M}_q(\mathrm{Cdim}_R) \subset \mathsf{Mod}\, A$ be the localizing subcategory from Definition 2.8. The filtration by dimension of support $\{\mathsf{M}_q\}$ of $\mathsf{Mod}\, A$ is called the *niveau filtration* in commutative algebraic geometry. For each q the quotient category $\mathsf{M}_q/\mathsf{M}_{q-1}$ is a locally noetherian abelian category, and the full subcategory $\mathsf{M}_{q,f}/M_{q-1,f}$ is noetherian (see [ASZ, Lemma 1.1]). By symmetry we have corresponding localizing subcategories $\mathsf{M}^\circ_{q,f} \subset \mathsf{M}^\circ_q \subset \mathsf{Mod}\, B^\circ$.

Recall from [ASZ, Section 1] that two abelian categories C and D are said to be *dual* if they are anti-equivalent, i.e., if C is equivalent to the opposite category D°. Two categories C and D are said to be in *step duality* if there are filtrations by dense abelian subcategories

$$0 = \mathsf{C}_{n_0-1} \subset \mathsf{C}_{n_0} \subset \cdots \subset \mathsf{C}_{n_1} = \mathsf{C} \quad \text{and}$$
$$0 = \mathsf{D}_{n_0-1} \subset \mathsf{D}_{n_0} \subset \cdots \subset \mathsf{D}_{n_1} = \mathsf{D}$$

such that the quotient categories $\mathsf{C}_i/\mathsf{C}_{i-1}$ and $\mathsf{D}_i/\mathsf{D}_{i-1}$ are dual for all $i = n_0, \ldots, n_1$. Now Theorem 0.2 is a special case of:

THEOREM 2.15. *Suppose $R \in \mathsf{D}(\mathsf{Mod}(A \otimes B^\circ))$ is an Auslander dualizing complex. Then $\mathsf{Mod}_f\, A$ and $\mathsf{Mod}_f\, B^\circ$ are in step duality. More precisely, for every q the functors $\mathrm{Ext}^q_A(-, R)$ and $\mathrm{Ext}^q_{B^\circ}(-, R)$ induce a duality between the quotient categories $\mathsf{M}_{q,f}/\mathsf{M}_{q-1,f}$ and $\mathsf{M}^\circ_{q,f}/\mathsf{M}^\circ_{q-1,f}$.*

Proof. Use Theorem 2.14 and the proof of [ASZ, Theorem 1.2]. ▮

COROLLARY 2.16. *For each q the category $\mathsf{M}_{q,f}/\mathsf{M}_{q-1,f}$ is artinian, i.e., every object has finite length.*

Proof. By [ASZ, Lemma 1.1] the categories $\mathsf{M}_{q,f}/\mathsf{M}_{q-1,f}$ and $\mathsf{M}^\circ_{q,f}/\mathsf{M}^\circ_{q-1,f}$ are noetherian, hence by Theorem 2.15 they are also artinian. ▮

At last here is:

Proof of Theorem 2.10. We already verified axioms (i) and (ii). The fact that Cdim_R is finitely partitive is immediate from Corollary 2.16, and this in turn easily implies axiom (iii)—cf. [MR, Corollary 8.3.6]. ▮

COROLLARY 2.17. *Every finitely generated A-module has a Cdim_R-critical composition series.*

Proof. This is by Theorem 2.10 and [MR, Proposition 6.2.20]. ∎

Let

$$d_0 := \inf\{\mathrm{Cdim}_R\, M \mid M \in \mathsf{Mod}\, A,\, M \neq 0\}$$
$$d_1 := \mathrm{Cdim}_R\, A = \sup\{\mathrm{Cdim}_R\, M \mid M \in \mathsf{Mod}\, A\}.$$

COROLLARY 2.18. *Suppose $R \in \mathsf{D}(\mathsf{Mod}(A \otimes B^\circ))$ is an Auslander dualizing complex. Then*

$$\mathrm{Kdim}\, M \leq \mathrm{Cdim}_R\, M - d_0$$

for all finitely generated A-modules M. In particular if $\mathrm{Cdim}_R\, M = d_0$ then M is artinian.

Proof. By induction, starting with $q = d_0$, Theorem 2.14 and Corollary 2.16 show that $\mathrm{Kdim}\, M \leq q - d_0$ for all $M \in \mathsf{M}_{q,f}$. ∎

Here is a generalization of [Bj, Theorem 1.14].

THEOREM 2.19 (Gabber's Maximality Principle). *Let A and B be k-algebras, with A left noetherian and B right noetherian, and let $R \in \mathsf{D}(\mathsf{Mod}(A \otimes B^\circ))$ be an Auslander dualizing complex. Suppose N is a Cdim_R-pure A-module with $\mathrm{Cdim}_R\, N = n$, and M is a finitely generated submodule. Then there is a unique maximal module $\tilde{M}$ such that $M \subset \tilde{M} \subset N$, $\tilde{M}$ is finitely generated, and $\mathrm{Cdim}_R\, \tilde{M}/M \leq n - 2$.*

Proof. Note that we do not assume N is finitely generated. The uniqueness is clear because Cdim_R is an exact dimension function. So it remains to show existence. If $\tilde{M}$ is any finitely generated submodule of N containing M, such that $\mathrm{Cdim}_R\, \tilde{M}/M \leq n - 2$, then $\mathrm{Ext}_A^{-n}(M, R) \cong \mathrm{Ext}_A^{-n}(\tilde{M}, R)$. Hence, by Theorem 2.14(3), the module $\tilde{M}$ embeds functorially into the finitely generated A-module $\mathrm{Ext}_{B^\circ}^{-n}(\mathrm{Ext}_A^{-n}(M, R), R)$. This implies there is a maximal such $\tilde{M}$. ∎

The next two theorems generalize [GL, Theorems 1.4 and 1.6] by eliminating the Gorenstein and Cohen–Macaulay conditions. From here on we consider a single noetherian algebra A (i.e., $A = B$).

DEFINITION 2.20. Let dim be an exact dimension function.

(1) dim is called *symmetric* if $\dim_A M = \dim_{A^\circ} M$ for every bimodule M finitely generated on both sides.

(2) dim is called *weakly symmetric* if $\dim_A M = \dim_{A^\circ} M$ for every bimodule M which is a subquotient of A.

LEMMA 2.21. *Let* dim *be a weakly symmetric exact dimension function.*

(1) *Let M be a finitely generated* dim*-pure A-module, and let* $I := \mathrm{Ann}_A(M)$. *If* $\dim M = \dim A/I$ *then A/I is* dim*-pure.*

(2) *Let I be an ideal of A such that A/I is* dim*-pure and let $\mathfrak{q}$ be a prime ideal of A that is minimal over I. Then* $\dim A/\mathfrak{q} = \dim A/I$.

Proof. This is completely analogous to [KL, 9.6 and 9.5]. ∎

Recall from [GL] that Spec A is said to have *normal separation* provided that for any pair prime ideals $\mathfrak{p} \subsetneqq \mathfrak{q}$, the factor $\mathfrak{q}/\mathfrak{p}$ contains a nonzero normal element of $A/\mathfrak{p}$. Under the assumptions of the lemma we say that *Tauvel's height formula* holds in A provided

$$\text{height } \mathfrak{p} + \dim A/\mathfrak{p} = \dim A$$

for all primes $\mathfrak{p}$.

THEOREM 2.22. *Suppose that A is a noetherian k-algebra, R is an Auslander dualizing complex over A, and* Cdim_R *is weakly symmetric. Let $\mathfrak{p} \subsetneqq \mathfrak{q}$ be prime ideals of A with* height $\mathfrak{q}/\mathfrak{p} = 1$. *If there exists an element $a \in \mathfrak{q} - \mathfrak{p}$ that is normal modulo $\mathfrak{p}$, then* $\mathrm{Cdim}_R A/\mathfrak{p} = \mathrm{Cdim}_R A/\mathfrak{q} + 1$.

Proof. Use the proof of [GL, Theorem 1.4] but replace GKdim by Cdim_R, and use Lemma 2.21 instead of [Len, Lemmas 2 and 3].

THEOREM 2.23. *Suppose that A is a noetherian k-algebra, R is an Auslander dualizing complex over A, and* Cdim_R *is weakly symmetric. If* Spec A *is normally separated, then A is catenary. If in addition A is prime, then Tauvel's height formula holds.*

Proof. The proof of [GL, Theorem 1.6] works here after we replace GKdim by Cdim_R. ∎

We call attention to Question 3.15 regarding the possible symmetry of Cdim_R.

The Macaulay property of [SZ] is adapted in the following way, to be used later in the paper.

DEFINITION 2.24. Suppose R is an Auslander dualizing complex over A. Let dim be an exact dimension function on finitely generated A-modules. If there is some integer c such that

$$\dim M = \mathrm{Cdim}_R M + c$$

for all $M \in \mathsf{Mod}_f A$, then we say R is *Macaulay with respect to* dim, or that R is dim-*Macaulay.*

Note that if $R = A$, then ‘‘GKdim-Macaulay’’ is equivalent to ‘‘Cohen–Macaulay’’ as it is used in [Bj, Lev, ASZ, SZ]. This is because GKdim $M + j_R(M) = c = \dim A$ in this case.

EXAMPLE 2.25. If A is a commutative affine k-algebra and R is any central dualizing complex over A, then R is Auslander and GKdim-Macaulay. In this case we also have Kdim M = GKdim M for all finitely generated A-modules in M.

3. RIGID DUALIZING COMPLEXES

In this section we consider dualizing complexes which satisfy a special condition discovered by Van den Bergh [VdB]. Rigid dualizing complexes are unique and even functorial. Furthermore if R is an Auslander rigid dualizing complex then the canonical dimension Cdim_R is particularly well behaved (as examples indicate; see Question 3.15). By default A and B denote noetherian k-algebras.

First we shall need some more notation for bimodules. Suppose A and B are k-algebras. For an element $a \in A$ we denote by $a^\circ \in A^\circ$ the same element. Thus for $a_1, a_2 \in A$, $a_1^\circ \cdot a_2^\circ = (a_2 \cdot a_1)^\circ \in A^\circ$. With this notation if M is a right A-module then the left A° action is $a^\circ \cdot m = m \cdot a, m \in M$. The algebra A^e has an involution $A^e \to (A^e)^\circ, a_1 \otimes a_2^\circ \mapsto a_2 \otimes a_1^\circ$ which allows us to regard every left A^e-module M as a right A^e-module in a consistent way:

$$(a_1 \otimes a_2^\circ) \cdot m = (a_2 \otimes a_1^\circ)^\circ \cdot m = m \cdot (a_2 \otimes a_1^\circ) = a_1 \cdot m \cdot a_2.$$

Given an $(A \otimes B^\circ)$-module M and a $(B \otimes A^\circ)$-module N we define a mixed action of $A^e \otimes B^e$ on the tensor product $M \otimes N$ as follows. A^e acts on $M \otimes N$ by the outside action

$$(a_1 \otimes a_2^\circ) \cdot (m \otimes n) := (a_1 \cdot m) \otimes (n \cdot a_2),$$

whereas B^e acts on $M \otimes N$ by the inside action

$$(b_1 \otimes b_2^\circ) \cdot (m \otimes n) := (m \cdot b_2) \otimes (b_1 \cdot n).$$

By default we regard the outside action as a left action and the inside action as a right action. If $A = B$ and $M = N$ then the two actions of A^e on $M \otimes M$ are interchanged by the involution $m_1 \otimes m_2 \mapsto m_2 \otimes m_1$. However, for the sake of definiteness in this case, given an A^e-module L, $\mathrm{Hom}_{A^e}(L, M \otimes M)$ shall refer to the homomorphisms $L \to M \otimes M$ which are A^e-linear with respect to the outside action.

DEFINITION 3.1 [VdB, Definition 8.1]. Suppose R is a dualizing complex over A. If there is an isomorphism

$$\phi : R \xrightarrow{\simeq} \mathrm{R\,Hom}_{A^e}(A, R \otimes R)$$

in $\mathsf{D}(\mathsf{Mod}\ A^e)$, we call (R, ϕ) a *rigid dualizing complex*.

It is obvious that if R is rigid, then any shift $R[n]$, for $n \neq 0$, is no longer rigid. Van den Bergh proved that a rigid dualizing complex (R, ϕ) over A is unique, up to an isomorphism in $\mathsf{D}(\mathsf{Mod}\ A^e)$ (see [VdB, Proposition 8.2]). Below we extend this result by proving that rigid dualizing complexes are functorial, in a suitable sense.

Let $A \to B$ be a k-algebra homomorphism. Given $M \in \mathsf{D}^+(\mathsf{Mod}\ B)$, $N \in \mathsf{D}^+(\mathsf{Mod}\ A)$, and a morphism $\psi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\ A)$, ψ factors naturally through $\mathrm{R\,Hom}_A(B, N)$. This can be seen by replacing N with an injective resolution I in $\mathsf{D}^+(\mathsf{Mod}\ A)$, and then we can take ψ to be a homomorphism of complexes. The image of ψ will then land inside $\mathrm{Hom}_A(B, I)$. The same fact is true for bimodules.

We say a k-algebra homomorphism $A \to B$ if *finite* if B is finitely generated as a left and as a right A-module.

THEOREM 3.2. *Let $A \to B$ be a finite k-algebra homomorphism. Suppose A and B have rigid dualizing complexes (R_A, ϕ_A) and (R_B, ϕ_B), respectively. Then there is at most one morphism $\psi : R_B \to R_A$ in $\mathsf{D}(\mathsf{Mod}\ A^e)$ satisfying conditions* (i) *and* (ii) *below*:

(i) *ψ induces an isomorphism*

$$R_B \cong \mathrm{R\,Hom}_A(B, R_A) \cong \mathrm{R\,Hom}_{A^\circ}(B, R_A)$$

in $\mathsf{D}(\mathsf{Mod}\ A^e)$.

(ii) *The diagram*

$$\begin{array}{ccc} R_B & \xrightarrow{\phi_B} & \mathrm{R\,Hom}_{B^e}(B, R_B \otimes R_B) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi\otimes\psi} \\ R_A & \xrightarrow{\phi_A} & \mathrm{R\,Hom}_{A^e}(A, R_A \otimes R_A) \end{array}$$

in $\mathsf{D}(\mathsf{Mod}\ A^e)$ *is commutative*.

The theorem is proved after this lemma. Given a k-algebra homomorphism $A \to B$, denote by $Z_B(A) \subset B$ the centralizer of A.

LEMMA 3.3. *Let $A \to B$ be a finite k-algebra homomorphism. Suppose A and B have dualizing complexes R_A and R_B, respectively, and $\psi : R_A \to R_B$ is a morphism in $\mathsf{D}(\mathsf{Mod}\ A^e)$ satisfying condition* (i) *of the theorem. Then $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\ A^e)}(R_B, R_A)$ is a free left and right $Z_B(A)$-module with basis ψ.*

Proof. Denote by $D := \mathrm{R}\operatorname{Hom}_A(-, R_A)$ and $D^\circ := \mathrm{R}\operatorname{Hom}_{A^\circ}(-, R_A)$ the dualizing functors. By assumption $R_B \cong D^\circ B$. Applying the functor D we get isomorphisms of left $Z_B(A)$-modules

$$\begin{aligned}\operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\, A^e)}(R_B, R_A) &\cong \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\, A^e)}(D^\circ B, D^\circ A)\\ &\cong \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\, A^e)}(A, B)\\ &\cong \operatorname{Hom}_{A^e}(A, B) \cong Z_B(A),\end{aligned}$$

and likewise for the right action. ▮

Proof of the Theorem. Assume ψ' is another such isomorphism. According to the lemma above,

$$\psi' = (b_1 \otimes 1)\psi = (1 \otimes b_2^\circ)\psi$$

for suitable $b_i \in Z_B(A)^\times$. So

$$\psi' \otimes \psi' = (b_1 \otimes b_2^\circ)(\psi \otimes \psi).$$

Now the diagram in condition (ii) consists of morphisms in $\mathsf{D}(\mathsf{Mod}\, A^e)$. Since multiplications by b_1 and b_2° are A^e-linear, we see that

$$(b_1 \otimes 1)\psi = (b_1 \otimes b_2^\circ)\psi \in \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\, A^e)}(R_B, R_A).$$

Hence dividing by the unit $b_1 \otimes 1$, we see that $b_2 = 1$. ▮

COROLLARY 3.4. *A rigid dualizing complex (R, ϕ) over A is unique up to a unique isomorphism.*

Proof. Suppose (R', ϕ') is another rigid dualizing complex. We claim that there is an isomorphism $\psi : R \xrightarrow{\simeq} R'$ in $\mathsf{D}(\mathsf{Mod}\, A^e)$ which satisfies condition (ii) of the theorem. By Theorem 3.2 such ψ is unique.

To produce ψ, choose any isomorphism $\psi' : R \xrightarrow{\simeq} R'$, which we know exists by [VdB, Proposition 8.2]. Then by Lemma 3.3 there are $a_i \in Z(A)^\times$ such that

$$(a_1 \otimes 1)^{-1}\psi' = (1 \otimes a_2^\circ)^{-1}\psi' = \phi'^{-1}(\psi' \otimes \psi')\phi.$$

The isomorphism

$$\psi := (a_1 \otimes 1)\psi' = (1 \otimes a_2^\circ)\psi'.$$

will satisfy condition (ii). ▮

Thus we may speak of *the* rigid dualizing complex of A (if it exists).

Lemma 3.3 can be sharpened when $A = B$.

PROPOSITION 3.5. *Let (R, ϕ) be a rigid dualizing complex over A. Then the two k-algebra homomorphisms*

$$\lambda, \rho : Z(A) \to \mathrm{End}_{\mathsf{D}(\mathsf{Mod}\, A^e)}(R)$$

from the center of A, namely left and right multiplication, are both isomorphisms, and are equal.

Proof. By Lemma 3.3 with $A = B$ and $\psi = 1$, we see that λ and ρ are isomorphisms. Take $a \in Z(A)$, and let $a' := \rho^{-1}\lambda(a) \in Z(A)$. Using the definition of the mixed action on $R \otimes R$ and the rigidification isomorphism ϕ, the commutation (or conjugation) of a across R is transferred to commutation of a' across A. Since a' does commute with A it follows that $\lambda(a) = \rho(a)$ (and so in fact $a' = a$). ∎

We will often omit reference to the rigidifying isomorphism ϕ.

COROLLARY 3.6. *If R is a rigid dualizing complex over A then for any q the cohomology bimodule $H^q R$ is central over $Z(A)$.*

DEFINITION 3.7. Let $A \to B$ be a finite homomorphism of k-algebras. Assume the rigid dualizing complexes (R_A, ϕ_A) and (R_B, ϕ_B) exist. If there is a morphism ψ satisfying the conditions of Theorem 3.2 then we call it the *trace morphism* and denote it by $\mathrm{Tr}_{B/A}$.

The next corollary is obvious.

COROLLARY 3.8. *Let $A \to B$ and $B \to C$ be finite k-algebra homomorphisms. Assume the rigid dualizing complexes (R_A, ϕ_A), (R_B, ϕ_B), and (R_C, ϕ_C) and the trace morphisms $\mathrm{Tr}_{B/A}$ and $\mathrm{Tr}_{C/B}$ exist. Then $\mathrm{Tr}_{C/A}$ exists too, and*

$$\mathrm{Tr}_{C/A} = \mathrm{Tr}_{B/A}\mathrm{Tr}_{C/B}.$$

The existence of the trace morphism allows us to transfer good properties of R_A to R_B.

PROPOSITION 3.9. *Let $A \to B$ be a finite homomorphism of k-algebras, and assume the rigid dualizing complexes R_A and R_B and the trace morphism $\mathrm{Tr}_{B/A}$ exist.*

(1) *Let C be any k-algebra. Then for $M \in \mathsf{D}(\mathsf{Mod}(B \otimes C^{\circ}))$ there is a functorial isomorphism*

$$\mathrm{R\,Hom}_B(M, R_B) \cong \mathrm{R\,Hom}_A(M, R_A)$$

in $\mathsf{D}(\mathsf{Mod}(C \otimes A^{\circ}))$.

(2) *If* R_A *is Auslander then so is* R_B.

(3) *For any* B*-module* M, $\operatorname{Cdim}_{R_A;\,A} M = \operatorname{Cdim}_{R_B;\,B} M$.

(4) *If* R_A *is* GKdim-*Macaulay, then so is* R_B.

(5) *Suppose* $A \to B$ *is surjective. If* R_A *is* Kdim-*Macaulay then so is* R_B.

Proof. (1) We can assume R_A and R_B are complexes of injectives over A^e and B^e, respectively, and $\operatorname{Tr}_{B/A}$ is a homomorphism of complexes. Then we get a functorial morphism $\mathrm{R}\operatorname{Hom}_B(M, R_B) \to \mathrm{R}\operatorname{Hom}_A(M, R_A)$ in $\mathsf{D}(\mathsf{Mod}(C \otimes A^\circ))$. To prove it's an isomorphism we can forget the C-module structure. Because the two functors are way-out in both directions (see [RD, Sect. 1.7]) and they send direct sums to direct products, it suffices to check for an isomorphism when $M = B$. But that's given.

(2), (3) Let M be a finitely generated B-module and $N \subset \operatorname{Ext}^q_B(M, R_B)$ a B°-submodule. Then by part (1), $N \subset \operatorname{Ext}^q_A(M, R_A)$ as A°-modules, and for every p, $\operatorname{Ext}^p_{B^\circ}(N, R_B) \cong \operatorname{Ext}^p_{A^\circ}(N, R_A)$ as A-modules. This proves the Auslander condition for B and the dimension equality for finitely generated B-modules.

(4) This follows from part (3) and the fact $\operatorname{GKdim}_A M = \operatorname{GKdim}_B M$.

(5) This is similar to (4). ∎

Definition 3.10. Suppose A has an Auslander rigid dualizing complex R. Then we denote the canonical dimension Cdim_R by Cdim.

Example 3.11. Suppose A is an affine k-algebra and finite over its center. Then we can find a smooth integral commutative k-algebra C (e.g., a polynomial algebra), and a finite homomorphism $C \to Z(A)$. Say $\operatorname{Kdim} C = n$. Since $\Omega^n_{C/K}[n]$ is a rigid dualizing complex over C, it follows from [Ye3, Proposition 5.9] that

$$R := \mathrm{R}\operatorname{Hom}_C\big(A, \Omega^n_{C/k}[n]\big)$$

is a rigid dualizing complex over A.

Example 3.12. Let A be the algebra $\left(\begin{smallmatrix} k & V \\ 0 & k \end{smallmatrix}\right)$ where V is a finite rank k-module. The rigid dualizing complex is $A^* := \operatorname{Hom}_k(A, k)$. Now A is hereditary, hence Gorenstein, so the bimodule A is a dualizing complex. When $V \neq 0$ the dualizing complexes A and A^* are not isomorphic, so A is not rigid then.

Example 3.13. Let $t_1, t_2, \ldots$ be a countable sequence of commuting indeterminates and let $C := k(t_1, t_2, \ldots)$ be the field of rational functions. We claim that as k-algebra, C has no rigid dualizing complex. Since any dualizing complex over C has to be of the form $R = C^{\sigma}[n]$ for an automorphism σ and an integer n (by [Ye3, Corollary 4.6 and Propositions 3.4 and 3.5]), it suffices to prove that $\mathrm{Ext}^i_{C^e}(C, C^e) = 0$ for all i. This follows from the next lemma with $n = 0$.

LEMMA 3.14. *Let $D_n := C^e/I$ where I is the ideal generated by the elements $f_j := x_j \otimes 1 - 1 \otimes x_j$ for $j = 1, \dots, n$. Then $\mathrm{Ext}^i_{C^e}(C, D_n) = 0$ for all $i, n \geq 0$.*

Proof. Assume on the contrary that $\mathrm{Ext}^i_{C^e}(C, D_n) \neq 0$ for some i and n. Let i_0 be the smallest such i. Since f_{n+1} is nonzero in the domain D_n there is a short exact sequence

$$0 \to D_n \xrightarrow{f_{n+1}} D_n \to D_{n+1} \to 0$$

of C^e-modules. That induces an exact sequence

$$0 = \mathrm{Ext}^{i_0 - 1}_{C^e}(C, D_{n+1}) \to \mathrm{Ext}^{i_0}_{C^e}(C, D_n) \xrightarrow{f_{n+1}} \mathrm{Ext}^{i_0}_{C^e}(C, D_n)$$

But f_{n+1} annihilates the C^e-module C, which implies $\mathrm{Ext}^{i_0}_{C^e}(C, D_n) = 0$, contradicting the choice of i_0. ∎

We end the section with a basic question.

QUESTION 3.15. *Let R be a rigid dualizing complex and M an A-bimodule finitely generated on both sides. Is there a functorial isomorphism* $\mathrm{R\,Hom}_A(M, R) \cong \mathrm{R\,Hom}_{A^\circ}(M, R)$? *Or, at least, is* $\mathrm{Cdim}_{R;\,A} M = \mathrm{Cdim}_{R;\,A^\circ} M$?

For a partial answer turn to Section 6, where the presence of auxiliary filtrations allows us to take advantage of results in Sections 4–5 on graded algebras.

4. DUALIZING COMPLEXES OVER GRADED ALGEBRAS

In this section we consider connected $\mathbb{Z}$-graded k-algebras, namely algebras $A = \bigoplus_{n \geq 0} A_n$ with $A_0 \cong k$ and A_n a finitely generated k-module.

For such an algebra A let $\mathsf{GrMod}\, A$ be the category of $\mathbb{Z}$-graded left modules, with degree 0 homomorphisms. For $M, N \in \mathsf{GrMod}\, A$ we write $M(n)$ for the shifted module with $M(n)_i = M_{n+i}$, and

$$\mathrm{Hom}^{\mathrm{gr}}_A(M, N) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}^{\mathrm{gr}}_{\mathsf{GrMod}\, A}(M, N(n)).$$

There is a forgetful functor $\mathsf{GrMod}\, A \to \mathsf{Mod}\, A$. Observe that $\mathrm{Hom}^{\mathrm{gr}}_A(M, N) \subset \mathrm{Hom}_A(M, N)$ with equality when M is finitely generated.

$\mathsf{GrMod}\, A$ is an abelian category with direct and inverse limits, enough injectives, and enough projectives. Let $\mathsf{D}(\mathsf{GrMod}\, A)$ be the derived category. The derived functors $\mathrm{R\,Hom}_A^{\mathrm{gr}}(M, N)$ and $M \otimes_A^L N$ are calculated just as in the ungraded case, see Section 1, but using graded-projectives or graded-injectives.

We say $M \in \mathsf{GrMod}\, k$ is *locally finite* if each M_n is a finitely generated k-module. Let $M^* := \mathrm{Hom}_k^{\mathrm{gr}}(M, k)$. Denote by $\mathsf{D}_{\mathrm{lf}}(\mathsf{GrMod}\, A)$ the subcategory of complexes with locally finite cohomologies. Matlis duality says that $M \cong M^{**}$ for $M \in \mathsf{D}_{\mathrm{lf}}(\mathsf{GrMod}\, A)$.

We denote by $\mathfrak{m}$ the augmentation ideal $\bigoplus_{n>0} A_n$ of A, and we write $\Gamma_{\mathfrak{m}} M$ for the $\mathfrak{m}$-torsion submodule of a graded A-module M. There is a derived functor $\mathrm{R}\Gamma_{\mathfrak{m}}$, which is calculated by graded-injectives (see [Ye1]). The cohomology modules are $\mathrm{H}^i \mathrm{R}\Gamma_{\mathfrak{m}} M = \lim_{n \to} \mathrm{Ext}_A^i(A/A_{\geq n}, M)$. We write $\mathfrak{m}^\circ$ for the augmentation ideal of A°.

The definitions and results of the previous sections can all be translated to the graded category by adding the adjective "graded'' where needed, like "graded dualizing complex,'' "graded Auslander property,'' etc. The proofs of the graded variants of these results are identical to the ungraded ones, so there is no need to repeat them. In the rest of the paper we shall refer to such a result by writing something like "according to the graded variant of Theorem ....''

Remark 4.1. Let G be any finitely generated abelian group. Fix an isomorphism $G \cong \mathbb{Z}^r \times T$, where T is a finite group, and a basis $g_1, \ldots, g_r$ of $\mathbb{Z}^r$. Let G_+ be the semigroup generated by 0 and the elements $g_i + t, 1 \leq i \leq r, t \in T$. For $g, g' \in G$ we write $g \geq g'$ if $g - g' \in G_+$, and this defines a partial order on G. A G-graded k-algebra A is called *connected* if $A = \bigoplus_{g \in G_+} A_g$, $A_0 = k$, and each A_g is finitely generated as a module over k. The augmentation ideal of A is $\mathfrak{m} := \bigoplus_{g>0} A_g$.

Note that the group homomorphisms $\phi : G \to \mathbb{Z}$ sending $g_i \mapsto 1$ makes A into a connected $\mathbb{Z}$-graded algebra, with $A_n = \bigoplus_{\phi(g)=n} A_g, n \in \mathbb{Z}$.

It is not hard to see that all results in this paper which are stated for connected $\mathbb{Z}$-graded algebras are also valid for connected G-graded k-algebras, for any G as above.

Throughout this section A and B are connected graded noetherian k-algebras.

DEFINITION 4.2 [Ye1, Definition 4.1]. A *balanced dualizing complex* over A is a graded dualizing complex R such that

$$\mathrm{R}\Gamma_{\mathfrak{m}} R \cong \mathrm{R}\Gamma_{\mathfrak{m}^\circ} R \cong A^*$$

in $\mathsf{D}(\mathsf{GrMod}\, A^e)$.

A balanced dualizing complex R is unique up to isomorphism in $\mathsf{D}(\mathsf{GrMod}\, A^e)$, and its endomorphisms are just elements of k. Thus if we choose an isomorphism $\phi : \mathrm{R}\Gamma_{\mathfrak{m}} R \xrightarrow{\simeq} A^*$ in $\mathsf{D}(\mathsf{GrMod}\, A^e)$, the pair (R, ϕ) is unique up to a unique isomorphism.

It had been known for some time (by [Ye1]) that balanced dualizing complexes exist for Artin−Schelter Gorenstein algebras, twisted homogeneous coordinate algebras, and algebras finite over their centers. Recently additional existence results became available, due to the work of Van den Bergh. First recall the following definition taken from [AZ].

DEFINITION 4.3. The condition χ holds for a noetherian connected graded k-algebra A if for every $M \in \mathsf{GrMod}_f A$ and integer i, $\mathrm{Ext}^i_A(k, M)$ is a finitely generated k-module.

In view of [AZ, Proposition 3.1(3)], this definition is equivalent to [AZ, Definition 3.2]; and by [AZ, Proposition 3.11(2)] it is equivalent to [AZ, Definition 3.7]. The next lemma provides further characterization of the condition χ. Recall that a graded module M is said to be *right bounded* if $M_n = 0$ for $n \gg 0$.

LEMMA 4.4. *Let A be a noetherian connected graded k-algebra and $M \in \mathsf{GrMod}_f\, A$. Then the following are equivalent*:

(i) $\mathrm{Ext}^i_A(k, M)$ *is a finitely generated k-module for all i.*

(ii) $\mathrm{H}^i \mathrm{R}\Gamma_{\mathfrak{m}} M$ *is right bounded for all i.*

(iii) $\mathrm{H}^i \mathrm{R}\Gamma_{\mathfrak{m}} M$ *is an artinian A-module for all i.*

Proof. (i) ⇔ (ii) is by [AZ, Corollary 3.6(3)]. (iii) ⇒ (ii) is immediate, since the socle of $\mathrm{H}^i \mathrm{R}\Gamma_{\mathfrak{m}} M$ is a finitely generated k-module, hence bounded. Finally assume (i), and let I be a minimal graded-injective resolution of M. From [Ye1, Lemma 4.3] it follows that $\Gamma_{\mathfrak{m}} I^i \cong A^* \otimes \mathrm{Ext}^i_A(k, M)$, which is artinian. Hence $H^i R\Gamma_{\mathfrak{m}} M$ is artinian. ∎

In an earlier paper we proved the next theorem.

THEOREM 4.5 [YZ1, Theorem 4.2]. *Let A be a noetherian connected graded k-algebra. If A has a balanced dualizing complex then the condition χ holds for A and A°, and the functors $\Gamma_{\mathfrak{m}}$ and $\Gamma_{\mathfrak{m}^\circ}$ have finite cohomological dimensions.*

The converse, which is quite harder, was proved by Van den Bergh.

THEOREM 4.6 [VdB, Theorem 6.3]. *Let A be a noetherian connected graded k-algebra. Assume the condition χ holds for A and A°, and the functors $\Gamma_{\mathfrak{m}}$ and $\Gamma_{\mathfrak{m}^\circ}$ have finite cohomological dimensions. Then*

$$R_A := (\mathrm{R}\Gamma_{\mathfrak{m}} A)^*$$

is a balanced dualizing complex.

Let us summarize some other known results related to the balanced dualizing complexes.

THEOREM 4.7 (Local Duality). *Let R be a balanced dualizing complex over a noetherian connected graded k-algebra A. Then for any graded k-algebra B and any $M \in \mathsf{D}(\mathsf{GrMod}(A \otimes B^{\circ}))$ there is a functorial isomorphism*

$$\mathrm{R}\,\mathrm{Hom}_A^{\mathrm{gr}}(M, R) \cong (\mathrm{R}\Gamma_{\mathfrak{m}} M)^{*}.$$

This is proved by combining [VdB, Theorems 5.1 and 6.3]. The theorem was first proved in [Ye1], but only for $M \in \mathsf{D}_f^b(\mathsf{GrMod}\, A)$.

PROPOSITION 4.8 [VdB, Proposition 8.2(2)]. *A balanced dualizing complex R is rigid in the graded sense.*

Remark 4.9. According to an exercise in [VdB] (whose only proof we know is quite involved), if I is a graded-injective A-module, then I has injective dimension ≤ 1 in $\mathsf{Mod}\, A$. An immediate consequence of this fact is that a graded dualizing complex R over A is also an ungraded dualizing complex. The special case we need, namely that a balanced dualizing complex R is rigid in the ungraded sense, is proved by other means in Corollary 6.7.

Here is another result from [VdB]. Let us write $\mathfrak{m}_{A^e}$ for the augmentation ideal of A^e, so $\mathfrak{m}_{A^e} = \mathfrak{m} \otimes A^{\circ} + A \otimes \mathfrak{m}^{\circ}$.

THEOREM 4.10 [VdB, Corollary 4.8]. *Assume A has a balanced dualizing complex R. Let $M \in \mathsf{D}(\mathsf{GrMod}\, A^e)$ have finitely generated cohomology modules on both sides. Then there is a functorial isomorphism*

$$\mathrm{R}\Gamma_{\mathfrak{m}} M \cong \mathrm{R}\Gamma_{m_{A^e}} M \cong \mathrm{R}\Gamma_{\mathfrak{m}^{\circ}} M.$$

We obtain the following interesting result:

COROLLARY 4.11. *Let R be a balanced dualizing complex over A. Then there is a functorial isomorphism*

$$\mathrm{R}\,\mathrm{Hom}_A(M, R) \cong \mathrm{R}\,\mathrm{Hom}_{A^{\circ}}(M, R)$$

for $M \in \mathsf{D}(\mathsf{GrMod}\, A^e)$ with finitely generated cohomology modules on both sides.

We shall write Cdim_A instead of $\mathrm{Cdim}_{R;\,A}$ when R is the balanced dualizing complex, and when we are working in $\mathsf{GrMod}\, A$. Since a balanced dualizing complex is rigid in the ungraded sense (by Corollary 6.7), this is consistent with Definition 3.10.

DEFINITION 4.12. If A has a graded Auslander balanced dualizing complex R we say A is *graded Auslander*. Furthermore if dim is an exact dimension function on graded modules, and if R is graded dim-Macaulay, then we say A is *graded Auslander* dim-*Macaulay*.

According to [Ye1, Theorem 3.9], any two graded dualizing complexes R_1, R_2 satisfy $R_2 \cong R_1 \otimes_A A^{\sigma}(m)[n]$, for an automorphism σ and integers n, m. It follows that R_1 is graded Auslander iff R_2 is so. In particular, if A is graded Gorenstein, then A is graded Auslander–Gorenstein in the usual sense iff it is graded Auslander in the sense of Definition 4.12.

Taking cohomologies in the previous corollary we get:

COROLLARY 4.13. *Suppose A is graded Auslander. Then* Cdim *is symmetric on graded modules. That is to say, if M is a graded A-bimodule, finitely generated on both sides, then* $\operatorname{Cdim}_A M = \operatorname{Cdim}_{A^\circ} M$.

If A is graded Auslander we have a bound on Krull dimension of graded modules:

THEOREM 4.14. *Suppose A is graded Auslander. Then*

$$\operatorname{Kdim} M \leq \operatorname{Cdim} M = \sup\{q \mid \mathrm{H}^q \mathrm{R}\Gamma_{\mathfrak{m}} M \neq 0\} < \infty$$

for all finitely generated graded A-modules M.

Proof. By Theorem 4.7, if $M \neq 0$, we have $\operatorname{Cdim} M = \sup\{q | \mathrm{H}^q \mathrm{R}\Gamma_{\mathfrak{m}} M \neq 0\}$. Next for a finitely generated graded A-module M the Krull dimension is the same when computed in $\mathsf{GrMod}\, A$ and in $\mathsf{Mod}\, A$. By the graded variant of Corollary 2.18 we get $\operatorname{Kdim} M \leq \operatorname{Cdim}_R M$, since clearly $d_0 = 0$. ∎

LEMMA 4.5. *Let $A \to B$ be a finite homomorphism of noetherian connected graded k-algebras, with augmentation ideals $\mathfrak{m}_A, \mathfrak{m}_B$. Assume A satisfies condition χ. Then there is a functorial isomorphism*

$$\mathrm{R}\Gamma_{\mathfrak{m}_B} M \cong \mathrm{R}\Gamma_{\mathfrak{m}_A} M$$

for $M \in \mathsf{D}^+(\mathsf{GrMod}\, B)$.

Proof. For any homomorphism $A \to B$ of graded algebras there is a functorial morphism $\mathrm{R}\Gamma_{\mathfrak{m}_B} M \to \mathrm{R}\Gamma_{\mathfrak{m}_A} M$ in $\mathsf{D}(\mathsf{GrMod}\, A)$. By [AZ, Lemma 8.2], the extra assumptions guarantee that $\mathrm{H}^p \mathrm{R}\Gamma_{\mathfrak{m}_B} M \to \mathrm{H}^p \mathrm{R}\Gamma_{\mathfrak{m}_A} M$ is bijective for all p. ∎

The following theorem is a generalization of [Jo1, Theorem 3.3].

THEOREM 4.16. *Let $A \to B$ be a finite homomorphism of graded k-algebras and let R_A be a balanced dualizing complex over A. Then*:

(1) *B has a balanced dualizing complex R_B.*

(2) *There is a morphism $\mathrm{Tr}_{B/A} : R_B \to R_A$ in $\mathsf{D}(\mathsf{GrMod}\, A^e)$, which satisfies conditions* (i) *and* (ii) *of Theorem* 3.2.

Proof. From [VdB, Theorem 6.3] we know that A and A° satisfy χ, and $\Gamma_{\mathfrak{m}_A}$ and $\Gamma_{\mathfrak{m}_A^\circ} = \Gamma_{\mathfrak{m}_{A^\circ}}$ have finite cohomological dimensions. So by Lemma 4.15 the same is true for B. Thus B has a balanced dualizing complex $R_B \cong (\mathrm{R}\Gamma_{\mathfrak{m}_B} B)^*$.

The morphism $A \to B$ in $\mathsf{D}(\mathsf{GrMod}\, A^e)$ induces a morphism $(\mathrm{R}\Gamma_{\mathfrak{m}_A} B)^* \to (\mathrm{R}\Gamma_{\mathfrak{m}_A} A)^*$, also in $\mathsf{D}(\mathsf{GrMod}\, A^e)$. But $(\mathrm{R}\Gamma_{\mathfrak{m}_A} B)^* \cong (\mathrm{R}\Gamma_{\mathfrak{m}_B} B)^*$ and we get $\mathrm{Tr}_{B/A} : R_B \to R_A$. The isomorphism of functors $\mathrm{R}\Gamma_{\mathfrak{m}_A} \cong \mathrm{R}\Gamma_{\mathfrak{m}_A^\circ}$ of [VdB, Corollary 4.8] shows that $\mathrm{Tr}_{B/A}$ is the same when calculated on the right, i.e., using $\mathrm{R}\Gamma_{\mathfrak{m}_A^\circ}$.

Condition (i) of Theorem 3.2 is a consequence of local duality. To verify condition (ii) we again view $A \to B$ as a morphism in $\mathsf{D}(\mathsf{GrMod}\, A^e)$. By [VdB, Theorems 4.7 and 5.1] we get a commutative diagram

$$\begin{array}{ccccc} (\mathrm{R}\Gamma_{\mathfrak{m}_B} B)^* & \xrightarrow{\cong} & (\mathrm{R}\Gamma_{\mathfrak{m}_{B^e}} B)^* & \xrightarrow{\cong} & \mathrm{R}\,\mathrm{Hom}_{B^e}(B, (\mathrm{R}\Gamma_{\mathfrak{m}_{B^e}} B^e)^*) \\ \downarrow & & \downarrow & & \downarrow \\ (\mathrm{R}\Gamma_{\mathfrak{m}_A} A)^* & \xrightarrow{\cong} & (\mathrm{R}\Gamma_{\mathfrak{m}_{A^e}} A)^* & \xrightarrow{\cong} & \mathrm{R}\,\mathrm{Hom}_{A^e}(A, (\mathrm{R}\Gamma_{\mathfrak{m}_{A^e}} A^e)^*). \end{array}$$

Finally by [VdB, Theorem 7.1]

$$\left(\mathrm{R}\Gamma_{\mathfrak{m}_{A^e}} A^e\right)^* \cong \left(\mathrm{R}\Gamma_{\mathfrak{m}_A} A\right)^* \otimes \left(\mathrm{R}\Gamma_{m_A^\circ} A\right)^*,$$

and of course the same for B. ▮

Applying the graded variant of Proposition 3.9 we obtain the following corollary.

COROLLARY 4.17. *Let A and B be as in Theorem* 4.16

(1) *There is a functorial isomorphism*

$$\mathrm{R}\,\mathrm{Hom}^{\mathrm{gr}}_B(M, R_B) \cong \mathrm{R}\,\mathrm{Hom}^{\mathrm{gr}}_A(M, R_A)$$

for all $M \in \mathsf{D}(\mathsf{GrMod}\, B)$.

(2) *If A is graded Auslander then so is B.*

(3) $\mathrm{Cdim}_A M = \mathrm{Cdim}_B M$ *for $M \in \mathsf{GrMod}\, B$.*

(4) *If A is graded Auslander* GKdim-*Macaulay then so is B.*

(5) *Suppose $A \to B$ is surjective. If A is graded Auslander* Kdim-*Macaulay, then so is B.*

The next three propositions show that the graded Auslander property can be transferred from one algebra to a related algebra.

PROPOSITION 4.18. *Suppose A has a balanced dualizing complex. Let* dim *stand for either* Kdim *or* GKdim.

(1) *Let* $\mathfrak{a}, \mathfrak{b}$ *be graded ideals. If the quotient algebras* $A/\mathfrak{a}$ *and* $A/\mathfrak{b}$ *are graded Auslander (resp. and graded* dim-*Macaulay), then so is* $A/\mathfrak{a}\mathfrak{b}$.

(2) *Let* $\mathfrak{a}$ *be a nilpotent graded ideal of* A. *If* $A/\mathfrak{a}$ *is graded Auslander (resp. and graded* dim-*Macaulay) then so is* A.

(3) *If for every minimal graded prime ideal* $\mathfrak{p}$ *the quotient algebra* $A/\mathfrak{p}$ *is graded Auslander (resp. and graded* dim-*Macaulay), then so is* A.

Proof. (1) As usual we write $D := \mathrm{R}\,\mathrm{Hom}_A^{\mathrm{gr}}(-, R)$ where R is the balanced dualizing complex. We may assume $\mathfrak{a}\mathfrak{b} = 0$. Given a finitely generated graded module M consider the exact sequence $0 \to \mathfrak{b}M \to M \to M/\mathfrak{b}M \to 0$, and note that $\mathfrak{b}M$ is an $A/\mathfrak{a}$-module. For any i there is an exact sequence $\mathrm{H}^i D(M/\mathfrak{b}M) \to \mathrm{H}^i DM \to \mathrm{H}^i D(\mathfrak{b}M)$. Since $A/\mathfrak{a}$ and $A/\mathfrak{b}$ have the graded Auslander property, the subquotients of $\mathrm{H}^i D(M/\mathfrak{b}M)$ and $\mathrm{H}^i D(\mathfrak{b}M)$ have Cdim no more than i. Observe that here we are using Corollary 4.17. Hence by the long exact sequence of duality, submodules of $\mathrm{H}^i DM$ have Cdim no more than i. The assertion about the Macaulay property is clear.

(2) Use part (1) and induction.

(3) Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the minimal prime ideals of A. Then $(\prod_i \mathfrak{p}_i)^n = 0$ for some n, and we can use parts (1) and (2). ∎

PROPOSITION 4.19. *Let* $A \to B$ *be a finite homomorphism of connected graded algebras, and assume* $B \cong A \oplus L$ *as graded* A-*bimodules. If* B *has a balanced dualizing complex then so does* A.

Proof. Let M be a finitely generated graded A-module. Then by Lemma 4.15 we get

$$\mathrm{R}\Gamma_{\mathfrak{m}_B}(B \otimes_A M) \cong \mathrm{R}\Gamma_{\mathfrak{m}_A}(B \otimes_A M) \cong \mathrm{R}\Gamma_{\mathfrak{m}_A} M \oplus \mathrm{R}\Gamma_{\mathfrak{m}_A}(L \otimes_A M).$$

By Theorem 4.5 and Lemma 4.4 applied to B, we see that the graded B-module $\mathrm{H}^i \mathrm{R}\Gamma_{\mathfrak{m}_B}(B \otimes_A M)$ is right bounded and vanishes for large i. Hence the same is true for the A-module $\mathrm{H}^i \mathrm{R}\Gamma_{\mathfrak{m}_A} M$. Now apply Theorem 4.6 ∎

We do not know if under the assumptions of the proposition above the Auslander property can be transferred from B to A. However, as shown to us by Van den Bergh this is true in a special case:

PROPOSITION 4.20. *Let G be a finite group of order prime to* char k, *acting on B by graded k-algebra automorphisms, and let $A := B^G$ be the fixed ring. If B is graded Auslander then so is A.*

Proof. Given a finitely generated A-module M and a graded A°-submodule $N \subset \mathrm{Ext}_A^q(M, R_A)$ we want to prove that $\mathrm{Ext}_{A^\circ}^p(N, R_A) = 0$ for all $p < q$. Let $L := \{b \in B | \sum_{g \in G} g(b) = 0\}$, so $B = A \oplus L$. We have isomorphisms of graded A°-modules

$$\begin{aligned}&\mathrm{Ext}_A^q(M, R_A) \oplus \mathrm{Ext}_A^q(L \otimes_A M, R_A)\\ &\quad\cong \mathrm{Ext}_A^q(B \otimes_A M, R_A) \cong \mathrm{Ext}_B^q(B \otimes_A M, R_B)\end{aligned}$$

that respect the G-action. Note that G acts trivially on $\mathrm{Ext}_A^q(M, R_A)$. Consider the graded B°-module $N \cdot B \subset \mathrm{Ext}_B^q(B \otimes_A M, R_B)$. Clearly $N \cdot B = N + N \cdot L$. But if

$$n = \sum_i n_i l_i \in N \cap (N \cdot L)$$

with $n_i \in N$ and $l_i \in L$, then

$$n = |G|^{-1} \sum_{g \in G} g\left(\sum_i n_i l_i\right) = \sum_i n_i \cdot |G|^{-1} \sum_{g \in G} g(l_i) = 0.$$

We conclude that $N \cdot B = N \oplus N \cdot L$ as graded A°-modules. Therefore

$$\begin{aligned}&\mathrm{Ext}_{A^\circ}^p(N, R_A) \oplus \mathrm{Ext}_{A^\circ}^p(N \cdot L, R_A)\\ &\quad\cong \mathrm{Ext}_{A^\circ}^p(N \cdot B, R_A) \cong \mathrm{Ext}_{B^\circ}^p(N \cdot B, R_B) = 0.\end{aligned}$$

∎

5. GRADED ALGEBRAS WITH SOME COMMUTATIVITY HYPOTHESIS

In this section we continue the discussion of balanced dualizing complexes over connected graded noetherian k-algebras, but now we look at algebras which have some commutativity hypothesis, like PI, FBN, or enough normal elements. The main result here is:

THEOREM 5.1. *Let A be a noetherian connected graded k-algebra. Suppose $t \in A$ is a homogeneous normal element of positive degree, and let $B := A/(t)$.*

(1) *If B has a balanced dualizing complex, then so does A.*

(2) *If in addition B is graded Auslander, then so is A.*

(3) *If in addition B is graded* Kdim (*resp.* GKdim)-*Macaulay, then so is A.*

The "classical" case of this theorem i.e., when B is Gorenstein and t is a regular element (i.e., a non-zero-divisor), is [Lev, Theorem 3.6]. Part (1) of the theorem is a trivial consequence of Theorems 4.5 and 4.6, and [AZ, Theorem 8.8].

The proof of parts (2) and (3) appears after a series of lemmas. In these lemmas we assume that A has a balanced dualizing complex (by part (1)) and B is graded Auslander. The modules $M, N, \dots$ will be finitely generated graded by default. By Proposition 4.18 we can assume A is prime, hence t is a regular element. The same proposition tells us that

$$A/(t^n) \text{ is graded Auslander for all } n \geq 1. \tag{5.2}$$

We denote by D the duality functor $\mathrm{R}\operatorname{Hom}_A(-, R)$, where R is the balanced dualizing complex of A. Recall that according to Theorem 4.7, $\mathrm{H}^{-i}DM \cong (\mathrm{R}^i\Gamma_{\mathfrak{m}} M)^*$. By definition $\operatorname{Cdim} M = \sup\{i | \mathrm{H}^{-i}DM \neq 0\}$, so trivially

$$\operatorname{Cdim} M \leq \max\{\operatorname{Cdim} M', \operatorname{Cdim} M/M'\} \tag{5.3}$$

for all $M' \subset M$.

LEMMA 5.4. *If M is t-torsion-free then* $\operatorname{Cdim} M = \operatorname{Cdim} M/tM + 1$. *If* $d = \operatorname{Cdim} M$ *then $H^{-d}DM$ is t-torsion-free.*

Proof. Let σ be the automorphism of A such that $t \cdot a = \sigma(a) \cdot t$, and let ${}^{\sigma}M$ be the corresponding twisted module. Then we have an exact sequence

$$0 \to {}^{\sigma^{-1}} M(-l) \xrightarrow{t\cdot} M \to M/tM \to 0,$$

where l is the degree of t. It is easy to see that $\mathrm{H}^iD({}^{\sigma^{-1}}M(-l)) \cong (\mathrm{H}^iDM)^{\sigma^{-1}}(l)$, so there is a long exact sequence

$$\mathrm{H}^iD(M/tM) \to \mathrm{H}^iDM \xrightarrow{\cdot t} (\mathrm{H}^iDM)^{\sigma^{-1}}(l) \to \mathrm{H}^{i+1}D(M/tM). \tag{5.5}$$

If $\mathrm{H}^{i+1}D(M/tM) = 0$ then by the graded Nakayama Lemma we get $\mathrm{H}^iDM = 0$. Therefore

$$\operatorname{Cdim} M/tM \leq \operatorname{Cdim} M \leq \operatorname{Cdim} M/tM + 1.$$

Now let $d = \operatorname{Cdim} M$. We need to show that $\mathrm{H}^{-d}D(M/tM) = 0$. If not, then $\operatorname{Cdim} M/tM = d$, and hence also $\operatorname{Cdim} M/t^nM = \operatorname{Cdim} t^{n-1}M/t^nM = d$ for all $n \geq 1$. According to Proposition 4.18 the algebra $A/(t)^n$ has the graded Auslander property. This implies that

$$\operatorname{Cdim} \mathrm{H}^{-d}D\left(t^{n-1}M/t^nM\right) = d \qquad \text{and}$$
$$\operatorname{Cdim} \mathrm{H}^{-d+1}D\left(M/t^{n-1}M\right) \leq d - 1.$$

Looking at the exact sequence

$$\begin{aligned} 0 &\to \mathrm{H}^{-d}D\big(M/t^{n-1}M\big) \to \mathrm{H}^{-d}D\big(M/t^{n}M\big) \\ &\to \mathrm{H}^{-d}D\big(t^{n-1}M/t^{n}M\big) \xrightarrow{\phi} \mathrm{H}^{-d+1}D\big(M/t^{n-1}M\big) \end{aligned}$$

we see that ϕ cannot be an injection. Therefore

$$\mathrm{H}^{-d}D\big(M/t^{n-1}M\big) \subsetneqq \mathrm{H}^{-d}D\big(M/t^{n}M\big) \subset \mathrm{H}^{-d}D(M).$$

But this is true for all $n \geq 1$, contradicting the noetherian property of $\mathrm{H}^{-d}D(M)$. The upshot is that $\mathrm{H}^{d}D(M/tM) = 0$. Taking $i = -d$ in (5.5) we conclude that $\mathrm{H}^{d}DM$ is t-torsion-free. ∎

LEMMA 5.6. *Let N be the t-torsion submodule of M. Then*

$$\operatorname{Cdim} M = \max\{\operatorname{Cdim} N, \operatorname{Cdim} M/N\}.$$

Proof. Let d be the right hand side. Trivially $\operatorname{Cdim} M \leq d$ holds. Assume $\operatorname{Cdim} M < d$. Dualizing the exact sequence $0 \to N \to M \to M/N \to 0$ we obtain a long exact sequence

$$\begin{aligned} \cdots &\to \mathrm{H}^{i}D(M/N) \to \mathrm{H}^{i}DM \to \mathrm{H}^{i}DN \to \mathrm{H}^{i+1}D(M/N) \\ &\to \mathrm{H}^{i+1}DM \to \cdots \end{aligned}$$

and taking $i = -d-1$ we see that $\mathrm{H}^{-d}D(M/N) = 0$, so $\operatorname{Cdim} M/N < d$. Hence $\operatorname{Cdim} N = d$. Also taking $i = -d$ we see that $\mathrm{H}^{-d}DN \subset \mathrm{H}^{-d+1}D(M/N)$. Now $\mathrm{H}^{-d}DN$ is t-torsion, yet by Lemma 5.4, $\mathrm{H}^{-d+1}D(M/N)$ is t-torsion-free. We conclude that $\mathrm{H}^{-d}DN = 0$, which is a contradiction. ∎

LEMMA 5.7. $\operatorname{Cdim} M = \max\{\operatorname{Cdim} L, \operatorname{Cdim} M/L\}$ *for every* $L \subset M$.

Proof. By (5.3) it remains to prove $\geq$. Let N be the t-torsion submodule of M. By (5.2) we have $\operatorname{Cdim} N \geq \operatorname{Cdim} N \cap L$. On the other hand by Lemma 5.6, $\operatorname{Cdim} M = \max\{\operatorname{Cdim} N, \operatorname{Cdim} M/N\}$ and $\operatorname{Cdim} L = \max\{\operatorname{Cdim} N \cap L, \operatorname{Cdim} L/N \cap L\}$. Hence it suffices to prove that $\operatorname{Cdim} M/N = \max\{\operatorname{Cdim} L/N \cap L, \operatorname{Cdim} M/L\}$. So we may assume $M = M/N$ is t-torsion-free.

If $P := M/\mathrm{IL}$ is also t-torsion-free then we get a short exact sequence

$$0 \to L/tL \to M/tM \to P/tP \to 0.$$

Hence the assertion follows from Lemma 5.4.

In general let $L' \supset L$ be such that L'/L is the t-torsion submodule of M/L. So $t^{n}L' \subset L$ for some n. By Lemma 5.4,

$$\operatorname{Cdim} L'/L \leq \operatorname{Cdim} L'/t^{n}L' = \operatorname{Cdim} L' - 1$$

and hence, from the long exact sequence of duality, we get Cdim $L =$ Cdim L'. Applying the previous paragraph and Lemma 5.6 we have

$$\begin{aligned}\text{Cdim } M &= \max\{\text{Cdim } L', \text{Cdim } M/L'\}\\ &= \max\{\text{Cdim } L, \text{Cdim } M/L'\}\\ &= \max\{\text{Cdim } L, \text{Cdim } L'/L, \text{Cdim } M/L'\}\\ &= \max\{\text{Cdim } L, \text{Cdim } M/L\}\end{aligned}$$

and we finish the proof. ∎

LEMMA 5.8. *Suppose t^n kills the t-torsion submodule of M. Then*

$$\text{Cdim } M \le \text{Cdim } M/t^nM + 1.$$

Proof. Let N be the t-torsion submodule of M and $P := M/N$. Then $t^nM \cong t^nP$ is a t-torsion-free submodule of M and $P/t^nP \cong M/(N \oplus t^nM)$. By Lemmas 5.4 and 5.6,

$$\text{Cdim } M = \max\{\text{Cdim } N, \text{Cdim } P\} = \max\{\text{Cdim } N, \text{Cdim } P/t^nP + 1\}.$$

On the other hand by lemma 5.7,

$$\text{Cdim } M/t^nM = \max\{\text{Cdim } N, \text{Cdim } P/t^nP\}.$$

It remains to combine these equalities. ∎

Proof of Theorem 5.1. As mentioned above part (1) is a consequence of Theorems 4.5 and 4.6, and [AZ, Theorem 8.8].

(2) Recall that the Auslander condition says that if $N \subset \mathrm{H}^{-i}DM$ then Cdim $N \le i$. We can assume A is prime and t is regular. By Lemma 5.7 it suffices to prove that $\text{Cdim}\,\mathrm{H}^{-i}DM \le i$ for all i. According to (5.2) the inequality holds for t-torsion modules, so using the long exact sequence of duality we may assume that M is t-torsion-free.

Choose n such that t^n kills the t-torsion submodule of $\mathrm{H}^{-i}DM$. The short exact sequence

$$0 \to M \xrightarrow{t^n\cdot} {}^{\sigma^n}M(nl) \to {}^{\sigma^n}(M/t^nM)(nl) \to 0$$

gives rise to a long exact sequence

$$\begin{aligned}\cdots \to (\mathrm{H}^{-i}DM)^{\sigma^n}(-nl) &\xrightarrow{t^n\cdot} \mathrm{H}^{-i}DM \to P\\ &= (\mathrm{H}^{-i+1}D(M/t^nM))^{\sigma^n}(-nl) \to \cdots.\end{aligned}$$

Since M/t^nM is t-torsion, P is also t-torsion, and every submodule of P has Cdim $\le i - 1$, So according to Lemma 5.8, $\text{Cdim}\,\mathrm{H}^{-i}DM \le (i-1) + 1 = i$.

(3) Assume B is graded GKdim-Macaulay. Since $A/(t^n)$ is graded GKdim-Macaulay for $n \geq 1$ (by Proposition 4.18), it suffices to show that $\operatorname{Cdim} M = \operatorname{GKdim} M$ for t-torsion-free modules. We know that in this case $\operatorname{GKdim} M = \operatorname{GKdim} N/tM + 1$, and by Lemma 5.4 this is true also for $\operatorname{Cdim} M$.

Finally assume B is graded Kdim-Macaulay. Again we need only consider M which is t-torsion-free. It is clear that $\operatorname{Kdim} M \geq \operatorname{Kdim} M/tM + 1$. Hence it follows from Lemmas 5.4 and 5.7 that $\operatorname{Kdim} M \geq \operatorname{Cdim} M$. But by Theorem 4.14, $\operatorname{Kdim} M \leq \operatorname{Cdim} M$. ∎

EXAMPLE 5.9. Let us consider a simple case of Theorem 7.1. Let $A = B[t]$ where t is a central variable of degree 1. Let R_B be the balanced dualizing complex of B. Then the balanced dualizing complex of A is nothing but

$$R_A = R_B \otimes_{k[t]} \Omega^1_{k[t]/k}[1] \cong R_B[t](-1)[1],$$

where $\Omega^1_{k[t]/k} = k[t] \cdot dt \cong k[t](-1)$ as graded $k[t]$-modules. This follows from [VdB, Theorem 7.1], which states that if B, C are noetherian connected graded k-algebras with balanced dualizing complexes R_B and R_C, respectively, and if $B \otimes C$ is noetherian, then $R_B \otimes R_C$ is a balanced dualizing complex over $B \otimes C$. By Theorem 5.1, A is graded Auslander and Kdim (resp. GKdim)-Macaulay if and only if B is.

COROLLARY 5.10. *Let A be a connected graded k-algebra with a balanced dualizing complex. Suppose that every graded prime quotient $A/\mathfrak{p}$ satisfies one of the following conditions*:

(i) *$A/\mathfrak{p}$ is graded Auslander* (*resp. and graded* Kdim (*or* GKdim)-*Macaulay*); *or*

(ii) *$A/\mathfrak{p}$ has a normal element of positive degree.*

Then A is graded Auslander (*resp. and graded* Kdim (*or* GKdim)-*Macaulay*).

Proof. Use Proposition 4.18, Theorem 5.1, and induction on $\operatorname{Kdim} A$. ∎

Recall that A has *enough normal elements* if every graded prime quotient $A/\mathfrak{p}$ (except for $\mathfrak{p} = \mathfrak{m}$) contains a normal element of positive degree. By Corollary 5.10, a noetherian connected graded algebra with enough normal elements is Auslander and GKdim-Macaulay. In the rest of this section we generalize this statement.

COROLLARY 5.11. *Let A and B be noetherian connected graded k-algebras. Assume A has a balanced dualizing complex and is graded Auslander* (*resp. and graded* Kdim (*or* GKdim)-*Macaulay*), *and B has enough normal* *elements. Then* $A \otimes B$ *is a noetherian connected graded* k*-algebra with a balanced dualizing complex, and it is graded Auslander* (*resp. and graded* Kdim (*or* GKdim)*-Macaulay*).

Proof. We first prove that $A \otimes B$ is noetherian by induction on Kdim B. Suppose $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ is the complete set of minimal graded primes of B. Then some product $\prod_{i=1}^{t} \mathfrak{p}_{n_i}$ is zero. For any $s \geq 1$ let $W_s := \prod_{i=1}^{s-1} \mathfrak{p}_{n_i} / \prod_{i=1}^{s} \mathfrak{p}_{n_i}$, where $W_1 = B/\mathfrak{p}_{n_1}$, and let $B_s = B/\mathfrak{p}_{n_s}$. Since W_s is a finitely generated B_s-module, $A \otimes W_s$ is a finitely generated $A \otimes B_s$-module. Hence it suffices to show that each $A \otimes B_s$ is noetherian. This reduces to the case when B is prime. Hence we may assume B has a regular normal element t of positive degree. It is obvious that $1 \otimes t$ is a regular normal element in $A \otimes B$. By induction hypothesis, $A \otimes B/(t)$ is noetherian, and therefore $A \otimes B$ is noetherian by [ATV, Theorem 8.2].

The graded Auslander and Macaulay properties follow from the same inductive procedure and Theorem 5.1 ∎

In [SZ, Theorem 3.10] it was shown that a connected graded PI algebra of finite injective dimension (i.e., a Gorenstein algebra) is graded Auslander−Gorenstein and graded GKdim-Macaulay. This was extended in [Zh] to an algebra A having enough normal elements. Corollary 5.11 (when $A = k$) generalizes these theorems by eliminating the Gorenstein condition. We extend the result further in Theorems 5.13 and 5.14 below.

DEFINITION 5.12. Let R be a balanced dualizing complex over A.

(1) We say two graded A-modules M and N are *similar* if there are isomorphisms $M \cong N$ and $\mathrm{R\,Hom}_A^{\mathrm{gr}}(M, R) \cong \mathrm{R\,Hom}_A^{\mathrm{gr}}(N, R)$ in $\mathsf{D}(\mathsf{GrMod}\, k)$.

(2) The algebra A satisfies the *similar submodule condition* if every nonzero, $\mathfrak{m}$-torsion-free, finitely generated, graded A-module M has graded submodules $N' \subsetneqq N \subset M$ with N' similar to $N(-l)$ for some $l > 0$.

We remark that two complexes $M, N \in \mathsf{D}_f^b(\mathsf{GrMod}\, A)$ are isomorphic in $\mathsf{D}(\mathsf{GrMod}\, k)$ if and only if they have equal Hilbert functions, namely $\mathrm{rank}_k(\mathrm{H}^i M)_j = \mathrm{rank}_k(\mathrm{H}^i N)_j$ for all $i, j \in \mathbb{Z}$. This definition of a similar submodule condition is equivalent to the definition given in [Zh, Section 2] when A is AS-Gorenstein, which is the case considered there. Also, as mentioned in [Zh, Sect. 2] there are algebras which do not satisfy the similar submodule condition.

THEOREM 5.13. *Assume A is a noetherian connected graded k-algebra and satisfies one of the following*:

(i) *A is a PI algebra*.

(ii) *A is graded-FBN*.

(iii) *A has enough normal elements*.

Then A has a balanced dualizing complex, and the similar submodule condition holds.

Note that if A is PI, then A is graded FBN and has enough normal elements.

Proof. The existence of a balanced dualizing complex follows Theorem 4.6, together with [AZ, Theorems 8.8 and 8.13]. The statement about the similar submodule condition is proved like in [Zh, Sect. 2, first paragraph, and Proposition 2.3]. ∎

We now prove a generalized version of [Zh, Theorem 3.1].

THEOREM 5.14. *Let A be a noetherian connected graded k-algebra which has a balanced dualizing complex R and satisfies the similar submodule condition. Then*

(1) *A is graded Auslander*.

(2) *A is graded* Kdim *and* GKdim-*Macaulay. More precisely*

$$\operatorname{Cdim}_R M = \operatorname{Kdim} M = \operatorname{GKdim} M$$

for every finitely generated left or right A-module M.

Proof. First we observe that [Zh, Lemma 2.2] holds (same proof), and hence $\operatorname{Kdim} M \geq \operatorname{GKdim} M$ and $\operatorname{GKdim} M < \infty$ for every finitely generated graded A-module M. Therefore, replacing $\operatorname{Ext}^i(-, A)$ with $\operatorname{Ext}^i_A(-, R)$, the proof of [Zh, Theorem 3.1] can be copied verbatim. Let us just mention the key point of the proof. We prove by induction on $\operatorname{GKdim} M$ that for every finitely generated graded A-module (resp. A°-module) M:

(a) $j_R(M) = -\operatorname{GKdim} M$.

(b) $\operatorname{GKdim} \operatorname{Ext}^j_A(M, R) = \operatorname{GKdim} M$, where $j = j_R(M)$.

(c) For every $j_R(M) \leq i \leq 0$ one has $\operatorname{GKdim} \operatorname{Ext}^i_A(M, R) \leq -i$.

This implies that A is graded Auslander and graded GKdim-Macaulay. By [Zh, Lemma 2.2], $\operatorname{Kdim} M \geq \operatorname{GKdim} M = \operatorname{Cdim} M$, and by Theorem 4.14, $\operatorname{Kdim} M \leq \operatorname{Cdim} M$. Hence A is graded Kdim-Macaulay. ∎

Theorem 0.6 is an immediate consequence of Theorems 5.13 and 5.14. Another consequence is Proposition 0.9 which we now prove.

Proof of Proposition 0.9. If A is connected graded, i.e., $A_0 = k$, the assertion follows from Theorems 5.13 and 5.14. Now assume A is not connected. Let $B := k + A_{\geq 1}$, which is connected. Since A and B differ by a finite rank k-module, B is noetherian. It remains to verify the following statements.

(a) If M is a finitely generated graded A-module, then $\mathrm{Kdim}_A M = \mathrm{Kdim}_B M$ and $\mathrm{GKdim}_A M = \mathrm{GKdim}_B M$.

(b) If A has enough normal elements, then so does B.

(c) If A is graded FBN, then so is B.

(a) The statement about GKdim is obvious because GKdim is determined by the degree of the Hilbert function of M (see [Zh, Lemma 2.2(1)]). Next we consider Kdim. Clearly $\mathrm{Kdim}_A M = 0$ if and only if M is a finitely generated k-module, if and only if $\mathrm{Kdim}_B M = 0$. For higher dimension we consider the quotient category $\mathsf{QGr}\, A := \mathsf{GrMod}\, A/\mathsf{M}_0$, where $\mathsf{M}_0 = \mathsf{M}_0(\mathrm{Kdim})$ is the localizing subcategory consisting of Kdim $= 0$ modules (torsion modules in the terminology of [AZ]). For any $M \in \mathsf{GrMod}_f\, A$ one has

$$\mathrm{Kdim}_A M = \mathrm{Kdim}_{\mathsf{GrMod}\, A} M = \mathrm{Kdim}_{\mathsf{QGr}\, A} M + 1.$$

Now since A and B differ by a finite rank k-module, $\mathsf{QGr}\, A$ is equivalent to $\mathsf{QGr}\, B$ by [AZ, Proposition 2.5], so $\mathrm{Kdim}_{\mathsf{QGr}\, A} M = \mathrm{Kdim}_{\mathsf{QGr}\, B} M$.

(b) Let $\mathfrak{p}$ be a graded prime ideal of B which is not $\mathfrak{m}_B = B_{\geq 1}$. Then $A\mathfrak{p}A$ and $\mathfrak{p}$ differ by a finitely generated k-module, and hence $\mathrm{GKdim}\, A/A\mathfrak{p}A = \mathrm{GKdim}\, B/\mathfrak{p}$. Let $\mathfrak{q}$ be a graded prime of A minimal over $A\mathfrak{p}A$ such that $\mathrm{GKdim}\, A/\mathfrak{q} = \mathrm{GKdim}\, B/\mathfrak{p}$. Then the map $B/\mathfrak{p} \to A/\mathfrak{q}$ is injective, and bijective in positive degrees. Thus normal elements of positive degree in $A/\mathfrak{q}$ are also normal elements in $B/\mathfrak{p}$.

(c) The proof is similar to (b) and we leave it to the reader. ∎

6. NOETHERIAN CONNECTED FILTRATIONS

In this section we use filtrations to transfer results of Sections 4–5 on connected graded algebras to non-graded algebras. Throughout the section A denotes a noetherian k-algebra.

Suppose a k-module M is given an increasing filtration $F = \{F_n M\}_{n \in \mathbb{Z}}$ with $\bigcap_n F_n M = 0$ and $\bigcup_n F_n M = M$. The *Rees module* is the graded $k[t]$-module

$$\mathrm{Rees}^F M := \bigoplus_{n \in \mathbb{Z}} F_n M \cdot t^n \subset M[t, t^{-1}],$$

where t is a central indeterminate. It is easy to check that

$$(\mathrm{Rees}^F M)/(t-1)\cdot(\mathrm{Rees}^F M) \cong M$$

and

$$(\mathrm{Rees}^F M)/t\cdot(\mathrm{Rees}^F M) \cong \mathrm{gr}^F M = \bigoplus_{n\in\mathbb{Z}} F_n M/F_{n-1}M.$$

Now let $F = \{F_n A\}_{n\in\mathbb{Z}}$ be a filtration of A such that $F_n A\cdot F_m A \subset F_{n+m}A$. The graded k-algebra $\mathrm{Rees}^F A$ is called the *Rees algebra*.

DEFINITION 6.1. If the Rees algebra $\mathrm{Rees}^F A$ is a noetherian connected graded k-algebra then F is called a *noetherian connected filtration*.

Observe that $\mathrm{Rees}^F A$ is connected graded iff $F_0 A \cong k$, $F_{-1}A = 0$, and $\mathrm{rank}_k F_n A < \infty$. If $\mathrm{Rees}^F A$ is noetherian then so is the associated graded algebra $\mathrm{gr}^F A$. By [ATV, Theorem 8.2] the converse is also true—if $\mathrm{gr}^F A$ is noetherian then so is $\mathrm{Rees}^F A$.

A filtration $\{F_n A\}$ on an A-module M with $F_n A\cdot F_m M \subset F_{n+m}M$ gives a graded module $\mathrm{Rees}^F M$ over $\mathrm{Rees}^F A$. We say $\{F_n M\}$ is a *good filtration* if $\mathrm{Rees}^F M$ is a finitely generated $(\mathrm{Rees}^F A)$-module.

The main result of this section is the following theorem. Part (1) is due to Van den Bergh [VdB, Theorem 8.6], but there was a subtle flaw in his statement: the shift by -1 was missing.

THEOREM 6.2. *Let F be a noetherian connected filtration on A and let* $\tilde{A} := \mathrm{Rees}^F A$.

(1) *If $\tilde{A}$ has a balanced dualizing complex $\tilde{R}$ then*

$$R := \left(A \otimes_{\tilde{A}} \tilde{R} \otimes_{\tilde{A}} A\right)[-1]$$

is a rigid dualizing complex over A.

(2) *If $\tilde{R}$ is graded Auslander then R is Auslander.*

(3) *Suppose $\tilde{R}$ is graded Auslander. If $\tilde{R}$ is graded* GKdim*-Macaulay then R is* GKdim*-Macaulay.*

The proof comes after this lemma.

Consider the functor $\pi : \mathsf{GrMod}\, k[t] \to \mathsf{Mod}\, k$, $\tilde{M} \mapsto \tilde{M}/(t-1)\tilde{M}$. Write $\tilde{A}_t := A[t, t^{-1}]$.

LEMMA 6.3. (1) *The functor π is exact.*

(2) *If $\tilde{I} \in \mathsf{GrMod}\,\tilde{A}$ is injective then $\pi\tilde{I} \in \mathsf{Mod}\, A$ is injective.*

(3) *There is a functorial isomorphism*

$$\mathrm{R\,Hom}_A(\pi\tilde{M}, \pi\tilde{N}) \cong \pi\, \mathrm{R\,Hom}^{\mathrm{gr}}_{\tilde{A}}(\tilde{M}, \tilde{N})$$

for $\tilde{M}, \tilde{N} \in \mathsf{D}^b_f(\mathsf{GrMod}\,\tilde{A})$.

Proof. (1), (2) π is a composition of the functors "localization by t" $(-)_t : \mathsf{GrMod}\,\tilde{A} \to \mathsf{GrMod}\,\tilde{A}_t$ and "taking degree 0 component" $(-)_0 : \mathsf{GrMod}\,\tilde{A}_t \to \mathsf{Mod}\,A$, both of which are exact. The second is even an equivalence, so injectives go to injectives. Since $\tilde{A}_t$ is noetherian a standard argument shows that the $\tilde{A}_t$-module $\tilde{I}_t$ is graded-injective.

(3) There is a functorial morphism $\psi : \pi\, \mathrm{R}\,\mathrm{Hom}^{\mathrm{gr}}_{\tilde{A}}(\tilde{M}, \tilde{N}) \to \mathrm{R}\,\mathrm{Hom}_A(\pi\tilde{M}, \pi\tilde{N})$. Fixing $\tilde{N}$ these are way-out right contravariant functors of $\tilde{M}$. By [RD, proposition I.7.1(iv)]—reversed—it's enough to check to that ψ is an isomorphism when $\tilde{M} = \tilde{A}$, and then it's trivial. ∎

Proof of Theorem 6.2. (1) By Proposition 4.8 and the graded version of Corollary 3.6 each cohomology module $\mathrm{H}^q\tilde{R}$ is $k[t]$-central (cf. Remark 6.6 below). Define

$$\tilde{R}_t := \tilde{A}_t \otimes_{\tilde{A}} \tilde{R} \otimes_{\tilde{A}} \tilde{A}_t \in \mathsf{D}\Big(\mathsf{GrMod}\big(\tilde{A}_t\big)^e\Big).$$

We see that the homomorphisms $\tilde{A}_t \otimes_{\tilde{A}} \tilde{R} \to \tilde{R}_t$ and $\tilde{R} \otimes_{\tilde{A}} \tilde{A}_t \to \tilde{R}_t$ are quasi-isomorphisms.

Because $\tilde{A} \to A = \pi\tilde{A}$ is flat we also have

$$A \otimes_{\tilde{A}} \tilde{R}[-1] \cong \tilde{R}[-1] \otimes_{\tilde{A}} A \cong R \in \mathsf{D}(\mathsf{Mod}\,\tilde{A}^e).$$

Considering only $\tilde{A}$-modules we have $R \cong \pi\tilde{R}[-1]$, so by Lemma 6.2, R has finite injective dimension and finitely generated cohomologies over A. By symmetry this is true also over A°. Part (3) of the lemma implies that $\mathrm{R}\,\mathrm{Hom}_A(R, R) \cong \pi\tilde{A} = A$, and likewise $\mathrm{R}\,\mathrm{Hom}_{A^\circ}(R, R) = A$. We conclude that R is dualizing.

Now let's prove R is rigid. There is a functorial isomorphism $\pi\tilde{M} \cong (\tilde{M}_t)_0$ for $\tilde{M} \in \mathsf{GrMod}\,k[t]$, and the algebra A^e is $\mathbb{Z}^2$-graded. Therefore we get

$$A^e \otimes_{\tilde{A}^e} (\tilde{R} \otimes \tilde{R}) \cong \big(A \otimes_{\tilde{A}} \tilde{R}\big) \otimes \big(\tilde{R} \otimes_{\tilde{A}} A\big) \cong \big(\tilde{R}_t\big)_0 \otimes \big(\tilde{R}_t\big)_0$$
$$\cong \big(\tilde{R}_t \otimes \tilde{R}_t\big)_{(0,0)}.$$

The algebra $(\tilde{A}_t)^e$ is strongly $\mathbb{Z}$-graded, and its degree 0 component is

$$\big(\tilde{A}_t\big)^e_0 \cong A^e[s, s^{-1}] \cong k[s, s^{-1}] \otimes A^e,$$

where $s := t \otimes t^{-1}$. Applying $(\tilde{A}_t)^e \otimes_{\tilde{A}^e} -$ to

$$\tilde{R} \cong \mathrm{R}\,\mathrm{Hom}^{\mathrm{gr}}_{\tilde{A}^e}\big(\tilde{A}, \tilde{R} \otimes \tilde{R}\big)$$

we obtain

$$\tilde{R}_t \cong \mathrm{R}\,\mathrm{Hom}^{\mathrm{gr}}_{(\tilde{A}_t)^e}\big(\tilde{A}_t, \tilde{R}_t \otimes \tilde{R}_t\big)$$

and taking degree 0 components we get

$$R[1] \cong \mathrm{R\,Hom}^{\mathrm{gr}}_{(\tilde{A}_t)^e_0}\Big(A, \big(\tilde{R}_t \otimes \tilde{R}_t\big)_0\Big)$$
$$\cong \mathrm{R\,Hom}_{k[s,s^{-1}]\otimes A^e}\big(A, k[s,s^{-1}] \otimes (R[1] \otimes R[1])\big).$$

But

$$k \cong \big(k[s,s^{-1}] \xrightarrow{s-1} k[s,s^{-1}]\big) \in \mathsf{D}\big(\mathsf{Mod}\, k[s,s^{-1}]\big),$$

and A is a finitely presented A^e-module, so

$$\mathrm{R\,Hom}_{k[s,s^{-1}]\otimes A^e}\big(k \otimes A, k[s,s^{-1}] \otimes M\big) \cong \mathrm{R\,Hom}_{A^e}(A, M)[-1]$$

for any complex of A^e-modules M.

(2) Let $\tilde{M}$ be a t-torsion-free finitely generated graded $\tilde{A}$-module. We claim that

$$j_{R;A}(\pi\tilde{M}) = j_{\tilde{R};\tilde{A}}\tilde{M} + 1. \tag{6.4}$$

First by Lemma 6.3 we get for any q,

$$\mathrm{Ext}_A^{-q+1}(\pi\tilde{M}, R) \cong \pi\, \mathrm{Ext}_{\tilde{A}}^{-q}(\tilde{M}, \tilde{R}), \tag{6.5}$$

so $j_{R;A}(\pi\tilde{M}) \geq j_{\tilde{R};\tilde{A}}\tilde{M} + 1$. Now take $q := -j_{\tilde{R};\tilde{A}}\tilde{M}$ and write $\tilde{N} := \mathrm{Ext}_{\tilde{A}}^{-q}(\tilde{M}, \tilde{R})$. If $\pi\tilde{N} = 0$ then $t^l\tilde{N} = 0$ for some $l > 0$. But in the Ext spectral sequence converging to $\tilde{M}$ (see proof of Theorem 2.10) the dominant term is $\mathrm{Ext}_{\tilde{A}^\circ}^{-q}(\tilde{N}, \tilde{R})$ which is killed by t^l. We get $\mathrm{Cdim}_{\tilde{R}}\, t^l\tilde{M} < \mathrm{Cdim}_{\tilde{R}}\, \tilde{M} = q$, which is absurd since $t^l\tilde{M} \cong \tilde{M}(-l)$.

Given finitely generated A-modules $M \subset N$, take any good filtration $\{F_n N\}$, and let $F_n M := M \cap F_n N$, $\tilde{M} := \mathrm{Rees}^F M$, and $\tilde{N} := \mathrm{Rees}^F N$. Since $\tilde{M}$ and $\tilde{N}$ are t-torsion-free we see that

$$\mathrm{Cdim}_R M = \mathrm{Cdim}_{\tilde{R}}\, \tilde{M} - 1 \leq \mathrm{Cdim}_{\tilde{R}}\, \tilde{N} - 1 = \mathrm{Cdim}_R N.$$

Finally let $L := \mathrm{Ext}_A^{-q}(M, R)$. Then, with $\tilde{M} := \mathrm{Rees}^F M$ and $\tilde{L} := \mathrm{Ext}_{\tilde{A}}^{q-1}(\tilde{M}, \tilde{R})$, we see that $j_{R;A}L \geq j_{\tilde{R};\tilde{A}}\tilde{L} + 1 \geq -q$, verifying the Auslander condition on one side. By symmetry it holds also on the other side.

(3) By the proof of part (2), given a finitely generated A-module M one has $\mathrm{Cdim}_{R;A} M + 1 = \mathrm{Cdim}_{\tilde{R};\tilde{A}}\, \tilde{M}$ with $\tilde{M} := \mathrm{Rees}^F M$ w.r.t. any good filtration $\{F_n M\}$. But because $\tilde{M}_t \cong M \otimes k[t, t^{-1}]$ we also have

$$\mathrm{GKdim}_A M + 1 = \mathrm{GKdim}_{\tilde{A}_t}\, \tilde{M}_t = \mathrm{GKdim}_{\tilde{A}}\, \tilde{M}.$$

∎

Remark 6.6. It is not too hard to show that R could be chosen to be a $k[t]$-central complex of graded A-bimodules.

Observe that the rigid dualizing complex R has $\mathrm{H}^q R = 0$ for $q > 0$, since $\mathrm{H}^0 \tilde{R}$ is t-torsion and $\mathrm{H}^q \tilde{R} = 0$ for $q > 0$.

COROLLARY 6.7. *Suppose A is connected graded and R is a balanced dualizing complex over A. Then R is a rigid dualizing complex over A in the ungraded sense.*

Proof. First let us note that for any graded A-module M we can define a filtration $F_n M := \bigoplus_{i \leq n} M_i$. Then we have a functorial isomorphism of graded $A[t]$-modules $\mathrm{Rees}^F M \xrightarrow{\simeq} M[t]$.

In particular we get $\tilde{A} \cong A[t]$, so by Example 5.9 we know that $\tilde{R} = R[t](-1)[1]$ is the balanced dualizing complex of $\tilde{A}$. But then $\pi\tilde{R}[-1] = \pi R[t](-1) \cong R$ in $\mathsf{D}(\mathsf{Mod}\, A^e)$, and this is rigid by Theorem 6.2(1). ∎

The next corollary implies Theorem 0.7.

COROLLARY 6.8. *Suppose A has a noetherian connected filtration F, and let $\bar{A} := \mathrm{gr}^F A$.*

(1) *If $\bar{A}$ has a balanced dualizing complex $\bar{R}$, then A has a rigid dualizing complex R.*

(2) *If $\bar{R}$ is graded Auslander then R is Auslander.*

(3) *If $\bar{R}$ is also graded* GKdim-*Macaulay, then R is* GKdim-*Macaulay.*

Proof. According to Theorem 5.1, the Rees algebra $\tilde{A} = \mathrm{Rees}^F A$ inherits these properties from $\bar{A}$. And by Theorem 6.2 they pass to A. ∎

COROLLARY 6.9. *Suppose A has a noetherian connected filtration F, and $\bar{A} := \mathrm{gr}^F A$ satisfies either of the following*:

(i) *$\bar{A}$ is a PI algebra.*

(ii) *$\bar{A}$ is graded-FBN.*

(iii) *$\bar{A}$ has enough normal elements.*

Then A has an Auslander, GKdim-*Macaulay, rigid dualizing complex.*

Proof. Combine Corollary 6.8 and Theorems 5.13 and 5.14. ∎

Here are some examples of algebras which admit noetherian connected filtrations.

EXAMPLE 6.10. If A is a noetherian connected graded algebra, then the filtration $F_n A = \bigoplus_{i \leq n} A_i$ is a noetherian connected filtration.

EXAMPLE 6.11. Suppose A is generated by elements $x_1, \ldots, x_n$, and for every $i \neq j$ there is some relation

$$x_j x_i = q_{i,j} x_i x_j + a_{i,j} x_i + b_{i,j} x_j + c_{i,j} \tag{6.12}$$

with $q_{i,j}, a_{i,j}, b_{i,j}, c_{i,j} \in k$. Let $V := k + \sum k \cdot x_i \subset A$ and define a filtration $F_n A := V^n$. Then $\mathrm{gr}^F A$ is a quotient of the skew polynomial algebra $k_q[x_1, \ldots, x_n]$, so F is a noetherian connected filtration. Furthermore, $\mathrm{gr}^F A$ has enough normal elements (namely the x_i); so Corollary 6.9 holds. It is easy to check that the relations (6.11) are satisfied in the following classes of algebras:

(i) Commutative affine algebras.

(ii) Weyl algebras, enveloping algebras of finite dimensional Lie algebras and their quotients.

(iii) Most classes of quantum algebras listed in [GL].

Recall that a homomorphism $f: A \to B$ is called finite if B is a finite left and right A-module. f is *centralizing* if $B = A \cdot Z_B(A)$. Thus f is finite centralizing iff there exist $b_1, \ldots, b_m \in Z_B(A)$ such that $B = \sum A \cdot b_i$.

LEMMA 6.13. *Suppose $f: A \to B$ is a finite centralizing homomorphism and F is a noetherian connected filtration on A. Then there is a noetherian connected filtration F on B such that f preserves the filtrations and $\mathrm{Rees}^F(f): \mathrm{Rees}^F A \to \mathrm{Rees}^F B$ is finite.*

Proof. Let $b_1, \ldots, b_m$ be elements of B which commute with A and $B = \sum A \cdot b_i$. Choose elements $a_{i,j,l} \in A$ such that $b_i b_j = \sum_l a_{i,j,l} b_l$. Let $n_0 > 0$ be large enough such that $a_{i,j,l}$ are in $F_{n_0} A$. Define

$$F_n B := F_n A \cdot 1 + \sum_i F_{n-n_0} A \cdot b_i \subset B.$$

Clearly this is a connected filtration. Since the elements $(1, b_1, \ldots, b_m)$ determine a surjective bimodule homomorphism

$$\mathrm{Rees}^F A \oplus (\mathrm{Rees}^F A)^m(-n_0) \twoheadrightarrow \mathrm{Rees}^F B$$

we see that $\mathrm{Rees}^F B$ is noetherian. ∎

EXAMPLE 6.14. If A is an affine k-algebra finite over its center, then there is a finite centralizing homomorphism $k[t_1, \ldots, t_n] \to A$ from a commutative polynomial algebra. By the lemma and Example 6.10, A has a noetherian connected filtration. Thus A satisfies the assumptions of Corollary 6.9.

EXAMPLE 6.15. Here is an example of a prime PI algebra A which is not finite over its center yet has a noetherian connected filtration (Schelter's Example, [Ro, p. 492, Exercise 27]). Let t, λ_1, λ_2 be commuting indeterminates of degree 1. Define

$$\tilde{C} := \mathbb{Q}\left[\sqrt{2}, \sqrt{3}, \lambda_1, \lambda_2, t\right]$$
$$\tilde{C}_1 := \mathbb{Q}\left[\sqrt{6}, t\sqrt{2} + \lambda_1, \lambda_2, \lambda_2\sqrt{2}, t\right]$$
$$\tilde{C}_2 := \mathbb{Q}\left[\sqrt{6}, t\sqrt{3} + \lambda_1, \lambda_2, \lambda_2\sqrt{2}, t\right]$$
$$\tilde{M} := \tilde{C}_1\lambda_2 + \tilde{C}_1\lambda_2\sqrt{2} \subset \tilde{C}$$
$$\tilde{A} := \begin{bmatrix} \tilde{C}_1 & \tilde{M} \\ \tilde{M} & \tilde{C}_2 \end{bmatrix}.$$

Then $\tilde{A}$ is a noetherian graded algebra and $\tilde{A}_0$ is finite over $k := \mathbb{Q}$. The quotient $A := \tilde{A}/(t-1)$ acquires a filtration $\{F_n A\}$, and if we modify it by setting $F_0 A := \mathbb{Q}$ this becomes a connected filtration. But A is not finite over its center.

QUESTION 6.16. *Does every noetherian affine* PI *k-algebra admit a noetherian connected filtration? This seems to be a hard question. A similar one was posed by M. Lorenz over ten years ago* (*see* [Lo, *p.* 436]).

In Section 3 we found that rigid dualizing complexes are sometimes functorial w.r.t. finite algebra homomorphisms, via the trace morphism. Here is such an instance:

THEOREM 6.17. *Let $A \to B$ be a finite centralizing homomorphism. Suppose A has a noetherian connected filtration F and* $\mathrm{gr}^F A$ *has a balanced dualizing complex. Then A and B have rigid dualizing complexes R_A and R_B, respectively, and the trace morphism* $\mathrm{Tr}_{B/A} : R_B \to R_A$ *of Definition* 3.7 *exists.*

Proof. By Lemma 6.13 we get a finite homomorphism of graded algebras $\tilde{A} = \mathrm{Rees}^F A \to \tilde{B} = \mathrm{Rees}^F B$. So according to Theorem 4.16 the trace morphism $\mathrm{Tr}_{\tilde{B}/\tilde{B}} : R_{\tilde{B}} \to R_{\tilde{A}}$ exists. Now apply the functor π and use Theorem 6.2. ∎

For applications of this result see Proposition 3.9.

Let σ be a k-algebra automorphism A. Recall that A^σ is the invertible bimodule with basis e satisfying $e \cdot a = \sigma(a) \cdot e$. A noetherian connected graded k-algebra B is called *AS-Gorenstein* if B satisfies χ and the bimodule B has finite injective dimension on both sides. Here AS stands for Artin–Schelter. We say B is *AS-regular* if B is AS-Gorenstein and gl. dim $B < \infty$.

PROPOSITION 6.18. *Suppose A has a noetherian connected filtration F and* $\mathrm{gr}^F A$ *is AS-Gorenstein* (*resp. AS-regular*). *Then the following statements hold*.

(1) *A is a Gorenstein* (*resp. regular*) *algebra*.

(2) *The rigid dualizing complex of A is $R = A^{\sigma}[n]$ where n is an integer and σ is some k-algebra automorphism of A.*

(3) *If* $\mathrm{gr}^F A$ *is graded Auslander* (*resp. and graded* GKdim-*Macaulay*), *then A is Auslander–Gorenstein* (*resp. and Cohen–Macaulay*) *in the sense of* [Bj].

(4) *Let $B = A/\mathfrak{a}$ be any quotient algebra, M any B-module, and q an integer. Then the twisted module $\mathrm{Ext}^q_A(M, A)^{\sigma}$ is a B°-module.*

Proof. (1), (2) Let n be the injective dimension of $\mathrm{gr}^F A$, and let $\tilde{A} := \mathrm{Rees}^F A$. By [Lev, Theorem 3.6] the injective dimension of $\tilde{A}$ is $n + 1$. Since $\tilde{A}$ satisfies χ it is AS-Gorenstein. So the balanced dualizing complex of $\tilde{A}$ is $\tilde{R} = \tilde{A}^{\tilde{\sigma}}(d)[n + 1]$ for some graded automorphism $\tilde{\sigma}$ and for some integer d. By Theorem 6.2 the rigid dualizing complex of A is $R = \pi\tilde{A}^{\tilde{\sigma}}[n]$. Since $\tilde{A}^{\tilde{\sigma}} = \mathrm{H}^{-n-1}\tilde{R}$ this is $k[t]$-central; so $\tilde{\sigma}(t) = t$. We see there is an induced automorphism σ of A and $R = A^{\sigma}[n]$.
If $\mathrm{gr}\, A$ has finite global dimension, then so does A.

(3) This follows from (2) and Theorems 5.1 and 6.2.

(4) It follows from Theorem 6.17 and Proposition 3.9(1) that

$$\mathrm{Ext}^q_A(M, A)^{\sigma} \cong \mathrm{Ext}^{q-n}_A(M, A^{\sigma}[n]) \cong \mathrm{Ext}^{q-n}_A(M, R) \cong \mathrm{Ext}^{q-n}_B(M, R_B),$$

where R_B is the rigid dualizing complex of B. ∎

Example 2.3 shows that Proposition 6.18 (2), (3) might fail even if A is Gorenstein. The next example shows that σ could be nontrivial.

EXAMPLE 6.19. Let A be the quantum plane $k_q[x, y] := k\langle x, y\rangle/(yx - qxy)$ for $q \in k^{\times}$ with $q^2 \neq 1$. The automorphism σ in Proposition 6.18(4) is $\sigma(x) = qx$, $\sigma(y) = q^{-1}y$ (cf. [Ye1, Examples 6.21 and 7.14]). Consider the ideal $I = A \cdot f \cdot A$ where $f := x - y$, which is not normal. An easy computation shows that $B \cong k[\epsilon]$ with $\epsilon^2 = 0$, and $\epsilon \equiv x \equiv y \pmod I$. Now consider the graded A^e-module $N := \mathrm{Ext}^2_A(B, A)$. One has $N^{\sigma}(-2) \cong B^*$ as A^e-modules, so N is killed by $qx - q^{-1}y = \sigma(f) \in A^{\circ}$, and hence N cannot be a B°-module.

EXAMPLE 6.20. Let A be the Weyl algebra $k\langle x, y\rangle/(xy - yx - 1)$. Take the standard filtration $F_n A = (k + kx + ky)^n$. Then the Rees algebra $\tilde{A}$ is generated by x, y, t with t central and $xy - yx = t^2$. $\tilde{A}$ is an Artin–Schelter regular algebra of global dimension 3, so its balanced dualizing complex is $\tilde{A}^{\tilde{\sigma}}(-3)[3]$ for some automorphism $\tilde{\sigma}$. Let $\tilde{B} := \tilde{A}/(t^2)$, which is a commutative AS-Gorenstein algebra. As in [Ye1, Theorem 7.18] we find that $\tilde{\sigma} = 1$. Therefore the rigid dualizing complex of A is $A[2]$. Observe that $\operatorname{Cdim} A = 2 = \operatorname{GKdim} A$. In [Ye4] we prove the more general statement that if C is any smooth integral commutative k-algebra of dimension n, $\operatorname{char} k = 0$, and $A := \mathscr{D}(C)$ is the ring of differential operators, then the rigid dualizing complex of A is $A[2n]$.

Suppose A has a noetherian connected filtration F. A *two-sided good filtration* on a bimodule M is a filtration $\{F_n M\}$ such that $F_n A \cdot F_m M \subset F_{n+m} M$, $F_n M \cdot F_m A \subset F_{n+m} M$ and $\operatorname{Rees}^F M$ is a finitely generated $(\operatorname{Rees}^F A)$-module on both sides.

PROPOSITION 6.21. *Assume A has a noetherian connected filtration and* gr *A has a balanced dualizing complex. Let R be the rigid dualizing complex of A. If a bimodule M has a two-sided good filtration then*

$$\mathrm{R}\operatorname{Hom}_A(M, R) \cong \mathrm{R}\operatorname{Hom}_{A^\circ}(M, R).$$

Proof. Let $\tilde{M} := \operatorname{Rees}^F M$. According to Corollary 4.17 there is an isomorphism

$$\mathrm{R}\operatorname{Hom}_{\tilde{A}}(\tilde{M}, \tilde{R}) \cong \mathrm{R}\operatorname{Hom}_{\tilde{A}^\circ}(\tilde{M}, \tilde{R})$$

in $\mathsf{D}(\mathsf{GrMod}\, A^e)$, where $\tilde{R}$ is the balanced dualizing complex of $\tilde{A}$. Since $\tilde{M}$ is $k[t]$-central we can apply the functor π. ∎

Recall the notion of weakly symmetric dimension function (Definition 2.20).

COROLLARY 6.22. *Assume A has an Auslander rigid dualizing complex, and a noetherian connected filtration such that* gr *A has a balanced dualizing complex. Then* Cdim *is weakly symmetric.*

Proof. As can be readily verified, the class of A-bimodules which admit two-sided good filtrations is closed under submodules, quotients, and finite direct sums. Given a bimodule M which is a subquotient of A, Proposition 6.21 applies and hence $\operatorname{Cdim}_A M = \operatorname{Cdim}_{A^\circ} M$. ∎

We can now give the

Proof of Theorem 0.1. By Corollary 6.9, A has an Auslander rigid dualizing complex R, and by Corollary 6.22, Cdim_R is weakly symmetric (this also follows from the GKdim-Macaulay property). Now use Theorem 2.23. ∎

The next theorem has the same conclusions as [ASZ, Theorem 6.1], but our assumptions are much more focused. Let $\mathfrak{n}$ be the prime radical of A. Recall that $\mathfrak{n}$ is said to be *weakly invariant* w.r.t. an exact dimension function dim if $\dim \mathfrak{n} \otimes_A M < \dim A/\mathfrak{n} = \dim A$ for every finitely generated A-module M with $\dim M < \dim A$ (and the same for right modules); cf. [MR, 6.8.13]. A ring is called *quasi-Frobenius* if it is artinian and self injective.

THEOREM 6.23. *Let A be an Auslander–Gorenstein noetherian k-algebra of injective dimension n. Assume A has a filtration such that* $\operatorname{gr} A$ *is an AS-Gorenstein noetherian connected graded k-algebra. Then*

(1) *The prime radical $\mathfrak{n}$ is weakly invariant.*

(2) *If $\mathfrak{p}$ is a minimal prime then* $\operatorname{Cdim} A/\mathfrak{p} = n$.

(3) *A has a quasi-Frobenius ring of fractions.*

Proof. By Proposition 6.18 the rigid dualizing complex of A is $R = A^{\sigma}[n]$. According to Corollary 6.22, Cdim_R is weakly symmetric. Now the function denoted δ in [ASZ, Theorem 6.1] coincides with Cdim_R, so all assumptions of that theorem hold. ∎

The next theorem is due to Gabber in the case when $\operatorname{gr} A$ is Auslander–Gorenstein, and an elegant proof was communicated to us by Van den Bergh. We extend the result by dropping the Gorenstein condition.

THEOREM 6.24. *Let A be a filtered k-algebra such that* $\operatorname{gr} A$ *is a noetherian connected graded k-algebra with graded Auslander balanced dualizing complex. Given a* Cdim*-pure A-module M, there is a good filtration $\{F_n M\}$ on it such that* $\operatorname{gr}^F M$ *is* Cdim*-pure.*

Proof. The basic idea is to start with an arbitrary good filtration on M and to modify it to get purity.

Let $\tilde{A}$ be the Rees algebra of A, and let $\tilde{R}$ be its balanced dualizing complex. So $\tilde{R}$ has the graded Auslander property.

Let $n := \operatorname{Cdim} M + 1$. Choose any good filtration F' on M and let $\tilde{M} := \operatorname{Rees}^{F'} M$. Since $\tilde{M}$ is t-torsion-free and $M = \pi \tilde{M}$, by (6.4) we have $\operatorname{Cdim} \tilde{M} = n$. If $\tilde{M}' \subset \tilde{M}$ is any nonzero graded submodule then because $\pi \tilde{M}' \subset M$, and because M is pure, we see that $\operatorname{Cdim} \tilde{M}' = n$. Thus $\tilde{M}$ is pure.

Set

$$E(\tilde{M}) = \operatorname{Ext}_{\tilde{A}^{\circ}}^{-n}\left(\operatorname{Ext}_{\tilde{A}}^{-n}(M, \tilde{R}), \tilde{R}\right).$$

As in Theorem 2.14(3) there is an exact sequence

$$0 \to \tilde{M} \to (E\tilde{M}) \to \tilde{Q} \to 0, \tag{6.25}$$

where Cdim $\tilde{Q} \leq n - 2$. Consider the module $\tilde{N}$ which is the t-saturation of $\tilde{M}$ in $E(\tilde{M})$, i.e.,

$$\tilde{N} := \left\{ x \in E(\tilde{M}) | t^i x \in \tilde{M} \text{ for some } i \geq 0 \right\}.$$

Since $\tilde{N}/\tilde{M}$ is t-torsion it follows that $M = \pi\tilde{M} \to \pi\tilde{N}$ is bijective; but the filtration F on $\pi\tilde{N}$ may be different from F'. Observe that $\tilde{N}$ is t-torsion-free, so by lifting back the filtration we obtain $\tilde{N} \cong \text{Rees}^F \pi\tilde{N}$. By (6.25) we get Cdim $\tilde{N}/\tilde{M} \leq n - 2$, which implies that $E(\tilde{M}) \to E(\tilde{N})$ is bijective. Therefore by changing the good filtration on M from F' to F, we can assume that in (6.25) the module $\tilde{Q}$ is t-torsion-free.

Having done so we get a short exact sequence

$$0 \to \pi_0 \tilde{M} \to \pi_0 E(\tilde{M}) \to \pi_0 \tilde{Q} \to 0,$$

where π_0 denotes the functor $M \mapsto M/tM$. Hence in order to prove that $\text{gr}\, M = \pi_0 \tilde{M}$ is pure it suffices to prove that $\pi_0 E(\tilde{M})$ is pure. Now, using the duality functor $D = \text{R}\,\text{Hom}^{\text{gr}}_{\tilde{A}}(-, \tilde{R})$ we have $\tilde{L} := \text{Ext}^{-n}_{\tilde{A}}(\tilde{M}, \tilde{R}) = \text{H}^{-n} D\tilde{M}$, which is t-torsion-free by Lemma 5.4. Therefore by the same lemma, Cdim $\pi_0 \tilde{L} = n - 1$. Now $E(\tilde{M}) = \text{H}^{-n} D^\circ \tilde{L}$, and by formula (5.5) when $i = -n$ we get

$$\pi_0 \, \text{H}^{-n} D^\circ \tilde{L} \subset \left(\text{H}^{-(n-1)} D^\circ \pi_0 \tilde{L} \right)(-1).$$

But by Theorem 2.14, the module $\text{H}^{-(n-1)} D^\circ \pi_0 \tilde{L}$ is pure of dimension $n - 1$. ∎

If $\text{gr}\, A$ is a commutative affine k-algebra and M is a finitely generated A-module, define $I(M) \subset \text{gr}\, A$ to be the prime radical of $\text{Ann}_{\text{gr}\, A} \text{gr}^F M$ for some good filtration F on M. By [MR, Proposition 8.6.17], $I(M)$ is independent of the choice of good filtration. The *characteristic variety* of M is defined to be

$$\text{Ch}(M) = \text{Spec}\, \text{gr}\, A / I(M)$$

(cf. [Co, p. 98] for the case when A is a Weyl algebra).

Proof of Theorem 0.4. Recall that a variety is called pure if all its irreducible components have the same dimension. The support of a Kdim-pure finitely generated module N over a commutative affine k-algebra B is pure. But for a commutative algebra $\text{Kdim}_B = \text{GKdim}_B = \text{Cdim}_B$. Here we take $B := \text{gr}\, A$ and $N := \text{gr}^F M$.

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