ADELIC CHERN FORMS AND APPLICATIONS

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Abstract. Let X be a variety over a field of characteristic 0. Given a vector bundle E on X we construct Chern forms $c_i(E;\nabla) \in \Gamma(X, \mathcal{A}_X^{2i})$. Here $\mathcal{A}_X^{\cdot}$ is the sheaf of *Beilinson adeles* and ∇ is an *adelic connection*. When X is smooth $\mathrm{H}^p\Gamma(X, \mathcal{A}_X^{\cdot}) = \mathrm{H}^p_{\mathrm{DR}}(X)$, the algebraic De Rham cohomology, and $c_i(E) = [c_i(E;\nabla)]$ are the usual Chern classes. We include three applications of the construction: (1) existence of adelic secondary (Chern-Simons) characteristic classes on any smooth X and any vector bundle E; (2) proof of the Bott Residue Formula for a vector field action; and (3) proof of a Gauss-Bonnet Formula on the level of differential forms, namely in the De Rham-residue complex.

0. Introduction. Let X be a scheme of finite type over a field k. According to Beilinson [Be], given any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ and an integer q, there is a flasque $\mathcal{O}_X$-module $\underline{\mathbb{A}}^q_{\mathrm{red}}(\mathcal{M})$, called the *sheaf of adeles*. This is a generalization of the classical adeles of number theory (cf. Example 2.3). Moreover, there are homomorphisms $\partial\colon \underline{\mathbb{A}}^q_{\mathrm{red}}(\mathcal{M}) \to \underline{\mathbb{A}}^{q+1}_{\mathrm{red}}(\mathcal{M})$ which make $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{M})$ into a complex, and $\mathcal{M} \to \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{M})$ is quasi-isomorphism.

Now let $\Omega^{\cdot}_{X/k}$ be the algebra of Kähler differential forms on X. In [HY] we proved that the sheaf

$$\mathcal{A}_X^{\cdot} = \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\Omega^{\cdot}_{X/k}) = \bigoplus_{p,q} \underline{\mathbb{A}}^q_{\mathrm{red}}(\Omega^p_{X/k})$$

is a resolution of $\Omega^{\cdot}_{X/k}$ as differential graded algebras (DGAs). Therefore when X is smooth, $\mathcal{A}_X^{\cdot}$ calculates the algebraic De Rham cohomology: $\mathrm{H}^p_{\mathrm{DR}}(X) \cong \mathrm{H}^p\Gamma(X, \mathcal{A}_X^{\cdot})$. We see that there is an analogy between $\mathcal{A}_X^{\cdot}$ and the Dolbeault sheaves of smooth forms on a complex-analytic manifold.

Carrying this analogy further, in this paper we show that when $\operatorname{char} k = 0$, any vector bundle E on X admits an adelic connection ∇. Given such a connection one can assign adelic Chern forms $c_i(E,\nabla) \in \Gamma(X, \mathcal{A}_X^{2i})$, whose classes $c_i(E) := [c_i(E,\nabla)] \in \mathrm{H}^{2i}_{\mathrm{DR}}(X)$ are the usual Chern classes. We include three applications of our adelic Chern-Weil theory, to demonstrate its effectiveness and potential.

The idea of using adeles for an algebraic Chern-Weil theory goes back to Parshin, who constructed a Chern form $c_i(E) \in \underline{\mathbb{A}}^i(\Omega^i_{X/k})$ using an i-cocycle on $\mathrm{Gl}(\underline{\mathbb{A}}^1(\mathcal{O}_X))$ (see [Pa]). Unfortunately we found it quite difficult to perform

Manuscript received February 5, 1996.

Research of the first author supported in part by the Deutsche Forschungsgemeinschaft; research of the second author supported by an Allon Fellowship.

American Journal of Mathematics 121 (1999), 797–839.

calculations with Parshin's forms. Indeed, there is an inherent complication to any Chern-Weil theory based on $\mathcal{A}_X^{\cdot}$. The DGA $\mathcal{A}_X^{\cdot}$, with its Alexander-Whitney product, is not (graded) commutative. This means that even if one had some kind of "curvature matrix" R with entries in $\mathcal{A}_X^2$, one could not simply evaluate invariant polynomials on R.

The problem of noncommutativity was encountered long ago in algebraic topology, and was dubbed the "commutative cochain problem." The solution, by Thom and Sullivan, was extended to the setup of cosimplicial DGAs by Bousfield-Gugenheim and Hinich-Schechtman (see [BG], [HS1], [HS2]). In our framework this gives a sheaf of commutative DGAs $\tilde{\mathcal{A}}_X^{\cdot}$ on X, called the sheaf of *Thom-Sullivan adeles*, and a homomorphism of complexes ("integration on the fiber") $\int_\Delta\colon \tilde{\mathcal{A}}_X^{\cdot} \to \mathcal{A}_X^{\cdot}$. This map induces an isomorphism of graded algebras $\mathrm{H}^{\cdot}(\int_\Delta)\colon \mathrm{H}^{\cdot}\Gamma(X, \tilde{\mathcal{A}}_X^{\cdot}) \to \mathrm{H}^{\cdot}\Gamma(X, \mathcal{A}_X^{\cdot})$. We should point out that $\int_\Delta$ involves denominators, so it is necessary to work in characteristic 0.

Bott discovered a way of gluing together connections defined locally on a manifold (see [Bo1]). This method was imported to algebraic geometry by Zhou (in [Zh]), who used Čech cohomology. When we tried to write the formulas in terms of adeles, it became evident that they gave a connection on the Thom-Sullivan adeles $\tilde{\mathcal{A}}_X^{\cdot}$. Later we realized that a similar construction was used by Dupont in the context of simplicial manifolds (see [Du]).

In the remainder of the Introduction we outline the main results of our paper.

Adelic connections. Let k be a field of characteristic 0 and X a finite type scheme over it. The definition of Beilinson adeles on X and their properties will be reviewed in Section 2. For now let us just note that the sheaf of adeles $\tilde{\mathcal{A}}_X^{\cdot}$ is a commutative DGA, and $\Omega_{X/k}^{\cdot} \to \tilde{\mathcal{A}}_X^{\cdot}$ is a DGA quasi-isomorphism.

Let $\mathcal{E}$ be the locally free $\mathcal{O}_X$-module of rank r associated to the vector bundle E. An *adelic connection* on $\mathcal{E}$ is by definition a connection

$$\nabla\colon\ \tilde{\mathcal{A}}_X^0 \otimes_{\mathcal{O}_X} \mathcal{E} \to \tilde{\mathcal{A}}_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}$$

over the algebra $\tilde{\mathcal{A}}_X^0$.

Such connections are abundant. One way to get an adelic connection is by choosing, for every point x, a basis (or frame; we use these terms interchangeably) $\boldsymbol{e}_x = (e_{x,1}, \ldots, e_{x,r})$ for the $\mathcal{O}_{X,x}$-module $\mathcal{E}_x$. We then get a Levi-Civita connection

$$\nabla_x\colon\ \mathcal{E}_x \to \Omega_{X/k,x}^1 \otimes_{\mathcal{O}_{X,x}} \mathcal{E}_x$$

over the k-algebra $\mathcal{O}_{X,x}$. The Bott gluing mentioned above produces an adelic connection ∇ (see Proposition 3.11).

Adelic Chern-Weil homomorphism. Since $\tilde{\mathcal{A}}_X^{\cdot}$ is a (graded) commutative DGA, an adelic connection ∇ on $\mathcal{E}$ gives a curvature form

$$R := \nabla^2 \in \Gamma(X, \tilde{\mathcal{A}}_X^2 \otimes_{\mathcal{O}_X} \mathcal{E}nd(\mathcal{E})).$$

Denote by $\mathrm{S}(\mathrm{M}_r(k)^*) = \mathcal{O}(\mathrm{M}_r \times k)$ the algebra of polynomial functions on $r \times r$ matrices, and let $I_r(k) := \mathrm{S}(\mathrm{M}_r(k)^*)^{\mathrm{Gl}_r(k)}$ be the subalgebra of conjugation-invariant functions. Denote by P_i the ith elementary invariant polynomial, so $P_1 = \mathrm{tr}, \ldots, P_r = \det$.

For any $P \in I_r(k)$ the form $P(R) \in \Gamma(X, \tilde{\mathcal{A}}_X^{\cdot})$ is closed. So there is a k-algebra homomorphism $w_{\mathcal{E}}\colon\ I_r(k) \to \mathrm{H}^{\cdot}\Gamma(X, \tilde{\mathcal{A}}_X^{\cdot})$, $P \mapsto [P(R)]$, called the *adelic Chern-Weil homomorphism*. In Theorem 3.7 we prove that $P(\mathcal{E}) = w_{\mathcal{E}}(P)$ is independent of the connection ∇ (this is true even if X is singular). Defining the ith *adelic Chern form* to be

$$c_i(\mathcal{E}; \nabla) := \int_{\Delta} P_i(R) \in \Gamma(X, \mathcal{A}_X^{2i}),$$

we show the three axioms of Chern classes are satisfied (Theorem 3.17). Hence when X is smooth over k,

$$c_i(\mathcal{E}) := [c_i(\mathcal{E}; \nabla)] \in \mathrm{H}^{2i}_{\mathrm{DR}}(X)$$

is the usual ith Chern class.

Secondary characteristic classes. Suppose now that X is a smooth scheme over k, and let $\mathcal{E}$ be a locally free sheaf of rank r on X. Let $P \in I_r(k)$ be an invariant polynomial function of degree $m \geq 2$. In [BE], Bloch and Esnault showed that given an algebraic connection $\nabla\colon\ \mathcal{E} \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{E}$, there is a Chern-Simons class $\mathrm{T}P(\mathcal{E}, \nabla)$ satisfying $\mathrm{dT}P(\mathcal{E}, \nabla) = P(\mathcal{E}) \in \mathrm{H}^{2m}_{\mathrm{DR}}(X)$. (We are using the notation of [Es].)

Since adelic connections always exist, we can construct adelic secondary characteristic classes on *any* smooth k-scheme X and *any* locally free sheaf $\mathcal{E}$. Theorem 4.1 says that given an adelic connection ∇ there is a class

$$\mathrm{T}P(\mathcal{E}; \nabla) \in \Gamma\left(X, \mathcal{A}_X^{2m-1}/\mathrm{D}(\mathcal{A}_X^{2m-2})\right)$$

satisfying

$$\mathrm{DT}P(\mathcal{E}; \nabla) = P(\mathcal{E}) \in \mathrm{H}^{2m}_{\mathrm{DR}}(X).$$

The existence of adelic secondary characteristic classes, combined with the action of adeles on the residue complex (see below, and Theorem 6.1), opens new possibilities for research on vanishing of cohomology classes (cf. [Es]).

Bott Residue Formula. The adeles of differential forms can be integrated. If $\dim X = n$, each maximal chain of points $\xi = (x_0, \dots, x_n)$ determines a local integral $\mathrm{Res}_\xi\colon \Gamma(X, \mathcal{A}_X^{2n}) \to k$ (cf. [Be], [Ye1]). If X is smooth and proper then the global map

$$\int_X := \sum_\xi \mathrm{Res}_\xi\colon \mathrm{H}^{2n}_{\mathrm{DR}}(X) = \mathrm{H}^{2n}\Gamma(X, \mathcal{A}_X^{\cdot}) \to k$$

coincides with the usual "algebraic integral" of, say, [Ha1].

Assume X is a smooth projective variety of dimension n. Let $P \in I_r(k)$ be a homogeneous polynomial of degree n, so that $P = Q(P_1, \dots, P_r)$ for some polynomial Q in r variables. Let $v \in \Gamma(X, \mathcal{T}_X)$ be a vector field with isolated zeroes, and assume v acts on the locally free sheaf $\mathcal{E}$. For each zero z of v there is a local invariant $P(v, \mathcal{E}, z) \in k$, which has an explicit expression in terms of local coordinates. Theorem 5.1 says that

$$\int_X Q(c_1(\mathcal{E}), \dots, c_r(\mathcal{E})) = \sum_{v(z)=0} P(v, \mathcal{E}, z).$$

The proof of the theorem follows the steps of Bott's original proof in [Bo1], translated to adeles and algebraic residues. Example 5.4 provides an explicit illustration of the result in the case of a nonreduced zero z.

We should of course mention the earlier algebraic proof of the Bott Residue Formula for isolated zeroes, by Carrell-Lieberman [CL], which uses Grothendieck's global duality.

There is also a Bott Residue Formula for group actions, which is best stated as a localization formula in equivariant cohomology (cf. [AB]). Recently this formula was used in enumerative geometry, see for instance [ES] and [Ko]. Edidin-Graham [EG] proved Bott's formula in the equivariant intersection ring.

The Gauss-Bonnet formula. Let k be a perfect field of any characteristic, and let X be a finite type k-scheme. The residue complex $\mathcal{K}_X^{\cdot}$ is by definition the Cousin complex of $\pi^! k$, where $\pi\colon X \to \operatorname{Spec} k$ is the structural morphism (cf. [RD]). Each $\mathcal{K}_X^q$ is a quasi-coherent sheaf. Let

$$\mathcal{F}_X^{\cdot} := \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}^{\cdot}, \mathcal{K}_X^{\cdot})$$

which is a graded sheaf in the obvious way. According to [EZ] or [Ye3], there is an operator $\mathrm{D}\colon \mathcal{F}_X^i \to \mathcal{F}_X^{i+1}$ which makes $\mathcal{F}_X^{\cdot}$ into a DG $\Omega_{X/k}^{\cdot}$-module. $\mathcal{F}_X^{\cdot}$ is called the *De Rham-residue complex*. When X is smooth, $\mathrm{H}_i^{\mathrm{DR}}(X) = \mathrm{H}^{-i}\Gamma(X, \mathcal{F}_X^{\cdot})$.

In [Ye5] it is proved that there is a natural structure of right DG $\mathcal{A}_X^{\cdot}$-module on $\mathcal{F}_X^{\cdot}$, extending the $\Omega_{X/k}^{\cdot}$-module structure (cf. Theorem 6.1). The action is "by taking residues." When $f\colon X \to Y$ is proper then $\mathrm{Tr}_f\colon f_*\mathcal{F}_X^{\cdot} \to \mathcal{F}_Y^{\cdot}$ is a

homomorphism of DG $\mathcal{A}_Y^{\cdot}$-modules. If we view the adeles $\mathcal{A}_X^{p,q}$ as an algebraic analog of the smooth forms of type (p,q) on a complex-analytic manifold, then $\mathcal{F}_X^{-p,-q}$ is the analog of the currents of type (p,q).

Suppose $\operatorname{char} k = 0$, X is smooth irreducible of dimension n, $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of rank r, v is a regular section of $\mathcal{E}$ and Z is its zero scheme. Let $C_X, C_Z \in \Gamma(X, \mathcal{F}_X^{\cdot})$ be the fundamental classes. In Theorem 6.5 we prove the following version of the Gauss-Bonnet Formula: there is an adelic connection ∇ on $\mathcal{E}$ satisfying

$$C_X \cdot c_r(\mathcal{E}, \nabla) = (-1)^m C_Z \in \mathcal{F}_X^{-2(n-r)}$$

with $m = nr + \binom{r+1}{2}$.

Observe that this formula is on the level of differential forms. Passing to (co)homology we recover the familiar formula $c_r(\mathcal{E}) \smile [X] = [Z] \in \mathrm{H}^{\mathrm{DR}}_{(2n-r)}(X)$ (cf. [Fu] Section 14.1).

Acknowledgments. The authors wish to thank S. Kleiman who suggested studying the adelic approach to the Bott Residue Formula, J. Lipman for calling our attention to Zhou's work, V. Hinich for explaining to us the construction of the Thom-Sullivan cochains, D. Blanc for help with cosimplicial groups, and P. Sastry for discussions. Special thanks also go to P. Golginger and W. Krinner for collecting the available material on algebraic connections ([Go], [Kr]). Part of the research was done while the first author was visiting the Weizmann Institute. He wishes to express his gratitude towards this institution for its hospitality during his stay there.

1. Cosimplicial algebras and their normalizations. In this section we review some well-known facts about cosimplicial objects, and also discuss the less well-known Thom-Sullivan normalization. Our sources are [Ma], [ML], [BG] and [HS1].

Denote by Δ the category whose objects are the finite ordered sets $[n] := \{0, \dots, n\}$, and whose morphisms are the monotone nondecreasing functions $[m] \to [n]$. Let $\partial^i\colon [n-1] \to [n]$ stand for the ith coface map, and let $s^i\colon [n+1] \to [n]$ stand for the ith codegeneracy map. These maps generate by composition all morphisms in Δ. Let Δ° denote the opposite category. By definition a *simplicial object* in a category C is a functor $S\colon \Delta^\circ \to \mathsf{C}$. Often one writes S_n instead of the object $S[n] \in \mathsf{C}$. A *cosimplicial object* is a functor $S\colon \Delta \to \mathsf{C}$. Denote by $\Delta^\circ\mathsf{C}$ (resp. $\Delta\mathsf{C}$) the category of simplicial (resp. cosimplicial) objects in C.

Example 1.1. Let P be a partially ordered set. A simplex (or chain) of length n in P is a sequence $\sigma = (x_0, \dots, x_n)$, with $x_i \le x_{i+1}$. More generally, if P is a category, then an n-simplex is a functor $\sigma\colon [n] \to P$. Letting $S(P)_n$ be the set of n-simplices in P, we see that $S(P)$ is a simplicial set.

Example 1.2. If we take $P = [n]$, then we get the standard simplicial complex $\Delta^n \in \Delta^\circ\mathsf{Sets}$. As a functor $\Delta^\circ \to \mathsf{Sets}$ one has $\Delta^n = \mathrm{Hom}_\Delta(-, [n])$. Observe that

$$\Delta^n_m = \mathrm{Hom}_\Delta([m], [n]) = \{(i_0, \ldots, i_m) \mid 0 \leq i_0 \leq \cdots \leq i_m \leq n\}.$$

Example 1.3. Given a scheme X, specialization defines a partial ordering on its underlying set of points: $x \leq y$ if $y \in \overline{\{x\}}$. We denote by $S(X)$ the resulting simplicial set.

Example 1.4. Let Δ^n_{top} be the standard realization of Δ^n, i.e. the compact topological space

$$\{(t_0, \ldots, t_n) \mid t_i \geq 0 \text{ and } \sum t_i = 1\} \subset \mathbb{R}^{n+1}.$$

Then $\Delta_{\mathrm{top}} = \{\Delta^n_{\mathrm{top}}\}$ is a cosimplicial topological space.

Let k be any commutative ring. By a differential graded (DG) k-module we mean a cochain complex, namely a graded module $M = \bigoplus_{q \in \mathbb{Z}} M^q$ with an endomorphism d of degree 1 satisfying $\mathrm{d}^2 = 0$. By a differential graded algebra (DGA) over k we mean a DG module A with a DG homomorphism $A \otimes_k A \to A$. So A is neither assumed to be commutative nor associative.

Example 1.5. Let $t_0, t_1, \ldots$ be indeterminates (of degree 0). Define

$$R_n := k[\, t_0, \ldots, t_n \,]/(t_0 + \cdots + t_n - 1)$$

and $\Delta^n_k := \operatorname{Spec} R_n$. Then as in the previous example, $\Delta_k = \{\Delta^n_k\}$ is a cosimplicial scheme. Letting $\Omega^\cdot(\Delta^n_k) := \Omega^\cdot_{R_n/k}$, we see that $\Omega^\cdot(\Delta_k) := \{\Omega^\cdot(\Delta^n_k)\}$ is a simplicial DGA over k.

Consider a cosimplicial k-module $M = \{M^q\} \in \Delta\mathsf{Mod}(k)$. Its standard normalization is the DG module $(\mathrm{N}M, \partial)$ whose degree q piece is $\mathrm{N}^q M := \bigcap \mathrm{Ker}(s^i) \subset M^q$, and $\partial := \sum (-1)^i \partial^i$. Now suppose $M = \{M^{\cdot,q}\} \in \Delta\mathsf{DGMod}(k)$, i.e. a cosimplicial DG k-module. Each $M^{\cdot,q} = \bigoplus_{p \in \mathbb{Z}} M^{p,q}$ is a DG module with operator d: $M^{p,q} \to M^{p+1,q}$, and each $M^{p,\cdot}$ is a cosimplicial k-module. Define $\mathrm{N}^{p,q}M := \mathrm{N}^q M^{p,\cdot}$, $\mathrm{N}^i M := \bigoplus_{p+q=i} \mathrm{N}^{p,q} M$, and $\mathrm{N}M := \bigoplus_i \mathrm{N}^i M$. Then $\mathrm{N}M$ is a DG module with coboundary operator $\mathrm{D} := \mathrm{D}' + \mathrm{D}''$, where $\mathrm{D}' := (-1)^q \mathrm{d}$: $\mathrm{N}^{p,q}M \to \mathrm{N}^{p+1,q}M$ and $\mathrm{D}'' := \partial$: $\mathrm{N}^{p,q}M \to \mathrm{N}^{p,q+1}M$. Another way to visualize this is by defining for each q a DG module $\mathrm{N}^{\cdot,q}M := \bigcap \mathrm{Ker}(s^i) \subset M^{\cdot,q}[-q]$ (the shift by $-q$), so the operator is indeed D'. Then $\mathrm{D}'' = \partial$: $\mathrm{N}^{\cdot,q}M \to \mathrm{N}^{\cdot,q+1}M$ has degree 1 and $\mathrm{N}M = \bigoplus_q \mathrm{N}^{\cdot,q}M$.

If A is a cosimplicial DGA, that is $A \in \Delta\,\mathsf{DGA}(k)$, then $\mathrm{N}A$ is a DGA with

the Alexander-Whitney product. For any $a \in \mathrm{N}^{p,q}A$ and $b \in \mathrm{N}^{p',q'}A$ one has

$$a \cdot b = \partial^-(a) \cdot \partial^+(b) \in \mathrm{N}^{p+p',q+q'}A, \tag{1.6}$$

where $\partial^-\colon [q] \to [q+q']$ is the simplex $(0, 1, \ldots, q)$ and $\partial^+\colon [q'] \to [q+q']$ is the simplex $(q, q+1, \ldots, q+q')$ (cf. Example 1.2). Note that if each algebra $A^{\cdot,q}$ is associative, then so is $\mathrm{N}A$; however $\mathrm{N}A$ is usually not commutative. If M is a cosimplicial DG left A-module then $\mathrm{N}M$ is a DG left $\mathrm{N}A$-module.

We shall need another normalization of cosimplicial objects. The definition below is extracted from the work of Hinich-Schechtman, cf. [HS1], [HS2]. Fix a commutative $\mathbb{Q}$-algebra k.

Definition 1.7. Suppose $M = \{M^q\}$ is a cosimplicial k-module. Let

$$\tilde{\mathrm{N}}^q M \subset \prod_{l=0}^{\infty} \left(\Omega^q(\Delta^l_{\mathbb{Q}}) \otimes_{\mathbb{Q}} M^l\right)$$

be the submodule consisting of all elements $\boldsymbol{u} = (u_0, u_1, \ldots)$, $u_l \in \Omega^q(\Delta^l_{\mathbb{Q}}) \otimes_{\mathbb{Q}} M^l$, s.t.

$$(1 \otimes \partial^i)u_l = (\partial_i \otimes 1)u_{l+1} \tag{1.8}$$

$$(s_i \otimes 1)u_l = (1 \otimes s^i)u_{l+1} \tag{1.9}$$

for all $0 \le l$, $0 \le i \le l+1$. Given a cosimplicial DG k-module $M = \{M^{\cdot,q}\}$, let $\tilde{\mathrm{N}}^{p,q}M := \tilde{\mathrm{N}}^q M^{p,\cdot}$, $\tilde{\mathrm{N}}^i M := \bigoplus_{p+q=i} \tilde{\mathrm{N}}^{p,q}M$ and $\tilde{\mathrm{N}}M := \bigoplus_i \tilde{\mathrm{N}}^i M$. Define $\mathrm{D}' := (-1)^q \otimes \mathrm{d}$, $\mathrm{D}'' := \mathrm{d} \otimes 1$ and $\mathrm{D} := \mathrm{D}' + \mathrm{D}''$. The resulting complex $(\tilde{\mathrm{N}}M, \mathrm{D})$ is called the *Thom-Sullivan normalization* of M. If A is a cosimplicial k-DGA, then $\tilde{\mathrm{N}}A$ inherits the component-wise multiplication from $\prod_l \Omega^q(\Delta^l_{\mathbb{Q}}) \otimes_{\mathbb{Q}} A^{p,l}$, so it is a DGA.

In the definition above the signs are in agreement with the usual conventions; keep in mind that $M^{p,q}$ is in degree p (cf. [ML] Ch. VI Section 7). It is clear that if each DGA $A^{\cdot,q}$ is commutative (resp. associative), then so is $\tilde{\mathrm{N}}A$.

Usual integration on the real simplex Δ^l_{top} yields a $\mathbb{Q}$-linear map of degree 0, $\int_{\Delta^l}\colon \Omega^{\cdot}(\Delta^l_{\mathbb{Q}}) \to \mathbb{Q}[-l]$, such that $\int_{\Delta^l}(\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_l) = \frac{1}{l!}$. By linearity, for any cosimplicial DG module M this extends to a degree 0 homomorphism

$$\int_{\Delta^l}\colon \Omega^{\cdot}(\Delta^l_{\mathbb{Q}}) \otimes_{\mathbb{Q}} M^{\cdot,l} \to \mathbb{Q}[-l] \otimes_{\mathbb{Q}} M^{\cdot,l} = M^{\cdot,l}[-l].$$

Note that $\int_{\Delta^l}$ sends $\mathrm{D}' := (-1)^q \otimes \mathrm{d}$ to $\mathrm{D}' := (-1)^l \mathrm{d}$. Define $\int_{\Delta}\colon \tilde{\mathrm{N}}M \to$

$\bigoplus_q M^{\cdot,q}[-q]$ by:

$$\int_\Delta (u_0, u_1, \ldots) := \sum_{l=0}^{\infty} \int_{\Delta^l} u_l. \tag{1.10}$$

LEMMA 1.11. *The image of* $\int_\Delta$ *lies inside* $\mathrm{N}M$*, and* $\int_\Delta \colon \tilde{\mathrm{N}}M \to \mathrm{N}M$ *is a* k*-linear homomorphism of complexes.*

Proof. A direct verification, amounting to Stoke's Theorem on Δ^l_{top}. □

What we have is a natural transformation $\int_\Delta \colon \tilde{\mathrm{N}} \to \mathrm{N}$ of functors $\Delta\mathsf{DGMod}(k) \to \mathsf{DGMod}(k)$.

THEOREM 1.12. (Simplicial De Rham Theorem) *Let* M *a cosimplicial DG* $\mathbb{Q}$*-module. Then* $\int_\Delta \colon \tilde{\mathrm{N}}M \to \mathrm{N}M$ *is a quasi-isomorphism.*

THEOREM 1.13. *Let* k *be a commutative* $\mathbb{Q}$*-algebra and* A *a cosimplicial DG* k*-algebra. Then* $\mathrm{H}(\int_\Delta) \colon \mathrm{H}\tilde{\mathrm{N}}A \to \mathrm{H}\mathrm{N}A$ *is an isomorphism of graded* k*-algebras. If* M *is a cosimplicial DG* A*-module, then* $\mathrm{H}(\int_\Delta) \colon \mathrm{H}\tilde{\mathrm{N}}M \to \mathrm{H}\mathrm{N}M$ *is an isomorphism of graded* $\mathrm{H}\tilde{\mathrm{N}}A$*-modules.*

The proofs are essentially contained in [BG] and [HS1]. For the sake of completeness we include proofs in Appendix A of this paper.

2. Adeles of differential forms. In this section we apply the constructions of Section 1 to the cosimplicial DGA $\underline{\mathbb{A}}(\Omega^{\cdot}_{X/k})$ on a scheme X. This will give two DGAs, $\mathcal{A}^{\cdot}_X$ and $\tilde{\mathcal{A}}^{\cdot}_X$, which are resolutions of $\Omega^{\cdot}_{X/k}$.

Let us begin with a review of *Beilinson adeles* on a noetherian scheme X of finite dimension. A chain of points in X is a sequence $\xi = (x_0, \ldots, x_q)$ of points with $x_{i+1} \in \overline{\{x_i\}}$. Denote by $S(X)_q$ the set of length q chains, so $\{S(X)_q\}_{q \geq 0}$ is a simplicial set. For $T \subset S(X)_q$ and $x \in X$ let

$$\hat{x}T := \{(x_1, \ldots, x_q) \mid (x, x_1, \ldots, x_q) \in T\}.$$

According to [Be] there is a unique collection of functors $\mathbb{A}(T, -) \colon \mathsf{Qco}(X) \to \mathsf{Ab}$, indexed by $T \subset S(X)_q$, each of which commuting with direct limits, and satisfying

$$\mathbb{A}(T, \mathcal{M}) = \begin{cases} \prod_{x \in X} \lim_{\leftarrow n} \mathcal{M}_x / \mathfrak{m}_x^n \mathcal{M}_x & \text{if } q = 0 \\ \prod_{x \in X} \lim_{\leftarrow n} \mathbb{A}(\hat{x}T, \mathcal{M}_x / \mathfrak{m}_x^n \mathcal{M}_x) & \text{if } q > 0 \end{cases}$$

for $\mathcal{M}$ coherent. Here $\mathfrak{m} \subset \mathcal{O}_{X,x}$ is the maximal ideal and $\mathcal{M}_x / \mathfrak{m}_x^n \mathcal{M}_x$ is treated as a quasi-coherent sheaf with support $\overline{\{x\}}$. Furthermore each $\mathbb{A}(T, -)$ is exact.

For a single chain ξ one also writes $\mathcal{M}_\xi := \mathbb{A}(\{\xi\}, \mathcal{M})$, and this is the *Beilinson completion* of $\mathcal{M}$ along ξ. Then

$$\mathbb{A}(T, \mathcal{M}) \subset \prod_{\xi \in T} \mathcal{M}_\xi \tag{2.1}$$

which permits us to consider the adeles as a "restricted product." For $q = 0$ and $\mathcal{M}$ coherent we have $\mathcal{M}_{(x)} = \widehat{\mathcal{M}}_x$, the $\mathfrak{m}_x$-adic completion, and (2.1) is an equality. In view of this we shall say that $\mathbb{A}(T, \mathcal{M})$ is the group of adeles combinatorially supported on T and with values in $\mathcal{M}$.

Define a presheaf $\underline{\mathbb{A}}(T, \mathcal{M})$ by

$$\Gamma(U, \underline{\mathbb{A}}(T, \mathcal{M})) := \mathbb{A}(T \cap S(U)_q, \mathcal{M}) \tag{2.2}$$

for $U \subset X$ open. Then $\underline{\mathbb{A}}(T, \mathcal{M})$ is a flasque sheaf. Also $\underline{\mathbb{A}}(T, \mathcal{O}_X)$ is a flat $\mathcal{O}_X$-algebra, and $\underline{\mathbb{A}}(T, \mathcal{M}) \cong \underline{\mathbb{A}}(T, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{M}$.

For every q define the sheaf of degree q *Beilinson adeles*

$$\underline{\mathbb{A}}^q(\mathcal{M}) := \underline{\mathbb{A}}(S(X)_q, \mathcal{M}).$$

Then $\underline{\mathbb{A}}(\mathcal{M}) = \{\underline{\mathbb{A}}^q(\mathcal{M})\}_{q \in \mathbb{N}}$ is a cosimplicial sheaf. The standard normalization $\mathrm{N}^q\underline{\mathbb{A}}(\mathcal{M})$ is canonically isomorphic to the sheaf $\underline{\mathbb{A}}^q_{\mathrm{red}}(\mathcal{M}) := \underline{\mathbb{A}}(S(X)^{\mathrm{red}}_q, \mathcal{M})$, where $S(X)^{\mathrm{red}}_q$ is the set of nondegenerate chains. Note that $\underline{\mathbb{A}}^q_{\mathrm{red}}(\mathcal{M}) = 0$ for all $q > \dim X$. A fundamental theorem of Beilinson says that the canonical homomorphism $\mathcal{M} \to \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{M})$ is a quasi-isomorphism. We see that $\mathrm{H}^q\Gamma(X, \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{M})) = \mathrm{H}^q(X, \mathcal{M})$.

The complex $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X)$ is a DGA, with the Alexander-Whitney product. For local sections $a \in \underline{\mathbb{A}}^q_{\mathrm{red}}(\mathcal{O}_X)$ and $b \in \underline{\mathbb{A}}^{q'}_{\mathrm{red}}(\mathcal{O}_X)$ the product is $a \cdot b = \partial^-(a) \cdot \partial^+(b) \in \underline{\mathbb{A}}^{q+q'}_{\mathrm{red}}(\mathcal{O}_X)$, where ∂^- and ∂^+ correspond respectively to the initial and final segments of $(0, \ldots, q, \ldots, q + q')$. This algebra is not (graded) commutative. For proofs and more details turn to [Hr], [Ye1] Chapter 3 and [HY] Section 1.

Example 2.3. Suppose X is a nonsingular curve. The relation to the classical ring of adeles $\mathbb{A}(X)$ of Chevalley and Weil is $\mathbb{A}(X) = \Gamma(X, \underline{\mathbb{A}}^1_{\mathrm{red}}(\mathcal{O}_X))$.

Now assume X is a finite type scheme over the noetherian ring k. In [HY] it was shown that given any differential operator (DO) $D\colon \mathcal{M} \to \mathcal{N}$ there is an induced DO $D\colon \underline{\mathbb{A}}^q(\mathcal{M}) \to \underline{\mathbb{A}}^q(\mathcal{N})$. Applying this to the De Rham complex $\Omega^{\cdot}_{X/k}$ we get a cosimplicial DGA $\underline{\mathbb{A}}(\Omega^{\cdot}_{X/k})$. The *De Rham adele complex* is the DGA

$$\mathcal{A}^{\cdot}_X := \mathrm{N}\underline{\mathbb{A}}(\Omega^{\cdot}_{X/k}).$$

Since

$$\mathcal{A}_X^{p,q} \cong \underline{\mathbb{A}}_{\mathrm{red}}^q(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \Omega_{X/k}^p$$

we see that $\mathcal{A}_X^{\cdot}$ is bounded. By a standard double complex spectral sequence argument (see [HY] Proposition 2.1) we get:

PROPOSITION 2.4. *The natural DGA map $\Omega_{X/k}^{\cdot} \to \mathcal{A}_X^{\cdot}$ is a quasi-isomorphism of sheaves. Hence* $\mathrm{H}^{\cdot}(X, \Omega_{X/k}^{\cdot}) \cong \mathrm{H}^{\cdot}\Gamma(X, \mathcal{A}_X^{\cdot})$.

Let us examine the DGA $\mathcal{A}_X^{\cdot}$ a little more closely. The operators are $\mathrm{D}' = (-1)^q \mathrm{d}\colon \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p+1,q}$, $\mathrm{D}'' = \partial\colon \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{p,q+1}$ and $\mathrm{D} = \mathrm{D}' + \mathrm{D}''$. As for the multiplication, consider local sections $a \in \underline{\mathbb{A}}_{\mathrm{red}}^q(\mathcal{O}_X)$, $b \in \underline{\mathbb{A}}_{\mathrm{red}}^{q'}(\mathcal{O}_X)$, $\alpha \in \Omega_{X/k}^p$ and $\beta \in \Omega_{X/k}^{p'}$. Then

$$(a \otimes \alpha) \cdot (b \otimes \beta) = (-1)^{q'p} \partial^-(a) \cdot \partial^+(b) \otimes \alpha \wedge \beta \in \mathcal{A}_X^{p+p',q+q'} \tag{2.5}$$

(cf. formula (1.6)).

Remark 2.6. In the analogy to sheaves of smooth forms on a complex-analytic manifold, our operators D', D'' play the roles of ∂, $\bar{\partial}$ respectively. Note however that here $\mathcal{A}_X^{p,q}$ is not a locally free $\mathcal{A}_X^0$-module for $0 < q \leq n$, even when X is smooth. The same is true also for the sheaves $\tilde{\mathcal{A}}_X^{p,q}$ defined below.

If k is a perfect field and X is an integral scheme of dimension n, then each maximal chain $\xi = (x_0, \ldots, x_n)$ defines a k-linear map $\mathrm{Res}_\xi\colon \Omega_{X/k,\xi}^n \to k$ called the *Parshin residue* (cf. [Ye1] Definition 4.1.3). By (2.1) we obtain $\mathrm{Res}_\xi\colon \Gamma(X, \mathcal{A}_X^{2n}) \to k$.

PROPOSITION 2.7. *Suppose k is a perfect field.*

(1) *Given $\alpha \in \Gamma(X, \mathcal{A}_X^{2n})$, one has $\mathrm{Res}_\xi \alpha = 0$ for all but finitely many ξ. Hence $\int_X := \sum_\xi \mathrm{Res}_\xi\colon \Gamma(X, \mathcal{A}_X^{2n}) \to k$ is well defined.*

(2) *If X is proper then $\int_X \mathrm{D}\beta = 0$ for all $\beta \in \Gamma(X, \mathcal{A}_X^{2n-1})$. Hence $\int_X\colon \mathrm{H}^{2n}(X, \Omega_{X/k}^{\cdot}) \to k$ is well defined.*

(3) *If X is smooth and proper then $\int_X\colon \mathrm{H}_{\mathrm{DR}}^{2n}(X) \to k$ coincides with the non-degenerate map of [Ha1].*

Proof. (1) See [HY] Proposition 3.4. (2) This follows from the Parshin-Lomadze Residue Theorem ([Ye1] Theorem 4.2.15). (3) By [HY] Theorem 3.1 and [Ye3] Corollary 3.8. □

Now assume k is any $\mathbb{Q}$-algebra. The Thom-Sullivan normalization determines a sheaf $\tilde{\mathrm{N}}^q \underline{\mathbb{A}}(\mathcal{M})$, where $\Gamma(U, \tilde{\mathrm{N}}^q \underline{\mathbb{A}}(\mathcal{M})) = \tilde{\mathrm{N}}^q \Gamma(U, \underline{\mathbb{A}}(\mathcal{M}))$. Applying this to the cosimplicial DGA $\underline{\mathbb{A}}(\Omega_{X/k}^{\cdot})$ we obtain:

Definition 2.8. The sheaf of *Thom-Sullivan adeles* is the sheaf of DGAs

$$\tilde{\mathcal{A}}_X^{\cdot} := \tilde{\mathrm{N}}\underline{\mathbb{A}}(\Omega_{X/k}^{\cdot}).$$

$(\tilde{\mathcal{A}}_X^{\cdot}, \mathrm{D})$ is an associative, commutative DGA. The natural map $\Omega_{X/k}^{\cdot} \to \tilde{\mathcal{A}}_X^{\cdot}$ is an injective DGA homomorphism. The coboundary operator on $\tilde{\mathcal{A}}_X^{\cdot}$ is $\mathrm{D} = \mathrm{D}' + \mathrm{D}''$, where $\mathrm{D}'\colon \tilde{\mathcal{A}}_X^{p,q} \to \tilde{\mathcal{A}}_X^{p+1,q}$ and $\mathrm{D}''\colon \tilde{\mathcal{A}}_X^{p,q} \to \tilde{\mathcal{A}}_X^{p,q+1}$. The "integral on the fibers" $\int_\Delta$ sheafifies to give a degree 0 DG $\Omega_{X/k}^{\cdot}$-module homomorphism $\int_\Delta\colon \tilde{\mathcal{A}}_X^{\cdot} \to \mathcal{A}_X^{\cdot}$. This is not an algebra homomorphism! However:

PROPOSITION 2.9. *For every open set $U \subset X$, $\mathrm{H}^{\cdot}(\int_\Delta)\colon \mathrm{H}^{\cdot}\Gamma(U, \tilde{\mathcal{A}}_X^{\cdot}) \to \mathrm{H}^{\cdot}\Gamma(U, \mathcal{A}_X^{\cdot})$ is an isomorphism of graded k-algebras.*

Proof. Apply Theorems 1.12 and 1.13 to the cosimplicial DGA $\Gamma(U, \underline{\mathbb{A}}(\Omega_{X/k}^{\cdot}))$. □

Remark 2.10. We do not know whether the sheaves $\tilde{\mathcal{A}}_X^{p,q}$ are flasque. The DGA $\tilde{\mathcal{A}}_X^{\cdot}$ is not bounded; however, letting $n := \sup\{p \mid \Omega_{X/k,x}^p \neq 0 \text{ for some } x \in X\}$, then $n < \infty$ and $\tilde{\mathcal{A}}_X^{p,q} = 0$ for all $p > n$.

COROLLARY 2.11. *If X is smooth over k, then the homomorphisms $\Omega_{X/k}^{\cdot} \to \tilde{\mathcal{A}}_X^{\cdot} \to \mathcal{A}_X^{\cdot}$ induce isomorphisms of graded k-algebras*

$$\mathrm{H}_{\mathrm{DR}}^{\cdot}(X) = \mathrm{H}^{\cdot}(X, \Omega_{X/k}^{\cdot}) \cong \mathrm{H}^{\cdot}\Gamma(X, \tilde{\mathcal{A}}_X^{\cdot}) \cong \mathrm{H}^{\cdot}\Gamma(X, \mathcal{A}_X^{\cdot}).$$

Given a quasi-coherent sheaf $\mathcal{M}$ set

$$\tilde{\mathcal{A}}_X^{p,q}(\mathcal{M}) := \tilde{\mathrm{N}}^q\underline{\mathbb{A}}(\Omega_{X/k}^p \otimes_{\mathcal{O}_X} \mathcal{M}). \tag{2.12}$$

In particular we have $\tilde{\mathcal{A}}_X^{p,q} = \tilde{\mathcal{A}}_X^{0,q}(\Omega_{X/k}^p)$.

LEMMA 2.13. (1) *Let $\mathcal{M}$ be a quasi-coherent sheaf. Then the complex*

$$0 \to \mathcal{M} \to \tilde{\mathcal{A}}_X^{0,0}(\mathcal{M}) \xrightarrow{\mathrm{D}''} \tilde{\mathcal{A}}_X^{0,1}(\mathcal{M}) \xrightarrow{\mathrm{D}''} \cdots$$

is exact.

(2) *If $\mathcal{E}$ is locally free of finite rank, then $\tilde{\mathcal{A}}_X^{p,q}(\mathcal{E}) \cong \tilde{\mathcal{A}}_X^{p,q} \otimes_{\mathcal{O}_X} \mathcal{E}$.*

(3) *Suppose* $\mathrm{d}\colon \mathcal{M} \to \mathcal{N}$ *is a k-linear DO. Then* d *extends to a DO* $\mathrm{d}\colon \tilde{\mathcal{A}}_X^{0,q}(\mathcal{M}) \to \tilde{\mathcal{A}}_X^{0,q}(\mathcal{N})$ *which commutes with* D''.

Proof. (1) Use the quasi-isomorphism

$$\int_\Delta\colon \tilde{\mathcal{A}}_X^{0,\cdot}(\mathcal{M}) = \tilde{\mathrm{N}}\underline{\mathbb{A}}(\mathcal{M}) \to \mathrm{N}\underline{\mathbb{A}}(\mathcal{M}) = \underline{\mathbb{A}}_{\mathrm{red}}^{\cdot}(\mathcal{M}).$$

(2) Multiplication induces a homomorphism $\tilde{\mathcal{A}}_X^{\cdot} \otimes_{\mathcal{O}_X} \mathcal{E} \to \tilde{\mathcal{A}}_X^{\cdot}(\mathcal{E})$. Choose a *local algebraic frame* $\boldsymbol{f} = (f_1, \ldots, f_r)^{\mathrm{t}}$ for $\mathcal{E}$ on a small open set U; i.e. an isomorphism $\boldsymbol{f}\colon \mathcal{O}_U^r \xrightarrow{\simeq} \mathcal{E}|_U$. Then we see that $\tilde{\mathcal{A}}_X^{\cdot} \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\simeq} \tilde{\mathcal{A}}_X^{\cdot}(\mathcal{E})$.
(3) The DO d: $\underline{\mathbb{A}}(\mathcal{M}) \to \underline{\mathbb{A}}(\mathcal{N})$ respects the cosimplicial structure. □

In order to clarify the algebraic structure of $\tilde{\mathcal{A}}_X^{\cdot}$ we introduce the following local objects. Given a chain $\xi = (x_0, \ldots, x_l)$ in X let

$$\tilde{\mathcal{A}}_\xi^{p,q} := \Omega^q(\Delta_{\mathbb{Q}}^l) \otimes_{\mathbb{Q}} \Omega_{X/k,\xi}^p. \tag{2.14}$$

As usual we set $\mathrm{D}' := (-1)^q \otimes \mathrm{d}$, $\mathrm{D}'' := \mathrm{d} \otimes 1$ and $\mathrm{D} := \mathrm{D}' + \mathrm{D}''$. The DGA $(\tilde{\mathcal{A}}_\xi^{\cdot}, \mathrm{D})$ is generated (as a DGA) by

$$\tilde{\mathcal{A}}_\xi^0 = \mathcal{O}_{X,\xi}[t_0, \ldots, t_l]/(\textstyle\sum t_i - 1). \tag{2.15}$$

When X is smooth of dimension n over k near x_0 then $\tilde{\mathcal{A}}_\xi^{p,q}$ is free of rank $\binom{n}{p}\binom{l}{q}$ over $\tilde{\mathcal{A}}_\xi^0$. Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ let $\tilde{\mathcal{A}}_\xi^{p,q}(\mathcal{M}) := \tilde{\mathcal{A}}_\xi^{p,q} \otimes_{\mathcal{O}_X} \mathcal{M}$.

LEMMA 2.16. (1) *For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ and open set $U \subset X$ there are natural commutative diagrams*

$$\begin{array}{ccc}
\Gamma(U, \tilde{\mathcal{A}}_X^{p,q}(\mathcal{M})) & \xrightarrow{\int_\Delta} & \Gamma(U, \mathcal{A}_X^{p,q}(\mathcal{M})) \\
{\scriptstyle \Phi_{\mathcal{M}}^{p,q}}\downarrow & & \downarrow \\
\prod_{\xi \in S(U)} \tilde{\mathcal{A}}_\xi^{p,q}(\mathcal{M}) & \xrightarrow{\int_\Delta} & \prod_{\xi \in S(U)_q^{\mathrm{red}}} (\Omega_{X/k,\xi}^p \otimes_{\mathcal{O}_X} \mathcal{M})
\end{array}$$

(2) $\Phi_{\mathcal{M}} := \sum \Phi_{\mathcal{M}}^{p,q}$ *is injective and commutes with* D' *and* D''.

(3) $\Phi_{\mathcal{O}_X}$ *is a DGA homomorphism and* $\Phi_{\mathcal{M}}$ *is* $\Gamma(U, \tilde{\mathcal{A}}_X^{\cdot})$*-linear.*

Proof. This is immediate from Definition 1.7 and formula (2.1). □

LEMMA 2.17. *Let $\mathcal{M}$ be a quasi-coherent sheaf. The natural homomorphism $\mathcal{M} \to \tilde{\mathcal{A}}_X^0(\mathcal{M})$ extends to an $\mathcal{O}_X$-linear homomorphism $\underline{\mathbb{A}}^0(\mathcal{M}) \to \tilde{\mathcal{A}}_X^0(\mathcal{M})$.*

Proof. Consider the ith covertex map $\sigma_i\colon [0] \to [l]$, which is the simplex $\sigma_i = (i) \in \Delta_0^l \cong \mathrm{Hom}_\Delta([0], [l])$ (cf. Example 1.2). There is a corresponding homomorphism $\sigma_i\colon \underline{\mathbb{A}}^0(\mathcal{M}) \to \underline{\mathbb{A}}^l(\mathcal{M})$. Given a local section $u \in \underline{\mathbb{A}}^0(\mathcal{M})$, send it to $(u_0, u_1, \ldots) \in \tilde{\mathrm{N}}^0\underline{\mathbb{A}}(\mathcal{M}) = \tilde{\mathcal{A}}_X^0(\mathcal{M})$, where $u_l := \sum_{i=0}^l t_i \otimes \sigma_i(u)$. □

Because of the functoriality of our constructions we have:

PROPOSITION 2.18. *Let $f\colon X \to Y$ be a morphism of k-schemes. Then the pullback homomorphism $f^*\colon \Omega^{\cdot}_{Y/k} \to f_*\Omega^{\cdot}_{X/k}$ extends to DGA homomorphisms $f^*\colon \tilde{\mathcal{A}}^{\cdot}_Y \to f_*\tilde{\mathcal{A}}^{\cdot}_X$ and $f^*\colon \mathcal{A}^{\cdot}_Y \to f_*\mathcal{A}^{\cdot}_X$ giving a commutative diagram*

$$\begin{array}{ccccc}
\Omega^{\cdot}_{Y/k} & \longrightarrow & \tilde{\mathcal{A}}^{\cdot}_Y & \xrightarrow{\int_\Delta} & \mathcal{A}^{\cdot}_Y \\
{\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow \\
f_*\Omega^{\cdot}_{X/k} & \longrightarrow & f_*\tilde{\mathcal{A}}^{\cdot}_X & \xrightarrow{f_*(\int_\Delta)} & f_*\mathcal{A}^{\cdot}_X.
\end{array}$$

Remark 2.19. One can show that $(\tilde{\mathcal{A}}^0_X)^\times = \mathcal{O}_X^\times$ (invertible elements). We leave this as an exercise to the interested reader.

3. Adelic Chern-Weil theory. Let us quickly review the notion of a connection on a module. For a full account see [GH] Section 0.5 and 3.3, [KL] Appendix B, [Ka], [Go] or [Kr]. In this section k is a field of characteristic 0. Suppose $A = A^0 \oplus A^1 \oplus \cdots$ is an associative, commutative DG k-algebra (i.e. $ab = (-1)^{ij}ba$ for $a \in A^i$, $b \in A^j$), with operator d. Given an A^0-module M, a connection on M is a k-linear map $\nabla\colon M \to A^1 \otimes_{A^0} M$ satisfying the Leibniz rule $\nabla(am) = \mathrm{d}a \otimes m + a\nabla m$, $a \in A^0$. ∇ extends uniquely to an operator $\nabla\colon A \otimes_{A^0} M \to A \otimes_{A^0} M$ of degree 1 satisfying the graded Leibnitz rule.

The curvature of ∇ is the operator $R := \nabla^2\colon M \to A^2 \otimes_{A^0} M$, which is A^0-linear. The connection is flat, or integrable, if $R = 0$. If B is another DGA and $A \to B$ is a DGA homomorphism, then by extension of scalars there is an induced connection $\nabla_B\colon B^0 \otimes_{A^0} M \to B^1 \otimes_{A^0} M$ over B^0.

If M is free of rank r, choose a frame $\boldsymbol{e} = (e_1, \ldots, e_r)^{\mathrm{t}}\colon (A^0)^r \xrightarrow{\simeq} M$. Notice that we write $\boldsymbol{e}$ as a column. This gives a connection matrix $\boldsymbol{\theta} = (\theta_{i,j})$, $\theta_{i,j} \in A^1$, determined by $\nabla\boldsymbol{e} = \boldsymbol{\theta} \otimes \boldsymbol{e}$ (i.e. $\nabla e_i = \sum_j \theta_{i,j} \otimes e_j$). In this case $R \in A^2 \otimes_{A^0} \mathrm{End}\,(M)$, so we get a curvature matrix $\Theta = (\Theta_{i,j})$ satisfying $R = \sum_{i,j} \Theta_{i,j} \otimes (e_i \otimes e_j^\vee)$. Here $\boldsymbol{e}^\vee := (e_1^\vee, \ldots, e_r^\vee)$ is the dual basis and $\Theta_{i,j} \in A^2$. One has $\Theta = \mathrm{d}\boldsymbol{\theta} - \boldsymbol{\theta} \wedge \boldsymbol{\theta}$. If $\boldsymbol{f}$ is another basis of M, with transition matrix $\boldsymbol{g} = (g_{i,j})$, $\boldsymbol{e} = \boldsymbol{g} \cdot \boldsymbol{f}$, then the matrix of ∇ with respect to $\boldsymbol{f}$ is $\boldsymbol{g}^{-1}\boldsymbol{\theta}\boldsymbol{g} - \boldsymbol{g}^{-1}\mathrm{d}\boldsymbol{g}$, and the curvature matrix is $\boldsymbol{g}^{-1}\Theta\boldsymbol{g}$.

Example 3.1. The Levi-Civita connection on M determined by $\boldsymbol{e}$, namely $\nabla = (\mathrm{d}, \ldots, \mathrm{d})$, has matrix $\boldsymbol{\theta} = 0$ and so is integrable. In terms of another basis $\boldsymbol{f} = \boldsymbol{g} \cdot \boldsymbol{e}$ the matrix will be $-\boldsymbol{g}^{-1}\mathrm{d}\boldsymbol{g}$.

Denote by $\mathrm{M}_r(k)$ the algebra of matrices over the field k and $\mathrm{M}_r(k)^* := \mathrm{Hom}_k(\mathrm{M}_r(k), k)$. Then the symmetric algebra $\mathrm{S}(\mathrm{M}_r(k)^*)$ is the algebra of polynomial functions on $\mathrm{M}_r(k)$. The algebra $I_r(k) := \mathrm{S}(\mathrm{M}_r(k)^*)^{\mathrm{Gl}_r(k)}$ of conjugation-invariant functions is generated by the elementary invariant polynomials $P_1 = \mathrm{tr}, \ldots, P_r = \det$, with P_i homogeneous of degree i.

LEMMA 3.2. *Assume that* $A^1 = A^0 \cdot \mathrm{d}A^0$. *Given any matrix* $\boldsymbol{\theta} \in \mathrm{M}_r(A^1)$ *let* $\Theta := \mathrm{d}\boldsymbol{\theta} - \boldsymbol{\theta} \cdot \boldsymbol{\theta}$. *Then for any* $P \in I_r(k)$ *one has* $\mathrm{d}P(\Theta) = 0$.

Proof. By assumption we can write $\theta_{i,j} = \sum_l b_{i,j,l} \mathrm{d}a_l$ for suitable $a_l, b_{i,j,l} \in A^0$. Let A_{u} be the universal algebra for this problem: A^0_{u} is the polynomial algebra $k[\boldsymbol{a}, \boldsymbol{b}]$, where $\boldsymbol{a} = \{a_l\}$ and $\boldsymbol{b} = \{b_{i,j,l}\}$ are finite sets of indeterminates; $A_{\mathrm{u}} = \Omega^{\cdot}_{A^0_{\mathrm{u}}/k}$; and $\boldsymbol{\theta}_{\mathrm{u}} \in \mathrm{M}_r(A^1_{\mathrm{u}})$ is the obvious connection matrix. The DG k-algebra homomorphism $A_{\mathrm{u}} \to A$ sends $\boldsymbol{\theta}_{\mathrm{u}} \mapsto \boldsymbol{\theta}$, and hence it suffices to prove the case $A = A_{\mathrm{u}}$.

Write $X := \operatorname{Spec} A^0$, which is nothing but affine space $\mathbf{A}^N_k$ for some N. We want to show that the form $\mathrm{d}P(\Theta) = 0 \in \Gamma(X, \Omega^{\cdot}_{X/k})$. For a closed point $x \in X$ the residue field $k(x)$ is a finite separable extension of k. This implies that the unique k-algebra lifting $k(x) \to \widehat{\mathcal{O}}_{X,x} = \mathcal{O}_{X,(x)}$ has the property that $\mathrm{d}\colon \mathcal{O}_{X,(x)} \to \Omega^1_{X/k,(x)}$ is $k(x)$-linear. Since X is smooth we have $\mathcal{O}_{X,(x)} \cong k(x)[[f_1, \ldots, f_N]]$. We see that the differential equation on page 401 of [GH] can be solved formally in $\mathrm{M}_r(\mathcal{O}_{X,(x)})$. Then the proof of the lemma on page 403 of [GH] shows that $\mathrm{d}P(\Theta)_{(x)} \in \mathfrak{m}_x \cdot \Omega^{\cdot}_{X/k,(x)}$. Since this is true for all closed points $x \in X$ and $\Omega^{\cdot}_{X/k,(x)}$ is a free $\mathcal{O}_X$-module, it follows that $\mathrm{d}P(\Theta) = 0$. □

Let us now pass to schemes. Assume X is a finite type k-scheme (not necessarily smooth), and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. We shall be interested in the sheaf of commutative DGAs $\tilde{\mathcal{A}}^{\cdot}_X$ and the locally free $\tilde{\mathcal{A}}^0_X$-module $\tilde{\mathcal{A}}^0_X(\mathcal{E}) \cong \tilde{\mathcal{A}}^0_X \otimes_{\mathcal{O}_X} \mathcal{E}$.

Definition 3.3. An *adelic connection* on $\mathcal{E}$ is a connection

$$\nabla\colon\ \tilde{\mathcal{A}}^0_X(\mathcal{E}) \to \tilde{\mathcal{A}}^1_X(\mathcal{E})$$

over the algebra $\tilde{\mathcal{A}}^0_X$.

Definition 3.4. The *adelic curvature form* associated to an adelic connection ∇ on $\mathcal{E}$ is

$$R := \nabla^2 \in \operatorname{Hom}_{\tilde{\mathcal{A}}^0_X}\left(\tilde{\mathcal{A}}^0_X(\mathcal{E}), \tilde{\mathcal{A}}^2_X(\mathcal{E})\right) \cong \Gamma\left(X, \tilde{\mathcal{A}}^2_X \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})\right).$$

Suppose ∇ is an adelic connection on $\mathcal{E}$ and $P \in I_r(k)$. Since $P\colon\ \mathcal{E}nd(\mathcal{E}) \to \mathcal{O}_X$ is well defined, we get an induced sheaf homomorphism $P\colon\ \tilde{\mathcal{A}}^{\cdot}_X \otimes_{\mathcal{O}_X} \mathcal{E}nd(\mathcal{E}) \to \tilde{\mathcal{A}}^{\cdot}_X$. In particular we have $P(R) \in \Gamma(X, \tilde{\mathcal{A}}^{\cdot}_X)$.

LEMMA 3.5. $P(R)$ *is closed, i.e* $\mathrm{D}P(R) = 0$.

Proof. This can be checked locally on X, so let U be an open set on which $\mathcal{E}$ admits an algebraic frame $\boldsymbol{f}$. This frame induces isomorphisms of sheaves

$\boldsymbol{f}\colon (\tilde{\mathcal{A}}_U^{p,q})^r \xrightarrow{\simeq} \tilde{\mathcal{A}}_X^{p,q}(\mathcal{E})|_U$ for all p, q. If $\boldsymbol{\theta} \in \mathrm{M}_r(\Gamma(U, \tilde{\mathcal{A}}_X^1))$ is the matrix of the connection $\nabla\colon \Gamma(U, \tilde{\mathcal{A}}_X^0(\mathcal{E})) \to \Gamma(U, \tilde{\mathcal{A}}_X^1(\mathcal{E}))$ then $\Theta = \mathrm{D}\boldsymbol{\theta} - \boldsymbol{\theta} \cdot \boldsymbol{\theta} \in \mathrm{M}_r(\Gamma(U, \tilde{\mathcal{A}}_X^2))$ is the matrix of R, and we must show that $\mathrm{D}P(\Theta) = 0$.

According to Lemma 2.16, $\Phi\colon \Gamma(U, \tilde{\mathcal{A}}_X^{\cdot}) \to \prod_{\xi \in S(U)} \tilde{\mathcal{A}}_\xi^{\cdot}$ is an injective DGA homomorphism. Thus letting Θ_ξ be the ξ-component of Θ, it suffices to show that $\mathrm{D}P(\Theta_\xi) = 0$ for all ξ. Since $\tilde{\mathcal{A}}_\xi^1 = \tilde{\mathcal{A}}_\xi^0 \cdot \mathrm{D}\tilde{\mathcal{A}}_\xi^0$ we are done by Lemma 3.2. □

Recall that given a morphism of schemes $f\colon X \to Y$ there is a natural homomorphism of DGAs $f^*\colon \tilde{\mathcal{A}}_Y^{\cdot} \to f_*\tilde{\mathcal{A}}_X^{\cdot}$.

PROPOSITION 3.6. *Suppose $f\colon X \to Y$ is a morphism of schemes, $\mathcal{E}$ a locally free $\mathcal{O}_Y$-module and ∇ an adelic connection on $\mathcal{E}$. Then there is an induced adelic connection $f^*(\nabla)$ on $f^*\mathcal{E}$, and*

$$f^*(P(R_\nabla)) = P(R_{f^*(\nabla)}) \in \Gamma(X, \tilde{\mathcal{A}}_X^{\cdot}).$$

Proof. By adjunction there are homomorphisms

$$f^{-1}\mathcal{E} \xrightarrow{f^{-1}(\nabla)} f^{-1}\tilde{\mathcal{A}}_Y^1(\mathcal{E}) \to \tilde{\mathcal{A}}_X^1(f^*\mathcal{E})$$

of sheaves on X. Now $\tilde{\mathcal{A}}_X^0(f^*\mathcal{E}) = \tilde{\mathcal{A}}_X^0 \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{E}$, so by Leibnitz rule we get $f^*(\nabla)$. □

THEOREM 3.7. *Let X be a finite type k-scheme and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module. Choose an adelic connection ∇ on $\mathcal{E}$ and let R be the adelic curvature form. Then the k-algebra homomorphism* (*doubling degrees*)

$$\begin{aligned} w_{\mathcal{E}}\colon I_r(k) &\to \mathrm{H}^{\cdot}\Gamma(X, \tilde{\mathcal{A}}_X^{\cdot}) \cong \mathrm{H}^{\cdot}(X, \Omega_{X/k}^{\cdot}) \\ P &\mapsto [P(R)]. \end{aligned}$$

is independent of the connection ∇.

We call $w_{\mathcal{E}}$ the *adelic Chern-Weil homomorphism*, and we also write $P(\mathcal{E}) := w_{\mathcal{E}}(P)$.

Proof. Suppose ∇' is another adelic connection, with curvature form R'. We need to prove that $[P(R)] = [P(R')] \in \mathrm{H}^{\cdot}\Gamma(X, \tilde{\mathcal{A}}_X^{\cdot})$.

Consider the scheme $Y := X \times \Delta_{\mathbb{Q}}^1$, with projection morphism $s = s^0\colon Y \to X$ and two sections $\partial^0, \partial^1\colon X \to Y$ (cf. Example 1.5 for the notation). Since $s_*\Omega_{Y/k}^{\cdot} \cong \Omega_{X/k}^{\cdot} \otimes_{\mathbb{Q}} \Omega^{\cdot}(\Delta_{\mathbb{Q}}^1)$ and $\mathbb{Q} \to \Omega^{\cdot}(\Delta_{\mathbb{Q}}^1)$ is a quasi-isomorphism (Poincaré Lemma), we see that $s^*\colon \Omega_{X/k}^{\cdot} \to s_*\Omega_{Y/k}^{\cdot}$ is a quasi-isomorphism of sheaves on X. Because s is an affine morphism the sheaves $\Omega_{Y/k}^p$ are acyclic for s_*, and it follows that $s_*\Omega_{Y/k}^{\cdot} \to s_*\mathcal{A}_Y^{\cdot}$ is a quasi-isomorphism. We conclude (cf. Proposition 2.18) that $s^*\colon \mathcal{A}_X^{\cdot} \to s_*\mathcal{A}_Y^{\cdot}$ is a quasi-isomorphism. Passing to global cohomology we

also get $\mathrm{H}^{\cdot}(s^*)$: $\mathrm{H}^{\cdot}\Gamma(X, \mathcal{A}_X^{\cdot}) \xrightarrow{\simeq} \mathrm{H}^{\cdot}\Gamma(Y, \mathcal{A}_Y^{\cdot})$. Therefore

$$\mathrm{H}^{\cdot}(\partial^{0*}) = \mathrm{H}^{\cdot}(\partial^{1*})\colon\ \mathrm{H}^{\cdot}\Gamma(Y, \tilde{\mathcal{A}}_Y^{\cdot}) \xrightarrow{\simeq} \mathrm{H}^{\cdot}\Gamma(X, \tilde{\mathcal{A}}_X^{\cdot}) \tag{3.8}$$

with inverse $\mathrm{H}^{\cdot}(s^*)$.

Let $\mathcal{E}_Y := s^*\mathcal{E}$, with two induced adelic connections $s^*\nabla$ and $s^*\nabla'$. Define the mixed adelic connection

$$\nabla_Y := t_0 s^*\nabla + t_1 s^*\nabla'$$

on $\mathcal{E}_Y$, with curvature R_Y. Now $\partial^{0*}(t_0) = 0$, so

$$\partial^{0*}\nabla_Y = \partial^{0*}(t_0 s^*\nabla) + \partial^{0*}(t_1 s^*\nabla') = \nabla'$$

as connections on $\mathcal{E}$. Therefore $\partial^{0*}(P(R_Y)) = P(R')$ and likewise $\partial^{1*}(P(R_Y)) = P(R)$. Finally use (3.8). □

Next we show how to construct adelic connections.

Recall that to every chain $\xi = (x_0, \ldots, x_l)$ of length l there is attached a DGA

$$\tilde{\mathcal{A}}_\xi^{\cdot} = \Omega^{\cdot}(\Delta^l_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \Omega^{\cdot}_{X/k,\xi}$$

(cf. formula (2.14)). Set $\tilde{\mathcal{A}}_\xi^i(\mathcal{E}) := \tilde{\mathcal{A}}_\xi^i \otimes_{\mathcal{O}_X} \mathcal{E}$, so $\tilde{\mathcal{A}}_\xi^0(\mathcal{E})$ is a free $\tilde{\mathcal{A}}_\xi^0$-module of rank r. If $l = 0$ and $\xi = (x)$ then $\tilde{\mathcal{A}}_{(x)}^{\cdot} = \Omega^{\cdot}_{X/k,(x)}$ and $\tilde{\mathcal{A}}_{(x)}^0 = \mathcal{O}_{X,(x)} = \widehat{\mathcal{O}}_{X,x}$, the complete local ring. For $0 \le i \le l$ there is a DGA homomorphism

$$\tilde{\mathcal{A}}_{(x_i)}^{\cdot} = \Omega^{\cdot}_{X/k,(x_i)} \xrightarrow{\sigma_i} \Omega^{\cdot}_{X/k,\xi} \subset \tilde{\mathcal{A}}_\xi^{\cdot}$$

(cf. proof of Lemma 2.17).

Suppose we are given a set $\{\nabla_{(x)}\}_{x\in X}$ where for each point x

$$\nabla_{(x)}\colon\ \mathcal{E}_{(x)} \to \Omega^1_{X/k,(x)} \otimes_{\mathcal{O}_{X,(x)}} \mathcal{E}_{(x)} \tag{3.9}$$

is a connection over $\mathcal{O}_{X,(x)}$. Since $\tilde{\mathcal{A}}_\xi^i(\mathcal{E}) \cong \tilde{\mathcal{A}}_\xi^i \otimes_{\mathcal{O}_{X,(x)}} \mathcal{E}_{(x)}$, each connection $\nabla_{(x_i)}$ induces, by extension of scalars, a connection

$$\nabla_{\xi,i}\colon\ \tilde{\mathcal{A}}_\xi^0(\mathcal{E}) \to \tilde{\mathcal{A}}_\xi^1(\mathcal{E})$$

over the algebra $\tilde{\mathcal{A}}_\xi^0$. Define the "mixed" connection

$$\nabla_\xi := \sum_{i=0}^{l} t_i \nabla_{\xi,i}\colon\ \tilde{\mathcal{A}}_\xi^0(\mathcal{E}) \to \tilde{\mathcal{A}}_\xi^1(\mathcal{E}). \tag{3.10}$$

PROPOSITION 3.11. *Given a set of connections* $\{\nabla_{(x)}\}_{x\in X}$ *as above, there is a unique adelic connection* ∇ *on* $\mathcal{E}$*, such that under the embedding*

$$\Phi_{\mathcal{E}}\colon \Gamma(U, \tilde{\mathcal{A}}_X^{\cdot}(\mathcal{E})) \subset \prod_{\xi\in S(U)} \tilde{\mathcal{A}}_\xi^{\cdot}(\mathcal{E})$$

of Lemma 2.16*, one has* $\nabla e = (\nabla_\xi e_\xi)$ *for every local section* $e = (e_\xi) \in \Gamma(U, \tilde{\mathcal{A}}_X^0(\mathcal{E}))$*. Moreover,* $\nabla(\mathcal{E}) \subset \tilde{\mathcal{A}}_X^{1,0}(\mathcal{E})$*.*

Proof. The product

$$\nabla := \prod_\xi \nabla_\xi\colon \prod_{\xi\in S(X)} \tilde{\mathcal{A}}_\xi^0(\mathcal{E}) \to \prod_{\xi\in S(X)} \tilde{\mathcal{A}}_\xi^1(\mathcal{E})$$

is a connection over the algebra $\prod_\xi \tilde{\mathcal{A}}_\xi^0$. Since $\Phi_{\mathcal{E}}$ is injective and Φ is a DGA homomorphism, it suffices to show that $\nabla e \in \tilde{\mathcal{A}}_X^1(\mathcal{E})$ for every local section $e \in \tilde{\mathcal{A}}_X^0(\mathcal{E})$.

First consider a local section $e \in \mathcal{E}$. For every point x,

$$\nabla_{(x)} e \in \left(\Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{E}\right)_{(x)}.$$

Therefore, writing $\nabla_l := \prod_{\xi\in S(X)_l} \nabla_\xi$, we see that

$$\nabla_0 e \in \underline{\mathbb{A}}^0(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{E} \cong \underline{\mathbb{A}}^0(\Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{E}).$$

According to Lemma 2.17 we get a section

$$\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots) \in \tilde{\mathcal{A}}_X^0(\Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{E}) \cong \tilde{\mathcal{A}}_X^{1,0}(\mathcal{E})$$

with

$$\alpha_l = \sum_{i=0}^{l} t_i \otimes \sigma_i(\nabla_0 e) = \nabla_l e,$$

so $\boldsymbol{\alpha} = \nabla e$.

Finally, any section of $\tilde{\mathcal{A}}_X^0(\mathcal{E}) \cong \tilde{\mathcal{A}}_X^0 \otimes_{\mathcal{O}_X} \mathcal{E}$ is locally a sum of tensors $a \otimes e$ with $a \in \tilde{\mathcal{A}}_X^0$ and $e \in \mathcal{E}$, so by the Leibniz rule

$$\nabla(a \otimes e) = \mathrm{D}a \otimes e + a\nabla e \in \tilde{\mathcal{A}}_X^1(\mathcal{E}). \qquad \square$$

Observe that relative to a local algebraic frame $\boldsymbol{f}$ for $\mathcal{E}$, the matrix of a connection ∇ as in the proposition has entries in $\tilde{\mathcal{A}}_X^{1,0}$.

A *global adelic frame* for $\mathcal{E}$ is a family $\boldsymbol{e} = \{\boldsymbol{e}_{(x)}\}_{x \in X}$, where for each $x \in X$, $\boldsymbol{e}_{(x)}\colon \mathcal{O}_{X,(x)}^r \xrightarrow{\simeq} \mathcal{E}_{(x)}$ is a frame. In other words this is an isomorphism $\boldsymbol{e}\colon \underline{\mathbb{A}}^0(\mathcal{O}_X)^r \xrightarrow{\simeq} \underline{\mathbb{A}}^0(\mathcal{E})$ of $\underline{\mathbb{A}}^0(\mathcal{O}_X)$-modules.

The next corollary is inspired by the work of Parshin [Pa].

COROLLARY 3.12. *A global adelic frame $\boldsymbol{e}$ of $\mathcal{E}$ determines an adelic connection ∇.*

Proof. The frame $\boldsymbol{e}_{(x)}$ determines a Levi-Civita connection $\nabla_{(x)}$ on $\mathcal{E}_{(x)}$. Now use Proposition 3.11. □

We call such an connection *pointwise trivial*. In Sections 5 and 7 we shall only work with pointwise trivial connections.

Given a local section $\alpha \in \tilde{\mathcal{A}}_X^{\cdot}(\mathcal{M})$ we write $\alpha = \sum \alpha^{p,q}$ with $\alpha^{p,q} \in \tilde{\mathcal{A}}_X^{p,q}(\mathcal{M})$. For a chain ξ we write α_ξ for the ξ component of $\Phi_{\mathcal{M}}(\alpha)$ (see Lemma 2.16).

LEMMA 3.13. *Let ∇ be the pointwise trivial connection on $\mathcal{E}$ determined by an adelic frame $\boldsymbol{e}$. Let $\xi = (x_0, \ldots, x_l)$ be a chain, and let $\boldsymbol{f}$ be any frame of $\mathcal{E}_\xi$. Write $\boldsymbol{e}_{(x_i)} = \boldsymbol{g}_i \cdot \boldsymbol{f}$ for matrices $\boldsymbol{g}_i \in \mathrm{Gl}_r(\mathcal{O}_{X,\xi})$, $0 \le i \le l$. Then:*

(1) *The connection matrix of ∇_ξ w.r.t. the frame $\boldsymbol{f}$ is $\boldsymbol{\theta} = -\sum t_i \boldsymbol{g}_i^{-1} \mathrm{d}\boldsymbol{g}_i$.*

(2) *Let $\Theta^{1,1}$ be the matrix of the curvature form $R_\xi^{1,1}$ w.r.t. the frame $\boldsymbol{f} \otimes \boldsymbol{f}^\vee$. Then*

$$\Theta^{1,1} = -\sum \mathrm{d}t_i \wedge \boldsymbol{g}_i^{-1} \mathrm{d}\boldsymbol{g}_i.$$

Proof. Direct calculation. □

Definition 3.14. The ith Chern forms of $\mathcal{E}$ with respect to the adelic connection ∇ are

$$\begin{aligned} \tilde{c}_i(\mathcal{E}, \nabla) &:= P_i(R) \in \Gamma(X, \tilde{\mathcal{A}}_X^{2i}) \\ c_i(\mathcal{E}, \nabla) &:= \int_\Delta P_i(R) \in \Gamma(X, \mathcal{A}_X^{2i}). \end{aligned}$$

Let t be an indeterminate, and define $P_t := \sum_{i=1}^r P_i t^i \in I_r(k)[t]$.

PROPOSITION 3.15. (Whitney Sum Formula) *Let X be a finite type k-scheme. Suppose $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is a short exact sequence of locally free $\mathcal{O}_X$-modules. Then there exist adelic connections $\nabla', \nabla, \nabla''$ on $\mathcal{E}', \mathcal{E}, \mathcal{E}''$ respectively, with corresponding curvature forms R', R, R'', s.t.*

$$P_t(R) = P_t(R') \cdot P_t(R'') \in \Gamma(X, \tilde{\mathcal{A}}_X^{\cdot})[t].$$

Proof. For any point $x \in X$ choose a splitting $\sigma_{(x)}\colon \mathcal{E}''_{(x)} \to \mathcal{E}_{(x)}$ of the sequence, as modules over $\mathcal{O}_{X,(x)}$. Also choose frames $\boldsymbol{e}'_{(x)}$, $\boldsymbol{e}''_{(x)}$ for $\mathcal{E}'_{(x)}$, $\mathcal{E}''_{(x)}$ respectively.

Let $\boldsymbol{e}_{(x)} := (\boldsymbol{e}'_{(x)}, \sigma_{(x)}(\boldsymbol{e}''_{(x)}))$ be the resulting frame of $\mathcal{E}_{(x)}$. Use the global adelic frame $\boldsymbol{e} = \{\boldsymbol{e}_{(x)}\}$ to define an adelic connection ∇ on $\mathcal{E}$; and likewise define ∇' and ∇''.

In order to check that $P_t(R) = P_t(R') \cdot P_t(R'')$ it suffices, according to Lemma 2.16, to look separately at each chain $\xi = (x_0, \ldots, x_q)$. Let $\boldsymbol{g}_i \in \mathrm{Gl}_r(\mathcal{O}_{X,\xi})$ be the transition matrix $\boldsymbol{e}_{(x_i)} = \boldsymbol{g}_i \cdot \boldsymbol{e}_{(x_q)}$. Because of our special choice of frames the initial segment of each frame $\boldsymbol{e}_{(x_i)}$ is a frame for the submodule $\mathcal{E}'_\xi \subset \mathcal{E}_\xi$, which implies that $\boldsymbol{g}_i = \begin{pmatrix} \boldsymbol{g}'_i & * \\ 0 & \boldsymbol{g}''_i \end{pmatrix}$, where $\boldsymbol{g}'_i, \boldsymbol{g}''_i$ are the obvious transition matrices.

Now with respect to the frame $\boldsymbol{e}_{(x_q)}$ of $\mathcal{E}_\xi$, the connection matrix of $\nabla_{(x_i)}$ is $\boldsymbol{\theta}_i = -\boldsymbol{g}_i^{-1}\mathrm{d}\boldsymbol{g}_i$, so the matrices of ∇_ξ and R_ξ are

$$\begin{aligned} \boldsymbol{\theta} &= -(t_0\boldsymbol{g}_0^{-1}\mathrm{d}\boldsymbol{g}_0 + \cdots + t_{q-1}\boldsymbol{g}_{q-1}^{-1}\mathrm{d}\boldsymbol{g}_{q-1}) \\ \Theta &= \mathrm{D}\boldsymbol{\theta} - \boldsymbol{\theta} \wedge \boldsymbol{\theta}. \end{aligned}$$

It follows that $\Theta = \begin{pmatrix} \Theta' & * \\ 0 & \Theta'' \end{pmatrix}$. By linear algebra we conclude that $P_t(\Theta) = P_t(\Theta') \cdot P_t(\Theta'')$. □

PROPOSITION 3.16. *If $\mathcal{E}$ is an invertible $\mathcal{O}_X$-module and ∇ is an adelic connection on it, then the differential logarithm*

$$\mathrm{dlog}\colon\ \mathrm{Pic}\, X = \mathrm{H}^1(X, \mathcal{O}_X^*) \to \mathrm{H}^2(X, \Omega^{\cdot}_{X/k}) \cong \mathrm{H}^2(X, \mathcal{A}^{\cdot}_X)$$

sends

$$\mathrm{dlog}\,([\mathcal{E}]) = [c_1(\mathcal{E}; \nabla)].$$

Proof. (Cf. [HY] Proposition 2.6.) Suppose $\{U_i\}$ is a finite open cover of X s.t. $\mathcal{E}|_{U_i}$ is trivial with frame e_i. Let $g_{i,j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$ satisfy $e_i = g_{i,j}e_j$. Then the Čech cocycle $\{g_{i,j}\} \in C^1(\{U_i\}; \mathcal{O}_X^*)$ represents $[\mathcal{E}]$.

Choose a global adelic frame $\{e_{(x)}\}$ for $\mathcal{E}$ and let ∇ be the connection it determines. For a chain (x, y) let $g_{(x,y)} \in \mathcal{O}^*_{X,(x,y)}$ satisfy $e_{(x)} = g_{(x,y)}e_{(y)}$. By Lemma 3.13 we see that $c_1(\mathcal{E}; \nabla) = \{\mathrm{dlog}\, g_{(x,y)}\} \in \Gamma(X, \mathcal{A}^{1,1}_X)$.

For any point $x \in U_i$ define $g_{i,(x)} \in \mathcal{O}^*_{X,(x)}$ in the obvious way. Then $\{\mathrm{dlog}\, g_{i,(x)}\} \in C^0(\{U_i\}; \mathcal{A}^{1,0}_X)$, and

$$\mathrm{D}\{\mathrm{dlog}\, g_{i,(x)}\} = \{\mathrm{dlog}\, g_{i,j}\} - \{\mathrm{dlog}\, g_{(x,y)}\}.$$

Since $\Gamma(X, \mathcal{A}^{\cdot}_X) \to C^{\cdot}(\{U_i\}; \mathcal{A}^{\cdot}_X)$ is a quasi-isomorphism we are done. □

THEOREM 3.17. *Suppose X is smooth over k, so that* $\mathrm{H}^{\cdot}\Gamma(X, \tilde{\mathcal{A}}^{\cdot}_X) = \mathrm{H}^{\cdot}_{\mathrm{DR}}(X)$.

Then the Chern classes

$$c_i(\mathcal{E}) := [\tilde{c}_i(\mathcal{E}, \nabla)] \in \mathrm{H}^{2i}_{\mathrm{DR}}(X)$$

coincide with the usual ones.

Proof. By Theorem 3.7 and Propositions 3.6, 3.15 and 3.16 we see that the axioms of Chern classes (cf. [Ha2] Appendix A) are satisfied. □

Example 3.18. Consider the projective line $\mathbf{P} = \mathbf{P}^1_k$ and the sheaf $\mathcal{O}_{\mathbf{P}}(1)$. Let $v \in \Gamma(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1))$ have a zero at the point z. Define a global adelic frame $\{e_{(x)}\}$ by $e_{(x)} = v$ if $x \neq z$, and $e_{(z)} = w$, any basis of $\mathcal{O}_{\mathbf{P}}(1)_{(z)}$. So $v = aw$ for some regular parameter $a \in \mathcal{O}_{\mathbf{P},(z)}$. The local components of the Chern form $c_1(\mathcal{O}_{\mathbf{P}}(1); \nabla)$ are 0 unless $\xi = (z_0, z)$ (z_0 is the generic point), where we get $c_1(\mathcal{O}_{\mathbf{P}}(1); \nabla)_\xi = a^{-1}\mathrm{d}a$.

An algebraic connection on $\mathcal{E}$ is a connection $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/k}$. The connection ∇ is trivial if $(\mathcal{E}, \nabla) \cong (\mathcal{O}_X, \mathrm{d})^r$. ∇ is generically trivial if it is trivial on a dense open set. The next proposition explores the relation between adelic and algebraic connections.

PROPOSITION 3.19. *Assume X is smooth irreducible and k is algebraically closed. Let ∇ be an integrable adelic connection on $\mathcal{E}$.*

(1) *If $\nabla(\mathcal{E}) \subset \tilde{\mathcal{A}}^{1,0}_X(\mathcal{E})$ then ∇ is algebraic.*

(2) *If ∇ is algebraic and generically trivial then it is trivial.*

Proof. (1) By Lemma 2.15 (1) with $\mathcal{M} = \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{E}$, it suffices to prove that $\mathrm{D}''\nabla(\mathcal{E}) = 0$, which is a local statement. So choose a local algebraic frame $\boldsymbol{f}$ of $\mathcal{E}$ on some open set. Then we have a connection matrix $\boldsymbol{\theta}$ which is homogeneous of bidegree $(1, 0)$, and by assumption the curvature matrix $\Theta = \mathrm{D}\boldsymbol{\theta} - \boldsymbol{\theta} \cdot \boldsymbol{\theta}$ is zero. But since $\Theta^{1,1} = \mathrm{D}''\boldsymbol{\theta}$ we are done.

(2) The algebraic connection ∇ extends uniquely to an adelic connection with the same name (by Proposition 3.11). Let x_0 be the generic point, so by assumption we have a frame $\boldsymbol{e}_{(x_0)}$ for $\mathcal{E}_{(x_0)}$ which trivializes ∇. Now take any closed point x_1, so $\mathcal{O}_{X,(x_1)} \cong k[[t_1, \ldots, t_n]]$. It is well known that there is a frame $\boldsymbol{e}_{(x_1)}$ for $\mathcal{E}_{(x_1)}$ which trivializes ∇ (cf. [Ka]). Consider the chain $\xi = (x_0, x_1)$. With respect to the frame $\boldsymbol{e}_{(x_0)}$, the connection matrix of ∇_ξ is $\boldsymbol{\theta}_\xi = -t_1 \boldsymbol{g}^{-1}\mathrm{d}\boldsymbol{g}$, where $\boldsymbol{e}_{(x_1)} = \boldsymbol{g} \cdot \boldsymbol{e}_{(x_0)}$ and $\boldsymbol{g} \in \mathrm{Gl}_r(\mathcal{O}_{X,\xi})$.

Since ∇ is integrable we get

$$-\mathrm{d}t_1 \cdot \boldsymbol{g}^{-1}\mathrm{d}\boldsymbol{g} = \mathrm{D}''\boldsymbol{\theta}_\xi = \Theta^{1,1} = 0.$$

We conclude that

$$\mathrm{d}\boldsymbol{g} = 0 \in \mathrm{M}_r(\Omega^1_{X/k,\xi}).$$

But because X is smooth and k is algebraically closed, it follows that $\mathrm{H}^0\Omega^{\cdot}_{X/k,\xi} \subset \mathrm{H}^0\Omega^{\cdot}_{X/k,\eta} = k$, where η is any maximal chain containing ξ. So in fact $\boldsymbol{g} \in \mathrm{Gl}_r(k)$, and by faithful flatness we get

$$\boldsymbol{e}_{(x_0)} = \boldsymbol{g}^{-1} \cdot \boldsymbol{e}_{(x_1)} \in \mathcal{E}_{(x_1)} \cap \mathcal{E}_{(x_0)} = \mathcal{E}_{x_1}.$$

By going over all closed points x_1 we see that $\boldsymbol{e}_{(x_0)} \in \Gamma(X, \mathcal{E})$, which trivializes ∇. □

There do however exist integrable adelic connections which are not algebraic.

Example 3.20. Let X be any scheme of positive dimension, and let $\tilde{a} \in \Gamma(X, \tilde{\mathcal{A}}^0_X)$ be any element satisfying $\mathrm{D}''\mathrm{D}\tilde{a} \neq 0$. For instance, take a fixed point x_0 and an element $a_{(x_0)} \in \mathcal{O}_{X,(x_0)}$ s.t. $\mathrm{d}a_{(x_0)} \neq 0$. For any $x \neq x_0$ set $a_{(x)} := 0 \in \mathcal{O}_{X,(x)}$. Then $\{a_{(x)}\} \in \Gamma(X, \underline{\mathbb{A}}^0(\mathcal{O}_X))$, and by Lemma 2.17 we get $\tilde{a} \in \Gamma(X, \tilde{\mathcal{A}}^0_X)$. Now $\mathrm{D}''\mathrm{D}\tilde{a} = \mathrm{D}''\mathrm{D}'\tilde{a}$, and clearly $\mathrm{D}'\tilde{a}$ is not algebraic.

Take $\mathcal{E} = \mathcal{O}_X^2$. The matrix

$$\boldsymbol{e} = \begin{pmatrix} 1 & \tilde{a} \\ 0 & 1 \end{pmatrix} \in \mathrm{M}_2(\Gamma(X, \tilde{\mathcal{A}}^0_X))$$

is invertible, and we consider $\boldsymbol{e}$ as a frame for $\tilde{\mathcal{A}}^0_X(\mathcal{E})$. Define ∇ to be the Levi-Civita connection for $\boldsymbol{e}$. If we now consider the algebraic frame $\boldsymbol{f} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ of $\mathcal{E}$, then the connection matrix with respect to $\boldsymbol{f}$ is

$$-\boldsymbol{e}^{-1} \cdot \mathrm{D}\boldsymbol{e} = -\begin{pmatrix} 0 & \mathrm{D}\tilde{a} \\ 0 & 0 \end{pmatrix}.$$

So $\mathrm{D}''\nabla(\mathcal{E}) \neq 0$ and hence ∇ is not algebraic.

Question 3.21. Does there exist an adelic connection ∇ with curvature form R homogeneous of bidegree $(1, 1)$?

Remark 3.22. Theorem 3.7 works just as well for a relative situation: Y is a finite type k-scheme and $f\colon X \to Y$ is a finite type morphism. Then we can define $\mathcal{A}^{\cdot}_{X/Y} := \mathrm{N}\underline{\mathbb{A}}(\Omega^{\cdot}_{X/Y})$ and likewise $\tilde{\mathcal{A}}^{\cdot}_{X/Y}$. There are relative adelic connections on any locally free $\mathcal{O}_X$-module $\mathcal{E}$, and there is a Chern-Weil homomorphism

$$w_{\mathcal{E}}\colon\ I_r(k) \to \mathrm{H}^{\cdot} f_* \mathcal{A}^{\cdot}_{X/Y} \cong \mathrm{H}^{\cdot}\mathrm{R}f_*\Omega^{\cdot}_{X/Y}.$$

Remark 3.23. In [Du] a very similar construction is carried out to calculate characteristic classes of principal G-bundles, for a Lie group G. These classes are in the cohomology of the classifying space BG, which coincides with the cohomology of the simplicial manifold NG.

Remark 3.24. Suppose X is any finite type scheme over k. Then $R \in \Gamma(X, \tilde{\mathcal{A}}_X^{\cdot})$ is nilpotent and we may define

$$\tilde{\mathrm{ch}}(\mathcal{E}; \nabla) := \operatorname{tr} \exp R \in \Gamma(X, \tilde{\mathcal{A}}_X^{\cdot}).$$

Using the idea of the proof of Proposition 3.15 one can show that given a bounded complex $\mathcal{E}^{\cdot}$ of locally free sheaves, which is acyclic on an open set U, it is possible to find connections ∇_i on $\mathcal{E}_i$ s.t. $\sum_i (-1)^i \tilde{\mathrm{ch}}(\mathcal{E}_i; \nabla_i) = 0$ on U. In particular when $U = X$ we get a ring homomorphism

$$\mathrm{ch}\colon\ \mathrm{K}^0(X) \to \mathrm{H}^{\cdot}\Gamma(X, \tilde{\mathcal{A}}_X^{\cdot}) \cong \mathrm{H}^{\cdot}(X, \Omega_{X/k}^{\cdot}),$$

the *Chern character*. When X is smooth this is the usual Chern character into $\mathrm{H}_{\mathrm{DR}}^{\cdot}(X)$.

4. Secondary characteristic classes. Let k be a field of characteristic 0. In [BE], Bloch and Esnault show that given a locally free sheaf $\mathcal{E}$ on a smooth k-scheme X, an algebraic connection $\nabla\colon\ \mathcal{E} \to \Omega_{X/k}^1 \otimes_{\mathcal{O}_X} \mathcal{E}$ and an invariant polynomial $P \in I_r(k)$ of degree $m \geq 2$, there is a Chern-Simons class

$$\mathrm{T}P(\mathcal{E}; \nabla) \in \Gamma\left(X, \Omega_{X/k}^{2m-1}/\mathrm{d}(\Omega_{X/k}^{2m-2})\right)$$

satisfying

$$\mathrm{dT}P(\mathcal{E}; \nabla) = P(\mathcal{E}) \in \mathrm{H}_{\mathrm{DR}}^{2m}(X).$$

$\mathrm{T}P(\mathcal{E}; \nabla)$ is called the secondary, or Chern-Simons, characteristic class. The notation we use is taken from [Es]; the original notation in [BE] is $w_m(\mathcal{E}, \nabla, P)$.

Such algebraic connections exist when X is affine. However the authors of [BE] point out that any quasi-projective scheme X admits a vector bundle whose total space X' is affine, and then $\mathrm{H}_{\mathrm{DR}}^{\cdot}(X) \to \mathrm{H}_{\mathrm{DR}}^{\cdot}(X')$ is an isomorphism.

In Section 3 we proved that adelic connections always exist. In this section we define adelic Chern-Simons classes, which are global sections of sheaves on X itself:

THEOREM 4.1. *Suppose X is a smooth k-scheme, $\mathcal{E}$ a locally free $\mathcal{O}_X$-module, ∇ an adelic connection on $\mathcal{E}$ and $P \in I_r(k)$ homogeneous of degree $m \geq 2$. Then there is a class*

$$\mathrm{T}P(\mathcal{E}; \nabla) \in \Gamma\left(X, \mathcal{A}_X^{2m-1}/\mathrm{D}(\mathcal{A}_X^{2m-2})\right)$$

satisfying

$$\mathrm{DT}P(\mathcal{E};\nabla) = P(\mathcal{E}) \in \mathrm{H}^{2m}_{\mathrm{DR}}(X)$$

and commuting with pullbacks for morphisms of schemes $X' \to X$.

The proof is later in this section, after some preparation.

According to [BE] Theorem 2.2.1, for any commutative k-algebra B, invariant polynomial $P \in I_r(k)$ homogeneous of degree m and matrix $\boldsymbol{\theta} \in \mathrm{M}_r(\Omega^1_{B/k})$, there is a differential form $\mathrm{T}P(\boldsymbol{\theta}) \in \Omega^{2m-1}_{B/k}$. $\mathrm{T}P(\boldsymbol{\theta})$ is functorial on k-algebras, and satisfies

$$\mathrm{dT}P(\boldsymbol{\theta}) = P(\Theta) \in \Omega^{2m}_{B/k}, \tag{4.2}$$

where $\Theta = \mathrm{d}\boldsymbol{\theta} - \boldsymbol{\theta}\cdot\boldsymbol{\theta}$.

We shall need a slight generalization of [BE] Proposition 2.2.2. Consider M_r and GL_r as schemes over k. There is a universal invertible matrix

$$\boldsymbol{g} = \boldsymbol{g}_{\mathrm{u}} \in \Gamma(\mathrm{GL}_r, \mathrm{GL}_r(\mathcal{O}_{\mathrm{GL}_r})).$$

For an integer N there is a universal connection matrix

$$\boldsymbol{\theta} = \boldsymbol{\theta}_{\mathrm{u}} \in \Gamma(Y, \mathrm{M}_r(\Omega^1_{Y/k})),$$

where $Y := \mathrm{M}_r \times \mathbf{A}^N = \mathrm{Spec}\,[\boldsymbol{a}_{\mathrm{u}}, \boldsymbol{b}_{\mathrm{u}}]$ for a collection of indeterminates $\boldsymbol{a}_{\mathrm{u}} = \{a_p\}$ and $\boldsymbol{b}_{\mathrm{u}} = \{b_{i,j,p}\}$, $1 \le p \le N$, $1 \le i,j \le r$. The matrix is of course $\boldsymbol{\theta}_{\mathrm{u}} = (\theta_{i,j})$ with $\theta_{i,j} = \sum_p b_{i,j,p}\mathrm{d}a_p$. Then we get by pullback matrices $\boldsymbol{g}$ and $\boldsymbol{\theta}$ on $\mathrm{GL}_r \times Y$.

LEMMA 4.3. *Given an invariant polynomial P there is an open cover* $\mathrm{GL}_r = \bigcup U_i$ *and forms* $\beta_i = \beta_{\mathrm{u},i} \in \Gamma(U_i \times Y, \Omega^{2m-2}_{U_i\times Y})$ *s.t.*

$$\left(\mathrm{T}P(\boldsymbol{\theta}) - \mathrm{T}P(\mathrm{d}\boldsymbol{g}\cdot\boldsymbol{g}^{-1} + \boldsymbol{g}\cdot\boldsymbol{\theta}\cdot\boldsymbol{g}^{-1})\right)|_{U_i\times Y} = \mathrm{d}\beta_i.$$

Proof. Write

$$\alpha = \alpha_{\mathrm{u}} := \mathrm{T}P(\boldsymbol{\theta}) - \mathrm{T}P(\mathrm{d}\boldsymbol{g}\cdot\boldsymbol{g}^{-1} + \boldsymbol{g}\cdot\boldsymbol{\theta}\cdot\boldsymbol{g}^{-1}) \in \Gamma(\mathrm{GL}_r \times Y, \Omega^{2m-1}_{\mathrm{GL}_r\times Y/k}).$$

It is known that $\mathrm{d}\alpha = 0$. Let s: $\mathrm{GL}_r \to \mathrm{GL}_r \times Y$ correspond to any k-rational point of Y. Choose a point $x \in \mathrm{GL}_r$. By [BE] Proposition 2.2.2 there is an affine open neighborhood V of $s(x)$ in $\mathrm{GL}_r \times Y$ and a form $\beta' \in \Gamma(V, \Omega^{2m-2}_{\mathrm{GL}_r\times Y/k})$, s.t. $\alpha|_V = \mathrm{d}\beta'$. Define $U := s^{-1}(V)$, so $s^*(\alpha)|_U = \mathrm{d}\,s^*(\beta') \in \Gamma(U, \Omega^{2m-2}_{\mathrm{GL}_r/k})$. Since $\mathrm{H}(s^*)$: $\mathrm{H}^{\cdot}_{\mathrm{DR}}(U \times Y) \to \mathrm{H}^{\cdot}_{\mathrm{DR}}(U)$ is an isomorphism, it follows that there is some $\beta \in \Gamma(U \times Y, \Omega^{2m-2}_{U\times Y/k})$ with $\alpha|_{U\times Y} = \mathrm{d}\beta$. $\square$

Proof of the Theorem. Say $\dim X = n$. Let U be a sufficiently small affine open set of X s.t. $\mathrm{d}a_1, \ldots, \mathrm{d}a_n$ is an algebraic frame of $\Omega^1_{X/k}$, for some $a_1, \ldots, a_n \in \Gamma(U, \mathcal{O}_X)$; and there a local algebraic frame $\boldsymbol{f}$ for $\mathcal{E}$ on U.

We get an induced isomorphism $\boldsymbol{f}\colon (\tilde{\mathcal{A}}^0_U)^r \xrightarrow{\simeq} \tilde{\mathcal{A}}^0_X(\mathcal{E})|_U$. Let $\boldsymbol{\theta} = (\theta_{i,j}) \in \mathrm{M}_r(\Gamma(U, \tilde{\mathcal{A}}^1_U))$ be the connection matrix of ∇ with respect to $\boldsymbol{f}$. Define the commutative DGAs

$$A_l := \Omega^{\cdot}(\Delta^l_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \Gamma(U, \underline{\mathbb{A}}^l(\Omega^{\cdot}_{X/k})).$$

Then by Definition 1.7, $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2, \ldots)$ where $\boldsymbol{\theta}_l \in \mathrm{M}_r(A^1_l)$, and the various matrices $\boldsymbol{\theta}_l$ have to satisfy certain simplicial compatibility conditions.

Fix an index l. We have $A^0_l = \mathcal{O}(\Delta^l_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \Gamma(U, \underline{\mathbb{A}}^l(\mathcal{O}_X))$ which contains $\Gamma(U, \mathcal{O}_X)$. Thus we may uniquely write $(\theta_{i,j})_l = \sum_{p=1}^n b_{i,j,p}\mathrm{d}a_p + \sum_{p=1}^l b_{i,j,p+n}\mathrm{d}t_p$, with $b_{i,j,p} \in A^0_l$. It follows that for $N = n + l$ and $Y = \mathrm{M}_r \times \mathbf{A}^N$ there is a unique k-algebra homomorphism $\phi_l\colon \Gamma(Y, \mathcal{O}_Y) \to A^0_l$, with $\phi_l(\boldsymbol{a}_{\mathrm{u}}, \boldsymbol{b}_{\mathrm{u}}) = (\boldsymbol{a}, \boldsymbol{t}, \boldsymbol{b})$. This extends to a DGA homomorphism $\phi_l\colon \Gamma(Y, \Omega^{\cdot}_{Y/k}) \to A_l$, and sends the universal connection $\boldsymbol{\theta}_{\mathrm{u}}$ to $\boldsymbol{\theta}_l$. Define $\mathrm{T}P(\boldsymbol{\theta}_l) := \phi_l(\mathrm{T}P(\boldsymbol{\theta}_{\mathrm{u}})) \in A_l$.

Because the homomorphisms ϕ_l are completely determined by the matrices $\boldsymbol{\theta}_l$ it follows that the forms $\mathrm{T}P(\boldsymbol{\theta}_l)$ satisfy the simplicial compatibilities. So there is an adelic form $\mathrm{T}P(\boldsymbol{\theta}) \in \Gamma(U, \tilde{\mathcal{A}}^{2m-1}_X)$.

Now let $\boldsymbol{f}' = \boldsymbol{g} \cdot \boldsymbol{f}$ be another local algebraic frame for $\mathcal{E}$ on U, with $\boldsymbol{g} \in \Gamma(U, \mathrm{GL}_r(\mathcal{O}_X))$. Fix l as before and write

$$\alpha_l := \mathrm{T}P(\boldsymbol{\theta}_l) - \mathrm{T}P(\mathrm{d}\boldsymbol{g} \cdot \boldsymbol{g}^{-1} + \boldsymbol{g} \cdot \boldsymbol{\theta}_l \cdot \boldsymbol{g}^{-1}) \in \Gamma(U, \tilde{\mathcal{A}}^{2m-1}_X).$$

Let $h\colon U \to \mathrm{GL}_r$ be the scheme morphism s.t. $h^*(\boldsymbol{g}_{\mathrm{u}}) = \boldsymbol{g}$. The k-algebra homomorphism

$$\psi_l = h^* \otimes \phi_l\colon \Gamma(\mathrm{GL}_r \times Y, \mathcal{O}_{\mathrm{GL}_r \times Y}) \to A^0_l$$

extends to a DGA homomorphism and $\alpha_l = \psi_l(\alpha_{\mathrm{u}})$, where α_{u} is the obvious universal form. By the lemma, for every i there is a form

$$\beta_{i,l} := \psi_l(\beta_{\mathrm{u},i}) \in \Omega^{\cdot}(\Delta^l_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \Gamma(h^{-1}(U), \underline{\mathbb{A}}^l(\Omega^{\cdot}_{X/k}))$$

of degree $2m - 2$.

Since we are not making choices to define $\beta_{i,l}$ it follows that the simplicial compatibilities hold, and so we obtain an adele $\beta_i \in \Gamma(h^{-1}(U), \tilde{\mathcal{A}}^{2m-2}_X)$, which evidently satisfies

$$\alpha|_{h^{-1}(U_i)} = \mathrm{D}\beta_i \in \Gamma(h^{-1}(U_i), \tilde{\mathcal{A}}^{2m-1}_X).$$

This means that the element

$$\mathrm{T}\tilde{P}(\mathcal{E};\nabla) := \mathrm{T}P(\boldsymbol{\theta}) \in \Gamma\left(U, \tilde{\mathcal{A}}_X^{2m-1}/\mathrm{D}(\tilde{\mathcal{A}}_X^{2m-1})\right)$$

is independent of the local algebraic frame $\boldsymbol{f}$, and therefore glues to a global section on X. Finally set $\mathrm{T}P(\mathcal{E};\nabla) := \int_\triangle \mathrm{T}\tilde{P}(\mathcal{E};\nabla)$. □

Some of the deeper results of [BE] deal with integrable algebraic connections. Denote by $\mathcal{H}^i_{\mathrm{DR}}$ the sheafified De Rham cohomology, i.e., the sheaf associated to the presheaf $U \mapsto \mathrm{H}^i_{\mathrm{DR}}(U)$. Then

$$\begin{aligned}\mathcal{H}^i_{\mathrm{DR}} &= \mathrm{H}^i\Omega^{\cdot}_{X/k} = \mathrm{Ker}\left(\frac{\Omega^i_{X/k}}{\mathrm{d}(\Omega^{i-1}_{X/k})} \xrightarrow{\mathrm{d}} \Omega^{i+1}_{X/k}\right)\\ &\cong \mathrm{H}^i\mathcal{A}^{\cdot}_X = \mathrm{Ker}\left(\frac{\mathcal{A}^i_X}{\mathrm{D}(\mathcal{A}^{i-1}_X)} \xrightarrow{\mathrm{D}} \mathcal{A}^{i+1}_X\right).\end{aligned}$$

Because of formula (4.2), we get an adelic generalization of [BE] Proposition 2.3.2:

PROPOSITION 4.4. *If the adelic connection* ∇ *is integrable then* $\mathrm{T}P(\mathcal{E};\nabla) \in \Gamma(X, \mathcal{H}^{2m-1}_{\mathrm{DR}})$.

The next question is an extension of Basic Question 0.3.1 of [BE].

Question 4.5. Are the classes $\mathrm{T}P(\mathcal{E};\nabla)$ all zero for an integrable adelic connection ∇?

5. The Bott Residue Formula. Let X be smooth n-dimensional projective variety over the field k ($\mathrm{char}\, k = 0$). Suppose $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of rank r and $P \in I_r(k)$ is a homogeneous polynomial of degree n. The problem is to calculate the Chern number $\int_X P(\mathcal{E}) \in k$, where $\int_X\colon \mathrm{H}^{2n}_{\mathrm{DR}}(X) \to k$ is the nondegenerate map defined e.g. in [Ha1].

Assume $v \in \Gamma(X, \mathcal{T}_X)$ is a vector field which acts on $\mathcal{E}$. By this we mean there is a DO $\Lambda\colon \mathcal{E} \to \Omega^1_{X/k} \otimes_{\mathcal{O}_X} \mathcal{E}$ satisfying $\Lambda(ae) = v(a) \otimes e + a \otimes \Lambda e$ for local sections $a \in \mathcal{O}_X$ and $e \in \mathcal{E}$. Furthermore assume the zero scheme Z of v is finite (but not necessarily reduced). Then we shall define a local invariant $P(v, \mathcal{E}, z) \in k$ for every zero $z \in Z$, explicitly in terms of local coordinates, in equation (5.3). Our result is:

THEOREM 5.1. (Bott Residue Formula)

$$\int_X P(\mathcal{E}) = \sum_{z \in Z} P(v, \mathcal{E}, z).$$

The proof appears later in this section, after some preparation. It follows the original proof of Bott [Bo1], but using algebraic residues and adeles instead of complex-analytic methods. This is made possible by Proposition 2.7 and Theorem 3.17. We show that a good choice of adelic connection ∇ on $\mathcal{E}$ enables one to localize the integral to the zero locus Z. This is quite distinct from the proof of the Bott Residue Formula in [CL], where classes in Hodge cohomology $\mathrm{H}^q(X, \Omega^p_{X/k})$ are considered, and integration is done using Grothendieck's global duality theory.

Let us first recall the local cohomology residue map

$$\mathrm{Res}_{\mathcal{O}_{X,(z)}/k}\colon\ \mathrm{H}^n_z(\Omega^n_{X/k}) \to k$$

of [Li] and [HK]. Choose local coordinates $f_1, \ldots, f_n$ at z, so $\mathcal{O}_{X,(z)} \cong k(z)[[f_1, \ldots, f_n]]$. Local cohomology classes are represented by generalized fractions. Given $a = \sum_{\boldsymbol{i}} b_{\boldsymbol{i}} \boldsymbol{f}^{\boldsymbol{i}} \in \mathcal{O}_{X,(z)}$ where $\boldsymbol{i} = (i_1, \ldots, i_n)$, $b_{\boldsymbol{i}} \in k(z)$ and $\boldsymbol{f}^{\boldsymbol{i}} = f_1^{i_1} \cdots f_n^{i_n}$, the residue is

$$\mathrm{Res}_{\mathcal{O}_{X,(z)}/k} \left[\begin{array}{c} a \cdot \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \\ f_1^{i_1}, \ldots, f_n^{i_n} \end{array} \right] = \mathrm{tr}_{k(z)/k}\,(b_{i_1-1,\ldots,i_n-1}) \in k. \tag{5.2}$$

Let $a_1, \ldots, a_n \in \mathcal{O}_X$ be the unique local sections near z satisfying $v = \sum a_i \frac{\partial}{\partial f_i}$. Then by definition

$$\mathcal{O}_{Z,z} \cong k(z)[[f_1, \ldots, f_n]]/(a_1, \ldots, a_n).$$

The DO Λ restricts to an $\mathcal{O}_Z$-linear endomorphism $\Lambda|_Z$ of $\mathcal{O}_Z \otimes_{\mathcal{O}_X} \mathcal{E}$, giving an element $P(\Lambda|_Z) \in \mathcal{O}_{Z,z}$. Choose any lifting P' of $P(\Lambda|_Z)$ to $\mathcal{O}_{X,(z)}$, and define

$$P(v, \mathcal{E}, z) := (-1)^{\binom{n+1}{2}} \mathrm{Res}_{\mathcal{O}_{X,(z)}/k} \left[\begin{array}{c} P' \cdot \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \\ a_1, \ldots, a_n \end{array} \right]. \tag{5.3}$$

The calculation of (5.3), given the a_i and P', is quite easy: first express these elements as power series in $\boldsymbol{f}$. The rules for manipulating generalized fractions are the same as for ordinary fractions, so the denominator can be brought to be $\boldsymbol{f}^{\boldsymbol{i}}$. Now use (5.2).

Example 5.4. Let $X := \mathbf{P}^1_k$ and $\mathcal{E} := \mathcal{O}_X(1)$. Let $\operatorname{Spec} k[f] \subset X$ be the complement of one point (infinity), and let z be the origin (i.e. $f(z) = 0$). We embed $\mathcal{E}$ in the function field $k(X)$ as the subsheaf of functions with at most one pole at z. Now $v := f^2 \frac{\partial}{\partial f} \in \Gamma(X, \mathcal{T}_X)$ is a global vector field, and we see that its action on $k(X)$ preserves $\mathcal{E}$. So the theorem applies with $\Lambda = v$. Here is the calculation: the zero scheme of v is $Z = \operatorname{Spec} k[f]/(a)$ with $a = f^2$. Since $\Lambda(f^{-1}) = f^2 \frac{\partial(f^{-1})}{\partial f} = -1$

we see that with $P = P_1$, $P(\Lambda) = -f$, and so

$$P(v, \mathcal{E}, z) := (-1)\operatorname{Res}_{k[[f]]/k}\begin{bmatrix} -f\mathrm{d}f \\ f^2 \end{bmatrix} = 1,$$

as it should be.

Example 5.5. If z is a simple zero (that is to say $\mathcal{O}_{Z,z} = k(z)$) we recover the familiar formula of Bott. Denote by $\operatorname{ad}(v)$ the adjoint action of v on $\mathcal{T}_X$. Then $\operatorname{ad}(v)|_z$ is an invertible $k(z)$-linear endomorphism of $\mathcal{T}_X|_{k(z)}$. Its matrix with respect to the frame $(\frac{\partial}{\partial f_1}, \dots, \frac{\partial}{\partial f_n})^{\mathrm{t}}$ is $-\left(\frac{\partial a_i}{\partial f_j}\right)^{\mathrm{t}}$, and (5.3) becomes

$$P(v, \mathcal{E}, z) = (-1)^{\binom{n}{2}}\operatorname{tr}_{k(z)/k}\left(\frac{P(\Lambda|_{k(z)})}{\det(\operatorname{ad}(v)|_{k(z)})}\right).$$

In the previous example we could have chosen $v := f\frac{\partial}{\partial f}$. This has 2 simple zeroes: z, where $a = f$ and $P(\Lambda) = -1$, so $P(v, \mathcal{E}, z) = 1$; and infinity, where $P(\Lambda) = 0$.

Let us start our proofs by showing that the local invariant is indeed independent of choices.

LEMMA 5.6. *$P(v, \mathcal{E}, z)$ is independent of the coordinate system $f_1, \dots, f_n$ and the lifting P'.*

Proof. Let $g_1, \dots, g_n$ be another system of coordinates, and write $v = \sum b_i \frac{\partial}{\partial g_i}$. Then we get $a_i = \sum \frac{\partial f_i}{\partial g_j} b_j$. The formulas for changing numerator and denominator in generalized fractions imply that the value of (5.3) remains the same when computed relative to $g_1, \dots, g_n$. □

Remark 5.7. According to [HK] Theorems 2.3 and 2.4 one has $P(v, \mathcal{E}, z) = (-1)^{\binom{n}{2}}\tau_a^f P(\Lambda|_Z)$, where $\tau_a^f\colon \mathcal{O}_{Z,z} \to k$ is the trace of Scheja-Storch [SS].

LEMMA 5.8. *There exists an open subset $U \subset X$ containing Z, and sections $f_1, \dots, f_n \in \Gamma(U, \mathcal{O}_X)$, such that the corresponding morphism $U \to \mathbf{A}_k^n$ is unramified (and even étale), and the fiber over the origin is the reduced scheme Z_{red}. Thus $\mathcal{T}_X|_U$ has a frame $(\frac{\partial}{\partial f_1}, \dots, \frac{\partial}{\partial f_n})^{\mathrm{t}}$. Moreover, we can choose U s.t. there is a frame $(e_1, \dots, e_r)^{\mathrm{t}}$ for $\mathcal{E}|_U$.*

Proof. As X is projective and Z finite we can certainly find an affine open set $U = \operatorname{Spec} R$ containing Z. For each point $z \in Z$ we can find sections $f_{1,z}, \dots, f_{n,z} \in R$ and $e_{1,z}, \dots, e_{r,z} \in \Gamma(U, \mathcal{E})$ which satisfy the requirements at z. Choose a "partition of unity of U to order 1 near Z," i.e., a set of functions $\{\epsilon_z\}_{z\in Z} \subset R$

representing the idempotents of $R/\sum_z \mathfrak{m}_z^2$. Then define $f_i := \sum_z \epsilon_z f_{i,z}$ and $e_i := \sum_z \epsilon_z e_{i,z}$, and shrink U sufficiently. □

From here we continue along the lines of [Bo1], but of course we use adeles instead of smooth functions. The sheaf $\tilde{\mathcal{A}}_X^{p,q}$ plays the role of the sheaf of smooth (p,q) forms on a complex manifold. The operator D'' behaves like the anti-holomorphic derivative $\bar{\partial}$; specifically $\mathrm{D}''\alpha = 0$ for any $\alpha \in \Omega^{\cdot}_{X/k}$.

Fix an open set U and sections $f_1, \ldots, f_n$ as in Lemma 5.8. Then we get an algebraic frame $(\frac{\partial}{\partial f_1}, \ldots, \frac{\partial}{\partial f_n})^{\mathrm{t}}$ of $\mathcal{T}_X|_U$, and we can write the vector field $v = \sum a_i \frac{\partial}{\partial f_i}$ with $a_i \in \Gamma(U, \mathcal{O}_X)$. Choose a global adelic frame $\{\boldsymbol{e}_{(x)}\}_{x \in X}$ for $\mathcal{E}$ as follows:

$$\begin{aligned} \boldsymbol{e}_{(x)} &= (e_1, \ldots, e_r)^{\mathrm{t}} && \text{if } x \in U \\ \boldsymbol{e}_{(x)} &= \text{ arbitrary} && \text{if } x \notin U. \end{aligned} \tag{5.9}$$

Then we get a family of connections $\{\nabla_{(x)}\}_{x \in X}$, and a global connection $\nabla\colon \tilde{\mathcal{A}}_X^0(\mathcal{E}) \to \tilde{\mathcal{A}}_X^1(\mathcal{E})$ over the algebra $\tilde{\mathcal{A}}_X^0$. The curvature form $R \in \tilde{\mathcal{A}}_X^2(\mathcal{E}nd(\mathcal{E}))$ decomposes into homogeneous parts $R = R^{2,0} + R^{1,1}$. Since $\tilde{\mathcal{A}}_X^{p,q} = 0$ for $p > n$, we get $P(R) = P(R^{1,1})$; we will work with $R^{1,1}$.

LEMMA 5.10. *Applying the $\mathcal{O}_X$-linear homomorphism* $\mathrm{D}''\colon \tilde{\mathcal{A}}_X^{p,q}(\mathcal{E}nd(\mathcal{E})) \to \tilde{\mathcal{A}}_X^{p,q+1}(\mathcal{E}nd(\mathcal{E}))$ *one has* $\mathrm{D}''R^{1,1} = 0$.

Proof. This is a local statement. Passing to matrices using a local algebraic frame, it is enough to prove that $\mathrm{D}''\Theta^{1,1} = 0$. Now $\Theta = \mathrm{D}\boldsymbol{\theta} - \boldsymbol{\theta} \wedge \boldsymbol{\theta}$, and $\boldsymbol{\theta} \in \mathrm{M}_r(\tilde{\mathcal{A}}_X^{1,0})$, so $\Theta^{1,1} = \mathrm{D}''\boldsymbol{\theta}$. But $(\mathrm{D}'')^2 = 0$. □

Denote the canonical pairing $\mathcal{T}_X \otimes_{\mathcal{O}_X} \Omega^1_{X/k} \to \mathcal{O}_X$ by $\langle -, - \rangle$. It extends to a bilinear pairing $\tilde{\mathcal{A}}_X^0(\mathcal{T}_X) \otimes_{\mathcal{O}_X} \tilde{\mathcal{A}}_X^0(\Omega^1_{X/k}) \to \tilde{\mathcal{A}}_X^0$. For each point $x \in X$ we choose a form $\pi_{(x)} \in \Omega^1_{X/k,(x)}$ as follows:

(5.11)
(1) If $x \in Z$ set $\pi_{(x)} := 0$.
(2) If $x \in U - Z$, let j be the first index s.t. $a_j(x) \neq 0$, and set $\pi_{(x)} := a_j^{-1} \mathrm{d}f_j$.
(3) If $x \notin U$ take any form $\pi_{(x)} \in \Omega^1_{X/k,(x)}$ satisfying $\langle v, \pi_{(x)} \rangle = 1$.

Together we get a global section $\pi = \{\pi_{(x)}\}_{x \in X} \in \underline{\mathbb{A}}^0(\Omega^1_{X/k})$, and as indicated in Lemma 2.17, there is a corresponding global section $\pi \in \tilde{\mathcal{A}}_X^0(\Omega^1_{X/k}) = \tilde{\mathcal{A}}_X^{1,0}$.

LEMMA 5.12. *Considering $v \in \tilde{\mathcal{A}}_X^0(\mathcal{T}_X)$, one has the identity*

$$\langle v, \pi \rangle = 1 \in \tilde{\mathcal{A}}_X^0 \text{ on } X - Z. \tag{5.13}$$

Proof. Use the embedding of Lemma 2.16 to reduce formula (5.13) to the local formula $\langle v, \pi_\xi \rangle = \sum t_i = 1 \in \tilde{\mathcal{A}}^0_\xi$. □

In Bott's language (see [Bo1]) π is a projector for v.

Let $\iota_v = \langle v, - \rangle \colon \Omega^1_{X/k} \to \mathcal{O}_X$ be the interior derivative, or contraction along v. It extends to an $\mathcal{O}_X$-linear operator of degree -1 on $\Omega^{\cdot}_{X/k}$, and hence to an $\tilde{\mathcal{A}}^0_X$-linear operator of bidegree $(-1, 0)$ on $\tilde{\mathcal{A}}^{\cdot}_X$, which commutes (in the graded sense) with D'' and satisfies $\iota_v^2 = 0$.

LEMMA 5.14. *There exists a global section $L \in \tilde{\mathcal{A}}^0_X(\mathcal{E}nd(\mathcal{E}))$ satisfying*

$$\iota_v R^{1,1} = \mathrm{D}'' L \in \tilde{\mathcal{A}}^{0,1}_X(\mathcal{E}nd(\mathcal{E}))$$

and

$$L|_Z = \Lambda|_Z \in \mathcal{E}nd(\mathcal{E}|_Z).$$

Proof. Using Lemma 2.13, define

$$L := \Lambda - \iota_v \circ \nabla \colon \tilde{\mathcal{A}}^0_X(\mathcal{E}) \to \tilde{\mathcal{A}}^0_X(\mathcal{E}).$$

This is an $\tilde{\mathcal{A}}^0_X$-linear homomorphism. Let us distinguish between $\mathrm{D}''L$, which is the image of L under $\mathrm{D}''\colon \tilde{\mathcal{A}}^0_X(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E})) \to \tilde{\mathcal{A}}^{0,1}_X(\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{E}))$, and $\mathrm{D}'' \circ L$, which is the composed operator $\tilde{\mathcal{A}}^0_X(\mathcal{E}) \to \tilde{\mathcal{A}}^{0,1}_X(\mathcal{E})$. Both $\iota_v R^{1,1}$ and $\mathrm{D}''L$ can be thought of as $\mathcal{O}_X$-linear homomorphisms $\mathcal{E} \to \tilde{\mathcal{A}}^{0,1}_X(\mathcal{E})$. Since $\mathrm{D}''(\mathcal{E}) = 0$, one checks (using a local algebraic frame) that $\mathrm{D}''L = \mathrm{D}'' \circ L$ on $\mathcal{E}$. By the proof of Lemma 5.10, $\mathrm{D}'' \circ \nabla = R^{1,1}$ as operators $\mathcal{E} \to \tilde{\mathcal{A}}^{1,1}_X(\mathcal{E})$. Now $\mathrm{D}'' \circ \Lambda = \Lambda \circ \mathrm{D}''$. Therefore we get equalities

$$\mathrm{D}''L = \mathrm{D}'' \circ L - L \circ \mathrm{D}'' = -\mathrm{D}'' \circ \iota_v \circ \nabla = \iota_v \circ \mathrm{D}'' \circ \nabla = \iota_v R^{1,1}$$

of maps $\mathcal{E} \to \tilde{\mathcal{A}}^{0,1}_X(\mathcal{E})$. Finally the equality $L|_Z = \Lambda|_Z$ follows from the vanishing of ι_v on Z. □

Let t be an indeterminate, and define

$$\begin{aligned} \eta &:= P(L + tR^{1,1}) \cdot \pi \cdot (1 - t\mathrm{D}''\pi)^{-1} \\ &= P(L + tR^{1,1}) \cdot \pi \cdot (1 + t\mathrm{D}''\pi + (t\mathrm{D}''\pi)^2 + \cdots) \in \tilde{\mathcal{A}}^{\cdot}_X[\,t\,] \end{aligned}$$

(note that $(\mathrm{D}''\pi)^{n+1} = 0$, so this makes sense). Writing $\eta = \sum_i \eta_i t^i$ we see that $\eta_i \in \tilde{\mathcal{A}}^{i+1,i}_X$.

LEMMA 5.15. $\mathrm{D}''\eta_{n-1} + P(R^{1,1}) = 0$ *on* $X - Z$.

Proof. Using the multilinear polarization $\tilde{P}$ of P, Lemma 5.14 and the fact that $\iota_v - t\mathrm{D}''$ is an odd derivation, one sees that $(\iota_v - t\mathrm{D}'')P(L + tR^{1,1}) = 0$ (cf. [Bo1]). Since $\langle v, \pi \rangle = 1$ on $X - Z$ we get $(\iota_v - t\mathrm{D}'')\pi = (1 - t\mathrm{D}'')\pi$, $(\iota_v - t\mathrm{D}'')(1 - t\mathrm{D}''\pi) = 0$, and hence $(\iota_v - t\mathrm{D}'')\eta = P(L + tR^{1,1})$ on $X - Z$. Finally consider the coefficient of t^n in this expression, noting that $\eta_n = 0$, being a section of $\tilde{\mathcal{A}}_X^{n+1,n}$. □

The proof of the next lemma is easy.

LEMMA 5.16. *Let $P(M_1, \ldots, M_n)$ be a multilinear polynomial on $\mathrm{M}_r(k)$, invariant under permutations. Let $A = \bigoplus A^i$ be a commutative DG k-algebra and $A^- := \bigoplus A^{2i+1}$. Let $\alpha_1, \ldots, \alpha_n \in A^-$ and $M_1, \ldots, M_n \in \mathrm{M}_r(A^-)$. Then*

(1) *If $\alpha_i = \alpha_j$ or $M_i = M_j$ for two distinct indices i, j then*

$$P(\alpha_1 M_1, \ldots, \alpha_n M_n) = 0.$$

(2)

$$P\left(\sum_{i=1}^{n} \alpha_i M_i, \ldots, \sum_{i=1}^{n} \alpha_i M_i\right) = n! P(\alpha_1 M_1, \ldots, \alpha_n M_n).$$

Proof of Theorem 5.1. By definition $c_i(\mathcal{E}) = [\int_\Delta \tilde{c}_i(\mathcal{E}; \nabla)] \in \mathrm{H}^{2i}_{\mathrm{DR}}(X)$. It is known that $\tilde{\mathcal{A}}_X^{\cdot}$ is a commutative DGA, and that $\mathrm{H}(\int_\Delta)\colon \mathrm{H}\Gamma(X, \tilde{\mathcal{A}}_X^{\cdot}) \to \mathrm{H}^{\cdot}_{\mathrm{DR}}(X)$ is an isomorphism of graded algebras (see Corollary 2.11). Hence

$$Q(c_1(\mathcal{E}), \ldots, c_r(\mathcal{E})) = [\int_\Delta Q(\tilde{c}_1(\mathcal{E}; \nabla), \ldots, \tilde{c}_r(\mathcal{E}; \nabla))] = [\int_\Delta P(R)].$$

As mentioned before, $P(R) = P(R^{1,1}) \in \tilde{\mathcal{A}}_X^{2n}$. We must verify:

$$\int_X \int_\Delta P(R^{1,1}) = \sum_{z \in Z} P(v, \mathcal{E}, z).$$

Let

$$\Xi := S(U)_n^{\mathrm{red}} - S(U - Z)_n^{\mathrm{red}} = \{(x_0, \ldots, x_n) \mid x_n \in Z\}.$$

We are given that X is proper, so by [Be] (or by the Parshin-Lomadze Residue Theorem, [Ye1] Theorem 4.2.15)

$$\int_X \int_\Delta \mathrm{D}'' \eta_{n-1} = \int_X \mathrm{D}'' \int_\Delta \eta_{n-1} = 0.$$

Therefore by Lemma 5.15

$$\begin{aligned}\int_X\int_\Delta P(R^{1,1}) &= \int_X\int_\Delta (P(R^{1,1})+\mathrm{D}''\eta_{n-1})\\ &= \sum_{\xi\in\Xi}\mathrm{Res}_\xi\int_\Delta (P(R^{1,1})+\mathrm{D}''\eta_{n-1}).\end{aligned}$$

Let us look at what happens on the open set U. By construction the connection ∇ is integrable there (it is a Levi-Civita connection with respect to the algebraic frame $\boldsymbol{e}$), so $R=0$; hence $P(R^{1,1})=0$ and $\mathrm{D}''\eta_{n-1}=P(L)(\mathrm{D}''\pi)^n$. According to Lemma 5.14, $\mathrm{D}''L=0$, so by Lemma 2.13 one has $L\in\mathcal{E}nd(\mathcal{E})$. Therefore $P(L)\in\mathcal{O}_X$. Since $\int_\Delta$ is $\mathcal{O}_X$-linear we get $\int_\Delta \mathrm{D}''\eta_{n-1}=P(L)\int_\Delta(\mathrm{D}''\pi)^n$. All the above is on U. The conclusion is:

$$\int_X\int_\Delta P(R^{1,1})=\sum_{\xi\in\Xi}\mathrm{Res}_\xi\left(P(L)\int_\Delta(\mathrm{D}''\pi)^n\right). \tag{5.17}$$

Using the embedding of Lemma 2.16, for each $\xi\in\Xi$ one has $\pi_\xi=\sum t_i\pi_{(x_i)}$, and therefore $\mathrm{D}''\pi_\xi=\sum \mathrm{d}t_i\wedge\pi_{(x_i)}$. Let

$$\Xi_a:=\{\xi=(x_0,\dots,x_n)\mid a_1(x_i)=\cdots=a_i(x_i)=0 \text{ for } i=1,\dots,n\}.$$

If $\xi\notin\Xi_a$ then for at least one index $0\le i<n$, $\pi_{(x_i)}=\pi_{(x_{i+1})}$. So by Lemma 5.16 we get $(\mathrm{D}''\pi_\xi)^n=0$.

It remains to consider only $\xi\in\Xi_a$. Since $\pi_{(x_i)}=a_{i+1}^{-1}\mathrm{d}f_{i+1}$ for $0\le i<n$, and $\pi_{(x_n)}=0$, it follows from Lemma 5.16 that

$$(\mathrm{D}''\pi_\xi)^n=n!(-1)^{\binom{n}{2}}\mathrm{d}t_0\wedge\cdots\wedge\mathrm{d}t_{n-1}\wedge\frac{\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_n}{a_1\cdots a_n},$$

so

$$\int_\Delta(\mathrm{D}''\pi_\xi)^n=(-1)^{\binom{n+1}{2}}\frac{\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_n}{a_1\cdots a_n}\in\Omega^n_{X/k,\xi}.$$

Finally, according to [Hu2] Corollary 2.5 or [SY] Theorem 0.2.9, and our Lemma 5.6, and using the fact that $L|_Z=\Lambda|_Z$, it holds with $l=\binom{n+1}{2}$:

$$\sum_{\xi\in\Xi_a}(-1)^l\,\mathrm{Res}_\xi\frac{P(L)\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_n}{a_1\cdots a_n}$$

$$= \sum_{z\in Z} (-1)^l \operatorname{Res}_{\mathcal{O}_{X(z)}/k} \begin{bmatrix} P(L)\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \\ a_1, \ldots, a_n \end{bmatrix}$$

$$= \sum_{z\in Z} P(v, \mathcal{E}, z). \qquad \square$$

Remark 5.18. There is a sign error in [Ye1] Section 2.4. Let K and $L = K((t_1, \ldots, t_n))$ be topological local fields. Since the residue map $\operatorname{Res}_{L/K}\colon \Omega^{\cdot,\mathrm{sep}}_{L/k} \to \Omega^{\cdot,\mathrm{sep}}_{K/k}$ is an $\Omega^{\cdot,\mathrm{sep}}_{K/k}$-linear map of degree $-n$, it follows by transitivity that $\operatorname{Res}_{L/K}(t_1^{-1}\mathrm{d}t_1 \wedge \cdots \wedge t_n^{-1}\mathrm{d}t_n) = 1$, not $(-1)^{\binom{n}{2}}$. This error carried into [SY] Theorem 0.2.9.

Remark 5.19. Consider an action of an algebraic torus T on X and $\mathcal{E}$, with positive dimensional fixed point locus. A residue formula is known in this case (see [Bo1] and [AB]), but we were unable to prove it using our adelic method. The sticky part was finding an adelic model for the equivariant cohomology $\mathrm{H}^{\cdot}_T(X)$. An attempt to use an adelic version of the Cartan-De Rham complex did not succeed. Another unsuccessful try was to consider the classifying space BT as a cosimplicial scheme, and compute its cohomology via adeles.

Recently Edidin and Graham defined equivariant Chow groups, and proved the Bott formula in that context (see [EG]). The basic idea there is to approximate the classifying space BT by finite type schemes, "up to a given codimension." This approach is suited for global constructions, but again it did not help as far as adeles were concerned.

6. The Gauss-Bonnet Formula. Let k be a perfect field (of any characteristic) and X a finite type scheme, with structural morphism $\pi\colon X \to \operatorname{Spec} k$. According to Grothendieck Duality Theory [RD] there is a functor $\pi^!\colon \mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}(k)) \to \mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}(X))$ between derived categories, called the twisted inverse image. The object $\pi^! k$ is a dualizing complex on X, and it has a canonical representative, namely the residue complex $\mathcal{K}^{\cdot}_X := \mathrm{E}\pi^! k$. Here E is the Cousin functor. As graded sheaf $\mathcal{K}^{-q}_X = \bigoplus_{\dim \overline{\{x\}}=q} \mathcal{K}_X(x)$, where $\mathcal{K}_X(x)$ is a quasi-coherent sheaf, constant with support $\overline{\{x\}}$, and as $\mathcal{O}_{X,x}$-module it is an injective hull of the residue field $k(x)$.

$\mathcal{K}^{\cdot}_X$ enjoys some remarkable properties, which are deduced from corresponding properties of $\pi^!$. If X is pure of dimension n then there is a canonical homomorphism $C_X\colon \Omega^n_{X/k} \to \mathcal{K}^{-n}_X$, which gives a quasi-isomorphism $\Omega^n_{X/k}[n] \to \mathcal{K}^{\cdot}_X$ on the smooth locus of X. If $f\colon X \to Y$ is a morphism of schemes then there is a map of graded sheaves $\mathrm{Tr}_f\colon f_*\mathcal{K}^{\cdot}_X \to \mathcal{K}^{\cdot}_Y$, which becomes a map of complexes when f is proper.

For integers p, q define

$$\mathcal{F}_X^{p,q} := \mathcal{H}om_X(\Omega_{X/k}^{-p}, \mathcal{K}_X^q).$$

Clearly $\mathcal{F}_X^{\cdot}$ is a graded (left and right) $\Omega_{X/k}^{\cdot}$-module. Moreover according to [Ye3] Theorem 4.1, or [EZ], $\mathcal{F}_X^{\cdot}$ is in fact a DG $\Omega_{X/k}^{\cdot}$-module. The difficult thing is to define the dual operator Dual (d): $\mathcal{F}_X^{p,q} \to \mathcal{F}_X^{p+1,q}$, which is a differential operator of order 1. $\mathcal{F}_X^{\cdot}$ is called the *De Rham-residue complex*.

There is a special cocycle $C_X \in \Gamma(X, \mathcal{F}_X^{\cdot})$ called the *fundamental class*. When X is integral of dimension n then $C_X \in \mathcal{F}_X^{-n,-n}$ is the natural map $\Omega_{X/k}^n \to \mathcal{K}_X^{-n} = k(X) \otimes_{\mathcal{O}_X} \Omega_{X/k}^n$. For any closed subscheme $f\colon Z \to X$, the trace map $\mathrm{Tr}_f\colon f_*\mathcal{F}_Z^{\cdot} \to \mathcal{F}_X^{\cdot}$ is injective, which allows us to write $C_Z \in \mathcal{F}_X^{\cdot}$. If $Z_1, \dots, Z_r$ are the irreducible components of Z (with reduced subscheme structures) and $z_1, \dots, z_r$ are the generic points, one has $C_Z = \sum_i (\operatorname{length} \mathcal{O}_{Z,z_i}) C_{Z_i}$.

When X is pure of dimension n, C_X induces a map of complexes $C_X\colon \Omega_{X/k}^{\cdot}[2n] \to \mathcal{F}_X^{\cdot}$, $\alpha \mapsto C_X \cdot \alpha$. This is a quasi-isomorphism on the smooth locus of X ([Ye3] Proposition 5.8). Hence when X is smooth $\mathrm{H}^{-i}\Gamma(X, \mathcal{F}_X^{\cdot}) \cong \mathrm{H}_i^{\mathrm{DR}}(X)$, De Rham homology.

Given a morphism $f\colon X \to Y$, the trace $\mathrm{Tr}_f\colon f_*\mathcal{K}_X^{\cdot} \to \mathcal{K}_Y^{\cdot}$ and $f^*\colon \Omega_{Y/k}^{\cdot} \to f_*\Omega_{X/k}^{\cdot}$ induce a map of graded sheaves $\mathrm{Tr}_f\colon f_*\mathcal{F}_X^{\cdot} \to \mathcal{F}_Y^{\cdot}$. Given an étale morphism $g\colon U \to X$ there is a homomorphism of complexes $\mathrm{q}_g\colon \mathcal{F}_X^{\cdot} \to g_*\mathcal{F}_U^{\cdot}$, and $\mathrm{q}_g(C_X) = C_U$.

The next theorem summarizes a few theorems in [Ye5] about the action of $\mathcal{A}_X^{\cdot}$ on $\mathcal{F}_X^{\cdot}$.

THEOREM 6.1. *Let X be a finite type scheme over a perfect field k.*

(1) *$\mathcal{F}_X^{\cdot}$ is a right DG $\mathcal{A}_X^{\cdot}$-module, and the multiplication extends the $\Omega_{X/k}^{\cdot}$-module structure.*

(2) *If $f\colon X \to Y$ is proper then $\mathrm{Tr}_f\colon f_*\mathcal{F}_X^{\cdot} \to \mathcal{F}_Y^{\cdot}$ is $\mathcal{A}_Y^{\cdot}$-linear.*

(3) *If $g\colon U \to X$ is étale then $\mathrm{q}_g\colon \mathcal{F}_X^{\cdot} \to g_*\mathcal{F}_U^{\cdot}$ is $\mathcal{A}_X^{\cdot}$-linear.*

Note that from (1) it follows that if X is smooth of dimension n then $\mathcal{A}_X^{\cdot}[2n] \to \mathcal{F}_X^{\cdot}$, $\alpha \mapsto C_X \cdot \alpha$ is a quasi-isomorphism.

Let us say a few words about the multiplication $\mathcal{F}_X^{\cdot} \otimes \mathcal{A}_X^{\cdot} \to \mathcal{A}_X^{\cdot}$. Since $\mathcal{A}_X^{\cdot} \cong \underline{\mathbb{A}}_{\mathrm{red}}^{\cdot}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \Omega_{X/k}^{\cdot}$ and $\mathcal{F}_X^{\cdot} \cong \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}^{\cdot}, \mathcal{K}_X^{\cdot})$, it suffices to describe the product $\mathcal{K}_X^{\cdot} \otimes \underline{\mathbb{A}}_{\mathrm{red}}^{\cdot}(\mathcal{O}_X) \to \mathcal{K}_X^{\cdot}$. This requires the explicit construction of $\mathcal{K}_X^{\cdot}$ which we gave in [Ye3], and which we quickly review below.

The construction starts with the theory of *Beilinson completion algebras* (BCAs) developed in [Ye2]. A BCA A is a semilocal k-algebra with a topology and with valuations on its residue fields. Each local factor of A is a quotient of the ring of formal power series $L((s_n)) \cdots ((s_1))[[t_1, \dots, t_m]]$, where L is a finitely generated extension field of k, and $L((s_n)) \cdots ((s_1))$ is the field of iterated Laurent

series. One considers two kinds of homomorphisms between BCAs: morphisms $f\colon A \to B$ and intensifications $u\colon A \to \widehat{A}$.

Each BCA A has a *dual module* $\mathcal{K}(A)$, which is functorial with respect to these homomorphisms; namely there are maps $\mathrm{Tr}_f\colon \mathcal{K}(B) \to \mathcal{K}(A)$ and $\mathrm{q}_u\colon \mathcal{K}(A) \to \mathcal{K}(\widehat{A})$. If A is local with maximal ideal $\mathfrak{m}$ and residue field $K = A/\mathfrak{m}$, then a choice of coefficient field $\sigma\colon K \to A$ determines an isomorphism

$$\mathcal{K}(A) \cong \mathrm{Hom}_K^{\mathrm{cont}}(A, \Omega_{K/k}^{n,\mathrm{sep}}),$$

where $\Omega_{K/k}^{\cdot,\mathrm{sep}}$ is the separated algebra of differentials on K and $n = \mathrm{rank}_K\, \Omega_{K/k}^{1,\mathrm{sep}}$. In particular, algebraically $\mathcal{K}(A)$ is an injective hull of K.

Suppose $\xi = (x, \ldots, y)$ is a saturated chain of points in X (i.e. immediate specializations). Then the Beilinson completion $\mathcal{O}_{X,\xi}$ is a BCA. The natural algebra homomorphisms $\partial^-\colon \mathcal{O}_{X,(x)} \to \mathcal{O}_{X,\xi}$ and $\partial^+\colon \mathcal{O}_{X,(y)} \to \mathcal{O}_{X,\xi}$ are an intensification and a morphism, respectively. So there are homomorphisms on dual modules $\mathrm{q}_{\partial^-}\colon \mathcal{K}(\mathcal{O}_{X,(x)}) \to \mathcal{K}(\mathcal{O}_{X,\xi})$ and $\mathrm{Tr}_{\partial^+}\colon \mathcal{K}(\mathcal{O}_{X,\xi}) \to \mathcal{K}(\mathcal{O}_{X,(y)})$. The composition $\mathrm{Tr}_{\partial^+} \circ \mathrm{q}_{\partial^-}$ is denoted by δ_ξ. We regard $\mathcal{K}_X(x) := \mathcal{K}(\mathcal{O}_{X,(x)})$ as a quasi-coherent $\mathcal{O}_X$-module, constant on the closed set $\overline{\{x\}}$. Define

$$\mathcal{K}_X^q := \bigoplus_{\dim \overline{\{x\}} = -q} \mathcal{K}_X(x) \tag{6.2}$$

and

$$\delta = (-1)^{q+1} \sum_{(x,y)} \delta_{(x,y)}\colon \mathcal{K}_X^q \to \mathcal{K}_X^{q+1}. \tag{6.3}$$

Then the pair $(\mathcal{K}_X^{\cdot}, \delta)$ is the residue complex of X. That is to say, there is a canonical isomorphism $\mathcal{K}_X^{\cdot} \cong \pi^! k$ in the derived category $\mathsf{D}(\mathsf{Mod}(X))$ (see [Ye3] Corollary 2.5).

Let x be a point of dimension q in X, and consider a local section $\phi_x \in \mathcal{K}_X(x) \subset \mathcal{K}_X^{-q}$. Let $\xi = (x_0, \ldots, x_{q'})$ be any chain of length q' in X, and let $a_\xi \in \mathcal{O}_{X,\xi}$. Define $\phi_x \cdot a_\xi \in \mathcal{K}_X^{-q+q'}$ as follows. If $x = x_0$ and ξ is saturated, then

$$\phi_x \cdot a_\xi := \mathrm{Tr}_{\partial^+}(a_\xi \cdot \mathrm{q}_{\partial^-}(\phi_x)) \in \mathcal{K}_X(x_{q'}),$$

where the product $a_\xi \cdot \mathrm{q}_{\partial^-}(\phi_x)$ is in $\mathcal{K}(\mathcal{O}_{X,\xi})$. Otherwise set $\phi_x \cdot a_\xi := 0$.

It turns out that for local sections $\phi = (\phi_x) \in \mathcal{K}_X^{-q}$ and $a = (a_\xi) \in \underline{\mathbb{A}}_{\mathrm{red}}^{q'}(\mathcal{O}_X)$, one has $\phi_x \cdot a_\xi = 0$ for all but finitely many pairs x, ξ. Hence

$$\phi \cdot a := \sum_{x,\xi} \phi_x \cdot a_\xi \in \mathcal{K}_X^{-q+q'}$$

is well defined, and this is the product we use.

Example 6.4. Suppose X is integral of dimension n, x_0 is its generic point and $\phi \in \mathcal{K}_X^{-n} = \mathcal{K}_X(x_0) = \Omega^n_{k(x_0)/k}$. Consider a saturated chain $\xi = (x_0, \dots, x_q)$ and an element $a \in \mathcal{O}_{X,\xi} = k(x_0)_\xi$. We want to see what is $\psi := \phi \cdot a \in \mathcal{K}_X(x_q) = \mathcal{K}(\mathcal{O}_{X,(x_q)})$. Choose a coefficient field $\sigma\colon\ k(x_q) \to \mathcal{O}_{X,(x_q)}$, so that $\mathcal{K}(\mathcal{O}_{X,(x_q)}) \cong \mathrm{Hom}^{\mathrm{cont}}_{k(x_q)}(\mathcal{O}_{X,(x_q)}, \Omega^{n-q}_{k(x_q)/k})$. It is known that $k(x_0)_\xi = \prod L_i$, a finite product of topological local fields (TLFs), and $\sigma\colon\ k(x_q) \to L_i$ is a morphism of TLFs. Then for $b \in \mathcal{O}_{X,(x_q)}$ one has

$$\psi(b) = \sum_i \mathrm{Res}_{L_i/k(x_q)}(ba\phi) \in \Omega^{n-q}_{k(x_q)/k},$$

where $\mathrm{Res}_{L_i/k(x_q)}$ is the the residue of [Ye1] Theorem 2.4.3, and the product $ba\phi \in \Omega^{n,\mathrm{sep}}_{k(x_0)/k}$.

We can now state the main result of this section. From here to the end of Section 6 we assume $\mathrm{char}\, k = 0$.

THEOREM 6.5. (Gauss-Bonnet) *Assume* $\mathrm{char}\, k = 0$, *and let X be an integral, n-dimensional, quasi-projective k-variety* (*not necessarily smooth*). *Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank r. Suppose $v \in \Gamma(X, \mathcal{E})$ is a regular section, with zero scheme Z. Then there is an adelic connection ∇ on $\mathcal{E}$ satisfying*

$$C_X \cdot c_r(\mathcal{E}, \nabla) = (-1)^m C_Z \in \mathcal{F}_X^{-2(n-r)}$$

with $m = nr + \binom{r+1}{2}$.

Let U be an open subset such that $\mathcal{E}|_U$ is trivial and U meets each irreducible component of Z. Fix an algebraic frame $(v_1, \dots, v_r)^{\mathrm{t}}$ of $\mathcal{E}|_U$ and write

$$v = \sum_{i=1}^r a_i v_i,\ \ a_i \in \Gamma(U, \mathcal{O}_X). \tag{6.6}$$

For each $x \in X$ choose a local frame $\boldsymbol{e}_{(x)}$ of $\mathcal{E}_{(x)}$ as follows:

(6.7)
(1) If $x \notin Z \cup U$, take $\boldsymbol{e}_{(x)} = (v, *, \ldots, *)^{\mathrm{t}}$.
(2) If $x \in U - Z$, there is some $0 \leq i < r$ such that $a_1(x), \ldots, a_i(x) = 0$ but $a_{i+1}(x) \neq 0$. Then take $\boldsymbol{e}_{(x)} = (v, v_1, \ldots, v_i, v_{i+2}, \ldots, v_r)^{\mathrm{t}}$.
(3) If $x \in Z \cap U$, take $\boldsymbol{e}_{(x)} = (v_1, \ldots, v_r)^{\mathrm{t}}$.
(4) If $x \in Z - U$, take $\boldsymbol{e}_{(x)}$ arbitrary.

Let $\nabla_{(x)}$ be the resulting Levi-Civita connection on $\mathcal{E}_{(x)}$, let $\nabla\colon \tilde{\mathcal{A}}^0_X(\mathcal{E}) \to \tilde{\mathcal{A}}^1_X(\mathcal{E})$ be the induced adelic connection, and let $R \in \tilde{\mathcal{A}}^2_X(\mathcal{E}nd(\mathcal{E}))$ be the curvature. We get a top Chern form $P_r(R) = \det R \in \tilde{\mathcal{A}}^{2r}_X$. Under the embedding of DGAs $\Gamma(X, \tilde{\mathcal{A}}^{p,q}_X) \subset \prod_{\xi \in S(X)} \tilde{\mathcal{A}}^{p,q}_\xi$ of Lemma 2.16 we write $R = (R_\xi)$.

LEMMA 6.8. *Suppose $\xi = (x_0, \ldots, x_q)$ is a saturated chain of length q, with x_0 the generic point of X, and either:* (i) $q < r$; (ii) $x_q \notin Z$; *or* (iii) $q = r$ *and* $\boldsymbol{e}_{x_i} = \boldsymbol{e}_{x_{i+1}}$ *for some i. Then* $\int_{\Delta^q} \det R_\xi = 0$.

Proof. Let $\boldsymbol{g}_i \in \mathrm{GL}_r(\mathcal{O}_{X,\xi})$ be the transition matrix $\boldsymbol{e}_{x_i} = \boldsymbol{g}_i \cdot \boldsymbol{e}_{x_q}$, let $\boldsymbol{\theta}_i$ be the connection matrix of ∇_{x_i} with respect to the frame $\boldsymbol{e}_{x_q}$. Then the matrices $\boldsymbol{\theta}$, Θ of ∇_ξ, R_ξ are

$$\begin{aligned} \boldsymbol{\theta} &= -(t_0 \boldsymbol{g}_0^{-1} \mathrm{d}\boldsymbol{g}_0 + \cdots + t_{q-1} \boldsymbol{g}_{q-1}^{-1} \mathrm{d}\boldsymbol{g}_{q-1}) \\ \Theta &= \mathrm{D}\boldsymbol{\theta} - \boldsymbol{\theta} \wedge \boldsymbol{\theta}. \end{aligned}$$

In cases (i) and (ii), all $x_i \notin Z$, so

$$\boldsymbol{g}_i = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \ldots & * \\ * & * & \ldots & * \end{pmatrix} \qquad \boldsymbol{\theta}_i = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \ldots & * \\ * & * & \ldots & * \end{pmatrix}.$$

Therefore Θ has a zero first row too and $\det \Theta = 0$.

Now suppose $q = r$. Since $\Theta \in \mathrm{M}_r(\tilde{\mathcal{A}}^{1,1}_\xi) \oplus \mathrm{M}_r(\tilde{\mathcal{A}}^{2,0}_\xi)$, from degree considerations we conclude that $\int_{\Delta^r} \det \Theta = \int_{\Delta^r} \det(\Theta^{1,1})$. Let $\tilde{P}_r$ be the polarization of det. One has $\Theta^{1,1} = -\sum \mathrm{d}t_i \wedge \boldsymbol{\theta}_i$ (cf. Lemma 3.13), so by Lemma 5.16 we have

$$\det(\Theta^{1,1}) = \tilde{P}_r(-\mathrm{d}t_0 \wedge \boldsymbol{\theta}_0, \ldots, -\mathrm{d}t_{r-1} \wedge \boldsymbol{\theta}_{r-1}). \tag{6.9}$$

But in case (iii), using Lemma 2.16 again, we get $\det(\Theta^{1,1}) = 0$. □

LEMMA 6.10. *Suppose $\xi = (x_0, \ldots, x_r)$ is a saturated chain in U satisfying: x_0 is the generic point of X, and $a_1(x_i), \ldots, a_i(x_i) = 0$ for $0 \leq i \leq r$ (so in particular*

$x_r \in Z$). *Then*

$$\int_{\Delta^r} \det R_\xi = (-1)^{\binom{r+1}{2}} \frac{\mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_r}{a_1 \cdots a_r} \in \Omega^r_{X/k,\xi}.$$

Proof. Since the point x_r falls into case 3 of (6.7), and for every $i < r$, x_i falls into case 2, an easy linear algebra calculation shows that $\Theta^{1,1} = (\mathrm{d}t_{i-1} \wedge a_i^{-1}\mathrm{d}a_j)$ (i.e., $\mathrm{d}t_{i-1} \wedge a_i^{-1}\mathrm{d}a_j$ appears in the (i,j) position). By Lemma 5.16,

$$\begin{aligned} \det \Theta^{1,1} &= r!\mathrm{d}t_0 \wedge a_1^{-1}\mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}t_{r-1} \wedge a_r^{-1}\mathrm{d}a_r \\ &= r!(-1)^{\binom{r}{2}}\mathrm{d}t_0 \wedge \cdots \wedge \mathrm{d}t_{r-1} \wedge \frac{\mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_r}{a_1 \cdots a_r}. \end{aligned}$$

Now use the fact that $\int_{\Delta^r} \mathrm{d}t_0 \wedge \cdots \wedge \mathrm{d}t_{r-1} = (-1)^r (r!)^{-1}$. □

LEMMA 6.11. *Let $\xi = (x_0, \ldots, x_r = z)$ be a saturated chain, and let $\sigma\colon k(z) \to \mathcal{O}_{X,(z)}$ a coefficient field. Then the Parshin residue map* $\mathrm{Res}_{k(\xi)/k(z)}\colon \Omega^{\cdot,\mathrm{sep}}_{k(\xi)/k} \to \Omega^{\cdot}_{k(z)/k}$ (*see [Ye1] Definition* 4.1.3) *satisfies*

$$\mathrm{Res}_{k(\xi)/k(z)}(\mathrm{dlog}\, a_1 \wedge \cdots \wedge \mathrm{dlog}\, a_r \wedge \alpha) = 0$$

for all $a_1, \ldots, a_r \in \mathcal{O}_{X,(z)}$ and $\alpha \in (\mathfrak{m}_z + \mathrm{d}\mathfrak{m}_z)\Omega^{\cdot}_{X/k,(z)}$.

Proof. By induction on r. We start with $r = 1$. By the definition of residues, it suffices to prove that for any local factor L of $k(\xi)$, $\mathrm{Res}_{L/k(z)}(\mathrm{dlog}\, a_1 \wedge \alpha) = 0$. Now $L \cong K((t))$, with K a finite field extension of $k(z)$, and the image of $\mathcal{O}_{X,(z)} \to L$ lies in $K[[t]]$ (we are using the fact that $\mathrm{char}\, k = 0$). Note that $\Omega^{\cdot}_{X/k,(z)} = \Omega^{\cdot,\mathrm{sep}}_{\mathcal{O}_{X,(z)}/k}$, so $\alpha = t\beta + \mathrm{d}t \wedge \gamma$ for some $\beta, \gamma \in \Omega^{\cdot,\mathrm{sep}}_{K[[t]]/k}$. Also, $a_1 = t^e u$ with $u \in K[[t]]^*$ and $e \in \mathbb{Z}$, so $\mathrm{dlog}\, a_1 = e\,\mathrm{dlog}\, t + \mathrm{dlog}\, u$. But

$$\mathrm{Res}_{L/K}((e\,\mathrm{dlog}\, t + \mathrm{dlog}\, u)(t\beta + \mathrm{d}t \wedge \gamma)) = 0.$$

Now assume $r > 1$, and set $\partial_r \xi := (x_0, \ldots, x_{r-1})$ and $y := x_{r-1}$. First take $a_1, \ldots, a_r, \alpha$ algebraic, i.e., $a_i \in \mathcal{O}_{X,z}$ and $\alpha \in (\mathfrak{m}_z + \mathrm{d}\mathfrak{m}_z)\Omega^{\cdot}_{X/k,z}$. Let $\tau\colon k(y) \to \mathcal{O}_{X,(y)}$ be a lifting compatible with $\sigma\colon k(z) \to \mathcal{O}_{X,(z)}$ (cf. [Ye1] Definition 4.1.5; again we use $\mathrm{char}\, k = 0$), so by [Ye1] Corollary 4.1.16,

$$\mathrm{Res}_{k(\xi)/k(z)} = \mathrm{Res}_{k((y,z))/k(z)} \circ \mathrm{Res}_{k(\partial_r \xi)/k(y)}\colon \Omega^{\cdot}_{k(x)/k} \to \Omega^{\cdot}_{k(z)/k}.$$

The lifting τ determines a decomposition

$$\Omega^{\cdot,\mathrm{sep}}_{\mathcal{O}_{X,(y)}/k} = \Omega^{\cdot}_{k(y)/k} \oplus (\mathfrak{m}_y + \mathrm{d}\mathfrak{m}_y)\Omega^{\cdot,\mathrm{sep}}_{\mathcal{O}_{X,(y)}/k},$$

and we decompose $\alpha = \alpha_0 + \alpha_1$ and $\operatorname{dlog} a_r = \beta_0 + \beta_1$ (or rather their images in $\Omega^{\cdot,\mathrm{sep}}_{\mathcal{O}_{X,(y)}/k}$) accordingly. Using the $\Omega^{\cdot}_{k(y)/k}$-linearity of $\operatorname{Res}_{k(\partial_r\xi)/k(y)}$ and induction applied to $\beta_0 \wedge \alpha_1$, $\beta_1 \wedge \alpha_0$ and $\beta_1 \wedge \alpha_1$, we get

$$\operatorname{Res}_{k(\partial_r\xi)/k(y)}(\operatorname{dlog} a_1 \wedge \cdots \wedge \operatorname{dlog} a_{r-1} \wedge (\operatorname{dlog} a_r \wedge \alpha)) = m\beta_0 \wedge \alpha_0$$

with $m \in \mathbb{Z}$. Since β_0, α_0 are respectively the images of $\operatorname{dlog} a_r$, α in $\Omega^{\cdot,\mathrm{sep}}_{k((y,z))/k}$, again using induction we have

$$\operatorname{Res}_{k((y,z))/k(z)}(\beta_0 \wedge \alpha_0) = 0.$$

Finally by the continuity of $\operatorname{Res}_{k(\xi)/k(z)}$ the result holds for any $a_1, \ldots, a_r, \alpha$. $\square$

Proof of Theorem 6.5. By the definition of the product and by Lemma 6.8, for evaluating the product $C_X \cdot c_r(\mathcal{E}, \nabla)$ we need only consider the components $c_r(\mathcal{E}, \nabla)_\xi = \int_\Delta \det R_\xi$ of $c_r(\mathcal{E}, \nabla)$ for saturated chains $\xi = (x_0, \ldots, x_r)$, where x_0 is the generic point of X, $x_r = z$ is the generic point of some irreducible component Z' of Z, and $a_i(x_j) = 0$ for all $i \leq j$. Fix one such component of Z', and let Ξ_z be the set of all such chains ending with z. By definition of C_Z we must show that the map

$$(C_X \cdot \int_\Delta \det R)_z \colon\ \Omega^{n-r}_{X/k,z} \to \mathcal{K}_X(z) \tag{6.12}$$

factors through the maps

$$\Omega^{n-r}_{X/k,z} \twoheadrightarrow \Omega^{n-r}_{k(z)/k} \xrightarrow{(-1)^m l} \Omega^{n-r}_{k(z)/k} = \mathcal{K}_{Z'_{\mathrm{red}}}(z) \subset \mathcal{K}_X(z),$$

where Z'_{red} is the reduced scheme, l is the length of the artinian ring $\mathcal{O}_{Z',z} = \mathcal{O}_{X,(z)}/(a_1, \ldots, a_r)$, and $m = \begin{pmatrix} r+1 \\ 2 \end{pmatrix} + nr$.

Choose a coefficient field $\sigma\colon\ k(z) \to \mathcal{O}_{X,(z)}$. By Lemma 6.10 and Example 6.4 we have for any $\alpha \in \Omega^{n-r}_{X/k,z}$:

$$\begin{aligned} &(C_X \cdot c_r(\mathcal{E}, \nabla))_z(\alpha) \\ &\quad = (-1)^m \sum_{\xi \in \Xi_z} (\mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_r \wedge \alpha) \cdot \left(\frac{1}{a_1 \cdots a_r}\right)_\xi \\ &\quad = (-1)^m \sum_{\xi \in \Xi_z} \operatorname{Res}_{k(\xi)/k(z)} \left(\frac{\mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_r \wedge \alpha}{a_1 \cdots a_r}\right). \end{aligned} \tag{6.13}$$

By Lemma 6.11 we see that this expression vanishes for $\alpha \in \operatorname{Ker}(\Omega^{n-r}_{X/k,(z)} \twoheadrightarrow \Omega^{n-r}_{k(z)/k})$, so that we can assume $\alpha \in \operatorname{Im}(\sigma\colon \Omega^{n-r}_{k(z)/k} \to \Omega^{n-r}_{X/k,(z)})$. Now $\operatorname{Res}_{k(\xi)/k(z)}$ is a graded left $\Omega^{\cdot}_{k(z)/k}$-linear homomorphism of degree $-r$, so α may be extracted. On the other hand, by [Hu2] Corollary 2.5 or [SY] Theorem 0.2.5, and by [HK] Example 1.14.b, we get

$$\sum_{\xi\in\Xi_z} \operatorname{Res}_{k(\xi)/k(z)}\left(\frac{\mathrm{d}a_1\wedge\cdots\wedge\mathrm{d}a_r}{a_1\cdots a_r}\right) = \operatorname{Res}_{\mathcal{O}_{X,(z)}/k(z)}\left[\begin{array}{c}\mathrm{d}a_1\wedge\cdots\wedge\mathrm{d}a_r\\ a_1,\ldots,a_r\end{array}\right] = l.$$

This concludes the proof. □

Appendix A. Simplicial De Rham theorem. In this appendix we provide proofs to Theorems 1.12 and 1.13. These proofs are essentially contained in [BG] and [HS1], but not quite in the formulation needed here. For notation see Section 1.

Suppose $M \in \Delta\mathsf{Mod}(\mathbb{Q})$ and $N \in \Delta^\circ\mathsf{Mod}(\mathbb{Q})$. For any $\sigma\colon [m] \to [n]$ in Δ there are homomorphisms $\sigma^*\colon M^m \to M^n$ and $\sigma_*\colon N_n \to N_m$. Let

$$N \otimes_{\leftarrow} M \subset \prod_{n=0}^{\infty}\left(N_n \otimes_{\mathbb{Q}} M^n\right) \tag{A.1}$$

be the submodule consisting of all $(u_0, u_1, \ldots)$, $u_n \in N_n \otimes_{\mathbb{Q}} M^n$, s.t. for each $\sigma\colon [m] \to [n]$,

$$(1 \otimes \sigma^*)(u_m) = (\sigma_* \otimes 1)(u_n).$$

Of course it suffices to check this condition for $\sigma = \partial^i, s^i$. If $M \in \Delta\mathsf{DGMod}(\mathbb{Q})$ and $N \in \Delta^\circ\mathsf{DGMod}(\mathbb{Q})$ we set $(N \otimes_{\leftarrow} M)^{p,q} := N^{q,\cdot} \otimes_{\leftarrow} M^{p,\cdot}$, and in the usual way this yields $N \otimes_{\leftarrow} M \in \mathsf{DGMod}(\mathbb{Q})$.

Example A.2. Taking $N = \Omega^{\cdot}(\Delta_{\mathbb{Q}})$ we get

$$\tilde{\mathrm{N}}M = \Omega^{\cdot}(\Delta_{\mathbb{Q}}) \otimes_{\leftarrow} M.$$

Given a simplicial set S, $\operatorname{Hom}_{\mathsf{Sets}}(S, \mathbb{Q})$ is a cosimplicial $\mathbb{Q}$-algebra. Let

$$C^{\cdot}(S, \mathbb{Q}) := \mathrm{N}\operatorname{Hom}_{\mathsf{Sets}}(S, \mathbb{Q})$$

which is a DGA, called the algebra of normalized cochains on S. Observe that $C^q(S, \mathbb{Q}) \cong \operatorname{Hom}_{\mathsf{Sets}}(S^{\mathrm{red}}_q, \mathbb{Q})$, where S^{red}_q is the set of nondegenerate simplices.

In particular $C^{\cdot}(\Delta^n, \mathbb{Q})$ is a finite (noncommutative) $\mathbb{Q}$-algebra. Define a DGA

$$A^{\cdot}(S, \mathbb{Q}) := \tilde{\mathrm{N}}\,\mathrm{Hom}_{\mathsf{Sets}}(S, \mathbb{Q}).$$

Applying this to the cosimplicial simplicial set $\Delta = \{\Delta^n\}$ we get simplicial DGAs $C^{\cdot}(\Delta, \mathbb{Q})$ and $A^{\cdot}(\Delta, \mathbb{Q})$.

LEMMA A.3. (1) *For any* $M \in \Delta\mathsf{DGMod}(\mathbb{Q})$ *there an isomorphism of DG modules*

$$\mathrm{N}M \cong C^{\cdot}(\Delta, \mathbb{Q}) \otimes_{\leftarrow} M.$$

(2) *For any* $S \in \Delta^{\circ}\mathsf{Sets}$ *there are isomorphisms of DGAs*

$$\begin{aligned} A^{\cdot}(S, \mathbb{Q}) &\cong \mathrm{Hom}_{\Delta^{\circ}\mathsf{Sets}}(S, \Omega^{\cdot}(\Delta_{\mathbb{Q}})) \\ C^{\cdot}(S, \mathbb{Q}) &\cong \mathrm{Hom}_{\Delta^{\circ}\mathsf{Sets}}(S, C^{\cdot}(\Delta, \mathbb{Q})). \end{aligned}$$

(3) *There is an isomorphism of DGAs*

$$A^{\cdot}(\Delta^n, \mathbb{Q}) \cong \Omega^{\cdot}(\Delta^n_{\mathbb{Q}}).$$

Proof. (1) Straightforward; cf. [HS1] Prop. 3.1.3.

(2) For any finite $\mathbb{Q}$-module V one has $\mathrm{Hom}_{\mathsf{Sets}}(S_n, V) \cong \mathrm{Hom}_{\mathsf{Sets}}(S_n, \mathbb{Q}) \otimes_{\mathbb{Q}} V$. Apply this to $V = \Omega^q(\Delta^n_{\mathbb{Q}})$ and $V = C^q(\Delta^n, \mathbb{Q})$.

(3) This follows from part 2 for $S = \Delta^n$, and using the isomorphism $T_n \cong \mathrm{Hom}_{\Delta^{\circ}\mathsf{Sets}}(\Delta^n, T)$ for any $T \in \Delta^{\circ}\mathsf{Sets}$. □

Regarding notation, the DGA we call $C^{\cdot}(S, \mathbb{Q})$ (resp. $A^{\cdot}(S, \mathbb{Q})$) is denoted by $C^*(S)$ (resp. $A^*(S)$) in [BG]. Our simplicial DGA $C^{\cdot}(\Delta, \mathbb{Q})$ is denoted by Z^* in [HS1].

Define a homomorphism

$$\rho\colon\ \Omega^{\cdot}(\Delta^n_{\mathbb{Q}}) \to C^{\cdot}(\Delta^n, \mathbb{Q}) \tag{A.4}$$

in $\mathsf{DGMod}(\mathbb{Q})$ by $\rho(\alpha)(\sigma) = \int_{\Delta^m_{\mathrm{top}}} \sigma_*(\alpha) \in \mathbb{Q}$ for a differential form $\alpha \in \Omega^{\cdot}(\Delta^n_{\mathbb{Q}})$ and a simplex $\sigma\colon [m] \to [n]$ (cf. [BG] Section 2.1 or [HS1] Section 4.4.1). When $[n] \in \Delta$ varies ρ becomes a transformation of functors $\Delta^{\circ} \to \mathsf{DGMod}(\mathbb{Q})$, i.e., a morphism in $\Delta^{\circ}\mathsf{DGMod}(\mathbb{Q})$. One directly verifies:

LEMMA A.5. *Let* $M \in \Delta\mathsf{DGMod}(\mathbb{Q})$. *Under the identifications* $\tilde{\mathrm{N}}M \cong \Omega^{\cdot}(\Delta_{\mathbb{Q}}) \otimes_{\leftarrow} M$ (*Example* A.2) *and* $\mathrm{N}M \cong C^{\cdot}(\Delta, \mathbb{Q}) \otimes_{\leftarrow} M$ (*Lemma* A.3), *the homomorphism* $\int_{\Delta}\colon \tilde{\mathrm{N}}M \to \mathrm{N}M$ *corresponds to* $\rho \otimes_{\leftarrow} 1$.

Proof of Theorem 1.12. In the course of the proof of Theorem 2.2 of [BG] it

is shown, using an acyclic models argument, that the homomorphism

$$\rho\colon\ A^{\cdot}(S,\mathbb{Q}) \to C^{\cdot}(S,\mathbb{Q})$$

is a homotopy equivalence, of functors $\Delta^{\circ}\mathsf{Sets} \to \mathsf{DGMod}(\mathbb{Q})$. This means that there exist natural transformations $\phi\colon\ C^{\cdot}(S,\mathbb{Q}) \to A^{\cdot}(S,\mathbb{Q})$, h and h' satisfying $\rho\phi = h\mathrm{D} + \mathrm{D}h$ and $\phi\rho = h'\mathrm{D} + \mathrm{D}h'$. Taking $S = \Delta^n$ and using Lemma A.3 we conclude that the morphism ρ of equation (A.4) is a homotopy equivalence in $\Delta^{\circ}\mathsf{DGMod}(\mathbb{Q})$. Applying $(-)\otimes_{\leftarrow} M$ it follows that $\int_{\Delta}\colon\ \tilde{\mathrm{N}}M \to \mathrm{N}M$ is a homotopy equivalence in $\mathsf{DGMod}(\mathbb{Q})$. □

Proof of Theorem 1.13. According to [BG] Proposition 3.3 (and using our Lemma A.3), there exists a $\mathbb{Q}$-linear homomorphism

$$\rho_2\colon\ \Omega^{\cdot}(\Delta_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \Omega^{\cdot}(\Delta_{\mathbb{Q}}) \to C^{\cdot}(\Delta,\mathbb{Q})$$

of functors on Δ°, of degree -1, s.t.

$$\mathrm{d}\rho_2 + \rho_2\mathrm{d} = \rho\mu - \mu(\rho\otimes\rho).$$

Here μ is multiplication. Extend this bilinearly to a graded homomorphism

$$\rho_2\colon\ (\Omega^{\cdot}(\Delta_{\mathbb{Q}}) \otimes_{\mathbb{Q}} A)^{\otimes 2} \to C^{\cdot}(\Delta,\mathbb{Q}) \otimes_{\mathbb{Q}} A$$

i.e.,

$$\rho_2((\alpha\otimes a)\otimes(\beta\otimes b)) = (-1)^{pq'}\rho_2(\alpha\otimes\beta)\otimes a\cdot b$$

for $a \in A^{p,l}$, $b \in A^{p',l}$, $\alpha \in \Omega^q(\Delta^l_{\mathbb{Q}})$ and $\beta \in \Omega^{q'}(\Delta^l_{\mathbb{Q}})$. Setting

$$\rho_2(u\otimes v) := (\rho_2(u_0\otimes v_0), \rho_2(u_1\otimes v_1), \ldots)$$

for $u = (u_0, u_1, \ldots), v = (v_0, v_1, \ldots) \in \tilde{\mathrm{N}}A$ we get a $\mathbb{Q}$-linear homomorphism $\rho_2\colon\ (\tilde{\mathrm{N}}A)^{\otimes 2} \to \mathrm{N}A$. A simple calculation shows that for any pair of cocycles $u, v \in \tilde{\mathrm{N}}A$,

$$\rho(u\otimes v) - \rho(u)\cdot\rho(v) = \mathrm{D}\rho_2(u\otimes v).$$ □

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