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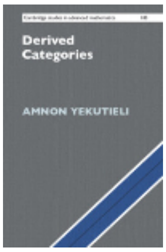
# Derived Categories

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MAA REVIEW TABLE OF CONTENTS

[Reviewed by Felipe Zaldivar, on 05/3/2020]

Introduced by J.-L. Verdier in his Ph. D. Thesis in the early 1960's, derived categories are a construction and tool of homological algebra which simplify and refine several important constructions that are otherwise very cumbersome. The original goal of the theory included a framework to formulate and prove a duality theorem of A. Grothendieck. Using derived categories, Verdier also obtained some other important duality theorems that generalized the classical duality theorems of Alexander and Poincaré.

The definition and construction of the derived category of an abelian category fits naturally in the program that treats homological algebra as the natural framework to formulate and prove results in large areas of mathematics, especially those close to algebraic geometry and algebraic topology. After their introduction, derived categories were only used by the algebraic geometers close to A. Grothendieck's school and their relations to other areas of mathematics were not pursued. Perhaps one reason for this is the fact that very few publications of that time gave an exposition of these new constructions. The first report of J.-L. Verdier was his 1963 IHES notes *Catégories Dérivées, État 0.* Grothendieck's proof, using derived caterories, of his duality theorem was first published in R. Hartshorne's seminar notes Residues and Duality (Lecture Notes in Mathematics 20, Springer-Verlag, 1966). A second reason may be the fact that the construction of the derived category of a given abelian category is a bit involved.

Starting with an abelian category A, one associates to A two other categories: First, the category C(A) of complexes in A, which is also an abelian category that contains A as a full subcategory in a natural fashion. The second category K(A) has the same objects as C(A) but its morphisms are homotopy classes of maps of complexes. The category K(A) is additive but it is not abelian and this is a problem because there is no intrinsic notion of an exact sequence in K(A) and thus there is no obvious way to do homological algebra in K(A). Verdier's insight was to distinguish some remarkable class of triangles in K(A) which play the role of short exact sequences. The construction of these distinguished triangles uses a translation functor in K(A) whose motivation comes from some important algebraic topological constructions, namely the cone of a map, the suspension functor and the homotopy fiber or cofiber sequences. The important fact, is that K(A)
has the structure of a triangulated category, a concept introduced by Verdier as an intermediate step towards the construction of the derived category, and that is now of interest in its own right.

The second step is to invert the quasi-isomorphisms of K(A) by considering an adequate category of fractions associated to a triangulated category, a vast generalization of the ring of fractions via the Ore conditions on a multiplicative set of a given ring. The resulting derived category D(A) inherits some properties of the various categories underlying its construction, for instance, D(A) is also a triangulated category and every short exact sequence in A gives rise to a distinguished triangle in D(A). One further remark is that the category of complexes C(A) has a natural gradation and there is a natural way to equip it with a (graded) differential, making C(A) a differential graded category. Hence, the passage from C(A) to the homotopy category K(A) can be abstracted starting with a DG category.

Triangulated categories and derived categories have nowadays a large range of applications outside its natural algebraic geometry birthplace: From intersection cohomology and perverse sheaves in algebraic topology, to D-modules in microlocal analysis or the Riemann-Hilbert correspondence, and the more abstract and ever growing derived algebraic geometry which encompasses a large part of algebraic topology and provides a setting for motivic homotopy theory.

Recent expositions of the theory come in chapters or sections of books whose goals are the corresponding applications. A. Neeman's *Triangulated Categories* is an exception, but its main topic is on triangulated categories per se. For the on-line savvy, the ever-growing Stacks Project has lots of information on derived categories and applications.

The book under review gives a detailed exposition of derived categories and derived functors in its first twelve chapters, starting from some basic knowledge of category theory. The next six chapters focus on some applications to commutative and non-commutative algebra, from dualizing complexes and perfect and tilting DG modules, to derived torsion over graded rings. The book is perfectly suited for the interested graduate student with plenty of explicit constructions, examples and exercises. In addition to being a thorough introduction to the subject, the book is a monograph filled with applications otherwise available only in research articles.

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