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Derived categories. (English) Zbl 1444.18001

Cambridge Studies in Advanced Mathematics 183. Cambridge: Cambridge University Press (ISBN 978-1-108-41933-8/hbk; 978-1-108-29282-5/ebook). xi, 607 p. (2020).

This book is concerned mostly with the category $\boldsymbol{C}(A, \mathrm{M})$ of differential graded (DG) A-modules in M, where A is a DG ring and M is an abelian category. The category $\boldsymbol{C}(A, \mathrm{M})$ is a DG category, determining the *homotopy category* $\boldsymbol{K}(A, \mathrm{M})$ with its *triangulated structure*. The *derived category* $\boldsymbol{D}(A, \mathrm{M})$ is obtained from $\boldsymbol{K}(A, \mathrm{M})$ by inverting the quasi-isomorphisms.

Now a synopsis of the book, consisting of 18 chapters, is in order. Chapter 1 and Chapter 2 are pretty much a review of the standard material on categories and functors, in particular abelian categories and additive functors. a fresh topic being *sheaf tricks* (§2.4), which are cheap substitute for the Freyd-Mitchell Theorem in the sense that they facilitate proofs of various results in abstract abelian categories. Chapter 3 presents a lot of basic material about DG algebras.

Chapter 4 investigates the translation functor and the standard cone of a strict morphism in the DG category of $\boldsymbol{C}(A, \mathrm{M})$, addressing properties of DG functors

$$F : \boldsymbol{C}(A, \mathrm{M}) \to \boldsymbol{C}(B, \mathrm{M})$$

between such DG categories.

Chapter 5 introduces triangulated categories and triangulated functors, proving that fora DG ring A and an abelian category M, the homotopy category $\boldsymbol{K}(A, \mathrm{M})$ is triangulated and that a DG-functor

$$F : \boldsymbol{C}(A, \mathrm{M}) \to \boldsymbol{C}(B, \mathrm{M})$$

gives rise to a triangulated functor

$$F : \boldsymbol{K}(A, \mathrm{M}) \to \boldsymbol{K}(B, \mathrm{M})$$

Chapter 6 is devoted to the general theory of *Ore localization of categories*, which is applied in Chapter 7 to triangulated categories, yielding the derived category $\boldsymbol{D}(A, \mathrm{M})$ of DG A-module M.

Given a DG ring A, an abelian category M, a triangualted category E and a triangulated functor

$$F : \boldsymbol{K}(A, \mathrm{M}) \to \mathrm{E}$$

Chapter 8 defines the left and right derived functors

$$\mathrm{L}F, \mathrm{R}F : \boldsymbol{D}(A, \mathrm{M}) \to \mathrm{E}$$

proving the uniqueness of the derived functors and their existence under certain suitable assumptions. Chapter 9 extends the theory of triangualted derived functors in Chapter 8 to *triangualted bifunctors*. The universal properties of the derived functors are best stated in 2-categorical languages, as is expounded in §8.1 and §8.2. Derived functors in the abstract setting, as opposed to the triangualted setup, are defined in §8.3, where the main results for them are presented. These results are specialized to triangualted functors in §8.4, to contravariant triangualted functors in §8.5, and to triangualted bifunctors in §9.2.

Chapter 10 defines K-*injective* and K-*projective* objects in $\boldsymbol{K}(A, \mathrm{M})$, and also K-*flat* objects in $\boldsymbol{K}(A)$. These constitute full triangualted subcategories of $\boldsymbol{K}(A, \mathrm{M})$ called *resolving subcategories*. The existence of K-injective, K-projective and K-flat resolutions is established in Chapter 11.

Chapter 12 discusses the derived Hom and tensor bifunctors in general situations, showing how these bifunctors are related in adjunction formulas.

Chapter 13 addresses *dualizing complexes* over commutative rings. Based on [*R. Hartshorne*, Residues and duality. Lecture notes of a seminar on the work of A. Grothendieck, given at Havard 1963/64. Appendix: Cohomology with supports and the construction of the $f^!$ functor by P. Deligne. Berlin-Heidelberg-New York: Springer-Verlag (1966; Zbl 0212.26101)], the first and third sections of this chapter deal with dualizing complexes and *residue complexes*, respectively, over commutative rings. The last two sections (§13.4 and §13.5) are concerned with rigid dualizing complexes in the sense of *M. Van den Bergh* [J. Algebra 195, No. 2, 662–679 (1997; Zbl 0894.16020)].

Chapter 14 studies more general concepts of perfect DG modules and tilting DG bimodules over noncommutative DG rings.

Chapter 15 addresses *algebraically graded rings* and the categories of *algebraically graded modules* over them. The subsequent two chapters (Chapter 16 and Chapter 17) concentrate on *connected graded rings*, which behave like complete local rings within the algebraically graded context. The interest in algebraically graded rings, and especially connected graded rings, stems from the prominent role played in noncommutative algebraic geometry, as it was developed by *M. Artin* et al. [Prog. Math. 86, 33–85 (1990; Zbl 0744.14024); Adv. Math. 109, No. 2, 228–287 (1994; Zbl 0833.14002); *J. T. Stafford* and *M. Van den Bergh*, Bull. Am. Math. Soc., New Ser. 38, No. 2, 171–216 (2001; Zbl 1042.16016)].

The goal of Chapter 18 is to introduce *rigid noncommutative dualizing complexes* in the sense of *M. Van den Bergh* [J. Algebra 195, No. 2, 662–679 (1997; Zbl 0894.16020)] and to prove their existence and uniqueness under certain conditions.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

`18-01` Introductory exposition (textbooks, tutorial papers, etc.) pertaining to category theory

`18G80` Derived categories, triangulated categories