# THE DERIVED PICARD GROUP AND REPRESENTATIONS OF QUIVERS

AMNON YEKUTIELI

Abstract. The derived Picard group $\mathrm{DPic}(A)$ of a ring $A$ is the group of auto-equivalences of the derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A)$ induced by tilting complexes. When $A$ is either local or commutative, $\mathrm{DPic}(A)$ is a product of the usual (noncommutative) Picard group $\mathrm{Pic}(A)$ and a cyclic group; in this sense $\mathrm{DPic}(A)$ has a noncommutative geometry content.

In this talk we consider the case where $A$ is the path algebra of a finite quiver $\vec{\Delta}$ over a field $k$ (i.e. $A$ is a finite dimensional hereditary $k$-algebra). There is a natural action of $\mathrm{DPic}(A)$ on a certain infinite quiver. This action is faithful when $\vec{\Delta}$ is a tree, and otherwise a connected linear algebraic group may occur as a factor of $\mathrm{DPic}(A)$. At any rate we get an effective description of $\mathrm{DPic}(A)$; examples will be shown.

In the hereditary case $\mathrm{DPic}(A)$ coincides with the group of triangle auto-equivalences of the derived category of $A$-modules. This means that we can calculate the groups of auto-equivalences of various derived categories occurring, for example, in noncommutative geometry. (Joint work with J. Miyachi.)

## 0. Introduction

Let $k$ be a field, $A$ a $k$-algebra, and let $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A)$ be the derived category of $A$-modules. The derived Picard group $\mathrm{DPic}_k(A)$ is the group of auto-equivalences of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A)$ generated by tilting complexes.

The group $\mathrm{DPic}_k(A)$ becomes interesting when $A$ is nonlocal and noncommutative. This takes us into the realm of noncommutative algebraic geometry.

Today I will talk about the case when $A$ is a finite dimensional $k$-algebra. For a hereditary algebra $A$ we can completely describe the structure of the group $\mathrm{DPic}_k(A)$. This is a survey of joint work with J. Miyachi. Full details are in the paper [MY].

## 1. Some background

Suppose $k$ is a field and $A$ is a (unital, associative) $k$-algebra. Let $\mathsf{Mod}\,A$ be the category of left $A$-modules. Recall that Morita Theory tells us what are all $k$-linear auto-equivalences $F : \mathsf{Mod}\,A \to \mathsf{Mod}\,A$. They are (up to isomorphism) $FM = P \otimes_A M$, where $P$ is a $k$-central invertible bimodule (i.e. $A \cong P \otimes_A Q \cong Q \otimes_A P$ for some bimodule $Q$). We call the group generated by the invertible bimodules $P$ (with operation $P \otimes_A Q$) the (noncommutative) Picard group of $A$, denoted $\mathrm{Pic}_k(A)$.

**Remark 1.1.** If $A$ is commutative then $\mathrm{Pic}_k(A)$ contains the automorphism group $\mathrm{Aut}_k(A)$ and the usual (commutative) Picard group $\mathrm{Pic}_A(A)$ as subgroups.

Complete version of the manuscript submitted to a journal.

Now let us look at the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$. We will not attempt to say too much about it now. All we need to know are a few facts. The objects of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ are the bounded complexes of $A$-modules

$$M^{\cdot} = \left( \cdots \to 0 \to M^p \xrightarrow{\delta} M^{p+1} \xrightarrow{\delta} \cdots \to M^q \to 0 \to \cdots \right)$$

where $\delta \circ \delta = 0$. The category $\mathsf{Mod}\, A$ embeds in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ as the complexes concentrated in degree 0. And $M[n]$ is the complex with $M$ in degree $-n$. If $T^{\cdot}$ is a complex of bimodules, then one can define the derived tensor product $T^{\cdot} \otimes^{\mathrm{L}}_A M^{\cdot}$. We call $T^{\cdot}$ a tilting complex if the functor $F M^{\cdot} = T^{\cdot} \otimes^{\mathrm{L}}_A M^{\cdot}$ is an auto-equivalence of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$. Tilting complexes were considered by Rickard and Keller.

**Example 1.2.** If $P$ is an invertible bimodule then $P[n]$ is a tilting complex.

**Definition 1.3.** The derived Picard group $\mathrm{DPic}_k(A)$ is the group of isomorphism classes of tilting complexes, with operation $T^{\cdot} \otimes^{\mathrm{L}}_A S^{\cdot}$.

One reason to study $\mathrm{DPic}_k(A)$ is:

**Theorem 1.4** (Y.)**.** $\mathrm{DPic}_k(A)$ *parameterizes the isomorphism classes of dualizing complexes over $A$.*

We note that $\mathrm{DPic}_k(A)$ contains two natural subgroups: a copy of $\mathbb{Z}$ (represented by the complexes $A[n]$), and $\mathrm{Pic}_k(A)$.

**Theorem 1.5** (Y., Zimmermann-Rouquier-...)**.** *If $A$ is either local or commutative, then* $\mathrm{DPic}_k(A) = \mathrm{Pic}_k(A) \times \mathbb{Z}$*.*

This however is not the general situation:

**Example 1.6** (Y)**.** Consider the smallest nonlocal, noncommutative $k$-algebra $A = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$. The bimodule $A^* := \mathrm{Hom}_k(A, k)$ turns out to be a tilting complex. Let $\tau$ be the class of $A^*[-1]$ in $\mathrm{DPic}(A)$. Then one can show that $\tau \notin \mathrm{Pic}_k(A) \times \mathbb{Z}$.

This raises:

**Problem 1.7.** Study the structure of the group $\mathrm{DPic}_k(A)$ and its relation to the geometry of the noncommutative ring $A$.

Let us write $\mathrm{Out}^{\mathrm{tr}}_k(\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A))$ for the group of $k$-linear triangle auto-equivalences of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$. The next problem is open:

**Problem 1.8.** Is $\mathrm{DPic}_k(A) = \mathrm{Out}^{\mathrm{tr}}_k(\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A))$?

Recall that $A$ is hereditary if $\mathrm{gl.dim}\, A = 1$, or if every left of right ideal is a projective module.

**Theorem 1.9** (Miyachi and Y.)**.** *If $A$ is hereditary then the answer to the previous problem is positive.*

## 2. Quivers and Path Algebras

We will concentrate on a finite dimensional $k$-algebra $A$. Let us assume that $k$ is algebraically closed, and that $A$ is basic (i.e. $A/\mathfrak{p} = k$ for every prime ideal). Since every $k$-algebra is Morita equivalent to a basic one the latter is really no restriction.

The geometry of $A$ is described by a quiver $\vec{\Delta}$. Here is a nonorthodox definition of $\vec{\Delta}$. The set of vertices of $\vec{\Delta}$ is $\operatorname{Spec} A$, namely the set of prime ideals. Let $\mathfrak{r}$ be the Jacobson radical of $A$, and for each $\mathfrak{p}_i \in \operatorname{Spec} A$ let $e_i \in A/\mathfrak{r}$ be the corresponding central idempotent. So

$$\mathfrak{r}/\mathfrak{r}^2 = \bigoplus_{i,j} e_i(\mathfrak{r}/\mathfrak{r}^2)e_j \cong \bigoplus_{i,j} \frac{\mathfrak{p}_i \cap \mathfrak{p}_j}{\mathfrak{p}_i\mathfrak{p}_j}$$

as $A$-$A$-bimodules. Define $d_{i,j} := \dim_k \frac{\mathfrak{p}_i \cap \mathfrak{p}_j}{\mathfrak{p}_i\mathfrak{p}_j}$. Then there are $d_{i,j}$ arrows $\mathfrak{p}_i \to \mathfrak{p}_j$. The connected components of $\vec{\Delta}$ are called cliques, and they control Öre localization.

**Example 2.1.** Take $A = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$. The prime ideals are $\mathfrak{p}_1 = \begin{bmatrix} 0 & k \\ 0 & k \end{bmatrix}$ and $\mathfrak{p}_2 = \begin{bmatrix} k & k \\ 0 & 0 \end{bmatrix}$. Clearly $\frac{\mathfrak{p}_1 \cap \mathfrak{p}_2}{\mathfrak{p}_1\mathfrak{p}_2} = \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}$. So the quiver $\vec{\Delta}$ is $\underset{\mathfrak{p}_1}{\bullet} \longrightarrow \underset{\mathfrak{p}_2}{\bullet}$, a Dynkin quiver of type $A_2$.

On the other hand, given a finite quiver $\vec{\Delta}$ we can construct an algebra, called the path algebra $k\vec{\Delta}$. A path in $\vec{\Delta}$ is either a trivial path $e_x$ for each vertex $x$, or a sequence of arrows $\alpha_1\alpha_2\cdots\alpha_m$, such that $\alpha_{i+1}$ starts where $\alpha_i$ ends:

$$\underset{x_1}{\bullet} \xrightarrow{\alpha_1} \underset{x_2}{\bullet} \xrightarrow{\alpha_2} \underset{x_3}{\bullet} \xrightarrow{\alpha_3} \underset{x_4}{\bullet}$$

The paths form a basis of $k\vec{\Delta}$ as vector space, and multiplication of paths is by concatenation if possible, or 0.

**Example 2.2.** Take the quiver $\vec{\Delta} = \underset{x_1}{\bullet} \xrightarrow{\alpha} \underset{x_2}{\bullet}$. Then $e_{x_1}, e_{x_2}$ are orthogonal idempotents; $e_{x_1}\alpha = \alpha = \alpha e_{x_2}$; and $\alpha e_{x_1} = \alpha^2 = e_{x_2}\alpha = 0$. Thus $k\vec{\Delta} \cong \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$, with

$$e_{x_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ e_{x_2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ \alpha = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Here is a classical structure theorem for finite dimensional algebras.

**Theorem 2.3** (Gabriel). *Let $A$ be a basic finite dimensional $k$-algebra with quiver $\vec{\Delta}$. Then*

1. *$A \cong k\vec{\Delta}/I$ for an ideal $I \subset \mathfrak{a}^2$, where $\mathfrak{a} \subset k\vec{\Delta}$ is the ideal generated by the arrows.*
2. *$I = 0$ iff $A$ is hereditary.*

Some remarks:

**Remark 2.4.** According to the definitions of Cuntz-Quillen, a finite dimensional algebra $A$ is smooth iff it is hereditary.

**Remark 2.5.** The surjection $\vec{\Delta} \twoheadrightarrow A$ should be viewed as a noncommutative analog of the Cohen theorem for commutative rings: $k[[t_1, \ldots, t_n]] \twoheadrightarrow A$. Indeed the arrows in $k\vec{\Delta}$ are lifts of a basis of $\mathfrak{r}/\mathfrak{r}^2$.

**Remark 2.6.** Beilinson showed that the derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\,\mathbf{P}^1)$ of coherent sheaves on the projective line $\mathbf{P}^1$ is equivalent to $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,k\vec{\Delta}_2)$, for a certain quiver $\vec{\Delta}_2$. According to Kontsevich-Rosenberg one has $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\,\mathbf{NP}^n) \approx \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,k\vec{\Delta}_{n+1})$, where $\mathbf{NP}^n$ is their noncommutative projective $n$-space, so $\mathrm{Out}_k^{\mathrm{tr}}(\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\,\mathbf{NP}^n)) \cong \mathrm{DPic}_k(k\vec{\Delta}_{n+1})$.

## 3. The Auslander-Reiten Quiver

From now on $A = k\vec{\Delta}$ and the quiver $\vec{\Delta}$ is connected. An $A$-module is also called a representation of $\vec{\Delta}$. As customary we write $\mathsf{mod}\,A$ for the category of finitely generated $A$-modules (=finite dimensional representations).

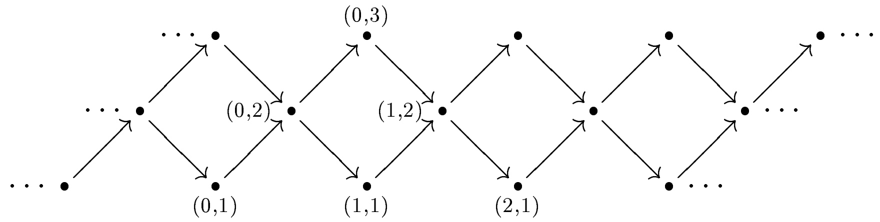
One of the important invariants of the algebra $A$ is the set of isomorphism classes of indecomposable modules. If this set is finite $A$ is said to have finite representation type.

**Theorem 3.1** (Gabriel)**.** *$A$ has finite representation type iff $\vec{\Delta}$ is a Dynkin quiver of type $A_n, D_n, E_6, E_7$ or $E_8$.*

The Auslander-Reiten quiver $\vec{\Gamma}(\mathsf{mod}\,A)$ is defined as follows. Its vertices are the isomorphism classes of indecomposable $A$-modules, and there are $d$ arrows $x \to y$, where $d$ is the dimension of the space of irreducible homomorphisms $\mathrm{Irr}(M_x, M_y)$ (this is like $\mathfrak{r}/\mathfrak{r}^2$).

Happel showed how to define the Auslander-Reiten quiver $\vec{\Gamma} = \vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A))$. This is an infinite quiver. Usually it is very complicated, but it contains a nice subquiver denoted $\vec{\mathbb{Z}}\vec{\Delta}$ that is easy to describe ($\vec{\mathbb{Z}}\vec{\Delta}$ was introduced by Riedtmann). When $\vec{\Delta}$ is a Dynkin quiver, then in fact $\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)) = \vec{\mathbb{Z}}\vec{\Delta}$.

**Example 3.2.** Take $\vec{\Delta} = \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet}$ . Then $\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)) = \vec{\mathbb{Z}}\vec{\Delta}$ is




The vertices in $\vec{\Gamma}(\mathsf{mod}\,A)$ are labeled.

The group $\mathrm{DPic}_k(A)$ acts naturally on the category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$, and hence on the quiver $\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A))$ (by quiver automorphisms). The element $\tau \in \mathrm{DPic}_k(A)$, represented by the tilting complex $A^*[-1]$, is called the translation. In the example above it is a shift by 1 to the left. Using this action we proved the following theorems. Let $\mathrm{Aut}(\vec{\mathbb{Z}}\vec{\Delta})$ denote the group of quiver automorphisms.

**Theorem 3.3** (Miyachi and Y.). *Suppose $A$ has finite representation type. Then*

$$\mathrm{DPic}_k(A) \cong \mathrm{Aut}(\mathbb{Z}\vec{\Delta})^{\langle\tau\rangle}.$$

**Example 3.4.** Let $\sigma \in \mathrm{DPic}_k(A)$ be represented by $A[1]$. If $\vec{\Delta}$ is of Dynkin type $A_n$ then $\mathrm{DPic}_k(A)$ is abelian, generated by $\tau$ and $\sigma$, with one relation $\tau^{n+1} = \sigma^{-2}$.

We denote by $\mathrm{Aut}((\mathbb{Z}\vec{\Delta})_0; d)$ the group of permutations of the vertex set $(\mathbb{Z}\vec{\Delta})_0$ that preserve the arrow-multiplicity $d$.

**Theorem 3.5** (Miyachi and Y.). *If $A$ has infinite representation type then there is an isomorphism of groups*

$$\mathrm{DPic}_k(A) \cong \big(\mathrm{Aut}((\mathbb{Z}\vec{\Delta})_0; d)^{\langle\tau\rangle} \ltimes \mathrm{Pic}^0_k(A)\big) \times \mathbb{Z}.$$

$\mathrm{Pic}^0_k(A)$ *is a connected linear algebraic group, and is trivial when $\vec{\Delta}$ is a tree.*

To conclude,

**Problem 3.6.** Can this analysis of $\mathrm{DPic}_k(A)$ be carried out for other finite dimensional algebras $A$? We note that that the quiver $\vec{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A))$ exists whenever $\mathrm{gl.dim}\, A < \infty$.

## References


[ARS] M. Auslander, I. Reiten and Sverre O. Smalø, "Representation Theory of Artin Algebras," Cambridge Studies in Advanced Math. **36**, Cambridge, UK 1997 (corrected paperback edition).

[Be] A.A. Beilinson, Coherent sheaves on $\mathbf{P}^n$ and problems of linear algebra, Func. Anal. Appl. **12** (1978), 214-216.

[BO] A.I. Bondal and D.O. Orlov, Reconstruction of a variety from the derived category and groups of autoequivalences, preprint; eprint: alg-geom/9712029.

[GS] F. Guil-Asensio and M. Saorin, The automorphism group and the Picard group of a monoidal algebra, Comm. Algebra **27** (1999), 857-887.

[GR] P. Gabriel and A.V. Roiter, "Representations of finite-dimensional algebras," Springer-Verlag, Berlin, 1997.

[Ha] D. Happel, "Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras," London Math. Soc. Lecture Notes **119**, University Press, Cambridge, 1987.

[KR] M. Kontsevich and A. Rosenberg, Noncommutative smooth spaces, eprint math.AG/9812158.

[MY] J. Miyachi and A. Yekutieli, Derived Picard groups of finite dimensional hereditary algebras, eprint: math.RA/9904006.

[RD] R. Hartshorne, "Residues and Duality", Lecture Notes in Math. **20**, Springer-Verlag, Berlin, 1966.

[Ri1] J. Rickard, Morita theory for derived categories, J. London Math. Soc. **39** (1989), 436-456.

[Ri2] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. **43** (1991), 37-48.

[Rl] C.M. Ringel, "Tame Algebras and Integral Quadratic Forms," Lecture Notes in Math. **1099**, Springer-Verlag, Berlin, 1984.

[Ye1] A. Yekutieli, Dualizing complexes over noncommutative graded algebras, J. Algebra **153** (1992), 41-84.

[Ye2] A. Yekutieli, Dualizing complexes, Morita equivalence and the derived Picard group of a ring, to appear in: J. London Math. Soc.; eprint: math.RA/9810134.

[Zi] A. Zimmermann, Derived equivalences of orders, Canadian Math. Soc. Conference Proceedings **18** (1996), 721-749.

Department of Mathematics and Computer Science, Ben Gurion University, Be'er Sheva 84105, ISRAEL

*E-mail address*: amyekut@math.bgu.ac.il

*WWW*: http://www.math.bgu.ac.il/∼amyekut