DUALITY AND TILTING FOR COMMUTATIVE DG RINGS

AMNON YEKUTIELI

Abstract. We consider commutative DG rings (better known as nonpositive strongly commutative associative unital DG algebras). For such a DG ring A we define the notions of *perfect, tilting, dualizing, Cohen-Macaulay and rigid DG A-modules*. *Geometrically perfect* DG modules are defined by a local condition on Spec $\bar{A}$, where $\bar{A}$ is the commutative ring $\mathrm{H}^0(A)$. *Algebraically perfect* DG modules are those that can be obtained from A by finitely many shifts, direct summands and cones. Tilting DG modules are those that have inverses w.r.t. the derived tensor product; their isomorphism classes form the derived Picard group $\mathrm{DPic}(A)$. Dualizing DG modules are a generalization of Grothendieck's original definition (and here A has to be *cohomologically pseudo-noetherian*). Cohen-Macaulay DG modules are the duals (w.r.t. a given dualizing DG module) of finite $\bar{A}$-modules. Rigid DG A-modules, relative to a commutative base ring $\mathbb{K}$, are defined using the *squaring operation*, and this is a generalization of Van den Bergh's original definition.

The techniques we use are the standard ones of derived categories, with a few improvements. We introduce a new method for studying DG A-modules: Čech resolutions of DG A-modules corresponding to open coverings of Spec $\bar{A}$.

Here are some of the new results obtained in this paper:

- A DG A-module is geometrically perfect iff it is algebraically perfect.
- The canonical group homomorphism $\mathrm{DPic}(A) \to \mathrm{DPic}(\bar{A})$ is bijective.
- The group $\mathrm{DPic}(A)$ acts simply transitively on the set of isomorphism classes of dualizing DG A-modules.
- Cohen-Macaulay DG modules are insensitive to cohomologically surjective DG ring homomorphisms.
- Rigid dualizing DG A-modules are unique up to unique rigid isomorphisms.

The functorial properties of Cohen-Macaulay DG modules that we establish here are needed for our work on rigid dualizing complexes over commutative rings, schemes and Deligne-Mumford stacks.

We pose several conjectures regarding existence and uniqueness of rigid DG modules over commutative DG rings.

Contents

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0. Introduction

In this paper we are interested in *commutative DG rings*. This is an abbreviation for strongly commutative nonpositive associative unital differential graded algebras over $\mathbb{Z}$. Thus a commutative DG ring is a graded ring $A = \bigoplus_{i \leq 0} A^i$, together with a differential d of degree 1, that satisfies the graded Leibniz rule. The multiplication satisfies $b \cdot a = (-1)^{ij} \cdot a \cdot b$ for all $a \in A^i$ and $b \in A^j$, and $a \cdot a = 0$ if i is odd. By default, the DG rings in the Introduction are commutative. Rings are viewed as DG rings concentrated in degree 0 (so they are commutative by default throughout the Introduction).

Commutative DG rings come up in the foundations of *derived algebraic geometry*, as developed by Toën-Vezzosi [TV]. Indeed, a *derived stack* is a stack of groupoids on the site of commutative DG rings (with its étale topology). Some precursors of this point of view are the papers [Hi2], [Ke2], [KoSo] and [Be]. Another role of commutative DG rings is as *resolutions of commutative rings*. We shall say more on this role below, since it was the immediate motivation for writing the present paper.

In our paper we study *perfect, tilting, dualizing, Cohen-Macaulay and rigid DG modules* over commutative DG rings. We give definitions that generalize the familiar definitions for rings. Many of the results that hold for rings, continue to hold in the much more complicated DG setting. Later in the Introduction we discuss the main definitions and results of the paper, and a couple of conjectures.

But first let us explain the problem that motivated our work on commutative DG rings. *Rigid dualizing complexes* over noetherian commutative rings are the foundation of a new approach to Grothendieck Duality on schemes and Deligne-Mumford stacks. See the papers [YZ1], [YZ2], [Ye3], [Ye6], [Ye8] and [Ye9]. Rigid dualizing complexes were introduced by Van den Bergh in the context of noncommutative rings over a base field $\mathbb{K}$; see [VdB]. But for the new approach to Grothendieck Duality in algebraic geometry, that should apply also to the arithmetic setup, we are interested in commutative rings over a base commutative ring $\mathbb{K}$ that is *not a field*.

In this context, the definition of a rigid complex relies on the more primitive notion of the *square of a complex*. Given a commutative $\mathbb{K}$-ring A, we choose a *K-flat commutative DG ring resolution* $\tilde{A} \to A$ relative $\mathbb{K}$. Such resolutions exist, because we work with strongly commutative DG rings (see Remark 4.3 for a discussion of this issue). For any complex of A-modules M, its square is the complex of A-modules

$$\operatorname{Sq}_{A/\mathbb{K}}(M) := \operatorname{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(A, M \otimes^{\mathrm{L}}_{\mathbb{K}} M). \tag{0.1}$$

The question of independence of $\operatorname{Sq}_{A/\mathbb{K}}(M)$ of the choice of resolution $\tilde{A}$ is very subtle; and in fact there was a mistake in the original proof in [YZ1] (which has since been corrected; see [Ye5]). A rigid dualizing complex over A relative to $\mathbb{K}$ is a pair (R, ρ), consisting of a dualizing complex R, and an isomorphism $\rho : R \xrightarrow{\simeq} \operatorname{Sq}_{A/\mathbb{K}}(R)$ in the derived category $\mathsf{D}(A)$.

In the course of writing the new paper [Ye7] – which corrects further mistakes in the earlier paper [YZ1], and extends it – we realized that we need Theorem 0.8, that deals with Cohen-Macaulay DG modules. This is explained in Remark 8.9.

The parts of the present paper leading to Section 8 set the stage for the definition of Cohen-Macaulay DG modules and the proof of Theorem 0.8.

From a wider perspective, we expect that the results in this paper shall find further applications in algebra and geometry. One such possible application would be the development of a theory of *rigid dualizing complexes in derived algebraic geometry* (of either flavor: Lurie's or Toën's). Indeed, a commutative DG ring is an affine derived scheme; so what is needed is a way to sheafify our constructions. That should be facilitated by the *étale descent* and *étale codescent* properties of *rigid residue complexes* (see [Ye6]).

Our work in the present paper should be easily accessible to anyone with a working knowledge of the derived category of modules over a ring (e.g. from the book [We]). This is because the methods we use are basically the same; there are only slight modifications. The necessary tools to upgrade from rings to DG rings (such as K-injective resolutions in place of injective resolutions) are recalled in Section 1 of our paper. We do not resort at all to the daunting technicalities of E_∞ rings. Likewise, we do not touch simplicial methods or Quillen model structures.

Let us now describe the work in this paper. Consider a commutative DG ring $A = \bigoplus_{i \leq 0} A^i$. Its cohomology $\mathrm{H}(A) = \bigoplus_{i \leq 0} \mathrm{H}^i(A)$ is a commutative graded ring. We use the notation $\bar{A} := \mathrm{H}^0(A)$. There is a canonical homomorphism of DG rings $A \to \bar{A}$. We say that A is *cohomologically pseudo-noetherian* if $\bar{A}$ is a noetherian ring, and $\mathrm{H}^i(A)$ is a finite (i.e. finitely generated) $\bar{A}$-module for every i. (Note that this is weaker than the condition that the ring $\mathrm{H}(A)$ is noetherian.)

The category of DG A-modules is denoted by $\mathsf{C}(A)$. It is a DG category, and its derived category, gotten by inverting the quasi-isomorphisms, is denoted by $\mathsf{D}(A)$. There are full triangulated subcategories $\mathsf{D}^+(A)$, $\mathsf{D}^-(A)$ and $\mathsf{D}^{\mathrm{b}}(A)$ of $\mathsf{D}(A)$, made up of the DG modules M with bounded below, bounded above and bounded cohomologies, respectively. The full subcategory of $\mathsf{D}(A)$ on the DG modules M, whose cohomology modules $\mathrm{H}^i(M)$ are finite over $\bar{A}$, is denoted by $\mathsf{D}_{\mathrm{f}}(A)$. As usual, for any boundedness condition $\star$ we let $\mathsf{D}^\star_{\mathrm{f}}(A) := \mathsf{D}_{\mathrm{f}}(A) \cap \mathsf{D}^\star(A)$. If A is cohomologically pseudo-noetherian, then the categories $\mathsf{D}^\star_{\mathrm{f}}(A)$ are triangulated, and $A \in \mathsf{D}^-_{\mathrm{f}}(A)$. In case A is a ring, then $\mathsf{D}(A) = \mathsf{D}(\mathsf{Mod}\, A)$, the derived category of A-modules.

In Section 1 we recall some facts on DG modules. We mention several kinds of resolutions of DG modules, and special attention is paid to semi-free resolutions. Section 2 is about various notions of cohomological dimension for DG modules and derived functors. For instance, in Definition 2.4 we introduce the projective and injective dimensions of a DG A-module M *relative to a subcategory* $\mathsf{E} \subseteq \mathsf{D}(A)$. In Section 3 we study the reduction functor $\mathsf{D}(A) \to \mathsf{D}(\bar{A})$, $M \mapsto \bar{A} \otimes^{\mathrm{L}}_A M$. We show that projective $\bar{A}$-modules can be lifted to DG A-modules.

In Section 4 we discuss localization of a commutative DG ring A on $\operatorname{Spec} \bar{A}$. We introduce the *Čech resolution* $\mathrm{C}(M; \boldsymbol{a})$ of a DG A-module M, associated to a *covering sequence* $\boldsymbol{a} = (a_1, \ldots, a_n)$ of $\bar{A}$. In case there is a decomposition $\operatorname{Spec} \bar{A} = \coprod_{i=1}^n \operatorname{Spec} \bar{A}_i$ into open-closed subsets, we show that there are canonically defined DG rings $A_1, \ldots, A_n$, and a DG ring quasi-isomorphism $A \to \prod_{i=1}^n A_i$, that in H^0 recovers the decomposition $\bar{A} \cong \prod_{i=1}^n \bar{A}_i$.

The topic of Section 5 is *perfect* DG modules. A DG A-module P is called *geometrically perfect* if locally on $\operatorname{Spec} \bar{A}$ it is isomorphic, in the derived category, to a finite semi-free DG module. See Definition 5.4 for the precise formulation. This definition appears to be completely new for DG rings. When A is a ring (so that $A = A^0 = \bar{A}$), this definition coincides with the one in [SGA 6, Exposé I], since a finite semi-free DG module over a ring is just a bounded complex of finite free modules.

A DG A-module P is called *algebraically perfect* if it can be finitely built from A by shifts, direct summands and cones. In other words, if P belongs to the epaisse subcategory of $\mathsf{D}(A)$ classically generated by A, in the sense of [BV]. This definition (without the qualification "algebraically") was already used in [ABIM].

When A is a ring, it is known that geometrically perfect complexes are the same as algebraically perfect complexes; see [SGA 6, Exposé I]. Therefore they are just called "perfect complexes".

Here is the main result Section 5. It is a combination of Theorem 5.11, Theorem 5.20 and Corollary 5.21.

Theorem 0.2. *Let A be a commutative DG ring, and let P be a DG A-module. The following four conditions are equivalent*:

(i) *The DG A-module P is geometrically perfect.*
(ii) *The DG A-module P is in $\mathsf{D}^{-}(A)$, and the DG $\bar{A}$-module $\bar{A} \otimes^{\mathrm{L}}_{A} P$ is geometrically perfect.*
(iii) *For any $M, N \in \mathsf{D}(A)$, the canonical morphism*
$$\mathrm{RHom}_{A}(P, M) \otimes^{\mathrm{L}}_{A} N \to \mathrm{RHom}_{A}(P, M \otimes^{\mathrm{L}}_{A} N)$$
in $\mathsf{D}(A)$ is an isomorphism.
(iv) *The DG A-module P is algebraically perfect.*

If A is cohomologically pseudo-noetherian, then the four conditions above are equivalent to:

(v) *The DG A-module P is in $\mathsf{D}^{-}_{\mathrm{f}}(A)$, and it has finite projective dimension relative to $\mathsf{D}(A)$.*

When A is a commutative ring, conditions (i) and (ii) are essentially the same; and (as already mentioned above) the equivalence of conditions (i) and (iv) was proved in [SGA 6]. But for DG rings this is a new result. In light of Theorem 0.2 we can unambiguously talk about "perfect DG modules".

Section 6 is about *tilting* DG modules. A DG A-module P is said to be tilting if there is some DG module Q such that $P \otimes^{\mathrm{L}}_{A} Q \cong A$ in $\mathsf{D}(A)$. The DG module Q is called a quasi-inverse of P. The next theorem is repeated as Theorem 6.5.

Theorem 0.3. *Let A be a commutative DG ring, and let P be a DG A-module. The following four conditions are equivalent*:

(i) *The DG A-module P is tilting .*
(ii) *The functor $P \otimes^{\mathrm{L}}_{A} -$ is an equivalence of $\mathsf{D}(A)$.*
(ii) *The functor $\mathrm{RHom}_{A}(P, -)$ is an equivalence of $\mathsf{D}(A)$.*
(iv) *The DG A-module P is perfect, and the adjunction morphism $A \to \mathrm{RHom}_{A}(P, P)$ in $\mathsf{D}(A)$ is an isomorphism.*

A combination of Theorems 0.3 and 0.2 implies that the DG A-module $Q := \mathrm{RHom}_{A}(P, A)$ is a quasi-inverse of the tilting DG module P.

As in [Ye2], we define the *commutative derived Picard group* $\mathrm{DPic}(A)$ to be the group whose elements are the isomorphism classes of tilting DG A-modules, and the multiplication is induced by $- \otimes^{\mathrm{L}}_{A} -$.

If $A \to B$ is a homomorphism of DG rings, then the operation $P \mapsto B \otimes^{\mathrm{L}}_{A} P$ induces a group homomorphism $\mathrm{DPic}(A) \to \mathrm{DPic}(B)$. The next result is Theorem 6.14 in the body of the paper.

Theorem 0.4. *Let A be a commutative DG ring. The canonical group homomorphism*
$$\mathrm{DPic}(A) \to \mathrm{DPic}(\bar{A})$$
is bijective.

In an earlier version of the paper we had a finiteness condition: we required $\operatorname{Spec} \bar{A}$ to have finitely many connected components. But, as noticed by Negron [Ng], this condition is superfluous.

It is known that the commutative derived Picard group of the ring $\bar{A}$ has this structure:

$$\operatorname{DPic}(\bar{A}) \cong \operatorname{Pic}(\bar{A}) \times \mathrm{F}_{\mathrm{lc}}(\operatorname{Spec} \bar{A}, \mathbb{Z}) .$$

Here $\mathrm{F}_{\mathrm{lc}}(\operatorname{Spec} \bar{A}, \mathbb{Z})$ is the group of locally constant functions $\operatorname{Spec} \bar{A} \to \mathbb{Z}$, and $\operatorname{Pic}(\bar{A})$ is the usual (commutative) Picard group. See Theorem 6.13, due to Negron, that refines earlier results in [Ye2], [RZ] and [Ye4].

In Section 7 we talk about *dualizing* DG modules. Here A is a cohomologically pseudo-noetherian commutative DG ring. A DG A-module $R \in \mathsf{D}^{+}_{\mathrm{f}}(A)$ is called dualizing if it has finite injective dimension relative to $\mathsf{D}(A)$, and the adjunction morphism $A \to \operatorname{RHom}_A(R, R)$ is an isomorphism. Note that when A is a ring, this is precisely the original definition found in [RD]; but for a DG ring there are several possible notions of injective dimension, and the correct one has to be used. See Definition 2.4(2) and Remark 2.9. Note also that R need not have bounded cohomology – see Corollary 7.3 and Example 7.26. For comparisons to dualizing DG modules, as defined previously in [Hi1], [FIJ] and [Lu2], see Example 7.23, Proposition 7.17 and Remark 7.27 respectively.

A DG ring A is called *tractable* if it is cohomologically pseudo-noetherian, and there is a homomorphism $\mathbb{K} \to A$ from a finite dimensional regular noetherian commutative ring $\mathbb{K}$, such that the induced homomorphism $\mathbb{K} \to \bar{A}$ is essentially finite type. Such a homomorphism $\mathbb{K} \to A$ is called a *traction* for A.

The next result is a combination of Theorem 7.9 and Corollary 7.11. When A is a ring, this was proved by Grothendieck [RD, Sections V.3 and V.10].

Theorem 0.5. *Let A be a tractable commutative DG ring. Then:*

(1) *A has a dualizing DG module.*
(2) *The operation $(P, R) \mapsto P \otimes^{\mathrm{L}}_A R$, for a tilting DG module P and a dualizing DG module R, induces a simply transitive action of the group $\operatorname{DPic}(A)$ on the set of isomorphism classes of dualizing DG A-modules.*

In particular, if $\bar{A}$ is a local ring, then by Theorems 0.4 and 6.13 we have $\operatorname{DPic}(A) \cong \mathbb{Z}$. Thus any two dualizing DG A-modules R, R' satisfy $R' \cong R[m]$ for an integer m.

A combination of Theorems 0.4 and 0.5 yields (see Corollary 7.12):

Corollary 0.6. *If A is a tractable commutative DG ring, then the operation $R \mapsto \operatorname{RHom}_A(\bar{A}, R)$ induces a bijection*

$$\frac{\{\text{dualizing DG } A\text{-modules}\}}{\text{isomorphism}} \xrightarrow{\simeq} \frac{\{\text{dualizing DG } \bar{A}\text{-modules}\}}{\text{isomorphism}} .$$

Here is a result that is quite surprising. It relies on a theorem of Jørgensen [Jo], who proved it in the local case (i.e. when $\bar{A}$ is a local ring).

Theorem 0.7. *Let A be a cohomologically bounded tractable commutative DG ring. If $\bar{A}$ is a perfect DG A-module, then the canonical homomorphism $A \to \bar{A}$ is a quasi-isomorphism.*

This is repeated (in slightly stronger form) as Theorem 7.21 in the body of the paper. See Remark 7.25 for an interpretation of this theorem.

Section 8 of the paper is about *Cohen-Macaulay* DG modules. The definition does not involve regular sequences of course; nor does it involve vanishing of local cohomologies as in [RD] (even though it could probably be stated in this language). Instead we use a fact discovered in [YZ3]: for a noetherian scheme X with dualizing complex $\mathcal{R}$, a complex $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\operatorname{Mod} \mathcal{O}_X)$ is CM (in the sense of [RD], for the dimension function determined by $\mathcal{R}$) iff $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},\mathcal{R})$ is (isomorphic to) a coherent sheaf. In [YZ3] the CM complexes inside $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ were also called *perverse coherent sheaves*.

With the explanation above, the next definition makes sense. Let R be a dualizing DG A-module. A DG module $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ is called *CM with respect to R* if $\mathrm{RHom}_A(M,R) \in \mathsf{D}^{0}_{\mathrm{f}}(A)$. Here $\mathsf{D}^{0}_{\mathrm{f}}(A)$ is the full subcategory of $\mathsf{D}(A)$ consisting of DG modules with finite cohomology concentrated in degree 0; and we know that it is equivalent to the category $\mathsf{Mod}_{\mathrm{f}}\,\bar{A}$ of finite $\bar{A}$-modules.

The next theorem is repeated, in slightly greater generality, as Theorem 8.7.

Theorem 0.8. *Let $f : A \to B$ be a homomorphism between tractable commutative DG rings, such that $\mathrm{H}^0(f) : \bar{A} \to \bar{B}$ is surjective. Let R_B be a dualizing DG B-module, and let $M, N \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(B)$.*

1. *If M is CM w.r.t. R_B, and there is an isomorphism $\mathrm{rest}_f(M) \cong \mathrm{rest}_f(N)$ in $\mathsf{D}(A)$, then N is also CM w.r.t. R_B.*
2. *If M and N are both CM w.r.t. R_B, then the homomorphism*
$$\mathrm{rest}_f : \mathrm{Hom}_{\mathsf{D}(B)}(M,N) \to \mathrm{Hom}_{\mathsf{D}(A)}\big(\mathrm{rest}_f(M), \mathrm{rest}_f(N)\big)$$
is bijective.

In the theorem, $\mathrm{rest}_f : \mathsf{D}(B) \to \mathsf{D}(A)$ is the restriction functor. As already mentioned, Theorem 0.8 is needed in [Ye7].

In our paper [Ye5] we introduce the *squaring operation* for commutative DG rings. The construction goes like this. The input is a homomorphism $A \to B$ of commutative DG rings. To this datum we associate a functor $\mathrm{Sq}_{B/A}$ from $\mathsf{D}(B)$ to itself. The formula is a bit more general than (0.1): we choose any K-flat DG ring resolution $\tilde{A} \to \tilde{B}$ of $A \to B$ (see Section 9), and we define

$$\mathrm{Sq}_{B/A}(M) := \mathrm{RHom}_{\tilde{B} \otimes_{\tilde{A}} \tilde{B}}(B, M \otimes^{\mathrm{L}}_{\tilde{A}} M) \in \mathsf{D}(B). \tag{0.9}$$

The proof that this definition does not depend on the resolution $\tilde{A} \to \tilde{B}$ is quite difficult.

A *rigid DG module* over B relative to A is a pair (M,ρ), consisting of a DG B-module M, and a *rigidifying isomorphism* $\rho : M \xrightarrow{\simeq} \mathrm{Sq}_{B/A}(M)$ in $\mathsf{D}(B)$. There is a notion of rigid morphism between rigid DG modules (Definition 9.3). Here are our results on rigid DG modules. The first is a DG version of [YZ1, Theorem 0.2], and it is repeated as Theorem 9.4.

Theorem 0.10. *Let $A \to B$ be a homomorphism of commutative DG rings, and let (M,ρ) be a rigid DG module over B relative to A. Assume that the adjunction morphism $B \to \mathrm{RHom}_B(M,M)$ in $\mathsf{D}(B)$ is an isomorphism. Then the only rigid automorphism of (M,ρ) is the identity.*

Next is a DG version of the uniqueness in [YZ2, Theorem 1.1](1). It is repeated as Theorem 9.7.

Theorem 0.11. *Let A be a tractable commutative DG ring, with traction $\mathbb{K} \to A$. Suppose (R,ρ) and (R',ρ') are rigid dualizing DG modules over A relative to $\mathbb{K}$. Then there is a unique rigid isomorphism $(R,\rho) \cong (R',\rho')$.*

Thus we can call (R,ρ) *the* rigid dualizing DG module of A relative to $\mathbb{K}$.

Conjecture 0.12. In the situation of Theorem 0.11, the rigid dualizing DG module over A relative to $\mathbb{K}$ exists.

Indeed, we give more detailed conjectures in Section 9, that would imply Conjecture 0.12. When A is a ring, these assertions were already proved in [YZ1] (with some corrections in [Ye7]). In case A has bounded cohomology, Conjecture 0.12 was very recently proved by Shaul [Sh2].

We end the Introduction with another conjecture on rigid DG modules.

Conjecture 0.13. In the situation of Theorem 0.11, let (M, ρ) be a rigid DG module over A relative to $\mathbb{K}$. Assume that $M \in \mathsf{D}^{+}_{\mathrm{f}}(A)$, and that M is nonzero on each connected component of $\operatorname{Spec} \bar{A}$. Then M is a dualizing DG A-module.

When A is a ring and $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$, this was proved in [YZ2] and [AIL]. More on this conjecture in Remark 9.9.

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1. DG Modules and their Resolutions

A *DG ring* (usually called an associative unital DG algebra over $\mathbb{Z}$) is a graded ring $A = \bigoplus_{i \in \mathbb{Z}} A^i$, with differential d of degree 1, satisfying the graded Leibniz rule

$$\mathrm{d}(a \cdot b) = \mathrm{d}(a) \cdot b + (-1)^i \cdot a \cdot \mathrm{d}(b)$$

for $a \in A^i$ and $b \in A^j$. A homomorphism of DG rings is a degree 0 ring homomorphism that commutes with the differentials. In this way we get a category, denoted by DGR. Rings are viewed as DG rings concentrated in degree 0. For a DG ring A, the cohomology $\mathrm{H}(A) = \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^i(A)$ is a graded ring.

The *opposite* of the DG ring A is the DG ring A^{op}, which is the same graded abelian group as A, with the same differential, but the multiplication $\cdot^{\mathrm{op}}$ is reversed in the graded sense. Namely, for elements $a \in A^i$ and $b \in A^j$, their product in A^{op} is

$$a \cdot^{\mathrm{op}} b := (-1)^{ij} \cdot b \cdot a.$$

A left DG A-module is a graded left A-module $M = \bigoplus_{i \in \mathbb{Z}} M^i$, with differential d satisfying the graded Leibniz rule. If M is a left DG A-module then $\mathrm{H}(M) = \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^i(M)$ is a graded left $\mathrm{H}(A)$-module. Note that a right DG A-module is the same as a left DG A^{op}-module.

Convention 1.1. By default, in this paper all DG modules are left DG modules.

From Section 4 onwards our DG rings will be commutative, and for them the distinction between left and right DG modules becomes negligible.

Definition 1.2. Let A be a DG ring. The category of (left) DG A-modules, with A-linear homomorphisms of degree 0 that commute with differentials, is denoted by $\mathsf{DGMod}\, A$, or by its abbreviation $\mathsf{C}(A)$. The derived category, gotten from $\mathsf{DGMod}\, A$ by inverting quasi-isomorphisms, is denoted by $\mathsf{D}(\mathsf{DGMod}\, A)$, or by the abbreviation $\mathsf{D}(A)$.

For information on $\mathsf{D}(A)$ see [BL, Section 10], [Ke1, Section 2] or [SP, Section 09KV]. If A is a ring, then $\mathsf{C}(A) = \mathsf{C}(\mathsf{Mod}\, A)$, the category of complexes of A-modules; and $\mathsf{D}(A) = \mathsf{D}(\mathsf{Mod}\, A)$, the usual derived category of the abelian category $\mathsf{Mod}\, A$.

Definition 1.3. Let $A = \bigoplus_{i \in \mathbb{Z}} A^i$ be a DG ring.

(1) We say that A is *a nonpositive DG ring* if $A^i = 0$ for all $i > 0$.
(2) If A is nonpositive DG ring, then we write $\bar{A} := \mathrm{H}^0(A)$, which is a ring. There is a canonical DG ring homomorphism $A \to \bar{A}$.

The full subcategory of DGR on the nonpositive DG rings is denoted by $\mathsf{DGR}^{\leq 0}$.

Convention 1.4. By default, in this paper all DG rings are nonpositive; i.e. we work inside $\mathsf{DGR}^{\leq 0}$.

One of the important advantages of nonpositive DG rings is that the differential d of any DG A-module M is A^0-linear. This implies that the two smart truncations

$$\operatorname{smt}^{\geq i}(M) := \big(\cdots \to 0 \to \operatorname{Coker}(\mathrm{d}|_{M^{i-1}}) \to M^{i+1} \to M^{i+2} \to \cdots \big) \tag{1.5}$$

and

$$\operatorname{smt}^{\leq i}(M) := \big(\cdots \to M^{i-2} \to M^{i-1} \to \operatorname{Ker}(\mathrm{d}|_{M^{i}}) \to 0 \to \cdots \big) \tag{1.6}$$

remain within $\mathsf{C}(A)$; and there are functorial homomorphisms $M \to \operatorname{smt}^{\geq i}(M)$ and $\operatorname{smt}^{\leq i}(M) \to M$ in $\mathsf{C}(A)$, inducing isomorphisms in $\mathrm{H}^{\geq i}$ and $\mathrm{H}^{\leq i}$ respectively. Note that these are the truncations $\tau_{\leq n}$ and $\tau_{\geq n}$ from [SP, Section 0118], that are variants of the truncations $\sigma_{>n}$ and $\sigma_{\leq n}$ from [RD, Section I.7, page 69]. Warning: the two stupid truncations might fail to work in this context – these truncated complexes of abelian groups might not be DG A-modules.

Recall that for a subset $S \subseteq \mathbb{Z}$, its infimum is $\inf(S) \in \mathbb{Z} \cup \{\pm\infty\}$, where $\inf(S) = +\infty$ iff $S = \varnothing$. Likewise the supremum is $\sup(S) \in \mathbb{Z} \cup \{\pm\infty\}$, where $\sup(S) = -\infty$ iff $S = \varnothing$. For $i, j \in \mathbb{Z} \cup \{\infty\}$, the expressions $i + j$ and $-i - j$ have obvious values in $\mathbb{Z} \cup \{\pm\infty\}$. And for $i, j \in \mathbb{Z} \cup \{\pm\infty\}$, the expression $i \leq j$ has an obvious meaning.

Let $M = \bigoplus_{i \in \mathbb{Z}} M^i$ be a graded abelian group. We write

$$\inf(M) := \inf \{ i \mid M^i \neq 0 \} \quad \text{and} \quad \sup(M) := \sup \{ i \mid M^i \neq 0 \}. \tag{1.7}$$

The amplitude of M is

$$\operatorname{amp}(M) := \sup(M) - \inf(M) \in \mathbb{N} \cup \{\pm\infty\}. \tag{1.8}$$

(For $M = 0$ this reads $\inf(M) = \infty$, $\sup(M) = -\infty$ and $\operatorname{amp}(M) = -\infty$.) Thus M is bounded (resp. bounded above, resp. bounded below) iff $\operatorname{amp}(M) < \infty$ (resp. $\sup(M) < \infty$, resp. $\inf(M) > -\infty$).

Given $i_0 \leq i_1$ in $\mathbb{Z} \cup \{\pm\infty\}$, the *integer interval* with these endpoints is the set of integers

$$[i_0, i_1] := \{ i \in \mathbb{Z} \mid i_0 \leq i \leq i_1 \}. \tag{1.9}$$

The integer interval $[i_0, i_1]$ is said to be bounded (resp. bounded above, resp. bounded below) if $i_0, i_1 \in \mathbb{Z}$ (resp. $i_1 \in \mathbb{Z}$, resp. $i_0 \in \mathbb{Z}$). The *length* of this interval is $i_1 - i_0 \in \mathbb{N} \cup \{\infty\}$. Of course the interval has finite length iff it is bounded. We write $-[i_0, i_1] := [-i_1, -i_0]$. Given a second integer interval $[j_0, j_1]$, we let

$$[i_0, i_1] + [j_0, j_1] := [i_0 + j_0, i_1 + j_1].$$

For the empty interval $\varnothing$, the sum is $[i_0, i_1] + \varnothing := \varnothing$.

Definition 1.10. Let $M = \bigoplus_{i \in \mathbb{Z}} M^i$ be a graded abelian group.

(1) We say that M is concentrated in an integer interval $[i_0, i_1]$ if
$$\{ i \in \mathbb{Z} \mid M^i \neq 0 \} \subseteq [i_0, i_1].$$
(2) The *concentration* of M is the smallest integer interval $\operatorname{con}(M)$ in which M is concentrated.

In other words, if $i_0 = \inf(M) \leq i_1 = \sup(M)$, then the concentration of M is the interval $\operatorname{con}(M) = [i_0, i_1]$, and the amplitude $\operatorname{amp}(M)$ is the length of $\operatorname{con}(M)$. Furthermore, $\operatorname{con}(M) = \varnothing$ iff $M = 0$.

Given an integer interval $[i_0, i_1]$, we denote by $\mathsf{D}^{[i_0,i_1]}(A)$ the full subcategory of $\mathsf{D}(A)$ consisting of the DG modules M whose cohomologies $\mathrm{H}(M)$ are concentrated in this interval; namely $\operatorname{con}(\mathrm{H}(M)) \subseteq [i_0, i_1]$. For $i \in \mathbb{Z}$ we write $\mathsf{D}^i(A) := \mathsf{D}^{[i,i]}(A)$. The subcategory $\mathsf{D}^{[i_0,i_1]}(A)$ is additive, but not triangulated. Similarly we have the subcategory $\mathsf{C}^{[i_0,i_1]}(A) \subseteq \mathsf{C}(A)$, consisting of the DG modules M that are concentrated in the integer interval $[i_0, i_1]$; namely $\operatorname{con}(M) \subseteq [i_0, i_1]$.

Definition 1.11. Let A be a DG ring.

(1) A DG A-module M is said to be *cohomologically bounded* (resp. *cohomologically bounded above*, resp. *cohomologically bounded below*) if the graded module $\mathrm{H}(M)$ is bounded (resp. bounded above, resp. bounded below). We denote by $\mathsf{D}^{\mathrm{b}}(A)$, $\mathsf{D}^{-}(A)$ and $\mathsf{D}^{+}(A)$ the corresponding full subcategories of $\mathsf{D}(A)$.
(2) The full subcategory of $\mathsf{D}(A)$ consisting of the DG modules M, whose cohomology modules $\mathrm{H}^i(M)$ are finite over $\bar{A}$, is denoted by $\mathsf{D}_{\mathrm{f}}(A)$.
(3) For any boundedness condition $\star$ we write $\mathsf{D}^{\star}_{\mathrm{f}}(A) := \mathsf{D}_{\mathrm{f}}(A) \cap \mathsf{D}^{\star}(A)$.

Thus we have

$$\mathsf{D}^{\mathrm{b}}(A) = \bigcup_{-\infty < i_0 \leq i_1 < \infty} \mathsf{D}^{[i_0,i_1]}(A)\,,$$

etc. The categories $\mathsf{D}^{\mathrm{b}}(A)$, $\mathsf{D}^{-}(A)$ and $\mathsf{D}^{+}(A)$ are triangulated. If $\bar{A}$ is left noetherian, then the categories $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$, $\mathsf{D}^{-}_{\mathrm{f}}(A)$ and $\mathsf{D}^{+}_{\mathrm{f}}(A)$ are also triangulated.

Given a DG A-module M, its shift by an integer i is the DG module $M[i]$, whose j-th graded component is $M[i]^j := M^{i+j}$. Elements of $M[i]^j$ are denoted by $m[i]$, with $m \in M^{i+j}$. The differential of $M[i]$ is $\mathrm{d}_{M[i]}(m[i]) := (-1)^i \cdot \mathrm{d}_M(m)[i]$. The left action of A on $M[i]$ is also twisted by ± 1, as follows: $a \cdot m[i] := (-1)^{ki} \cdot (a \cdot m)[i]$ for $a \in A^k$. The right action remains untwisted: $m[i] \cdot a := (m \cdot a)[i]$. See [Ye5, Section 1] for a detailed study of the shift operation, including an explanation of the sign that appears in the left action.

We now recall some resolutions of DG A-modules. A DG module N is called acyclic if $\mathrm{H}(N) = 0$. A DG A-module M is called *K-projective* (resp. *K-injective*), if for any acyclic DG A-module N, the DG $\mathbb{Z}$-module $\operatorname{Hom}_A(M, N)$ (resp. $\operatorname{Hom}_A(N, M)$) is also acyclic. The DG A-module M is called *K-flat* if for any acyclic DG A^{op}-module N, the DG $\mathbb{Z}$-module $N \otimes_A M$ is acyclic. It is easy to see that K-projective implies K-flat. More information about the operations $\operatorname{Hom}_A(-,-)$ and $- \otimes_A -$, and the various resolutions, see [Ye5, Section 1].

For a cardinal number r (possibly infinite) we denote by $M^{\oplus r}$ the direct sum of r copies of M. Recall that a DG A-module P is a *free DG module* if $P \cong \bigoplus_{i \in \mathbb{Z}} A[-i]^{\oplus r_i}$, where r_i are cardinal numbers. We say that P is a *finite free DG module* if $\sum_i r_i < \infty$ (for some such isomorphism).

Definition 1.12. Let P be a DG A-module. A *semi-free filtration* of P is an ascending filtration $\{\nu_j(P)\}_{j \in \mathbb{Z}}$ by DG A-submodules $\nu_j(P) \subseteq P$, such that $\nu_{-1}(P) = 0$, $P = \bigcup_j \nu_j(P)$, and each $\operatorname{gr}^{\nu}_j(P) := \nu_j(P)/\nu_{j-1}(P)$ is a free DG A-module.

Definition 1.13. Let P be a DG A-module, with semi-free filtration $\{\nu_j(P)\}_{j \in \mathbb{Z}}$.

(1) The filtration $\{\nu_j(P)\}_{j \in \mathbb{Z}}$ is said to have length l if

$$l = \inf\{j \in \mathbb{N} \mid \nu_j(P) = P\} \in \mathbb{N} \cup \{\infty\}.$$

(2) The filtration $\{\nu_j(P)\}_{j\in\mathbb{Z}}$ is called *pseudo-finite* if each free DG A-module $\mathrm{gr}^{\nu}_j(P)$ is finite, and $\lim_{j\to\infty}\sup(\mathrm{gr}^{\nu}_j(P)) = -\infty$.
(3) The filtration $\{\nu_j(P)\}_{j\in\mathbb{Z}}$ on P is called *finite* if it is pseudo-finite and has finite length.

Definition 1.14. A DG A-module P is called a *semi-free* (resp. *pseudo-finite semi-free*, resp. *finite semi-free*) *DG module* if it admits a semi-free (resp. pseudo-finite semi-free, resp. finite semi-free) filtration. The *semi-free length* of P is defined to be the minimum of the lengths of its semi-free filtrations.

Note that a semi-free DG module need not be bounded above. If P is a semi-free DG module then it is K-projective; see [Ke1, Section 3.1].

Recall that our DG rings are always nonpositive. The next proposition gives another characterization of pseudo-finite semi-free DG modules. The graded ring gotten from A by forgetting the differential is denoted by $A^{\natural}$. Likewise for DG modules.

Proposition 1.15. *Let P be a DG A-module.*
(1) *P is pseudo-finite semi-free iff there are $i_1 \in \mathbb{Z}$ and $r_i \in \mathbb{N}$, such that $P^{\natural} \cong \bigoplus_{i\leq i_1} A^{\natural}[-i]^{\oplus r_i}$ as graded $A^{\natural}$-modules.*
(2) *P is finite semi-free iff there is an isomorphism of graded $A^{\natural}$-modules as in item* (1) *above, and $i_0 \in \mathbb{Z}$, such that $r_i = 0$ for all $i < i_0$.*

Proof. Given an isomorphism $P^{\natural} \cong \bigoplus_{i\leq i_1} A^{\natural}[-i]^{\oplus r_i}$, define

$$\nu_j(P) := \bigoplus_{i_1-j\leq i\leq i_1} A[-i]^{\oplus r_i} \subseteq P.$$

This is a pseudo-finite semi-free filtration, of length $\leq i_1 - i_0$ in the finite case. The converse is clear, and so is item (2). $\square$

Definition 1.16. Let $[i_0, i_1]$ be an integer interval (possibly unbounded).
(1) Let P be a free graded $A^{\natural}$-module. We say that P has a *basis concentrated in* $[i_0, i_1]$ if there is an isomorphism of graded $A^{\natural}$-modules
$$P \cong \bigoplus_{i\in[i_0,i_1]} A^{\natural}[-i]^{\oplus r_i}$$
for some cardinal numbers r_i.
(2) A DG A-module M is said to be *generated* in the integer interval $[i_0, i_1]$ if there is a surjection of graded $A^{\natural}$-modules $P \to M^{\natural}$, where P is a free graded $A^{\natural}$-module with a basis concentrated in $[i_0, i_1]$.

Proposition 1.17. *Let M be a DG A-module generated in the integer interval $[i_0, i_1]$.*
(1) *If $N \in \mathsf{C}^{[j_0,j_1]}(A^{\mathrm{op}})$, then*
$$N \otimes_A M \in \mathsf{C}^{[j_0,j_1]+[i_0,i_1]}(\mathbb{Z}) = \mathsf{C}^{[j_0+i_0,j_1+i_1]}(\mathbb{Z}).$$
(2) *If $N \in \mathsf{C}^{[j_0,j_1]}(A)$, then*
$$\mathrm{Hom}_A(M,N) \in \mathsf{C}^{[j_0,j_1]-[i_0,i_1]}(\mathbb{Z}) = \mathsf{C}^{[j_0-i_1,j_1-i_0]}(\mathbb{Z}).$$

The easy proof is left to the reader.

Definition 1.18. Let A be a DG ring.
(1) A is called *cohomologically left noetherian* if the graded ring $\mathrm{H}(A)$ is left noetherian.
(2) A is called *cohomologically left pseudo-noetherian* if it satisfies these two conditions:

(i) The ring $\bar{A} = \mathrm{H}^0(A)$ is left noetherian.
(ii) For every i the left $\bar{A}$-module $\mathrm{H}^i(A)$ is finite (i.e. finitely generated).

We shall be mostly interested in cohomologically left pseudo-noetherian DG rings. Of course if A is cohomologically left noetherian, then it is cohomologically left pseudo-noetherian. These conditions are equivalent when A is cohomologically bounded.

As usual, by a semi-free resolution of a DG module M we mean a quasi-isomorphism $P \to M$ in $\mathsf{C}(A)$, where P is semi-free. Likewise we talk about K-injective resolutions $M \to I$.

Proposition 1.19. *Let M be a DG A-module.*

(1) *There is a semi-free resolution $P \to M$ such that* $\sup(P) = \sup(\mathrm{H}(M))$.
(2) *If the DG ring A is cohomologically left pseudo-noetherian, and if $M \in \mathsf{D}^-_{\mathrm{f}}(A)$, then there is a pseudo-finite semi-free resolution $P \to M$ such that* $\sup(P) = \sup(\mathrm{H}(M))$.
(3) *There is a K-injective resolution $M \to I$ such that* $\inf(I) = \inf(\mathrm{H}(M))$.

Proof. (1) See [Ke1, Theorem 3.1], [AFH], or [SP, Section 09KK], noting that $\sup(A) = 0$ (if A is nonzero).

(2) In this case the construction in item (1) can be made with finitely many basis elements in each degree. See [Ke1, Theorem 3.1] or [AFH].

(3) See [Ke1, Theorem 3.2], [AFH], or [SP, Section 09KQ]. The point is that any DG A-module can be embedded in a product of shifts of the DG A-module $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$. □

Remark 1.20. Suppose A is a ring. A DG A-module P is pseudo-finite semi-free iff it is a bounded above complex of finite free A-modules. Now according to [SGA 6] or [SP, Definition 064Q], a DG A-module M is called pseudo-coherent if it is quasi-isomorphic to some pseudo-finite semi-free DG module P. This explains the choice of the name "pseudo-finite semi-free DG module".

The name "pseudo-noetherian DG ring" was chosen due to the close relation to pseudo-finite semi-free resolutions; see Proposition 1.19(2).

2. Cohomological Dimension

We continue with the conventions of Section 1, namely our DG rings are nonpositive, and the DG modules are acted upon from the left. The concentration $\mathrm{con}(M)$ of a graded module M was introduced in Definition 1.10.

Definition 2.1. Let A and B be DG rings, and let $\mathsf{E} \subseteq \mathsf{D}(A)$ be a full subcategory.

(1) Let $F : \mathsf{E} \to \mathsf{D}(B)$ be an additive functor, and let $[d_0, d_1]$ be an integer interval. We say that F *has cohomological displacement at most* $[d_0, d_1]$ if
$$\mathrm{con}\big(\mathrm{H}(F(M))\big) \subseteq \mathrm{con}\big(\mathrm{H}(M)\big) + [d_0, d_1]$$
for every $M \in \mathsf{E}$.
(2) Let $F : \mathsf{E}^{\mathrm{op}} \to \mathsf{D}(B)$ be an additive functor, and let $[d_0, d_1]$ be an integer interval. We say that F *has cohomological displacement at most* $[d_0, d_1]$ if
$$\mathrm{con}\big(\mathrm{H}(F(M))\big) \subseteq -\,\mathrm{con}\big(\mathrm{H}(M)\big) + [d_0, d_1]$$
for every $M \in \mathsf{E}$.
(3) Let F be as in item (1) or (2). The *cohomological displacement of* F is the smallest integer interval $[d_0, d_1]$ for which F has cohomological displacement at most $[d_0, d_1]$. If $d_0 \in \mathbb{Z}$ (resp. $d_1 \in \mathbb{Z}$, resp. $d_0, d_1 \in \mathbb{Z}$) then F is said to have *bounded below* (resp. *bounded above*, resp. *bounded*) *cohomological displacement*.

(4) Let $[d_0, d_1]$ be the cohomological displacement of F. The *cohomological dimension* of F is $d := d_1 - d_0 \in \mathbb{N} \cup \{\infty\}$. If $d \in \mathbb{N}$, then F is said to have *finite cohomological dimension*.

Note that if $\mathsf{E}' \subseteq \mathsf{E}$, and F has cohomological displacement at most $[d_0, d_1]$, then $F|_{\mathsf{E}'}$ also has cohomological displacement at most $[d_0, d_1]$.

Example 2.2. Consider a commutative ring $A = B$, and let $\mathsf{E} := \mathsf{D}(A)$. For the covariant case (item (1) in Definition 2.1) take a nonzero projective module P, and let $F := \mathrm{Hom}_A(P \oplus P[1], -)$. Then F has cohomological displacement $[0, 1]$. For the contravariant case (item (2)) take a nonzero injective module I, and let $F := \mathrm{Hom}_A(-, I \oplus I[1])$. Then F has cohomological displacement $[-1, 0]$. In both cases the cohomological dimension of F is 1.

Remark 2.3. Suppose A and B are rings. If $\mathsf{E} = \mathsf{D}^+(A)$ and $F = \mathrm{R}F_0$ (or $\mathsf{E} = \mathsf{D}^-(A)$ and $F = \mathrm{L}F_0$) for some additive functor $F_0 : \mathsf{Mod}\, A \to \mathsf{Mod}\, B$, then the cohomological dimension of F is the usual cohomological dimension of F_0.

Assume that $\mathsf{E} = \mathsf{D}(A)$ and F is a triangulated functor. The functor F has bounded below (resp. above) cohomological displacement iff it is way-out right (resp. left), in the sense of [RD, Section I.7].

Definition 2.4. Let A be a DG ring, let $M \in \mathsf{D}(A)$, and let $[d_0, d_1]$ be an integer interval of length $d := d_1 - d_0$.

(1) Given a full subcategory $\mathsf{E} \subseteq \mathsf{D}(A)$, we say that M has *projective concentration* $[d_0, d_1]$ *and projective dimension* d *relative to* E if the functor
$$\mathrm{RHom}_A(M, -) : \mathsf{E} \to \mathsf{D}(\mathbb{Z})$$
has cohomological displacement $-[d_0, d_1]$ relative to E.

(2) Given a full subcategory $\mathsf{E} \subseteq \mathsf{D}(A)$, we say that M has *injective concentration* $[d_0, d_1]$ *and injective dimension* d *relative to* E if the functor
$$\mathrm{RHom}_A(-, M) : \mathsf{E}^{\mathrm{op}} \to \mathsf{D}(\mathbb{Z})$$
has cohomological displacement $[d_0, d_1]$ relative to E.

(3) Given a full subcategory $\mathsf{E} \subseteq \mathsf{D}(A^{\mathrm{op}})$, we say that M has *flat concentration* $[d_0, d_1]$ *and flat dimension* d *relative to* E if the functor
$$- \otimes^{\mathrm{L}}_A M : \mathsf{E} \to \mathsf{D}(\mathbb{Z})$$
has cohomological displacement $[d_0, d_1]$ relative to E.

Example 2.5. Continuing with the setup of Example 2.2, the DG module $P \oplus P[1]$ (resp. $I \oplus I[1]$) has projective (resp. injective) concentration $[-1, 0]$ relative to $\mathsf{D}(A)$.

Example 2.6. Let A be a DG ring, and consider the free DG module $P := A \in \mathsf{D}(A)$. The functor
$$F := \mathrm{RHom}_A(P, -) : \mathsf{D}(A) \to \mathsf{D}(\mathbb{Z})$$
is isomorphic to the forgetful functor $\mathrm{rest}_{A/\mathbb{Z}}$, so it has cohomological displacement $[0, 0]$ and cohomological dimension 0 relative to $\mathsf{D}(A)$. Thus the DG module P has projective concentration $[0, 0]$ and projective dimension 0 relative to $\mathsf{D}(A)$. Note however that the cohomology $\mathrm{H}(P)$ could be unbounded below.

The interval of generation of a DG module was introduced in Definition 1.16.

Proposition 2.7. *Let $M \in \mathsf{D}(A)$. Assume there is an isomorphism $P \cong M$ in $\mathsf{D}(A)$, where P is a K-flat* (*resp. K-projective*) *DG A-module generated in the integer interval $[d_0, d_1]$. Then M has flat* (*resp. projective*) *concentration at most $[d_0, d_1]$ relative to $\mathsf{D}(A^{\mathrm{op}})$* (*resp.* $\mathsf{D}(A)$).

Proof. First consider the K-flat case. Take any $N \in \mathsf{D}(A^{\mathrm{op}})$. After applying smart truncation, we can assume that $\mathrm{con}(N) = \mathrm{con}(\mathrm{H}(N))$. Now $N \otimes^{\mathrm{L}}_A M \cong N \otimes_A P$, and by Proposition 1.17 the bounds on $N \otimes_A P$ are as claimed.

Next consider the K-projective case. Take any $N \in \mathsf{D}(A)$. As above, we can assume that $\mathrm{con}(N) = \mathrm{con}(\mathrm{H}(N))$. We know that $\mathrm{RHom}_A(M, N) \cong \mathrm{Hom}_A(P, N)$. The bounds on the DG module $\mathrm{Hom}_A(P, N)$ are as claimed, by Proposition 1.17. □

Proposition 2.8. *If $M \in \mathsf{D}^{[d_0,d_1]}(A)$, then M has projective concentration at most $[-\infty, d_1]$ relative to $\mathsf{D}(A)$, injective concentration at most $[d_0, \infty]$ relative to $\mathsf{D}(A)$, and flat concentration at most $[-\infty, d_1]$ relative to $\mathsf{D}(A^{\mathrm{op}})$.*

Proof. First let's assume that $M \neq 0$ and $d_1 < \infty$. We know that M admits a semi-free resolution $P \to M$ with $\sup(P) = \sup(\mathrm{H}(M)) \leq d_1$. Now we can use Proposition 2.7 for the flat and projective concentrations.

Now let's assume that $M \neq 0$ and $d_0 > -\infty$. By Proposition 1.19 there is a K-injective resolution $M \to I$ with $\inf(I) = \inf(\mathrm{H}(M)) \geq d_0$. For any $N \in \mathsf{D}(A)$ we have $\mathrm{RHom}_A(N, M) \cong \mathrm{Hom}_A(N, I)$, and hence the bound on the injective concentration. □

Remark 2.9. Let us write <att> for either of the attributes projective, injective or flat. When A is a ring, $\mathsf{E} = \mathsf{D}^0(A) \approx \mathsf{Mod}\, A$ and $M \in \mathsf{Mod}\, A$, we recover the usual definition of <att> dimension of modules in ring theory. Furthermore, in the ring case, a DG A-module M has <att> concentration in a bounded integer interval $[d_0, d_1]$ relative to $\mathsf{D}^0(A)$, iff M isomorphic in $\mathsf{D}(A)$ to a complex P of <att> A-modules with $\mathrm{con}(P) \subseteq [d_0, d_1]$. This implies that M has finite <att> dimension relative to $\mathsf{D}(A)$. We do not know if this – namely the <att> dimension of M relative to $\mathsf{D}(A)$ is the same as that relative $\mathsf{D}^0(A)$ – is true when A is a genuine DG ring.

The next two theorems are variations of the opposite (in the categorical sense) of [RD, Proposition I.7.1], the "Lemma on Way-Out Functors". The canonical homomorphism $A \to \bar{A}$ lets us view any $\bar{A}$-module as a DG A-module.

Theorem 2.10. *Let A and B be DG rings, let $F, G : \mathsf{D}(A) \to \mathsf{D}(B)$ be triangulated functors, and let $\eta : F \to G$ be a morphism of triangulated functors. Assume that $\eta_M : F(M) \to G(M)$ is an isomorphism for every $M \in \mathsf{Mod}\, \bar{A}$.*

(1) *The morphism η_M is an isomorphism for every $M \in \mathsf{D}^{\mathrm{b}}(A)$.*
(2) *If F and G have bounded above cohomological displacements, then η_M is an isomorphism for every $M \in \mathsf{D}^{-}(A)$.*
(3) *If F and G have finite cohomological dimensions, then η_M is an isomorphism for every $M \in \mathsf{D}(A)$.*

Proof. (1) The proof is by induction on $j := \mathrm{amp}(\mathrm{H}(M))$. If $j = 0$ then M is isomorphic to a shift an object of $\bar{A}$, so η_M is an isomorphism. If $j > 0$, then using smart truncation we obtain a distinguished triangle $M' \to M \to M'' \xrightarrow{\triangle}$ in $\mathsf{D}(A)$ such that $\mathrm{amp}(\mathrm{H}(M'')) < j$ and $\mathrm{amp}(\mathrm{H}(M')) < j$. Since $\eta_{M''}$ and $\eta_{M'}$ are isomorphisms, so is η_M.

(2) Here we assume that F and G have cohomological displacements at most $[-\infty, d_1]$ for some integer d_1. Take any $M \in \mathsf{D}^{-}(A)$. In order to prove that η_M is an isomorphism it suffices to show that

$$\mathrm{H}^i(\eta_M) : \mathrm{H}^i(F(M)) \to \mathrm{H}^i(G(M))$$

is bijective for every $i \in \mathbb{Z}$.

Fix an integer i. Let $M' \to M \to M'' \xrightarrow{\triangle}$ be a distinguished triangle such that $\sup(\mathrm{H}(M')) \leq i - d_1 - 2$ and $\inf(\mathrm{H}(M'')) \geq i - d_1 - 1$. This can be obtained using smart truncation.

The cohomologies of $F(M')$ and $G(M')$ are concentrated in the degree range $\leq i - 2$. The distinguished triangle induces a commutative diagram of $\bar{A}$-modules with exact rows:

$$\begin{tikzcd}
\mathrm{H}^i(F(M')) \arrow[r] \arrow[d, "\mathrm{H}^i(\eta_{M'})"'] & \mathrm{H}^i(F(M)) \arrow[r] \arrow[d, "\mathrm{H}^i(\eta_{M})"'] & \mathrm{H}^i(F(M'')) \arrow[r] \arrow[d, "\mathrm{H}^i(\eta_{M''})"'] & \mathrm{H}^{i+1}(F(M')) \arrow[d, "\mathrm{H}^{i+1}(\eta_{M'})"'] \\
\mathrm{H}^i(G(M')) \arrow[r] & \mathrm{H}^i(G(M)) \arrow[r] & \mathrm{H}^i(G(M'')) \arrow[r] & \mathrm{H}^{i+1}(G(M'))\,.
\end{tikzcd}$$

The four terms involving M' are zero. Since M'' has bounded cohomology, we know by part (1) that $\mathrm{H}^i(\eta_{M''})$ is an isomorphism. Therefore $\mathrm{H}^i(\eta_M)$ is an isomorphism.

(3) Here we assume that F and G have finite cohomological dimensions. So F and G have cohomological displacements at most $[d_0, d_1]$ for some $d_0 \leq d_1$ in $\mathbb{Z}$. Take any $M \in \mathsf{D}(A)$, and fix $i \in \mathbb{Z}$. We want to show that $\mathrm{H}^i(\eta_M)$ is an isomorphism. Using smart truncations of M we obtain a distinguished triangle $M' \to M \to M'' \xrightarrow{\triangle}$ in $\mathsf{D}(A)$, such that $\sup(\mathrm{H}(M')) \leq i - d_0 + 1$ and $\inf(\mathrm{H}(M'')) \geq i - d_0 + 2$. The cohomologies of $F(M'')$ and $G(M'')$ are concentrated in degrees $\geq i + 2$. We have a commutative diagram of $\bar{A}$-modules with exact rows:

$$\begin{tikzcd}
\mathrm{H}^{i-1}(F(M'')) \arrow[r] \arrow[d, "\mathrm{H}^{i-1}(\eta_{M''})"'] & \mathrm{H}^i(F(M')) \arrow[r] \arrow[d, "\mathrm{H}^i(\eta_{M'})"'] & \mathrm{H}^i(F(M)) \arrow[r] \arrow[d, "\mathrm{H}^i(\eta_{M})"'] & \mathrm{H}^i(F(M'')) \arrow[d, "\mathrm{H}^i(\eta_{M''})"'] \\
\mathrm{H}^{i-1}(G(M'')) \arrow[r] & \mathrm{H}^i(G(M')) \arrow[r] & \mathrm{H}^i(G(M)) \arrow[r] & \mathrm{H}^i(G(M''))
\end{tikzcd}$$

The four terms involving M'' are zero here. Since M' has bounded above cohomology, we know by part (2) that $\mathrm{H}^i(\eta_{M'})$ is an isomorphism. Therefore $\mathrm{H}^i(\eta_M)$ is an isomorphism. $\square$

Theorem 2.11. *Let A and B be DG rings, let $F, G : \mathsf{D}(A) \to \mathsf{D}(B)$ be triangulated functors, and let $\eta : F \to G$ be a morphism of triangulated functors. Assume that A is cohomologically left pseudo-noetherian, and $\eta_A : F(A) \to G(A)$ is an isomorphism.*

1. *If F and G have bounded above cohomological displacements, then $\eta_M : F(M) \to G(M)$ is an isomorphism for every $M \in \mathsf{D}^-_{\mathrm{f}}(A)$.*
2. *If F and G have finite cohomological dimensions, then $\eta_M : F(M) \to G(M)$ is an isomorphism for every $M \in \mathsf{D}_{\mathrm{f}}(A)$.*

Proof. Step 1. Consider a finite free DG A-module P, i.e. $P \cong \bigoplus_{k=1}^{r} A[-i_k]$ in $\mathsf{C}(A)$ for some $i_1, \ldots, i_r \in \mathbb{Z}$. Because the functors F, G are triangulated, and η_A is an isomorphism, it follows that η_P is an isomorphism.

Step 2. Now let P be a finite semi-free DG A-module, with finite semi-free filtration $\{\nu_j(P)\}$ of length j_1 (see Definition 1.13). We prove that η_P is an isomorphism by induction on j_1. For $j_1 = 0$ this is step 1. Now assume $j_1 \geq 1$. Write $P' := \nu_{j_1 - 1}(P)$ and $P'' := \mathrm{gr}^{\nu}_{j_1}(P)$, so there is a distinguished triangle $P' \to P \to P'' \xrightarrow{\triangle}$ in $\mathsf{D}(A)$. According to step 1 and the induction hypothesis, the morphisms $\eta_{P''}$ and $\eta_{P'}$ are isomorphisms. Hence η_P is an isomorphism.

Step 3. Here we assume that F and G have cohomological displacements at most $[-\infty, d_1]$ for some integer d_1. Take any $M \in \mathsf{D}^-_{\mathrm{f}}(A)$. In order to prove that η_M is an isomorphism it suffices to show that

$$\mathrm{H}^i(\eta_M) : \mathrm{H}^i(F(M)) \to \mathrm{H}^i(G(M))$$

is bijective for every $i \in \mathbb{Z}$.

We may assume that M is nonzero. Let $i_1 := \sup(\mathrm{H}(M))$, which is an integer. There exists a pseudo-finite semi-free resolution $P \to M$ such that $\sup(P) = i_1$; see Proposition 1.19. We will prove that $\mathrm{H}^i(\eta_P)$ is an isomorphism for every i. Fix a pseudo-finite semi-free filtration $\{\nu_j(P)\}$ of P.

Take an integer j, and define $P' := \nu_j(P)$ and $P'' := P/\nu_j(P)$. So there is a distinguished triangle $P' \to P \to P'' \xrightarrow{\triangle}$ in $\mathsf{D}(A)$. The DG module P'' is concentrated in the degree range $\leq i_1 - j - 1$, and hence so is its cohomology. Thus the cohomologies of $F(P'')$ and $G(P'')$ are concentrated in the degree range $\leq i_1 - j - 1 + d_1$, or in other words $\mathrm{H}^i(F(P'')) = \mathrm{H}^i(G(P'')) = 0$ for all $i > i_1 - j - 1 + d_1$. On the other hand the DG module P' is finite semi-free. The distinguished triangle above induces a commutative diagram of $\bar{A}$-modules with exact rows:

(2.12)
$$\begin{tikzcd}
\mathrm{H}^{i-1}(F(P'')) \arrow[r] \arrow[d, "\mathrm{H}^{i-1}(\eta_{P''})"'] & \mathrm{H}^{i}(F(P')) \arrow[r] \arrow[d, "\mathrm{H}^{i}(\eta_{P'})"'] & \mathrm{H}^{i}(F(P)) \arrow[r] \arrow[d, "\mathrm{H}^{i}(\eta_{P})"'] & \mathrm{H}^{i}(F(P'')) \arrow[d, "\mathrm{H}^{i}(\eta_{P''})"'] \\
\mathrm{H}^{i-1}(G(P'')) \arrow[r] & \mathrm{H}^{i}(G(P')) \arrow[r] & \mathrm{H}^{i}(G(P)) \arrow[r] & \mathrm{H}^{i}(G(P''))
\end{tikzcd}$$

For any $i > i_1 + d_1 - j$ the modules in this diagram involving P'' are zero. By step 2 we know that $\mathrm{H}^i(\eta_{P'})$ is an isomorphism for every i. Therefore $\mathrm{H}^i(\eta_P)$ is an isomorphism for every $i > i_1 + d_1 - j$. Since j can be made arbitrarily large, we conclude that $\mathrm{H}^i(\eta_P)$ is an isomorphism for every i.

Step 4. Here we assume that F and G have finite cohomological dimensions. So F and G have cohomological displacements at most $[d_0, d_1]$ for some $d_0 \leq d_1$ in $\mathbb{Z}$. This step is very similar to the proof of Theorem 2.10(3).

Take any $M \in \mathsf{D}_{\mathrm{f}}(A)$. Fix $i \in \mathbb{Z}$. We want to show that $\mathrm{H}^i(\eta_M)$ is an isomorphism. Using smart truncations of M, we obtain a distinguished triangle $M' \to M \to M'' \xrightarrow{\triangle}$ in $\mathsf{D}_{\mathrm{f}}(A)$, such that $M' \in \mathsf{D}_{\mathrm{f}}^{[-\infty, i-d_0+1]}(A)$ and $M'' \in \mathsf{D}_{\mathrm{f}}^{[i-d_0+1, \infty]}(A)$. We get a commutative diagram like (2.12). The modules $\mathrm{H}^{i-1}(F(M''))$, $\mathrm{H}^i(F(M''))$, $\mathrm{H}^{i-1}(G(M''))$ and $\mathrm{H}^i(G(M''))$ are zero. Since $M' \in \mathsf{D}^-_{\mathrm{f}}(A)$, step 3 says that $\mathrm{H}^i(\eta_{M'})$ is an isomorphism. Therefore $\mathrm{H}^i(\eta_M)$ is an isomorphism. □

The next theorem is a variant of the opposite of [RD, Proposition I.7.3]. For a boundedness condition $\star$, we denote by $-\star$ the opposite boundedness condition. Thus if $\mathsf{D}^\star$ denotes either D, D^+, D^- or D^{b}, then $\mathsf{D}^{-\star}$ denotes either D, D^-, D^+ or D^{b} respectively.

Theorem 2.13. *Let A and B be DG rings, and let $F : \mathsf{D}(A)^{\mathrm{op}} \to \mathsf{D}(B)$ be a triangulated functor. Assume that A and B are cohomologically left pseudo-noetherian, and $F(A) \in \mathsf{D}^+_{\mathrm{f}}(B)$.*

(1) *If F has bounded below cohomological displacement, then $F(M) \in \mathsf{D}^+_{\mathrm{f}}(B)$ for every $M \in \mathsf{D}^-_{\mathrm{f}}(A)$.*
(2) *If F has finite cohomological dimension, then $F(M) \in \mathsf{D}^{-\star}_{\mathrm{f}}(B)$ for every $M \in \mathsf{D}^\star_{\mathrm{f}}(A)$.*

Proof. The proof is very similar to that of Theorem 2.11. We just outline the necessary changes.

Steps 1-2. P is a finite semi-free DG A-module. By induction on the length j_1 of a semi-free filtration, we prove that $F(P) \in \mathsf{D}^{+}_{\mathrm{f}}(B)$.

Step 3. Here F has cohomological displacement $[d_0, \infty]$ for some $d_0 \in \mathbb{Z}$, and $M \in \mathsf{D}^{-}_{\mathrm{f}}(A)$. Let $P \to M$ be a pseudo-finite semi-free resolution, with $\sup(P) = i_1$. Take any $j \in \mathbb{Z}$, and consider the distinguished triangle $P' \to P \to P'' \xrightarrow{\triangle}$ where $P' := \nu_j(P)$. There is an exact sequence of $\bar{B}$-modules

$$\mathrm{H}^{i-1}(F(P'')) \longrightarrow \mathrm{H}^{i}(F(P')) \longrightarrow \mathrm{H}^{i}(F(P)) \longrightarrow \mathrm{H}^{i}(F(P'')). \tag{2.14}$$

By step 2 we know that $\mathrm{H}^{i}(F(P')) \in \mathsf{Mod}_{\mathrm{f}}\, \bar{B}$ for every i. If $i \leq d_0 - i_1 + j$ then $\mathrm{H}^{i-1}(F(P'')) = \mathrm{H}^{i}(F(P'')) = 0$. Therefore $\mathrm{H}^{i}(F(P)) \in \mathsf{Mod}_{\mathrm{f}}\, \bar{B}$. But j can be made arbitrarily large. This proves that $F(M) \in \mathsf{D}_{\mathrm{f}}(B)$. But on the other hand we know that $\mathrm{H}^{i}(F(M)) = 0$ for all $i < -i_1 + d_0$. We conclude that $F(M) \in \mathsf{D}^{+}_{\mathrm{f}}(B)$.

Step 4. Here F has cohomological displacement at most $[d_0, d_1]$ for some $d_0 \leq d_1$ in $\mathbb{Z}$, $M \in \mathsf{D}_{\mathrm{f}}(A)$, and $i \in \mathbb{Z}$. We truncate M to obtain a distinguished triangle $M' \to M \to M'' \xrightarrow{\triangle}$ in $\mathsf{D}_{\mathrm{f}}(A)$, such that $M' \in \mathsf{D}^{[-\infty, j+1]}_{\mathrm{f}}(A)$ and $M'' \in \mathsf{D}^{[j+1, \infty]}_{\mathrm{f}}(A)$ for $j := i - d_1 - 3$. We get an exact sequence like (2.14). The modules $\mathrm{H}^{i-1}(F(M''))$ and $\mathrm{H}^{i}(F(M''))$ are zero, and $\mathrm{H}^{i}(F(M')) \in \mathsf{Mod}_{\mathrm{f}}\, \bar{B}$ by step 3. Therefore $\mathrm{H}^{i}(F(M)) \in \mathsf{Mod}_{\mathrm{f}}\, \bar{B}$. The condition on the boundedness of $\mathrm{H}(F(M))$ is established like in step 3. □

3. Reduction and Lifting

Recall that by default all DG modules are left DG modules, and all DG rings are nonpositive (Convention 1.4). In this section we study the canonical DG ring homomorphism $A \to \bar{A}$, and the corresponding reduction functor $\mathsf{D}(A) \to \mathsf{D}(\bar{A})$, $M \mapsto \bar{A} \otimes^{\mathrm{L}}_{A} M$. We do not make any finiteness assumptions on the cohomology modules $\mathrm{H}^{i}(M)$.

A triangulated functor F is called *conservative* if for any object M, $F(M) = 0$ implies $M = 0$; or equivalently, if for any morphism ϕ, $F(\phi)$ is an isomorphism implies ϕ is an isomorphism. Cf. [KaSc, Section 1.4]. The following result is analogous to the Nakayama Lemma (cf. Remark 7.24).

Proposition 3.1. *Let A be a DG ring. The reduction functor*

$$\bar{A} \otimes^{\mathrm{L}}_{A} - : \mathsf{D}^{-}(A) \to \mathsf{D}^{-}(\bar{A})$$

is conservative.

Proof. Take $M \in \mathsf{D}^{-}(A)$ not isomorphic to 0, and let $i_1 := \sup(\mathrm{H}(M))$. We can find a K-flat resolution (e.g. a semi-free resolution) $P \to M$ over A such that $\sup(P) = i_1$. Then $\bar{A} \otimes^{\mathrm{L}}_{A} M \cong \bar{A} \otimes_{A} P$, and (by the "Künneth trick")

$$\mathrm{H}^{i_1}(\bar{A} \otimes^{\mathrm{L}}_{A} M) \cong \mathrm{H}^{i_1}(\bar{A} \otimes_{A} P) \cong \bar{A} \otimes_{A} \mathrm{H}^{i_1}(P) \cong \mathrm{H}^{i_1}(M)$$

is nonzero. Hence $\bar{A} \otimes^{\mathrm{L}}_{A} M$ is nonzero. □

Given a homomorphism $\phi : P \to Q$ in $\mathsf{C}(A)$, we denote by $\mathrm{cone}(\phi)$ the corresponding cone, which is also an object of $\mathsf{C}(A)$.

Lemma 3.2. *Suppose $\phi : P \to Q$ is a homomorphism in $\mathsf{C}(A)$, where P and Q are pseudo-finite* (*resp. finite*) *semi-free DG modules. Then* $\mathrm{cone}(\phi)$ *is a pseudo-finite* (*resp. finite*) *semi-free DG module.*

Proof. Clear from Proposition 1.15. □

Proposition 3.3. *Let $M \in \mathsf{D}^{-}(A)$. We write $\bar{M} := \bar{A} \otimes^{\mathrm{L}}_{A} M \in \mathsf{D}^{-}(\bar{A})$.*

(1) *If $\bar{M}$ is isomorphic in $\mathsf{D}(\bar{A})$ to $\bar{A}^{\oplus r}$ for some cardinal number r, then M is isomorphic in $\mathsf{D}(A)$ to $A^{\oplus r}$.*
(2) *If $\bar{M}$ is isomorphic in $\mathsf{D}(\bar{A})$ to a semi-free DG $\bar{A}$-module $\bar{P}$ of semi-free length d, then M is isomorphic in $\mathsf{D}(A)$ to a semi-free DG A-module P of semi-free length d.*

Observe that the DG $\bar{A}$-module $\bar{P}$ in (2) above is nothing but a bounded complex of free $\bar{A}$-modules; cf. Proposition 1.15. The semi-free length was introduced in Definition 1.14.

Proof. Step 1. In view of Proposition 3.1 we can assume that $\bar{A}$ and M are nonzero. Define $i_1 := \sup(\mathrm{H}(M))$. By replacing M with a suitable resolution of it, we can assume that M is a K-flat DG A-module satisfying $\sup(M) = i_1$. After that we can also assume that $\bar{M} = \bar{A} \otimes_A M$. The Künneth formula says that

$$\mathrm{H}^{i_1}(\bar{M}) \cong \mathrm{H}^{i_1}(\bar{A} \otimes_A M) \cong \bar{A} \otimes_A \mathrm{H}^{i_1}(M) \cong \mathrm{H}^{i_1}(M)$$

as $\bar{A}$-modules. Therefore $\sup(\mathrm{H}(\bar{M})) = \sup(\bar{M}) = i_1$.

We are given an isomorphism $\bar{\phi} : \bar{P} \to \bar{M}$ in $\mathsf{D}(\bar{A}^0)$, where $\bar{P}$ is a bounded complex of free $\bar{A}$-modules. Since $\bar{P}$ is K-projective, we can assume that the isomorphism $\bar{\phi} : \bar{P} \to \bar{M}$ in $\mathsf{D}(\bar{A})$ is in fact a quasi-isomorphism in $\mathsf{C}(\bar{A})$. The proof continues by induction on $j := \mathrm{amp}(\bar{P}) \in \mathbb{N}$. Note that in item (1) we have $i_1 = 0$, $\bar{P} = \bar{A}^{\oplus r}$ and $j = 0$.

Step 2. In this step we assume that $j = 0$. This means that the only nonzero term of $\bar{P}$ is in degree i_1, and it is the free module $\bar{P}^{i_1} \cong \bar{A}^{\oplus r_1}$ for some cardinal number r_1. In other words, $\bar{P} \cong \bar{A}[-i_1]^{\oplus r_1}$ in $\mathsf{C}(\bar{A})$. So $\mathrm{H}^{i_1}(\bar{M}) \cong \bar{P}^{i_1}$, and $\mathrm{H}^{i}(\bar{M}) = 0$ for all $i \neq i_1$. Recall the isomorphism of $\bar{A}$-modules $\mathrm{H}^{i_1}(\bar{M}) \cong \mathrm{H}^{i_1}(M)$ from Step 1. There are canonical surjections $M^{i_1} \to \mathrm{H}^{i_1}(M)$ and $\bar{M}^{i_1} \to \mathrm{H}^{i_1}(\bar{M})$. We can write the quasi-isomorphism $\bar{\phi}$ as $\bar{\phi} : \bar{A}[-i_1]^{\oplus r_1} \to \bar{M}$.

Consider the nonderived reduction functor $F : \mathsf{C}(A) \to \mathsf{C}(\bar{A})$, $F(-) := \bar{A} \otimes_A -$. Let $P := A[-i_1]^{\oplus r_1}$, a free DG A-module satisfying $F(P) \cong \bar{P}$. There exists a homomorphism $\phi : P \to M$ in $\mathsf{C}(A)$ that lifts the quasi-isomorphism $\bar{\phi} : \bar{P} \to \bar{M}$, namely $\bar{\phi} = F(\phi)$. Now the DG modules P and M are K-flat, so $\bar{\phi} = \mathrm{L}F(\phi)$. Since $\bar{\phi}$ is an isomorphism, and since the functor $\mathrm{L}F$ is conservative, we conclude that ϕ is an isomorphism. This proves item (1).

Step 3. Here we suppose that $j \geq 1$. Let $i_2 := \sup(\bar{P})$, which is of course $\geq i_1$. Say $\bar{P}^{i_2} \cong \bar{A}^{\oplus r_2}$ for some natural number r_2. Define DG modules $\bar{P}' := \bar{A}[-i_2]^{\oplus r_2}$ and $P' := A[-i_2]^{\oplus r_2}$; these satisfy $\bar{P}' \cong \bar{A} \otimes_A P'$. The inclusion $\bar{P}^{i_2} \subseteq \bar{P}$ is viewed as a DG module homomorphism $\bar{\alpha} : \bar{P}' \to \bar{P}$. We also have a quasi-isomorphism $\bar{\phi} : \bar{P} \to \bar{M}$ and an equality $\bar{M} = \bar{A} \otimes_A M$ in $\mathsf{C}(\bar{A})$. In this way we obtain a homomorphism $\bar{\psi} : \bar{P}' \to \bar{M}$, $\bar{\psi} := \bar{\phi} \circ \bar{\alpha}$. Because P' is a free DG A-module, there is a homomorphism $\psi : P' \to M$ in $\mathsf{C}(A)$ lifting $\bar{\psi}$, namely $\bar{\psi} = F(\psi)$, where F is the functor $\bar{A} \otimes_A -$.

Let $M'' \in \mathsf{C}(A)$ be the cone of ψ, so there is a distinguished triangle

$$P' \xrightarrow{\psi} M \xrightarrow{\chi} M'' \to P'[1] \tag{3.4}$$

in $\mathsf{D}(A)$. Define $\bar{M}'' := F(M'')$ and $\bar{\chi} := F(\chi)$, which are an object and a morphism in $\mathsf{C}(\bar{A})$, respectively. Since all three DG modules in this triangle are K-flat, it follows that

$$\bar{P}' \xrightarrow{\bar{\psi}} \bar{M} \xrightarrow{\bar{\chi}} \bar{M}'' \to \bar{P}'[1]$$

is a distinguished triangle in $\mathsf{D}(\bar{A})$. On the other hand, let $\bar{P}''$ be the cokernel of the inclusion $\bar{\alpha} : \bar{P}' \to \bar{P}$. So there is a distinguished triangle

$$\bar{P}' \xrightarrow{\bar{\alpha}} \bar{P} \xrightarrow{\bar{\beta}} \bar{P}'' \to \bar{P}'[1]$$

in $\mathsf{D}(\bar{A})$. Consider the diagram of solid arrows in $\mathsf{D}(\bar{A})$:

$$\begin{tikzcd}
\bar{P}' \arrow[r, "\bar{\alpha}"] \arrow[d, "="] & \bar{P} \arrow[r, "\bar{\beta}"] \arrow[d, "\bar{\phi}"] & \bar{P}'' \arrow[r] \arrow[d, dashed, "\bar{\phi}''"] & \bar{P}'[1] \arrow[d, "="] \\
\bar{P}' \arrow[r, "\bar{\psi}"] & \bar{M} \arrow[r, "\bar{\chi}"] & \bar{M}'' \arrow[r] & \bar{P}'[1]
\end{tikzcd}$$

The square on the left is commutative, and therefore it extends to an isomorphism of distinguished triangles. So there is an isomorphism $\bar{\phi}'' : \bar{P}'' \to \bar{M}''$ in $\mathsf{D}(\bar{A})$.

Finally, the complex $\bar{P}''$ is a bounded complex of finite free $\bar{A}$-modules, of amplitude $j - 1 \geq 0$. According to the induction hypothesis (step 2 for $j > 1$, and step 1 for $j = 1$) there is an isomorphism $\phi'' : P'' \xrightarrow{\simeq} M''$ in $\mathsf{D}(A)$ for some finite semi-free DG A-module P''. From (3.4) we obtain a distinguished triangle

$$P' \xrightarrow{\psi} M \xrightarrow{\psi} P'' \xrightarrow{\gamma} P'[1]$$

in $\mathsf{D}(A)$. We can assume that γ is a homomorphism in $\mathsf{C}(A)$. Turning this triangle we get a distinguished triangle

$$P''[-1] \xrightarrow{-\gamma[-1]} P' \xrightarrow{\psi} M \xrightarrow{\triangle} .$$

Define P to be the cone on the homomorphism $-\gamma[-1] : P''[-1] \to P'$. Then $P \cong M$ in $\mathsf{D}(A)$, and by Lemma 3.2, P is a finite semi-free DG A-module. $\square$

We learned the next proposition from B. Antieau and J. Lurie. Cf. [Lu1, Corollary 7.2.2.19] for a more general statement.

Proposition 3.5. *Let $\bar{P}$ be a projective $\bar{A}$-module. Then there exists a DG A-module P with these properties:*

(i) *P is a direct summand, in $\mathsf{D}(A)$, of a direct sum of copies of A.*
(ii) *$\bar{A} \otimes^{\mathrm{L}}_{A} P \cong \bar{P}$ in $\mathsf{D}(\bar{A})$.*

Proof. Say $\bar{P}$ is a direct summand, in $\mathsf{Mod}\,\bar{A}$, of a free $\bar{A}$-module $\bar{F}$. So $\bar{P}$ is the image of an idempotent endomorphism $\bar{\phi} : \bar{F} \to \bar{F}$. Let F be the direct sum in $\mathsf{C}(A)$ of copies of A, as many as there are copies of $\bar{A}$ in $\bar{F}$. There is a canonical surjection $F \to \bar{F}$. Choose any homomorphism $\phi : F \to F$ in $\mathsf{C}(A)$ lifting $\bar{\phi}$. Note that many such ϕ exist; and they aren't necessarily idempotents. Let P be the homotopy colimit construction on ϕ. Namely we let

$$\Phi : \bigoplus\nolimits_{i \in \mathbb{N}} F \to \bigoplus\nolimits_{i \in \mathbb{N}} F$$

be the homomorphism

$$\Phi(a_0, a_1, a_2, \ldots) := (a_0, a_1 - \phi(a_0), a_2 - \phi(a_1), \ldots)$$

in $\mathsf{C}(A)$, and then we define $P := \mathrm{cone}(\Phi) \in \mathsf{C}(A)$. Because P is K-flat over A, we see that $\bar{A} \otimes^{\mathrm{L}}_{A} P \cong \bar{A} \otimes_{A} P \cong \mathrm{cone}(\bar{\Phi})$, where

$$\bar{\Phi} : \bigoplus\nolimits_{i \in \mathbb{N}} \bar{F} \to \bigoplus\nolimits_{i \in \mathbb{N}} \bar{F}$$

is the homotopy colimit construction on $\bar{\phi}$. An easy calculation, using the fact that $\bar{\phi}$ is an idempotent with image $\bar{P}$, shows that $\mathrm{cone}(\bar{\Phi}) \cong \bar{P}$ in $\mathsf{D}(\bar{A})$. This proves (ii).

As for (i): say $\bar{Q}$ is the other direct summand of $\bar{F}$ in $\mathsf{Mod}\,\bar{A}$. We can lift it to a DG A-module Q as above. Now $\bar{P} \oplus \bar{Q} \cong \bar{F}$ in $\mathsf{D}(\bar{A})$. By Proposition 3.3(1) we deduce that $P \oplus Q \cong F$ in $\mathsf{D}(A)$. □

4. Localization of Commutative DG Rings

In this section we specialize to DG rings satisfying a commutativity condition. Such a DG ring A can be localized on $\operatorname{Spec} \bar{A}$.

Definition 4.1. Let $A = \bigoplus_{i \in \mathbb{Z}} A^i$ be a DG ring.

(1) A is called *weakly commutative* if $b \cdot a = (-1)^{ij} \cdot a \cdot b$ for all $a \in A^i$ and $b \in A^j$.
(2) A is called *strongly commutative* if it is weakly commutative, and if $a \cdot a = 0$ for all $a \in A^i$ with i odd.
(3) A is called a *commutative DG ring* if it is strongly commutative and nonpositive.

If A is weakly commutative, then any left DG A-module M can be viewed as a right DG A-module. The formula for the right action is this: $m \cdot a := (-1)^{ij} \cdot a \cdot m$ for $a \in A^i$ and $m \in M^j$.

The full subcategory of DGR on the commutative DG rings is denoted by $\mathsf{DGR}^{\leq 0}_{\mathrm{sc}}$.

Convention 4.2. By default, all DG rings from here on in the paper are commutative (Definition 4.1(3)), unless explicitly stated otherwise; i.e. we work inside $\mathsf{DGR}^{\leq 0}_{\mathrm{sc}}$. In particular all rings are commutative by default.

Since our DG rings are now commutative, we can talk about cohomologically pseudo-noetherian DG rings, instead of cohomologically left pseudo-noetherian ones (Definition 1.18).

Remark 4.3. In [YZ1], and in earlier versions of this paper, we used the name "super-commutative" for what we now call "strongly commutative". The book [ML] uses the term "strictly commutative".

Of course when 2 is invertible in the ring A^0 (e.g. in characteristic 0), there is no difference between weakly commutative and strongly commutative DG rings.

The reader may wonder why we chose to work in the category $\mathsf{DGR}^{\leq 0}_{\mathrm{sc}}$ of strongly commutative nonpositive DG rings, and not in the category $\mathsf{DGR}^{\leq 0}_{\mathrm{wc}}$ of weakly commutative ones. Here is the reason.

Consider any nontrivial commutative base ring $\mathbb{K}$ (e.g. $\mathbb{K} = \mathbb{Z}$), and the corresponding slice category $\mathsf{DGR}^{\leq 0}_{\mathrm{sc}} / \mathbb{K}$. Let X be a set of nonpositive graded variables. The strongly commutative polynomial ring in X (see [Ye5, Definition 3.10]) is denoted by $\mathbb{K}_{\mathrm{sc}}[X]$. It is easy to see that $\mathbb{K}_{\mathrm{sc}}[X]$ is a flat $\mathbb{K}$-module. A DG ring $A \in \mathsf{DGR}^{\leq 0}_{\mathrm{sc}} / \mathbb{K}$ is called semi-free if $A^{\natural} \cong \mathbb{K}_{\mathrm{sc}}[X]$ for some X. Such a semi-free DG ring A is a bounded above complex of flat $\mathbb{K}$-modules, and thus it is a K-flat DG $\mathbb{K}$-module. We had shown in [YZ1] that there are enough semi-free resolutions in $\mathsf{DGR}^{\leq 0}_{\mathrm{sc}} / \mathbb{K}$. Therefore there are enough K-flat resolutions in $\mathsf{DGR}^{\leq 0}_{\mathrm{sc}} / \mathbb{K}$.

On the other hand, let us examine the category $\mathsf{DGR}^{\leq 0}_{\mathrm{wc}} / \mathbb{Z}$. The weakly commutative polynomial ring $\mathbb{Z}_{\mathrm{wc}}[x]$ over $\mathbb{Z}$, in a single variable x of degree -1, has this structure as a graded $\mathbb{Z}$-module:

$$\mathbb{Z}_{\mathrm{wc}}[x] = \mathbb{Z} \oplus (\mathbb{Z} \cdot x) \oplus \big(\tfrac{\mathbb{Z}}{(2)} \cdot x^2 \big) \oplus \big(\tfrac{\mathbb{Z}}{(2)} \cdot x^3 \big) \oplus \cdots .$$

We see that it is not flat over $\mathbb{Z}$. Moreover, it is not hard to see that if A is any object in $\mathsf{DGR}^{\leq 0}_{\mathrm{wc}} / \mathbb{Z}$ such that $A^{\natural} \cong \mathbb{Z}_{\mathrm{wc}}[x]$, then A is not a K-flat DG $\mathbb{Z}$-module. Similarly, it seems that any semi-free object $A \in \mathsf{DGR}^{\leq 0}_{\mathrm{wc}}$ that has a nontrivial degree -1 component, also fails to be a K-flat DG $\mathbb{Z}$-module. The upshot is that, most likely, *there are not enough K-flat resolutions in* $\mathsf{DGR}^{\leq 0}_{\mathrm{wc}} / \mathbb{Z}$.

Proposition 4.4. *Let A be a DG ring. The action of A^0 on $\mathsf{C}(A)$ makes $\mathsf{D}(A)$ into an $\bar{A}$-linear category.*

Proof. Let $\phi : M \to N$ be a morphism in $\mathsf{C}(A)$; namely ϕ is a degree 0 cocycle in the DG abelian group $\mathrm{Hom}_A(M, N)$. For any $a \in A^{-1}$, the homomorphism $\mathrm{d}(a) \cdot \phi = \mathrm{d}(a \cdot \phi)$ is a degree 0 coboundary in $\mathrm{Hom}_A(M, N)$, and so it vanishes in the homotopy category $\mathsf{K}(A)$. Therefore $\mathsf{K}(A)$ is an $\bar{A}$-linear category, and hence so is its localization $\mathsf{D}(A)$. □

Definition 4.5. Let A be a DG ring, with canonical homomorphism $\pi : A \to \bar{A} = \mathrm{H}^0(A)$. Given a multiplicatively closed subset S of $\bar{A}$, the set $\tilde{S} := \pi^{-1}(S) \cap A^0$ is a multiplicatively closed subset of A^0. Define the ring $A^0_S := \tilde{S}^{-1} \cdot A^0$ (this is the usual localization), and the DG ring $A_S := A^0_S \otimes_{A^0} A$. There is a canonical DG ring homomorphism $\lambda_S : A \to A_S$.

If $S = \{s^i\}_{i \in \mathbb{N}}$ for some element $s \in \bar{A}$, then we also use the notation $A_s := A_S$.

There is the usual localization $\bar{A}_S = S^{-1} \cdot \bar{A}$ of the ring $\bar{A}$ w.r.t. S. We get a graded ring $\mathrm{H}(A)_S := \bar{A}_S \otimes_{\bar{A}} \mathrm{H}(A)$. There are graded ring homomorphisms $\mathrm{H}(A) \to \mathrm{H}(A)_S$ and $\mathrm{H}(\lambda_S) : \mathrm{H}(A) \to \mathrm{H}(A_S)$.

Proposition 4.6. *Let A be a DG ring and $S \subseteq \bar{A}$ a multiplicatively closed subset.*

(1) *There is a unique isomorphism of graded $\mathrm{H}(A)$-rings $\mathrm{H}(A_S) \cong \mathrm{H}(A)_S$.*

(2) *For any DG A-module M there is a unique isomorphism of graded $\mathrm{H}(A_S)$-modules*
$$\mathrm{H}(A_S \otimes_A M) \cong \mathrm{H}(A)_S \otimes_{\mathrm{H}(A)} \mathrm{H}(M)$$
that is compatible with the homomorphisms from $\mathrm{H}(M)$.

(3) *If A is cohomologically pseudo-noetherian, then so is A_S.*

Proof. Items (1-2) are true because the ring homomorphism $\lambda_S : A^0 \to A^0_S$ is flat. Item (3) is an immediate consequence of (1-2). □

Definition 4.7. Let A be a ring, and let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a sequence of elements of A. We call $\boldsymbol{a}$ a *covering sequence* of A if $\sum_{i=1}^{n} A \cdot a_i = A$.

Consider the spectrum $\operatorname{Spec} A$ of the ring A. For an element $a \in A$ we identify the principal open set $\{\mathfrak{p} \in \operatorname{Spec} A \mid a \notin \mathfrak{p}\}$ with the scheme $\operatorname{Spec} A_a$, where A_a is the localization of A w.r.t. a. Clearly a sequence $\boldsymbol{a} = (a_1, \ldots, a_n)$ in A is a covering sequence iff $\operatorname{Spec} A = \bigcup_{i=1}^{n} \operatorname{Spec} A_{a_i}$.

Let A be a DG ring, and let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a covering sequence of $\bar{A}$. For any strictly increasing sequence $\boldsymbol{i} = (i_0, \ldots, i_p)$ of length p in the integer interval $[0, n]$, i.e. $1 \leq i_0 < \cdots < i_p \leq n$, we define the ring

$$\mathrm{C}(A^0; \boldsymbol{a})(\boldsymbol{i}) := A^0_{a_{i_0}} \otimes_{A^0} \cdots \otimes_{A^0} A^0_{a_{i_p}}, \tag{4.8}$$

where $A^0_{a_0}, \ldots, A^0_{a_p}$ are the localizations from Definition 4.5. Next, for any $p \in [0, n-1]$ we let

$$\mathrm{C}^p(A^0; \boldsymbol{a}) := \bigoplus_{\boldsymbol{i}} \mathrm{C}(A^0; \boldsymbol{a})(\boldsymbol{i}) , \tag{4.9}$$

where the sum is on all strictly increasing sequences $\boldsymbol{i}$ of length p. Finally we define the DG A^0-module

$$\mathrm{C}(A^0; \boldsymbol{a}) := \bigoplus_{p=0}^{n-1} \mathrm{C}^p(A^0; \boldsymbol{a}). \tag{4.10}$$

The differential $\mathrm{C}^p(A^0; \boldsymbol{a}) \to \mathrm{C}^{p+1}(A^0; \boldsymbol{a})$ is $\sum_{\boldsymbol{i},k} (-1)^k \cdot \lambda_{\boldsymbol{i},k}$, where $\boldsymbol{i}$ runs over the strictly increasing sequences of length $p+1$, $k \in [0, p+1]$, $\partial_k(\boldsymbol{i})$ is the sequence obtained from $\boldsymbol{i}$ by omitting i_k, and $\lambda_{\boldsymbol{i},k} : \mathrm{C}(A^0;\boldsymbol{a})(\partial_k(\boldsymbol{i})) \to \mathrm{C}(A^0;\boldsymbol{a})(\boldsymbol{i})$ is the canonical ring homomorphism.

Definition 4.11. Let A be a DG ring, and let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a covering sequence of $\bar{A}$. For a DG A-module M, the *Čech DG module* of M is the DG A-module

$$\mathrm{C}(M;\boldsymbol{a}) := \mathrm{C}(A^0;\boldsymbol{a}) \otimes_{A^0} M.$$

There is a canonical DG module homomorphism $\mathrm{c}_M : M \to \mathrm{C}(M;\boldsymbol{a})$, sending $m \in M$ to $\sum_i 1_i \otimes m \in \mathrm{C}(M;\boldsymbol{a})$, where 1_i is the element $1 \in A^0_{a_i} \subseteq \mathrm{C}^0(A^0;\boldsymbol{a})$.

Proposition 4.12. *Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a covering sequence of $\bar{A}$, and let M be a DG A-module. Then the homomorphism $\mathrm{c}_M : M \to \mathrm{C}(M;\boldsymbol{a})$ is a quasi-isomorphism.*

Proof. Since $\mathrm{C}(A;\boldsymbol{a})$ is a K-flat DG A-module, and $\mathrm{C}(M;\boldsymbol{a}) \cong \mathrm{C}(A;\boldsymbol{a}) \otimes_A M$, we see that $\mathrm{C}(-;\boldsymbol{a})$ is a triangulated functor from $\mathsf{D}(A)$ to itself. The homomorphism c_M is a quasi-isomorphism iff it is an isomorphism in $\mathsf{D}(A)$.

The cohomological dimension of the functor $\mathrm{C}(-;\boldsymbol{a})$ is finite: it is at most $n-1$. According to Theorem 2.10(3) it suffices to check that c_M is a quasi-isomorphism for $M \in \mathsf{Mod}\,\bar{A}$. But in this case $\mathrm{C}(M;\boldsymbol{a}) \cong \mathrm{C}(\bar{A};\boldsymbol{a}) \otimes_{\bar{A}} M$, so $\mathrm{C}(M;\boldsymbol{a})$ is the usual Čech complex for the covering of $\operatorname{Spec} \bar{A}$ determined by the sequence $\boldsymbol{a}$. In geometric language (cf. [Ha, Section III.4]), writing $X := \operatorname{Spec} \bar{A}$ and $U_i := \operatorname{Spec} \bar{A}_{a_i}$, and letting $\mathcal{M}$ denote the quasi-coherent $\mathcal{O}_X$-module corresponding to M, we have $M \cong \Gamma(X, \mathcal{M})$ and $\mathrm{C}(M;\boldsymbol{a}) \cong \mathrm{C}(\{U_i\}, \mathcal{M})$. By [SP, Lemma 01X9], the homomorphism $\mathrm{c}_M : M \to \mathrm{C}(M;\boldsymbol{a})$ is a quasi-isomorphism. □

Remark 4.13. Actually the DG A-module $\mathrm{C}(A;\boldsymbol{a})$ has more structure. There is a cosimplicial commutative ring $\mathrm{C}_{\mathrm{cos}}(A^0;\boldsymbol{a})$, whose degree p piece is

$$\mathrm{C}^p_{\mathrm{cos}}(A^0;\boldsymbol{a}) := \prod_{\boldsymbol{i}} \mathrm{C}(A^0;\boldsymbol{a})(\boldsymbol{i}),$$

where $\boldsymbol{i} = (i_0, \ldots, i_p)$ are weakly increasing sequences in $[1,n]$. The Čech DG module $\mathrm{C}(A^0;\boldsymbol{a})$ is the standard normalization of $\mathrm{C}_{\mathrm{cos}}(A^0;\boldsymbol{s})$, and as such it has a structure of noncommutative central DG A^0-ring (which is concentrated in non-negative degrees). Hence $\mathrm{C}(A;\boldsymbol{a})$ is a noncommutative DG ring, and $\mathrm{c}_A : A \to \mathrm{C}(A;\boldsymbol{a})$ is a DG ring quasi-isomorphism. See [PSY, Section 8].

Definition 4.14. Let A be a ring, and let $\boldsymbol{e} = (e_1, \ldots, e_n)$ be a sequence of elements of A. We call $\boldsymbol{e}$ an *idempotent covering sequence* if each e_i is an idempotent element of A, $e_i \cdot e_j = 0$ for $i \neq j$, and $1 = \sum_{i=1}^n e_i$.

Suppose $\boldsymbol{e} = (e_1, \ldots, e_n)$ is an idempotent covering sequence of the ring A. Of course $\boldsymbol{e}$ is a covering sequence in the sense of Definition 4.7. For any i there is a unique A-ring isomorphism $A_{e_i} \cong A/(1-e_i) \cdot A$; namely the localization of A with respect to the element e_i is also the quotient ring modulo the ideal generated by the complementary idempotent $1-e_i$. There is a ring isomorphism

$$A \xrightarrow{\simeq} \prod_{i=1}^n A_{e_i}\,. \tag{4.15}$$

The scheme $\operatorname{Spec} A_{e_i}$ is an open-closed subscheme of $\operatorname{Spec} A$, and

$$\operatorname{Spec} A = \coprod_{i=1}^n \operatorname{Spec} A_{e_i}\,. \tag{4.16}$$

Definition 4.17. Let $\boldsymbol{e} = (e_1, \ldots, e_n)$ be an idempotent covering sequence of the ring A. The *$\boldsymbol{e}$-induced decomposition* of A is the ring isomorphism (4.15).

Definition 4.18. Let A be a ring. The set of locally constant functions $f : \operatorname{Spec} A \to \mathbb{Z}$ (for the Zariski topology on $\operatorname{Spec} A$) shall be denoted by $\mathrm{F}_{\mathrm{lc}}(\operatorname{Spec} A, \mathbb{Z})$. It is an abelian group by pointwise addition.

Proposition 4.19. *Let A be a ring. There is a bijection between the set $\mathrm{F}_{\mathrm{lc}}(\operatorname{Spec} A, \mathbb{Z})$, and the set of pairs $(\boldsymbol{e}, \boldsymbol{k})$, consisting of an idempotent covering sequence $\boldsymbol{e} = (e_1, \ldots, e_n)$ of A, and a nondecreasing sequence of integers $\boldsymbol{k} = (k_1, \ldots, k_n)$. This bijection sends a function $f \in \mathrm{F}_{\mathrm{lc}}(\operatorname{Spec} A, \mathbb{Z})$ to the pair $(\boldsymbol{e}, \boldsymbol{k})$ that satisfies $f^{-1}(k_i) = \operatorname{Spec} A_{e_i}$.*

Proof. Consider a locally constant function $f : \operatorname{Spec} A \to \mathbb{Z}$. Then f is continuous for the discrete topology on $\mathbb{Z}$. Since the topological space $\operatorname{Spec} A$ is quasi-compact, the image of f must be finite. This gives rise to a finite decomposition of $\operatorname{Spec} A$ into open-closed subsets, which must be of the form (4.16) for some idempotent covering sequence $\boldsymbol{e}$. After suitable renumbering we get the pair $(\boldsymbol{e}, \boldsymbol{k})$.

The converse is clear. □

Example 4.20. Often $\operatorname{Spec} A$ of the ring A has finitely many connected components; say n of them. Prototypes are:

(1) A is a noetherian ring.
(2) A is a semilocal ring.
(3) A is the ring of continuous (resp. differentiable) functions $X \to \mathbb{R}$, where X is a connected topological space (resp. a connected differentiable manifold). Here $n = 1$.

An ordering of the connected components of $\operatorname{Spec} A$ gives rise to an idempotent covering sequence $\boldsymbol{e} = (e_1, \ldots, e_n)$. The sequence of delta functions $(\delta_1, \ldots, \delta_n)$ is a basis of the group $\mathrm{F}_{\mathrm{lc}}(\operatorname{Spec} A, \mathbb{Z})$, which is thus isomorphic to $\mathbb{Z}^n$.

Definition 4.21. Let A be a ring, and assume that $\operatorname{Spec} A$ has finitely many connected components. A *connected component idempotent covering sequence* of A is an idempotent covering sequence $\boldsymbol{e} = (e_1, \ldots, e_n)$, such that each $\operatorname{Spec} A_{e_i}$ is nonempty and connected.

Clearly a connected component idempotent covering sequence of A is unique up to permutation.

Proposition 4.22. *Let A be a DG ring, and let $\boldsymbol{e} = (e_1, \ldots, e_n)$ be an idempotent covering sequence of $\bar{A} = \mathrm{H}^0(A)$. For any i we have the localized DG ring $A_i := A_{e_i}$ from Definition 4.5, and the DG ring homomorphism $\lambda_i : A \to A_i$. Then the DG ring homomorphism*

$$(\lambda_1, \ldots, \lambda_n) : A \to A_1 \times \cdots \times A_n$$

is a quasi-isomorphism.

Proof. Let's write $\bar{A}_i := \bar{A}_{e_i} = \mathrm{H}^0(A)_{e_i}$, so $\bar{A} = \prod_{i=1}^n \bar{A}_i$. Using Proposition 4.6(1) we obtain canonical graded ring isomorphisms

$$\mathrm{H}(A) \cong \prod\nolimits_{i=1}^n \big(\bar{A}_i \otimes_{\bar{A}} \mathrm{H}(A)\big) \cong \prod\nolimits_{i=1}^n \mathrm{H}(A_i) \cong \mathrm{H}\big(\prod\nolimits_{i=1}^n A_i\big).$$

The composition of these isomorphisms is exactly $\mathrm{H}(\lambda_1, \ldots, \lambda_n)$. □

Corollary 4.23. *With $A_1, \ldots, A_n$ as in Proposition 4.22, the restriction functors $\operatorname{rest}_{\lambda_i} : \mathsf{D}(A_i) \to \mathsf{D}(A)$ induce an equivalence of triangulated categories $\bigoplus_{i=1}^n \mathsf{D}(A_i) \to \mathsf{D}(A)$.*

Proof. This is standard. Cf. [YZ1, Proposition 1.4]. □

Definition 4.24. Let A be a DG ring, and let $e \in \bar{A} = \mathrm{H}^0(A)$ be an idempotent element. Consider the localized DG ring A_e corresponding to e, as in Definition 4.5. The triangulated functor

$$E : \mathsf{D}(A) \to \mathsf{D}(A)\ ,\ E(M) := A_e \otimes_A M,$$

is called the *idempotent functor* corresponding to e.

Definition 4.25. Let A be a DG ring, and let $\boldsymbol{e} = (e_1, \ldots, e_n)$ be an idempotent covering sequence of $\operatorname{Spec} \bar{A}$.

(1) The corresponding DG ring quasi-isomorphism
$$(\lambda_1, \ldots, \lambda_n) : A \to A_1 \times \cdots \times A_n \,,$$
from Proposition 4.22 is called the $\boldsymbol{e}$*-induced decomposition* of A.

(2) The corresponding functors $E_1, \ldots, E_n$ from Definition 4.24 are called the $\boldsymbol{e}$*-induced idempotent functors* of A.

(3) In case $\operatorname{Spec} \bar{A}$ has finitely many connected components, and $\boldsymbol{e}$ is one of its connected component idempotent covering sequence, then the expression "$\boldsymbol{e}$-induced" above is sometimes replaced by "connected component".

Proposition 4.26. *Suppose $\boldsymbol{e} = (e_1, \ldots, e_n)$ is an idempotent covering sequence of $\bar{A}$. Let $E_1, \ldots, E_n$ be the $\boldsymbol{e}$-induced idempotent functors.*

(1) *We have $E_i \circ E_i \cong E_i$, $E_i \circ E_j = 0$ for $i \neq j$, and $\sum_{i=1}^{n} E_i \cong \mathrm{id}_{\mathsf{D}(A)}$.*

(2) *Under the equivalence of categories in Corollary* 4.23, $\mathsf{D}(A_i)$ *is the essential image of E_i.*

Proof. This is similar to the proof of Proposition 4.22, and is left to the reader. □

Proposition 4.27. *Let $f : A \to B$ be a DG ring homomorphism, and let $M, N \in \mathsf{D}(B)$. We write $\bar{f} := \mathrm{H}^0(f)$ and $F := \operatorname{rest}_f : \mathsf{D}(B) \to \mathsf{D}(A)$. Assume that the ring homomorphism $\bar{f} : \bar{A} \to \bar{B}$ is surjective. Let $\boldsymbol{e} = (e_1, \ldots, e_n)$ be an idempotent covering sequence of $\bar{B}$, let $E_1, \ldots, E_n$ be the $\boldsymbol{e}$-induced idempotent functors of B, and write $M_i := E_i(M)$ and $N_i := E_i(N)$. Then for any $i \neq j$ we have*
$$\operatorname{Hom}_{\mathsf{D}(A)}\big(F(M_i), F(N_j)\big) = 0.$$
Therefore we get a canonical isomorphism of $\bar{A}$-modules
$$\operatorname{Hom}_{\mathsf{D}(A)}\big(F(M), F(N)\big) \cong \bigoplus_{i=1}^{n} \operatorname{Hom}_{\mathsf{D}(A)}\big(F(M_i), F(N_i)\big).$$

Proof. For any i choose some element $a_i \in \bar{A}$ such that $\bar{f}(a_i) = e_i$. Consider the noncommutative rings $\operatorname{End}_{\mathsf{D}(B)}(M_i)$ and $\operatorname{End}_{\mathsf{D}(A)}(F(M_i))$. There is a commutative diagram (of noncommutative rings)
$$\begin{tikzcd}
\bar{A} \arrow[r] \arrow[d, "\bar{f}"'] & \operatorname{End}_{\mathsf{D}(A)}(F(M_i)) \\
\bar{B} \arrow[r] & \operatorname{End}_{\mathsf{D}(B)}(M_i) \arrow[u, "F"']\,.
\end{tikzcd}$$
Cf. Proposition 4.4.

Take two distinct indices i, j. We know that $e_i \cdot \mathrm{id}_{M_i} = \mathrm{id}_{M_i}$ in $\operatorname{End}_{\mathsf{D}(B)}(M_i)$, and $e_i \cdot \mathrm{id}_{M_j} = 0$ in $\operatorname{End}_{\mathsf{D}(B)}(M_j)$. Therefore $a_i \cdot \mathrm{id}_{F(M_i)} = \mathrm{id}_{F(M_i)}$ in $\operatorname{End}_{\mathsf{D}(A)}(F(M_i))$, and $a_i \cdot \mathrm{id}_{F(M_j)} = 0$ in $\operatorname{End}_{\mathsf{D}(A)}(F(M_j))$.

Consider any morphism $\phi : F(M_i) \to F(N_j)$ in $\mathsf{D}(A)$. Then
$$\begin{aligned}
\phi = \mathrm{id}_{F(M_j)} \circ \phi \circ \mathrm{id}_{F(M_i)} &= \mathrm{id}_{F(M_j)} \circ \phi \circ (a_i \cdot \mathrm{id}_{F(M_i)}) \\
&= (a_i \cdot \mathrm{id}_{F(M_j)}) \circ \phi \circ \mathrm{id}_{F(M_i)} = 0.
\end{aligned}$$
□

5. Perfect DG Modules

Recall that all DG rings are now commutative by default (Convention 4.2). In particular all rings are commutative. For a DG ring A, its reduction is $\bar{A} = \mathrm{H}^0(A)$.

Let A be a DG ring and $s \in \bar{A}$ an element. The localization A_s was defined in Definition 4.5. The notion of covering sequence of $\bar{A}$ was introduced in Definition 4.7, and finite semi-free DG modules were introduced in Definition 1.14.

If A is a ring, there are two definitions in the literature (see [SGA 6, Exposé I]) of a *perfect complex of A-modules*. Let us recall them; but in order to make a distinction, we shall add attributes to the name "perfect". The first definition is this: a complex $M \in \mathsf{D}(A)$ is *geometrically perfect* if there is a covering sequence $(s_1, \ldots, s_n)$ of A, and for every i there is an isomorphism $A_{s_i} \otimes_A M \cong P_i$ in $\mathsf{D}(A_{s_i})$, where P_i is a bounded complex of finite free A_{s_i}-modules. The second definition is: a complex $M \in \mathsf{D}(A)$ is *algebraically perfect* if there is an isomorphism $M \cong P$ in $\mathsf{D}(A)$, where P is a bounded complex of finite projective A-modules. It is known that the two definitions are equivalent - see [SGA 6, Exposé I] or [SP, [Section 08E4]. Therefore it is safe to use the name "perfect complex" without further qualification.

Here are our generalizations to DG rings.

Definition 5.1. Let A be a DG ring, and let M be a DG A-module. We say that M is *geometrically perfect* if there is a covering sequence $(s_1, \ldots, s_n)$ of $\bar{A}$, and for every i there is an isomorphism $A_{s_i} \otimes_A M \cong P_i$ in $\mathsf{D}(A_{s_i})$, for some finite semi-free DG A_{s_i}-module P_i.

Clearly when A is a ring, we recover the classical definition; but for DG rings this is a new definition.

Before stating the second definition, we need to recall some terminology on triangulated categories. Let D be a triangulated category. A full subcategory $\mathsf{E} \subseteq \mathsf{D}$ is called *epaisse* if it is closed under shifts, cones and direct summands (so E itself is triangulated). We say that an object $P \in \mathsf{E}$ is a *classical generator* of E if this is the smallest epaisse subcategory of D that contains P. In this case, any object of E can be obtained from P by finitely many shifts, direct summands and cones. See [Ri] and [BV].

Definition 5.2. Let A be a DG ring, and let M be a DG A-module. We say that M is *algebraically perfect* if M belongs to the epaisse subcategory of $\mathsf{D}(A)$ classically generated by A.

It is not hard to see that when A is a ring, this definition coincides with the definition of algebraically perfect complexes given before. Definition 5.2 is not new – it already appeared in [ABIM] (without the attribute "algebraically").

In our paper we are interested in geometrically perfect DG modules. However, as we shall prove in Corollary 5.21, these turn out to be the same as algebraically perfect DG modules.

Proposition 5.3. *Let $A \to B$ be a homomorphism of DG rings, and let M be a geometrically perfect DG A-module. Then $B \otimes^{\mathrm{L}}_A M$ is a geometrically perfect DG B-module.*

Proof. Let $\bar{f} : \bar{A} \to \bar{B}$ denote the induced ring homomorphism, and let $\boldsymbol{s} = (s_1, \ldots, s_n)$ and P_i be as in the definition. Define $t_i := \bar{f}(s_i)$ and $N := B \otimes^{\mathrm{L}}_A M$. Then $(t_1, \ldots, t_n)$ is a covering sequence of $\bar{B}$, $Q_i := B_{t_i} \otimes_{A_{s_i}} P_i$ is a finite semi-free DG B_{t_i}-module, and $B_{t_i} \otimes_B N \cong Q_i$ in $\mathsf{D}(B_{t_i})$. □

Lemma 5.4. *Let M be a geometrically perfect DG A-module.*

1. *M belongs to $\mathsf{D}^-(A)$.*
2. *If A is cohomologically pseudo-noetherian, then M belongs to $\mathsf{D}^-_{\mathrm{f}}(A)$.*

Proof. Step 1. Assume $M \cong P$, where P is a finite semi-free DG A-module. We know that $A \in \mathsf{D}^-(A)$, and that $A \in \mathsf{D}^-_{\mathrm{f}}(A)$ in the cohomologically pseudo-noetherian case. Now P is obtained from A by finitely many shifts and cones, and hence P also belongs to $\mathsf{D}^-(A)$, and to $\mathsf{D}^-_{\mathrm{f}}(A)$ in the cohomologically pseudo-noetherian case.

Step 2. Let $s_1, \ldots, s_n \in \bar{A}$ and $P_i \in \mathsf{D}(A_{s_i})$ be as in Definition 5.1. We know that for each i, $\bar{A}_{s_i} \otimes_{\bar{A}} \mathrm{H}(M) \cong \mathrm{H}(P_i)$ as graded modules over $\bar{A}_{s_i}$. Step 1 tells us that $P_i \in \mathsf{D}^-(A_{s_i})$, and that $P_i \in \mathsf{D}^-_{\mathrm{f}}(A_{s_i})$ in the cohomologically pseudo-noetherian case. From the faithfully flat ring homomorphism $\bar{A} \to \prod_i \bar{A}_{s_i}$ we deduce that $\mathrm{H}(M)$ is bounded above (cf. Proposition 4.6). In the cohomologically pseudo-noetherian case, descent implies that each $\mathrm{H}^j(M)$ is finite over $\bar{A}$. Cf. [SP, Lemma 066D], noting that an $\bar{A}$ module is finite iff it is 0-pseudo-coherent. □

Let $L, M, N \in \mathsf{D}(A)$. There is a canonical morphism

$$\psi_{L,M,N} : \mathrm{RHom}_A(L, M) \otimes^{\mathrm{L}}_A N \to \mathrm{RHom}_A(L, M \otimes^{\mathrm{L}}_A N) \tag{5.5}$$

in $\mathsf{D}(A)$, which is functorial in the three arguments. If we choose a K-projective resolution $\tilde{L} \to L$, and a K-flat resolution $\tilde{N} \to N$, then the morphism $\psi_{L,M,N}$ is represented by the homomorphism

$$\tilde{\psi}_{\tilde{L},M,\tilde{N}} : \mathrm{Hom}_A(\tilde{L}, M) \otimes_A \tilde{N} \to \mathrm{Hom}_A(\tilde{L}, M \otimes_A \tilde{N}) \tag{5.6}$$

in $\mathsf{C}(A)$, where

$$\tilde{\psi}_{\tilde{L},M,\tilde{N}}(\alpha \otimes n)(l) := (-1)^{jk} \cdot \alpha(l) \otimes n$$

for $\alpha \in \mathrm{Hom}_A(\tilde{L}, M)^i$, $n \in \tilde{N}^j$ and $l \in \tilde{L}^k$.

Lemma 5.7. *$L, M, N \in \mathsf{D}(A)$, and assume L is geometrically perfect. Then the morphism $\psi_{L,M,N}$ in formula* (5.5) *is an isomorphism.*

Proof. Step 1. Assume $L \cong \tilde{L}$ in $\mathsf{D}(A)$, where $\tilde{L}$ is a finite semi-free DG A-module. Choose a K-flat resolution $\tilde{N} \cong N$. Then the homomorphism $\tilde{\psi}_{\tilde{L},M,\tilde{N}}$ in (5.6) is in fact bijective.

Step 2. Let $\boldsymbol{s} = (s_1, \ldots, s_n)$ be a covering sequence of $\bar{A}$, and for every i let us write $A_i := A_{s_i}$. We assume that there are isomorphisms $A_i \otimes_A L \cong \tilde{L}_i$ in $\mathsf{D}(A_i)$, such that $\tilde{L}_i$ is a finite semi-free DG A_i-module; cf. Definition 5.1. In this step we assume that $M \cong A_i \otimes_A M$ in $\mathsf{D}(A)$ for some index i. Let's write $L_i := A_i \otimes_A L$, $M_i := A_i \otimes_A M$ and $N_i := A_i \otimes_A N$. Then, and using adjunction with respect to the homomorphism $A \to A_i$, we get isomorphisms

$$\begin{aligned} \mathrm{RHom}_A(L, M) \otimes^{\mathrm{L}}_A N &\cong \mathrm{RHom}_A(L, M_i) \otimes^{\mathrm{L}}_A N \\ &\cong \mathrm{RHom}_{A_i}(L_i, M_i) \otimes^{\mathrm{L}}_A N \cong \mathrm{RHom}_{A_i}(L_i, M_i) \otimes^{\mathrm{L}}_{A_i} N_i \end{aligned}$$

and

$$\mathrm{RHom}_A(L, M \otimes^{\mathrm{L}}_A N) \cong \mathrm{RHom}_A(L, M_i \otimes^{\mathrm{L}}_A N) \cong \mathrm{RHom}_{A_i}(L_i, M_i \otimes^{\mathrm{L}}_A N_i)$$

in $\mathsf{D}(A)$. By step 1, the morphism

$$\psi_{L_i,M_i,N_i} : \mathrm{RHom}_{A_i}(L_i, M_i) \otimes^{\mathrm{L}}_{A_i} N_i \to \mathrm{RHom}_{A_i}(L_i, M_i \otimes^{\mathrm{L}}_{A_i} N_i)$$

is an isomorphism.

Step 3. We keep the covering sequence $\boldsymbol{s} = (s_1, \ldots, s_n)$ from step 2. Since the Čech resolution $\mathrm{c}_N : M \to \mathrm{C}(M; \boldsymbol{s})$ is a quasi-isomorphism (Proposition 4.12), it suffices to prove that $\psi_{L,M',N}$ is an isomorphism, where $M' := \mathrm{C}(M; \boldsymbol{s})$.

The DG A^0-module $\mathrm{C}(A^0; \boldsymbol{s})$ is filtered by degree:

$$\mu^k(\mathrm{C}(A^0; \boldsymbol{s})) := \bigoplus_{j \geq k} \mathrm{C}^j(A^0; \boldsymbol{s}).$$

This is a decreasing filtration of finite length, because $\mu^0(\mathrm{C}(A^0; \boldsymbol{s})) = \mathrm{C}(A^0; \boldsymbol{s})$ and $\mu^n(\mathrm{C}(A^0; \boldsymbol{s})) = 0$. Now by definition $\mathrm{C}(M; \boldsymbol{s}) = \mathrm{C}(A^0; \boldsymbol{s}) \otimes_{A^0} M$, so we get an induced filtration of finite length $\{\mu^k(\mathrm{C}(M; \boldsymbol{s}))\}_{k \in \mathbb{Z}}$ on the DG module $\mathrm{C}(M; \boldsymbol{s})$, with

$$\mu^k\big(\mathrm{C}(M; \boldsymbol{s})\big) := \mu^k\big(\mathrm{C}(A^0; \boldsymbol{s})\big) \otimes_{A^0} M. \tag{5.8}$$

For every k the filtration gives rise to an exact sequence of DG A-modules, that becomes a distinguished triangle

$$\mu^{k+1}\big(\mathrm{C}(M; \boldsymbol{s})\big) \to \mu^k\big(\mathrm{C}(M; \boldsymbol{s})\big) \to \mathrm{gr}^k_\mu\big(\mathrm{C}(M; \boldsymbol{s})\big) \xrightarrow{\triangle} \tag{5.9}$$

in $\mathsf{D}(A)$. Thus to prove that $\psi_{L,M',N}$ is an isomorphism, it suffices to prove that $\psi_{L,M'_k,N}$ is an isomorphism, where

$$M'_k := \mathrm{gr}^k_\mu\big(\mathrm{C}(M; \boldsymbol{s})\big) \cong \mathrm{C}^k(A^0; \boldsymbol{s})[-k] \otimes_{A^0} M. \tag{5.10}$$

But M'_k is a finite direct sum of shifts of the DG modules

$$M''_{\boldsymbol{i}} := \mathrm{C}(A^0; \boldsymbol{s})(\boldsymbol{i}) \otimes_{A^0} M;$$

see formula (4.9). Thus we reduce the problem to proving that $\psi_{L,M''_{\boldsymbol{i}},N}$ is an isomorphism. Because $M''_{\boldsymbol{i}}$ satisfies the assumption in step 2, we are done. □

Theorem 5.11. *Let A be a DG ring, and let M be a DG A-module. The following two conditions are equivalent*:

(i) *M is geometrically perfect.*
(ii) *M belongs to $\mathsf{D}^-(A)$, and the DG $\bar{A}$-module $\bar{A} \otimes^{\mathrm{L}}_A M$ is geometrically perfect.*

If A is cohomologically pseudo-noetherian, then these two conditions are equivalent to:

(iii) *M is in $\mathsf{D}^-_{\mathrm{f}}(A)$, and it has finite projective dimension relative to $\mathsf{D}(A)$.*

When A is a ring, i.e. $A = \bar{A}$, the equivalence (i) ⇔ (ii) is almost a tautology, and the equivalence (i) ⇔ (iii) was already proved in [SGA 6, Exposé I]. But for a genuine DG ring this is a new result.

Proof. (i) ⇔ (ii): According to Lemma 5.4 we have $M \in \mathsf{D}^-(A)$. Write $\bar{M} := \bar{A} \otimes^{\mathrm{L}}_A M$. Consider a covering sequence $(s_1, \ldots, s_n)$ of $\bar{A}$. Let us write $A_i := A_{s_i}$, $M_i := A_i \otimes_A M$ and $\bar{A}_i := \mathrm{H}^0(A_{s_i})$. For every i there is an isomorphism $\bar{A}_i \otimes_{\bar{A}} \bar{M} \cong \bar{A}_i \otimes^{\mathrm{L}}_{A_i} M_i$ in $\mathsf{D}(A_{s_i})$. Using Proposition 3.3 we see that M_i is isomorphic in $\mathsf{D}(A_i)$ to a finite semi-free DG A_i-module iff $\bar{A}_i \otimes^{\mathrm{L}}_{A_i} M_i$ is isomorphic in $\mathsf{D}(\bar{A}_i)$ to a finite semi-free DG $\bar{A}_i$-module.

(i) ⇒ (iii): Here A is cohomologically pseudo-noetherian. Lemma 5.4 says that $M \in \mathsf{D}^-_{\mathrm{f}}(A)$. To prove that M has finite projective dimension relative to $\mathsf{D}(A)$, we have to bound $\mathrm{H}\big(\mathrm{RHom}_A(M, N)\big)$ in terms of $\mathrm{H}(N)$ for any $N \in \mathsf{D}(A)$. Choose a covering sequence $(s_1, \ldots, s_n)$ of $\bar{A}$ and finite semi-free DG A_i-modules P_i as in Definition 5.1, where $A_i := A_{s_i}$. Let $d_0 \leq d_1$ be integers such that every P_i is generated in the integer interval $[d_0, d_1]$ (see Definition 1.16). Using Lemma 5.7 for the isomorphism $\cong^\dagger$, and adjunction, we obtain isomorphisms

$$\begin{aligned} A_i \otimes_A \mathrm{RHom}_A(M, N) &\cong^\dagger \mathrm{RHom}_A(M, A_i \otimes_A N) \\ &\cong \mathrm{RHom}_{A_i}(A_i \otimes_A M, A_i \otimes_A N) \cong \mathrm{Hom}_{A_i}(P_i, A_i \otimes_A N) \end{aligned}$$

in $\mathsf{D}(A)$. Using Proposition 1.17, this proves that

$$\operatorname{con}\big(\mathrm{H}(\mathrm{RHom}_A(M,N))\big) \subseteq \operatorname{con}(\mathrm{H}(N)) - [d_0, d_1].$$

We conclude that the projective dimension of M relative to $\mathsf{D}(A)$ is $\leq d_1 - d_0$.

(iii) $\Rightarrow$ (ii): Here again A is cohomologically pseudo-noetherian. Let's write $\bar{M} := \bar{A} \otimes^{\mathsf{L}}_A M$. Because $M \in \mathsf{D}^-_{\mathrm{f}}(A)$, we can find a pseudo-finite semi-free resolution $P \to M$ in $\mathsf{C}(A)$. Thus $\bar{M} \cong \bar{A} \otimes_A P$ belongs to $\mathsf{D}^-_{\mathrm{f}}(\bar{A})$.

For every $\bar{N} \in \mathsf{D}(\bar{A})$ we have, by adjunction, $\mathrm{RHom}_{\bar{A}}(\bar{M}, \bar{N}) \cong \mathrm{RHom}_A(M, \bar{N})$. This shows that the projective dimension of $\bar{M}$ relative to $\mathsf{D}(\bar{A})$ is finite. But this just means that the complex $\bar{M}$ has finite projective dimension over the ring $\bar{A}$. In particular $\bar{M}$ belongs to $\mathsf{D}^-_{\mathrm{f}}(\bar{A})$. The usual syzygy argument shows that there is a quasi-isomorphism $\bar{P} \to \bar{M}$ in $\mathsf{C}(\bar{A})$, for some bounded complex of finite projective $\bar{A}$-modules $\bar{P}$. But locally on $\operatorname{Spec} \bar{A}$ each $\bar{P}^i$ is a free $\bar{A}$-module; and hence $\bar{M}$ is geometrically perfect. □

Remark 5.12. Possibly one could remove the pseudo-noetherian hypothesis on A in condition (iii) of Theorem 5.11. The new condition on M would most likely be this:

(iii') The DG A-module M is pseudo-coherent, and it has finite flat dimension relative to $\mathsf{D}^{\mathrm{b}}(A)$.

This would require a detailed study of pseudo-coherent DG A-modules. Cf. [SGA 6, Exposé I], [SP, Definition 0657] and [SP, Lemma 0658].

Recall that a DG A-module M is called a *compact object of* $\mathsf{D}(A)$ if for any collection $\{N_z\}_{z \in Z}$ of DG A-modules, the canonical homomorphism

$$\bigoplus_{z \in Z} \mathrm{Hom}_{\mathsf{D}(A)}(M, N_z) \to \mathrm{Hom}_{\mathsf{D}(A)}\Big(M, \bigoplus\nolimits_{z \in Z} N_z\Big) \tag{5.13}$$

is bijective. (In general this is only injective.) It is known that for a ring A, compact and perfect are the same (see [Ri, Section 6], [Ne, Example 1.13], or [SP, Proposition 07LT]). It turns out that this is also true for a DG ring.

First we need to know that being compact is a local property on $\operatorname{Spec} \bar{A}$. This is very similar to arguments found in [Ne].

Lemma 5.14. *Let A be a DG ring, let M be a DG A-module, and let $(s_1, \ldots, s_n)$ be a covering sequence of $\bar{A}$. The following conditions are equivalent.*

(i) *M is a compact object of $\mathsf{D}(A)$.*

(ii) *For every i the DG A_{s_i}-module $A_{s_i} \otimes_A M$ is a compact object of $\mathsf{D}(A_{s_i})$.*

Proof. (i) $\Rightarrow$ (ii): This is the easy implication. We write $A_i := A_{s_i}$ and $M_i := A_i \otimes_A M$. Let $F_i : \mathsf{D}(A_i) \to \mathsf{D}(A)$ be the restriction functor. It commutes with all direct sums. Given a collection $\{N_z\}_{z \in Z}$ in $\mathsf{D}(A_i)$, we have canonical isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathsf{D}(A_i)}\big(M_i, \textstyle\bigoplus_z N_z\big) &\cong \mathrm{Hom}_{\mathsf{D}(A)}\big(M, F_i(\textstyle\bigoplus_z N_z)\big) \\ &\cong \mathrm{Hom}_{\mathsf{D}(A)}\big(M, \textstyle\bigoplus_z F_i(N_z)\big) \cong \textstyle\bigoplus_z \mathrm{Hom}_{\mathsf{D}(A)}(M, F_i(N_z)) \\ &\cong \textstyle\bigoplus_z \mathrm{Hom}_{\mathsf{D}(A_i)}(M_i, N_z)\ . \end{aligned}$$

We use the adjunction for F_i and the fact that M is compact. The conclusion is that M_i is compact.

(ii) $\Rightarrow$ (i): For any DG A-module N we have the Čech resolution $\mathrm{c}_N : N \to \mathrm{C}(N; \boldsymbol{s})$ from Definition 4.11 and Proposition 4.12. Because

$$\mathrm{C}(N; \boldsymbol{s}) = \mathrm{C}(A^0; \boldsymbol{s}) \otimes_{A^0} N \cong \mathrm{C}(A; \boldsymbol{s}) \otimes_A N,$$

this functor commutes with all direct sums. Thus the canonical homomorphism

$$\bigoplus\nolimits_{z \in Z} \mathrm{C}(N_z; \boldsymbol{s}) \to \mathrm{C}\Big(\bigoplus\nolimits_{z \in Z} N_z \,; \boldsymbol{s}\Big) \tag{5.15}$$

is an isomorphism in $\mathsf{C}(A)$. Using (5.15) we obtain a commutative diagram of $\bar{A}$-modules

$$\begin{tikzcd} \bigoplus_z \mathrm{Hom}_{\mathsf{D}(A)}(M, N_z) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathsf{D}(A)}\Big(M, \bigoplus_z N_z\Big) \arrow[d, "\cong"] \\ \bigoplus_z \mathrm{Hom}_{\mathsf{D}(A)}\big(M, \mathrm{C}(N_z; \boldsymbol{s})\big) \arrow[r] & \mathrm{Hom}_{\mathsf{D}(A)}\Big(M, \bigoplus_z \mathrm{C}(N_z; \boldsymbol{s})\Big) \end{tikzcd} \tag{5.16}$$

where the vertical arrows are bijections. So it suffices to prove that the lower horizontal arrow is a bijection.

Consider the finite length filtration $\{\mu^k(\mathrm{C}(N_z; \boldsymbol{s}))\}_{k \in \mathbb{Z}}$ on the DG module $\mathrm{C}(N_z; \boldsymbol{s})$, as in formula (5.8). Passing to the associated distinguished triangles, and using induction on k, as was done in the proof of Lemma 5.7, we reduce the problem to the verification that

$$\bigoplus\nolimits_z \mathrm{Hom}_{\mathsf{D}(A)}(M, A_{\boldsymbol{i}} \otimes_A N_z) \to \mathrm{Hom}_{\mathsf{D}(A)}\Big(M, \bigoplus\nolimits_z A_{\boldsymbol{i}} \otimes_A N_z\Big) \tag{5.17}$$

is a bijection, where $A_{\boldsymbol{i}} := A^0_{s_{i_0}} \otimes_{A^0} \cdots \otimes_{A^0} A^0_{s_{i_k}}$ for some strictly increasing sequence $\boldsymbol{i} = (i_0, \ldots, i_k)$ in the integer interval $[1, n]$. Let's write $A' := A_{s_{i_0}}$. Adjunction for the DG ring homomorphism $A \to A'$ allows us to replace (5.17) with the homomorphism

(5.18)

$$\bigoplus\nolimits_z \mathrm{Hom}_{\mathsf{D}(A')}(A' \otimes_A M, A_{\boldsymbol{i}} \otimes_A N_z) \to \mathrm{Hom}_{\mathsf{D}(A')}\Big(A' \otimes_A M, \bigoplus\nolimits_z A_{\boldsymbol{i}} \otimes_A N_z\Big).$$

But we are assuming that $A' \otimes_A M$ is compact in $\mathsf{D}(A')$; so (5.18) is bijective. □

Lemma 5.19. *If M is a compact object of $\mathsf{D}(A)$, then it belongs to $\mathsf{D}^-(A)$.*

Proof. This is an argument from [Ri], slightly improved in the proof of [SP, Proposition 07LT].

Suppose $\{N_z\}_{z \in Z}$ is a collection of DG A-modules. Given a morphism $\psi : M \to \bigoplus_{z \in Z} N_z$ in $\mathsf{D}(A)$, there is a finite subset $Z_0 \subseteq Z$ such that ψ factors through $\bigoplus_{z \in Z_0} N_z$. So for any $z \notin Z_0$, the component $\psi_z : M \to N_z$ of ψ is zero.

For every $k \geq 0$ consider the smart truncation $\mathrm{smt}^{\geq k}(M)$ from (1.5). There is a canonical surjective homomorphism $\phi_k : M \to \mathrm{smt}^{\geq k}(M)$ in $\mathsf{C}(A)$, and we know that $\mathrm{H}^l(\phi_k)$ is an isomorphism for all $l \geq k$. Consider the homomorphism

$$\phi : M \to \bigoplus_{k \in \mathbb{N}} \mathrm{smt}^{\geq k}(M) \ , \ \phi := \sum \phi_k$$

in $\mathsf{C}(A)$. Let $\psi := \mathrm{Q}(\phi)$; so the k-th component of ψ is $\psi_k := \mathrm{Q}(\phi_k) : M \to \mathrm{smt}^{\geq k}(M)$. As explained in the paragraph above, there is an integer k_0 such that $\psi_{k_0+1} = 0$. Therefore

$$\mathrm{H}^l(\psi_{k_0+1}) = \mathrm{H}^l(\phi_{k_0+1}) : \mathrm{H}^l(M) \to \mathrm{H}^l(\mathrm{smt}^{\geq k_0+1}(M))$$

is zero for all l. We see that $\mathrm{H}^l(M) = 0$ for all $l \geq k_0 + 1$. □

Theorem 5.20. *Let A be a DG ring, and let L be a DG A-module. The following three conditions are equivalent*:

(i) *L is a geometrically perfect DG A-module.*
(ii) *L is a compact object of $\mathsf{D}(A)$.*

(iii) *For any* $M, N \in \mathsf{D}(A)$*, the canonical morphism*

$$\psi_{L,M,N} : \mathrm{RHom}_A(L, M) \otimes^{\mathrm{L}}_A N \to \mathrm{RHom}_A(L, M \otimes^{\mathrm{L}}_A N)$$

in $\mathsf{D}(A)$ *is an isomorphism.*

See the text just after formula (5.5) for a description of the morphism $\psi_{L,M,N}$ in condition (iii).

Proof. (i) $\Rightarrow$ (ii): Since a finite semi-free DG module is clearly compact, this follows from Lemma 5.14.

(ii) $\Rightarrow$ (i): Assume L is compact in $\mathsf{D}(A)$. Consider the DG $\bar{A}$-module $\bar{L} := \bar{A} \otimes^{\mathrm{L}}_A L$. Adjunction shows that

$$\mathrm{Hom}_{\mathsf{D}(\bar{A})}(\bar{L}, M) \cong \mathrm{Hom}_{\mathsf{D}(A)}(L, F(M))$$

functorially for $M \in \mathsf{D}(\bar{A})$. Here F is the forgetful functor, that commutes with all direct sums. Thus $\bar{L}$ is a compact object of $\mathsf{D}(\bar{A})$. Now by [Ri, Section 6], [Ne, Example 1.13] or [SP, Proposition 07LT]) the DG $\bar{A}$-module $\bar{L}$ is algebraically perfect. So there is an isomorphism $L \cong \bar{P}$ in $\mathsf{D}(\bar{A})$, where $\bar{P}$ is a bounded complex of finite projective $\bar{A}$-modules. But locally on Spec $\bar{A}$ each $\bar{P}^i$ is a free module. Thus $\bar{L}$ is geometrically perfect. By the lemma above we know that $L \in \mathsf{D}^-(A)$. The implication (ii) $\Rightarrow$ (i) in Theorem 5.11 says that L is geometrically perfect.

(i) $\Rightarrow$ (iii): This is Lemma 5.7.

(iii) $\Rightarrow$ (ii): Take any collection of DG A-modules $\{N_z\}_{z \in Z}$, and define $N := \bigoplus_{z \in Z} N_z$. By assumption, the any z the morphism

$$\psi_{L,A,N_z} : \mathrm{RHom}_A(L, A) \otimes^{\mathrm{L}}_A N_z \to \mathrm{RHom}_A(L, N_z)$$

is an isomorphism. Since derived tensor products commute with all direct sums, we get an isomorphism

$$\phi : \mathrm{RHom}_A(L, A) \otimes^{\mathrm{L}}_A N \xrightarrow{\simeq} \bigoplus_{z \in Z} \mathrm{RHom}_A(L, N_z).$$

Now the functor H^0 also commutes with all direct sums. So we get a commutative diagram of $\bar{A}$-modules

$$\begin{tikzcd}
\mathrm{H}^0\big(\mathrm{RHom}_A(L, A) \otimes^{\mathrm{L}}_A N\big) \arrow[r, "\mathrm{H}^0(\psi_{L,A,N})"] \arrow[d, "\mathrm{H}^0(\phi)"'] & \mathrm{H}^0\big(\mathrm{RHom}_A(L, N)\big) \arrow[d, "\cong"] \\
\bigoplus_{z \in Z} \mathrm{Hom}_{\mathsf{D}(A)}(L, N_z) \arrow[r, "\mathrm{can}"] & \mathrm{Hom}_{\mathsf{D}(A)}(L, N)
\end{tikzcd}$$

in which the vertical arrows are isomorphisms. Our assumption says that $\mathrm{H}^0(\psi_{L,A,N})$ is an isomorphism. Therefore the bottom arrow (marked "can") is an isomorphism too. But this is the morphism (5.13). $\square$

Corollary 5.21. *Let A be a DG ring and M a DG A-module. The following two conditions are equivalent:*

(i) *M is geometrically perfect* (*Definition* 5.1).
(ii) *M is algebraically perfect* (*Definition* 5.2).

Proof. By Theorem 5.20, M is geometrically perfect iff it is a compact object of $\mathsf{D}(A)$. On the other hand, it is well-known (see [BV, Proposition 2.2.4]) that M is algebraically perfect iff it is a compact object of $\mathsf{D}(A)$. $\square$

Convention 5.22. From here on we use the expression "perfect DG module", rather than the two longer yet equivalent expressions.

Remark 5.23. Suppose A is a noncommutative DG ring. Definition 5.1 is worthless here: even if A happens to be nonpositive, still the ring $\bar{A}$ is noncommutative, so we cannot localize on $\operatorname{Spec} \bar{A}$.

However, Definition 5.2 is fine when A is noncommutative, and also when A has nontrivial positive components. So we can talk about algebraically perfect DG A-modules for any $A \in \mathsf{DGR}$. Indeed, this is the definition of perfect DG module that was used in [ABIM]. Results of [Ri], [BV] and [ABIM] say that a DG A-module M is algebraically perfect iff it is a compact object of $\mathsf{D}(A)$, iff it is a direct summand, in $\mathsf{D}(A)$, of a finite semi-free DG A-module.

6. Tilting DG Modules

Recall that all DG rings here are commutative (Convention 4.2). In particular all rings are commutative.

Definition 6.1. Let A be a DG ring. A DG A-module P is called a *tilting DG module* if there exists some DG A-module Q such that $P \otimes^{\mathrm{L}}_A Q \cong A$ in $\mathsf{D}(A)$.

The DG module Q in the definition is called a *quasi-inverse* of P. Due to the symmetry of the operation $- \otimes^{\mathrm{L}}_A -$, Q is also tilting. It is easy to see that the quasi-inverse Q is unique, up to a nonunique isomorphism. If P_1 and P_2 are tilting then so is $P_1 \otimes^{\mathrm{L}}_A P_2$; this is because of the associativity of $- \otimes^{\mathrm{L}}_A -$. Hence the next definition makes sense.

Definition 6.2. The *commutative derived Picard group* of A is the abelian group $\operatorname{DPic}(A)$, whose elements are the isomorphism classes, in $\mathsf{D}(A)$, of tilting DG A-modules. The product is induced by the operation $- \otimes^{\mathrm{L}}_A -$, and the unit element is the class of A.

Lemma 6.3. *Let $f : A \to B$ be a homomorphism of DG rings.*

(1) *For any $M, N \in \mathsf{D}(A)$ there is an isomorphism*
$$(B \otimes^{\mathrm{L}}_A M) \otimes^{\mathrm{L}}_B (B \otimes^{\mathrm{L}}_A N) \cong B \otimes^{\mathrm{L}}_A (M \otimes^{\mathrm{L}}_A N)$$
in $\mathsf{D}(A)$.

(2) *If P is a tilting DG A-module, then $B \otimes^{\mathrm{L}}_A P$ is a tilting DG B-module.*

(3) *If f is a quasi-isomorphism and Q is a tilting DG B-module, then* $\operatorname{rest}_f(Q)$ *is a tilting DG A-module.*

Proof. (1) Choose K-flat resolutions $\tilde{M} \to M$ and $\tilde{N} \to N$ over A. Then $B \otimes_A \tilde{M}$ and $B \otimes_A \tilde{N}$ are K-flat over B, $\tilde{M} \otimes_A \tilde{N}$ is K-flat over A, and
$$\begin{aligned}(B \otimes^{\mathrm{L}}_A M) \otimes^{\mathrm{L}}_B (B \otimes^{\mathrm{L}}_A N) &\cong (B \otimes_A \tilde{M}) \otimes_B (B \otimes_A \tilde{N}) \\ &\cong B \otimes_A (\tilde{M} \otimes_A \tilde{N}) \cong B \otimes^{\mathrm{L}}_A (M \otimes^{\mathrm{L}}_A N).\end{aligned}$$

(2) Let $P, Q \in \mathsf{D}(A)$ be such that $P \otimes^{\mathrm{L}}_A Q \cong A$. By (1) we have
$$(B \otimes^{\mathrm{L}}_A P) \otimes^{\mathrm{L}}_B (B \otimes^{\mathrm{L}}_A Q) \cong B.$$

(3) Say $Q_1, Q_2 \in \mathsf{D}(B)$ satisfy $Q_1 \otimes^{\mathrm{L}}_B Q_2 \cong B$. Let $P_i := \operatorname{rest}_f(Q_i) \in \mathsf{D}(A)$. By the equivalence for DG ring quasi-isomorphisms (see [YZ1, Proposition 1.4]) we have
$$P_1 \otimes^{\mathrm{L}}_A P_2 \cong \operatorname{rest}_f(Q_1 \otimes^{\mathrm{L}}_B Q_2) \cong \operatorname{rest}_f(B) \cong A.$$
□

Proposition 6.4. *Let $f : A \to B$ be a homomorphism of DG rings.*

(1) *There is a group homomorphism*

$$\mathrm{DPic}(f) : \mathrm{DPic}(A) \to \mathrm{DPic}(B)$$

with formula $P \mapsto B \otimes^{\mathrm{L}}_{A} P$.

(2) *If* f *is a quasi-isomorphism then* $\mathrm{DPic}(f)$ *is bijective.*

Proof. (1) This follows from parts (1) and (2) of the lemma above.

(2) Part (3) of Lemma 6.3 shows that in case f is a quasi-isomorphism, the function $Q \mapsto \mathrm{rest}_f(Q)$ is an inverse of $\mathrm{DPic}(f)$. □

Theorem 6.5. *Let* A *be a DG ring and* $P \in \mathsf{D}(A)$. *The following four conditions are equivalent.*

(i) *The DG* A*-module* P *is tilting.*
(ii) *The functor* $P \otimes^{\mathrm{L}}_{A} -$ *is an equivalence of* $\mathsf{D}(A)$.
(iii) *The functor* $\mathrm{RHom}_A(P, -)$ *is an equivalence of* $\mathsf{D}(A)$.
(iv) *The DG* A*-module* P *is perfect, and the adjunction morphism* $A \to \mathrm{RHom}_A(P, P)$ *in* $\mathsf{D}(A)$ *is an isomorphism.*

Proof. (i) $\Rightarrow$ (ii): Let Q be a quasi-inverse of P. Then the functor $G(M) := Q \otimes^{\mathrm{L}}_{A} M$ is a quasi-inverse of the functor $F(M) := P \otimes^{\mathrm{L}}_{A} M$.

(ii) $\Rightarrow$ (i): The functor $F := P \otimes^{\mathrm{L}}_{A} -$ is essentially surjective on objects, so there is some $Q \in \mathsf{D}(A)$ such that $F(Q) \cong A$. Then Q is a quasi-inverse of P.

(ii) $\Leftrightarrow$ (iii): The functors $F := P \otimes^{\mathrm{L}}_{A} -$ and $G := \mathrm{RHom}_A(P, -)$ are adjoints, so F is an equivalence iff G is an equivalence.

(ii) $\Rightarrow$ (iv): Consider the auto-equivalence $F := P \otimes^{\mathrm{L}}_{A} -$ of $\mathsf{D}(A)$. Since A is compact and $P = F(A)$, it follows that P is compact. Now according to Theorem 5.20, perfect is the same as compact.

For any $M \in \mathsf{D}(A)$, the adjunction morphism $A \to \mathrm{RHom}_A(M, M)$ is an isomorphism iff the canonical graded ring homomorphism

$$\alpha_M : \mathrm{H}(A) \to \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{D}(A)}(M, M[k])$$

is bijective. The equivalence F induces a commutative diagram of graded rings

$$\begin{tikzcd}
\mathrm{H}(A) \arrow[d, "\alpha_M"'] \arrow[dr, "\alpha_{F(M)}"] & \\
\bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{D}(A)}(M, M[k]) \arrow[r, "F"] & \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{D}(A)}\big(F(M), F(M)[k]\big)
\end{tikzcd}$$

in which the horizontal arrow is an isomorphism. Now take $M := A$. Because α_A is an isomorphism, so is α_P.

(iv) $\Rightarrow$ (i): Define $Q := \mathrm{RHom}_A(P, A) \in \mathsf{D}(A)$. The implication (i) $\Rightarrow$ (iii) of Theorem 5.20 shows that

$$Q \otimes^{\mathrm{L}}_{A} P = \mathrm{RHom}_A(P, A) \otimes^{\mathrm{L}}_{A} P \cong \mathrm{RHom}_A(P, P) \cong A$$

in $\mathsf{D}(A)$. So P is tilting, with quasi-inverse Q. □

Corollary 6.6. *Let* P *be a tilting DG* A*-module. Then the DG* A*-module* $Q := \mathrm{RHom}_A(P, A)$ *is a quasi-inverse of* P.

Proof. This was shown in the proof of the implication (iv) $\Rightarrow$ (i) above. □

Remark 6.7. A DG A-module M for which the adjunction morphism $A \to \mathrm{RHom}_A(M, M)$ is an isomorphism, is sometimes called *semidualizing*; cf. [AIL]. Indeed, this condition is part of the definition of a *dualizing DG module* (see Definition 7.1 below). But as we saw in Theorem 6.5, this condition is also characteristic of tilting DG modules – so the name semidualizing might be confusing.

Corollary 6.8. *Let P be a tilting DG A-module, and let $F := P \otimes^{\mathrm{L}}_A -$ be the corresponding auto-equivalence of $\mathsf{D}(A)$.*

(1) *The functor F has finite cohomological dimension, and it preserves $\mathsf{D}^{+}(A)$, $\mathsf{D}^{-}(A)$ and $\mathsf{D}^{\mathrm{b}}(A)$.*
(2) *If A is cohomologically pseudo-noetherian, then the auto-equivalence F preserves the subcategory $\mathsf{D}_{\mathrm{f}}(A)$.*

Proof. (1) The theorem says that P is perfect. Let $s_1, \ldots, s_n \in \bar{A}$ and $P_1, \ldots, P_n$ be as in Definition 5.1. Let $d_0 \leq d_1$ be integers such that each P_i is generated in the integer interval $[d_0, d_1]$. Then the functor F has cohomological displacement at most $[d_0, d_1]$ relative to $\mathsf{D}(A)$, and cohomological dimension at most $d_1 - d_0$. The claim about $\mathsf{D}^{\star}(A)$ is now clear.

(2) Use Theorem 2.13(2), noting that $F(A) = P \in \mathsf{D}^{-}_{\mathrm{f}}(A)$, by Theorems 6.5 and 5.11. □

Proposition 6.9. *Let A be a DG ring, and let $\boldsymbol{e} = (e_1, \ldots, e_n)$ be an idempotent covering sequence of $\bar{A} = \mathrm{H}^0(A)$. For any i we have the localized DG ring $A_i = A_{e_i}$ from Definition 4.5, and the DG ring homomorphism $\lambda_i : A \to A_i$. Then the group homomorphisms*

$$\mathrm{DPic}(\lambda_i) : \mathrm{DPic}(A) \to \mathrm{DPic}(A_i)$$

induce a group isomorphism

$$\mathrm{DPic}(A) \xrightarrow{\simeq} \prod_{i=1}^{n} \mathrm{DPic}(A_i).$$

Proof. (1) According to Proposition 4.22 there is a DG ring quasi-isomorphism $\lambda : A \to \prod_{i=1}^{n} A_i$. By Proposition 6.4(2) there is a group isomorphism

$$\mathrm{DPic}(\lambda) : \mathrm{DPic}(A) \xrightarrow{\simeq} \mathrm{DPic}\Big(\prod_{i=1}^{n} A_i \Big).$$

And there is an obvious group isomorphism

$$\mathrm{DPic}\Big(\prod_{i=1}^{n} A_i \Big) \cong \prod_{i=1}^{n} \mathrm{DPic}(A_i).$$

□

From here to Theorem 6.13 we consider a ring A. The first lemma goes back to [RD, Lemma V.3.3], and it reappeared, in a noncommutative guise, in [Ye2] and [RZ]. But in the prior treatments the ring A was assumed to be noetherian. Therefore we give a full proof of the general case.

Lemma 6.10. *Let A be a local ring, and let P be a tilting DG A-module. Then $P \cong A[k]$ in $\mathsf{D}(A)$ for some integer k.*

Proof. Since P is a perfect complex of A-modules (Theorem 6.5), it is isomorphic in $\mathsf{D}(A)$ to a bounded complex of finite projective A-modules. Say $\mathrm{con}(\mathrm{H}(P)) = [i_0, i_1]$ for some integers $i_0 \leq i_1$. By replacing P with such a resolution, and then splitting off extra terms in high and low degrees, we can assume that P is a complex of finite projective A-modules, concentrated in the degree interval $[i_0, i_1]$. Then $P' := \mathrm{H}^{i_1}(P)$ is a nonzero finitely presented A-module.

Let Q be a quasi-inverse of P. By the same reasoning, we can assume that Q is a complex of finite projective A-modules, concentrated in a degree interval $[j_0, j_1]$, and $Q' := \mathrm{H}^{j_1}(Q)$ is a nonzero finitely presented A-module.

Now by the Künneth trick we have isomorphisms of A-modules

$$P' \otimes_A Q' \cong \mathrm{H}^{i_1+j_1}(P \otimes^{\mathrm{L}}_A Q) \cong \mathrm{H}^{i_1+j_1}(A).$$

On the other hand $P' \otimes_A Q' \neq 0$, by Nakayama. The conclusion is that $i_1 + j_1 = 0$ and $P' \otimes_A Q' \cong A$. This tells us that P' and Q' are flat A-modules. Again by Nakayama, we see that $P' \cong Q' \cong A$.

Finally, we can split the complexes P and Q into $P \cong P'[-i_1] \oplus P^{\mathrm{ex}}$ and $Q \cong Q'[-j_1] \oplus Q^{\mathrm{ex}}$, where P^{ex} and Q^{ex} are complexes of finite projective A-modules, concentrated in the degree intervals $[i_0, i_1 - 1]$ and $[j_0, j_1 - 1]$ respectively. Then $\mathrm{H}(P^{\mathrm{ex}} \otimes_A Q'[-j_1])$ is a direct summand of the graded module $\mathrm{H}(P \otimes^{\mathrm{L}}_A Q) \cong \mathrm{H}(A)$. Since $\mathrm{H}(P^{\mathrm{ex}} \otimes_A Q'[-j_1])$ is concentrated in the interval $[i_0 + j_1, -1]$, it must be zero. So $P^{\mathrm{ex}} = 0$ in $\mathsf{D}(A)$, and $P \cong P'[-i_1]$. □

Recall that an A-module P is called *invertible* if it is projective of rank 1. In other words, if P is locally free of rank 1. See [Bo, Section II.5.2, Theorem 1].

Proposition 6.11. *The following conditions are equivalent for an A-module P :*

(i) *P is invertible.*
(ii) *When viewed as a DG module, P is tilting.*

Proof. The implication (i) ⇒ (ii) is trivial. For the other direction: the arguments in the first paragraph of the proof of Lemma 6.10 show that P is a finitely presented A-module. Take any prime ideal $\mathfrak{p}$. By Lemma 6.3, the DG $A_{\mathfrak{p}}$-module $P_{\mathfrak{p}} := A_{\mathfrak{p}} \otimes_A P$ is tilting. Hence, by Lemma 6.10, we have $P_{\mathfrak{p}} \cong A_{\mathfrak{p}}$. According to [Bo, Section II.5.2, Theorem 1] the module P is invertible. □

Lemma 6.12. *Let A be a ring, and let P be a tilting DG A-module. Write $X :=$ Spec A, and for any integer i define the set*

$$Y_i := \{\mathfrak{p} \in X \mid \mathrm{H}^i(P)_{\mathfrak{p}} \neq 0\}.$$

Then:

(1) *Only finitely many of the Y_i are nonzero.*
(2) *$Y_i \cap Y_j = \varnothing$ if $i \neq j$.*
(3) *$X = \bigcup_i Y_i$.*
(4) *For any i, the set Y_i is open-closed.*

Proof. We may assume that $A \neq 0$, i.e. that $X \neq \varnothing$. This implies that $P \neq 0$. We know from Theorem 6.5 that P is a perfect complex of A-modules. Therefore the cohomology $\mathrm{H}(P)$ is bounded, and thus $\operatorname{con}(\mathrm{H}(P)) = [i_0, i_1]$ for some integers $i_0 \leq i_1$. This proves claim (1).

Take any prime $\mathfrak{p} \in X$. By Lemma 6.3 the DG $A_{\mathfrak{p}}$-module $P_{\mathfrak{p}} := A_{\mathfrak{p}} \otimes_A P$ is tilting. So by Lemma 6.10 we have $P_{\mathfrak{p}} \cong A_{\mathfrak{p}}[k]$ for some integer k. We see that $\mathfrak{p} \in Y_{-k}$, and $\mathfrak{p} \notin Y_i$ for $i \neq -k$. This proves claims (2) and (3).

It remains to prove claim (4). By induction on $i_1 - i_0 = \operatorname{amp}(\mathrm{H}(P))$, it suffices to prove that $Y' := Y_{i_1}$ is open-closed in X. The reason is this: once we know that Y' is open-closed, then so is its complement $Y'' := \bigcup_{i_0 \leq i < i_1} Y_i$. So $X = Y' \coprod Y''$ as schemes, and correspondingly $A = A' \times A''$ as rings, and $P \cong P' \oplus P''$ in $\mathsf{D}(A)$. But $\operatorname{amp}(\mathrm{H}(P'')) < i_1 - i_0$.

As explained in the proof of Lemma 6.10, we can assume that P is a complex of finite projective A-modules, concentrated in the degree interval $[i_0, i_1]$. So $Q := \mathrm{H}^{i_1}(P)$ is a finitely presented A-module. Because Y' is the support of the module Q, it is closed in X.

Take a prime $\mathfrak{p} \in Y'$. As we have already seen above, the $A_{\mathfrak{p}}$-module $Q_{\mathfrak{p}}$ is isomorphic to $A_{\mathfrak{p}}$. According to [Bo, Section II.5.1, Corollary] there is an open neighborhood U of $\mathfrak{p}$ in X such that $Q_{\mathfrak{q}} \cong A_{\mathfrak{q}}$ for all $\mathfrak{q} \in U$. This shows that $U \subset Y'$. Therefore Y' is open in X. □

For a ring A we have the following theorem, due to Negron [Ng]. It is a refinement of earlier results, that are due (independently) to the author [Ye2] and to Rouquier-Zimmermann [RZ]. The earlier results focused on noetherian rings, and hence there was an assumption that $\operatorname{Spec} A$ has finitely many connected components. Negron recently noticed that this assumption is superfluous.

As usual, $\operatorname{Pic}(A)$ denotes the (commutative) Picard group of A, whose elements are the isomorphism classes of invertible A-modules.

The abelian group $\mathrm{F}_{\mathrm{lc}}(\operatorname{Spec} A, \mathbb{Z})$ was introduced in Definition 4.18. As shown in Proposition 4.19, each function $f \in \mathrm{F}_{\mathrm{lc}}(\operatorname{Spec} A, \mathbb{Z})$ determines a decomposition $A = \prod_{i=1}^{n} A_i$ of the ring A, and a nondecreasing sequence of integers $\boldsymbol{k} = (k_1, \ldots, k_n)$. The relation is this: $f(\operatorname{Spec} A_i) = k_i$.

Theorem 6.13. *Let A be a commutative ring. There is a canonical group isomorphism*

$$\operatorname{DPic}(A) \cong \operatorname{Pic}(A) \times \mathrm{F}_{\mathrm{lc}}(\operatorname{Spec} A, \mathbb{Z}),$$

characterized as follows:

- *The homomorphism* $\operatorname{Pic}(A) \to \operatorname{DPic}(A)$ *sends the class of an invertible A-module P to its class as a tilting DG A-module.*
- *Let $f \in \mathrm{F}_{\mathrm{lc}}(\operatorname{Spec} A, \mathbb{Z})$, with corresponding ring decomposition $A = \prod_{i=1}^{n} A_i$ and integer sequence $(k_1, \ldots, k_n)$. The tilting DG A-module associated to f is $\bigoplus_{i=1}^{n} A_i[k_i]$.*

Proof. For an invertible A-module Q and a function $f \in \mathrm{F}_{\mathrm{lc}}(\operatorname{Spec} A, \mathbb{Z})$ let

$$G(Q, f) := Q \otimes_A \big(\textstyle\bigoplus_{i=1}^{n} A_i[k_i]\big),$$

which is a tilting DG A-module. We have to prove that the group homomorphism

$$\operatorname{Pic}(A) \times \mathrm{F}_{\mathrm{lc}}(\operatorname{Spec} A, \mathbb{Z}) \to \operatorname{DPic}(A)$$

induced by G is bijective. It is certainly injective: if $G(Q, f) \cong A$, then f must be the constant function 0, as can be checked in $\operatorname{DPic}(A_{\mathfrak{p}})$ for each $\mathfrak{p} \in \operatorname{Spec} A$. But then $Q \cong A$ in $\mathsf{D}(A)$, which implies that $Q \cong A$ in $\operatorname{\mathsf{Mod}} A$.

It remains to prove that the group homomorphism induced by G is surjective. Take any tilting DG A-module P. Let $\operatorname{Spec} A = \coprod_{i=i_0}^{i_1} Y_i$ be the decomposition into open-closed sets induced by P, from Lemma 6.12. Consider the locally constant function $f : X \to \mathbb{Z}$ defined by $f|_{Y_i} := i$. Then the tilting DG module $Q := P \otimes^{\mathrm{L}}_{A} G(A, f)$ has the property that $\mathrm{H}^i(Q) = 0$ for all $i \neq 0$. By Proposition 6.11 we know that Q is isomorphic to an invertible A-module, say Q'. But then $P \cong G(Q', -f)$. □

Let A be a noetherian ring, $\mathfrak{a}$-adically complete with respect to some ideal $\mathfrak{a}$, with reduction $\bar{A} := A/\mathfrak{a}$. It is known that the group homomorphism $\operatorname{Pic}(A) \to \operatorname{Pic}(\bar{A})$ is bijective. For a proof see [Ha, Exercises II.9.6 and III.4.6].

The next theorem is a DG analogue of this fact. Recall that for a DG ring A there is a canonical homomorphism $A \to \bar{A}$.

Theorem 6.14. *Let A be a commutative DG ring. Then the canonical group homomorphism*

$$\operatorname{DPic}(A) \to \operatorname{DPic}(\bar{A})$$

is bijective.

Proof. Let us denote by $\pi : A \to \bar{A}$ the canonical DG ring homomorphism. We begin by proving that the homomorphism $\mathrm{DPic}(\pi)$ is injective. Suppose P is a tilting DG A-module such that $\bar{A} \otimes^{\mathrm{L}}_A P \cong \bar{A}$ in $\mathsf{D}(\bar{A})$. By Corollary 6.8(1) we know that $P \in \mathsf{D}^-(A)$. Then Proposition 3.3(1) says that $P \cong A$ in $\mathsf{D}(A)$.

Now let us prove that $\mathrm{DPic}(\pi)$ is surjective. Take any tilting DG $\bar{A}$-module $\bar{P}$. By Theorem 6.13 there is an isomorphism $\bar{P} \cong G(\bar{P}_0, f)$ for some invertible $\bar{A}$-module $\bar{P}_0$ and some function $f \in \mathrm{F}_{\mathrm{lc}}(\mathrm{Spec}\, A, \mathbb{Z})$.

By Proposition 3.5, there exists some DG module $P_0 \in \mathsf{D}^-(A)$ such that $\bar{A} \otimes^{\mathrm{L}}_A P_0 \cong \bar{P}_0$ in $\mathsf{D}(\bar{A})$. Let $\bar{Q}_0 \in \mathsf{Mod}\, \bar{A}$ be a quasi-inverse of the invertible module $\bar{P}_0$. By the same reason, there exists $Q_0 \in \mathsf{D}^-(A)$ such that $\bar{A} \otimes^{\mathrm{L}}_A Q_0 \cong \bar{Q}_0$ in $\mathsf{D}(\bar{A})$. Now by Lemma 6.3(1) we have

$$\bar{A} \otimes^{\mathrm{L}}_A (P_0 \otimes^{\mathrm{L}}_A Q_0) \cong \bar{P}_0 \otimes^{\mathrm{L}}_{\bar{A}} \bar{Q}_0 \cong \bar{A}$$

in $\mathsf{D}(\bar{A})$. Since $P_0 \otimes^{\mathrm{L}}_A Q_0 \in \mathsf{D}^-(A)$, by Proposition 3.3(1) we know that $P_0 \otimes^{\mathrm{L}}_A Q_0 \cong A$ in $\mathsf{D}(A)$. This shows that P_0 is a tilting DG A-module.

Finally, let $\boldsymbol{e} = (e_1, \ldots, e_n)$ and $\boldsymbol{k} = (k_1, \ldots, k_n)$ be the data corresponding to f from Proposition 4.19, and let $A \to \prod_{i=1}^n A_i$ be the $\boldsymbol{e}$-induced decomposition of A from Definition 4.25. Consider the tilting DG A-module

$$P := P_0 \otimes_A \big(\textstyle\bigoplus_{i=1}^n A_i[k_i]\big).$$

Then $\bar{P} \cong \bar{A} \otimes^{\mathrm{L}}_A P$. □

Corollary 6.15. *Let A be a DG ring. There is a canonical group isomorphism*

$$\mathrm{DPic}(A) \cong \mathrm{Pic}(\bar{A}) \times \mathrm{F}_{\mathrm{lc}}(\mathrm{Spec}\, \bar{A}, \mathbb{Z})\,.$$

Proof. Combine Theorems 6.14 and 6.13. □

Definition 6.16. Let A be a DG ring. We define $\mathrm{DPic}^0(A)$ to be the subgroup of $\mathrm{DPic}(A)$ that corresponds to $\mathrm{Pic}(\bar{A})$, under the canonical group isomorphism of Corollary 6.15.

Corollary 6.17. *If $\bar{A}$ is a local ring, then* $\mathrm{DPic}(A) \cong \mathbb{Z}$.

Proof. We know that $\mathrm{Pic}(\bar{A})$ is trivial, and $\mathrm{Spec}\, \bar{A}$ is connected. Now use Corollary 6.15. □

7. Dualizing DG Modules

Recall that all our DG rings are now commutative (Convention 4.2), and $\bar{A} = \mathrm{H}^0(A)$. In this section we concentrate on cohomologically pseudo-noetherian DG rings (Definition 1.18).

Here is a generalization of Grothendieck's definition of dualizing complex. When A is a ring, this is identical to the definition in [RD, Section V.2].

Definition 7.1. Let A be a cohomologically pseudo-noetherian DG ring. A DG A-module R is called a *dualizing* if it satisfies these three conditions:

(i) Each $\mathrm{H}^i(R)$ is a finite $\bar{A}$-module.
(ii) R has finite injective dimension relative to $\mathsf{D}(A)$.
(iii) The adjunction morphism $A \to \mathrm{RHom}_A(R, R)$ in $\mathsf{D}(A)$ is an isomorphism.

In other words, condition (i) says that $R \in \mathsf{D}_{\mathrm{f}}(A)$; condition (ii) says that the functor $\mathrm{RHom}_A(-, R)$ has finite cohomological dimension relative to $\mathsf{D}(A)$, as in Definitions 2.1 and 2.4(2); and condition (iii) says that the canonical graded ring homomorphism

$$\mathrm{H}(A) \to \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{D}(A)}(R, R[k])$$

is bijective.

Proposition 7.2. *Suppose R is a dualizing DG module over A, and consider the functor*

$$D := \mathrm{RHom}_A(-, R) : \mathsf{D}(A) \to \mathsf{D}(A).$$

Let $\mathsf{D}^{\star}$ denote either D, D^{b}, D^{+} or D^{-}; and correspondingly let $\mathsf{D}^{-\star}$ denote either D, D^{b}, D^{-} or D^{+}.

1. *For any $M \in \mathsf{D}^{\star}_{\mathrm{f}}(A)$ we have $D(M) \in \mathsf{D}^{-\star}_{\mathrm{f}}(A)$. In particular, $R \in \mathsf{D}^{+}_{\mathrm{f}}(A)$.*
2. *For any $M \in \mathsf{D}_{\mathrm{f}}(A)$ the canonical morphism $M \to D(D(M))$ in $\mathsf{D}_{\mathrm{f}}(A)$ is an isomorphism.*
3. *The functor*
$$D : \mathsf{D}^{\star}_{\mathrm{f}}(A)^{\mathrm{op}} \to \mathsf{D}^{-\star}_{\mathrm{f}}(A)$$
is an equivalence of triangulated categories.

Proof. (1) The functor D has finite cohomological dimension, so we can apply Theorem 2.13(2). Since $A \in \mathsf{D}^{-}_{\mathrm{f}}(A)$, we get $R = D(A) \in \mathsf{D}^{+}_{\mathrm{f}}(A)$.

(2) There is a morphism of triangulated functors $\eta : \mathrm{id}_{\mathsf{D}_{\mathrm{f}}(A)} \to D \circ D$. Both functors have finite cohomological dimensions, and η_A is an isomorphism. We can apply Theorem 2.11(2).

(3) Combine items (1) and (2). □

Corollary 7.3. *Let A be a cohomologically pseudo-noetherian DG ring, and let R be a dualizing DG A-module. The following two conditions are equivalent*:

(i) *The DG ring A is cohomologically bounded.*
(ii) *The DG module R is cohomologically bounded.*

Proof. This is by Proposition 7.2, since $R \cong D(A)$ and $A \cong D(R)$. □

In Example 7.26 we demonstrate the unbounded option.

Given a homomorphism $f : A \to B$ of DG rings, we denote by $\bar{f} := \mathrm{H}^0(f)$ the induced ring homomorphism.

Definition 7.4. Let $f : A \to B$ be a homomorphism between cohomologically pseudo-noetherian DG rings. We say that f is a *cohomologically pseudo-finite homomorphism* if $\bar{f} : \bar{A} \to \bar{B}$ is a finite ring homomorphism, i.e. $\bar{f}$ makes $\bar{B}$ into a finite $\bar{A}$-module.

Clearly if f is cohomologically finite, then rest_f sends $\mathsf{D}^{\star}_{\mathrm{f}}(B)$ into $\mathsf{D}^{\star}_{\mathrm{f}}(A)$, where $\star$ is either $+, -,$ b or blank.

Proposition 7.5. *Let $f : A \to B$ be a cohomologically pseudo-finite homomorphism between cohomologically pseudo-noetherian DG rings.*

1. *If R_A is a dualizing DG A-module, then $R_B := \mathrm{RHom}_A(B, R_A)$ is a dualizing DG B-module.*
2. *If f is a quasi-isomorphism and R_B is a dualizing DG B-module, then $R_A := \mathrm{rest}_f(R_B)$ is a dualizing DG A-module.*

Proof. (1) The proof is almost the same as in the case of a ring; see [RD, Proposition V.2.4]. Viewing B as an object of $\mathsf{D}^{-}_{\mathrm{f}}(A)$, Proposition 7.2(1) tells us that $R_B \in \mathsf{D}^{+}_{\mathrm{f}}(A)$; and hence $R_B \in \mathsf{D}^{+}_{\mathrm{f}}(B)$. For any $N \in \mathsf{D}(B)$ we have

$$\mathrm{RHom}_B(N, R_B) \cong \mathrm{RHom}_A(N, R_A)$$

by adjunction, and hence the injective dimension of R_B relative to $\mathsf{D}(B)$ is at most the injective dimension of A relative to $\mathsf{D}(A)$, which is finite. And finally

$$\mathrm{RHom}_B(R_B, R_B) \cong \mathrm{RHom}_A\big(\mathrm{RHom}_A(B, R_A), R_A\big) \cong B$$

by Proposition 7.2(2). These isomorphisms are actually inside $\mathsf{D}(B)$, and they are compatible with the canonical morphism from B.

(2) This is because here $\operatorname{rest}_f : \mathsf{D}(B) \to \mathsf{D}(A)$ is an equivalence, preserving boundedness and finiteness of cohomology. □

In commutative ring theory, many good properties of a ring A can be deduced if it is "not far" from a "nice" ring $\mathbb{K}$ (such as a field). This is the underlying reason for the next definition.

Recall that a ring homomorphism $A \to B$ is called of *essentially finite type* if it can be factored as $A \to B_{\mathrm{ft}} \to B$, where $A \to B_{\mathrm{ft}}$ is finite type (i.e. B is finitely generated as A-ring), and $B_{\mathrm{ft}} \to B$ is the localization at some multiplicatively closed subset of B_{ft}.

Definition 7.6. Let $\mathbb{K}$ be a noetherian ring, let A be a cohomologically pseudo-noetherian DG ring, and let $u : \mathbb{K} \to A$ a homomorphism of DG rings. We say that u is of *cohomologically essentially finite type*, and that A is a *cohomologically essentially finite type DG $\mathbb{K}$-ring*, if the ring homomorphism $\bar{u} : \mathbb{K} \to \bar{A}$ is of essentially finite type.

Definition 7.7. A DG ring A is called *tractable* if it is cohomologically pseudo-noetherian, and there exists a cohomologically essentially finite type homomorphism $\mathbb{K} \to A$ from some finite dimensional regular noetherian ring $\mathbb{K}$. Such a homomorphism $\mathbb{K} \to A$ is called a *traction* for A.

Lemma 7.8. *Let A be a DG ring, let $\mathbb{K}$ be a noetherian ring, and suppose there is a cohomologically essentially finite type homomorphism $u : \mathbb{K} \to A$. Then there is a commutative diagram of DG rings*

$$\begin{tikzcd}
\mathbb{K} \arrow[r, "u"] \arrow[d, "v"'] & A \arrow[r, "\pi"] \arrow[d, "f"'] & \bar{A} \\
A_{\mathrm{eft}} \arrow[r, "g"] & A_{\mathrm{loc}} \arrow[ur] &
\end{tikzcd}$$

such that π is the canonical homomorphism; f and g are quasi-isomorphism; and $\mathbb{K} \to A^0_{\mathrm{eft}}$ is essentially finite type.

Proof. Let $S := \pi^{-1}(\bar{A}^{\times}) \cap A^0$, namely $s \in S$ iff $\pi(s)$ is invertible in the ring $\bar{A}$. Define the DG ring $A_{\mathrm{loc}} := (S^{-1} \cdot A^0) \otimes_{A^0} A$. Then π factors via f, and f is a quasi-isomorphism.

Since $\mathbb{K} \to \bar{A}$ is essentially finite type, there is a polynomial ring $\mathbb{K}[\boldsymbol{t}]$ in finitely many variables of degree 0, and a homomorphism $h : \mathbb{K}[\boldsymbol{t}] \to \bar{A}$ which is essentially surjective, i.e. it is surjective after a localization. Thus, letting $T := h^{-1}(\bar{A}^{\times}) \subseteq \mathbb{K}[\boldsymbol{t}]$, and defining $A^0_{\mathrm{eft}} := T^{-1} \cdot \mathbb{K}[\boldsymbol{t}]$, the homomorphism $h_T : A^0_{\mathrm{eft}} \to \bar{A}$ is surjective.

Now the ring A^0_{eft} is noetherian. The homomorphism $h_T : A^0_{\mathrm{eft}} \to \bar{A}$ factors via a homomorphism $g^0 : A^0_{\mathrm{eft}} \to A^0_{\mathrm{loc}}$, and the composed homomorphism $A^0_{\mathrm{eft}} \to \mathrm{H}^0(A_{\mathrm{loc}})$ is surjective. Since the the modules $\mathrm{H}^i(A_{\mathrm{loc}})$ are finite over A^0_{eft}, we can extend A^0_{eft} to a DG ring A_{eft}, and simultaneously extend g^0 to a quasi-isomorphism $g : A_{\mathrm{eft}} \to A_{\mathrm{loc}}$, by inductively introducing finitely many new variables (free ring generators) in negative degrees. The process is the same as in the proof of [YZ1, Proposition 1.7(2)]. □

Theorem 7.9. *Let A be a tractable DG ring. Then A has a dualizing DG module.*

Proof. Let $\mathbb{K} \to A$ be a traction for A. Consider the diagram of homomorphisms in Lemma 7.8. Since $A \to A_{\mathrm{loc}}$ is a quasi-isomorphism, and $A^0_{\mathrm{eft}} \to A_{\mathrm{eft}} \to A_{\mathrm{loc}}$ are cohomologically pseudo-finite, it suffices (by Proposition 7.5) to show that A^0_{eft} has a dualizing DG module (which is the same as a dualizing complex over this ring, in the sense of [RD]). But the ring homomorphism $\mathbb{K} \to A^0_{\mathrm{eft}}$ can be factored into $\mathbb{K} \to \mathbb{K}[\boldsymbol{t}] \to B \to A^0_{\mathrm{eft}}$, where $\mathbb{K}[\boldsymbol{t}]$ is a polynomial ring in n variables, $\mathbb{K}[\boldsymbol{t}] \to B$ is surjective, and $B \to A^0_{\mathrm{eft}}$ is a localization. Thus, using [RD, Theorem V.8.3] and Proposition 7.5(1) above, the DG module

$$A^0_{\mathrm{eft}} \otimes_B \mathrm{RHom}_{\mathbb{K}[\boldsymbol{t}]}(B, \Omega^n_{\mathbb{K}[\boldsymbol{t}]/\mathbb{K}}[n])$$

is a dualizing DG module over A^0_{eft}. □

Theorem 7.10. *Let A be a cohomologically pseudo-noetherian DG ring, and let R be a dualizing DG module over A.*

(1) *If P is a tilting DG module, then $P \otimes^{\mathrm{L}}_A R$ is a dualizing DG module.*
(2) *If R' is a dualizing DG module, then $P := \mathrm{RHom}_A(R, R')$ is a tilting DG module, and $R' \cong P \otimes^{\mathrm{L}}_A R$ in $\mathsf{D}(A)$.*
(3) *If P is a tilting DG module, and if $R \cong P \otimes^{\mathrm{L}}_A R$ in $\mathsf{D}(A)$, then $P \cong A$ in $\mathsf{D}(A)$.*

This is similar to [RD, Theorem V.3.1], and the strategy of the proof is the same; cf. also [Ye2, Theorem 4.5].

Proof. (1) Assume P is a tilting DG module, and let $R' := P \otimes^{\mathrm{L}}_A R$. According to Corollary 6.8 the functor $P \otimes^{\mathrm{L}}_A -$ is an auto-equivalence of $\mathsf{D}(A)$, it has finite cohomological dimension, and it preserves $\mathsf{D}^+_{\mathrm{f}}(A)$. Therefore the DG module R' is dualizing.

(2) Define the objects $P := \mathrm{RHom}_A(R, R')$ and $P' := \mathrm{RHom}_A(R', R)$, and the functors $D := \mathrm{RHom}_A(-, R)$, $D' := \mathrm{RHom}_A(-, R')$, $F := P \otimes^{\mathrm{L}}_A -$ and $F' := P' \otimes^{\mathrm{L}}_A -$. We know that the functors $D, D', D' \circ D, D \circ D'$ have finite cohomological dimensions relative to $\mathsf{D}(A)$; the DG modules $P, P' \in \mathsf{D}^-_{\mathrm{f}}(A)$; and the functors F, F' have bounded above cohomological displacements relative to $\mathsf{D}(A)$. For any $M \in \mathsf{D}(A)$ there is a canonical morphism

$$\mathrm{RHom}_A(R, R') \otimes^{\mathrm{L}}_A M \to \mathrm{RHom}_A(\mathrm{RHom}_A(M, R), R'),$$

so we get a morphism of triangulated functors $\eta : F \to D' \circ D$. By definition η_A is an isomorphism, and Theorem 2.11(1) says that η_M is an isomorphism for every $M \in \mathsf{D}^-_{\mathrm{f}}(A)$. Likewise there is an isomorphism $\eta'_M : F'(M) \to (D \circ D')(M)$ for every $M \in \mathsf{D}^-_{\mathrm{f}}(A)$.

Let us calculate $P \otimes^{\mathrm{L}}_A P'$:

$$\begin{aligned} P \otimes^{\mathrm{L}}_A P' &\cong F(P') \cong (D' \circ D)(P') \\ &\cong (D' \circ D)(F'(A)) \cong (D' \circ D \circ D \circ D')(A) \cong A. \end{aligned}$$

This proves P is tilting. And

$$P \otimes^{\mathrm{L}}_A R \cong F(R) \cong (D' \circ D)(D(A)) \cong D'(A) \cong R'.$$

(3) If P is tilting and $R \cong P \otimes^{\mathrm{L}}_A R$, then

$$\begin{aligned} A &\cong \mathrm{RHom}_A(R, R) \cong \mathrm{RHom}_A(P \otimes^{\mathrm{L}}_A R, R) \\ &\cong^* \mathrm{RHom}_A(P, \mathrm{RHom}_A(R, R)) \cong \mathrm{RHom}_A(P, A), \end{aligned}$$

where the isomorphism $\cong^*$ is by adjunction. But then

$$P \cong A \otimes^{\mathrm{L}}_A P \cong \mathrm{RHom}_A(P, A) \otimes^{\mathrm{L}}_A P \cong^{\dagger} \mathrm{RHom}_A(P, P) \cong^{\dagger\dagger} A,$$

where the isomorphism $\cong^{\dagger}$ is by a combination of Theorems 6.5 and 5.20, and the isomorphism $\cong^{\dagger\dagger}$ is by Theorem 6.5. □

Corollary 7.11. *Assume A has some dualizing DG module. The formula $R \mapsto P \otimes^{\mathrm{L}}_{A} R$ induces a simply transitive action of the group* $\mathrm{DPic}(A)$ *on the set of isomorphism classes of dualizing DG A-modules.*

Proof. Clear from the theorem. □

Corollary 7.12. *Assume A has some dualizing DG module* (*e.g. A is tractable*). *The formula $R \mapsto \mathrm{RHom}_A(\bar{A}, R)$ induces a bijection*

$$\frac{\{\text{dualizing DG } A\text{-modules}\}}{\text{isomorphism}} \xrightarrow{\simeq} \frac{\{\text{dualizing DG } \bar{A}\text{-modules}\}}{\text{isomorphism}}.$$

Proof. By Corollary 7.11 the actions of the groups $\mathrm{DPic}(A)$ and $\mathrm{DPic}(\bar{A})$ on these two sets, respectively, are simply transitive. And by Theorem 6.14 the group homomorphism $\mathrm{DPic}(A) \to \mathrm{DPic}(\bar{A})$ induced by $P \mapsto \bar{A} \otimes^{\mathrm{L}}_{A} P$ is bijective. Thus it suffices to prove that the function induced by $R \mapsto \mathrm{RHom}_A(\bar{A}, R)$ is equivariant for the action of $\mathrm{DPic}(A)$. Here is the calculation:

$$\begin{aligned} \mathrm{RHom}_A(\bar{A}, P \otimes^{\mathrm{L}}_{A} R) &\cong^{\ddagger} P \otimes^{\mathrm{L}}_{A} \mathrm{RHom}_A(\bar{A}, R) \\ &\cong (\bar{A} \otimes^{\mathrm{L}}_{A} P) \otimes^{\mathrm{L}}_{\bar{A}} \mathrm{RHom}_A(\bar{A}, R). \end{aligned}$$

The isomorphism $\cong^{\ddagger}$ comes from Lemma 7.13 below, noting that the tilting DG module P satisfies condition $(*)$ of the lemma, since it is perfect. □

We say that a DG A-module N has *bounded below generation* if it is generated in the integer interval $[i_0, \infty]$ for some integer i_0; see Definition 1.16.

Lemma 7.13. *Let $L \in \mathsf{D}^{-}_{\mathrm{f}}(A)$ and $M, N \in \mathsf{D}^{+}(A)$. Assume that N satisfies this condition*:

$(*)$ *There is a covering sequence $(s_1, \ldots, s_n)$ of $\bar{A}$, and for every i there is an isomorphism $A_{s_i} \otimes_A N \cong \tilde{N}_i$ in $\mathsf{D}(A_{s_i})$, where $\tilde{N}_i$ is a K-flat DG A_{s_i}-module with bounded below generation.*

Then the canonical morphism

$$\psi_{L,M,N} : \mathrm{RHom}_A(L, M) \otimes^{\mathrm{L}}_{A} N \to \mathrm{RHom}_A(L, M \otimes^{\mathrm{L}}_{A} N)$$

in $\mathsf{D}(A)$, from formula (5.5), *is an isomorphism.*

Proof. Step 1. Here we assume that $N \cong \tilde{N}$ in $\mathsf{D}(A)$, where $\tilde{N}$ is a K-flat DG A-module of bounded below generation. Using smart truncation if needed, we can assume that the DG B-module M is bounded below. Let $\tilde{L} \to L$ be a pseudo-finite semi-free resolution over A (see Proposition 1.19). The morphism $\psi_{L,M,N}$ is represented by the homomorphism

$$\tilde{\psi}_{\tilde{L},M,\tilde{N}} : \mathrm{Hom}_A(\tilde{L}, M) \otimes_A \tilde{N} \to \mathrm{Hom}_A(\tilde{L}, M \otimes_A \tilde{N})$$

in $\mathsf{C}(A)$. Because the semi-free DG A-module $\tilde{L}$ is bounded above and has finitely many basis elements in each degree, and both M and $M \otimes_A \tilde{N}$ are bounded below, we see that $\tilde{\psi}_{\tilde{L},M,\tilde{N}}$ is bijective.

Step 2. Here N satisfies condition $(*)$. We claim that the obvious morphisms

$$\big(\mathrm{RHom}_A(L, M) \otimes^{\mathrm{L}}_{A} N\big) \otimes_{A^0} A^0_{s_i} \to \mathrm{RHom}_A(L, M) \otimes^{\mathrm{L}}_{A} \tilde{N}_i \tag{7.14}$$

and

$$\mathrm{RHom}_A(L, M \otimes^{\mathrm{L}}_{A} N) \otimes_{A^0} A^0_{s_i} \to \mathrm{RHom}_A(L, M \otimes^{\mathrm{L}}_{A} \tilde{N}_i) \tag{7.15}$$

in $\mathsf{D}(A)$, that come from the given isomorphisms $A_{s_i} \otimes_A N \xrightarrow{\simeq} \tilde{N}_i$, are isomorphisms. That (7.14) is an isomorphism is trivial. As for the morphism (7.15): condition (∗) implies that $M \otimes^{\mathrm{L}}_A N$ belongs to $\mathsf{D}^+(A)$. Since $A^0_{s_i}$ is a K-flat DG module over A^0 generated in degree 0, we can use Step 1.

Step 3. Now we are in the general situation. Let ψ_i be the morphism gotten from $\psi_{L,M,N}$ by the localization $A^0_{s_i} \otimes_{A^0} -$, so ψ_i goes from the first object in (7.14) to the first object in (7.15). Since $\bar{A} \to \prod_i \bar{A}_{s_i}$ is faithfully flat, it suffices to prove that all the ψ_i are isomorphisms. But by step 2 it suffices to show that

$$\psi_{L,M,\tilde{N}_i} : \mathrm{RHom}_A(L, M) \otimes^{\mathrm{L}}_A \tilde{N}_i \to \mathrm{RHom}_A(L, M \otimes^{\mathrm{L}}_A \tilde{N}_i)$$

is an isomorphism. Since $\tilde{N}_i$ is a K-flat DG A-module with bounded below generalization, we can use step 1. □

Corollary 7.16. *If the ring $\bar{A}$ is local, then any two dualizing DG A-modules R and R' satisfy $R' \cong R[m]$ for some integer m.*

Proof. By Corollary 6.17 we have $\mathrm{DPic}(A) \cong \mathbb{Z}$, generated by the class of $A[1]$. Now use Corollary 7.11. □

Proposition 7.17. *Let A be a cohomologically noetherian DG ring, and let R be a dualizing DG A-module. Assume A is cohomologically bounded. Then R is dualizing in the sense of* [FIJ, Definition 1.8].

Proof. There are four conditions in [FIJ, Definition 1.8]. Condition (1) – the existence of resolutions – is trivial in our commutative situation. Condition (2) says that if $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ then $\mathrm{RHom}_A(M, R) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$; and this is true by Proposition 7.2(1). Condition (3) requires that for any $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$, letting N be either M or $M \otimes^{\mathrm{L}}_A R$, the adjunction morphisms

$$N \to \mathrm{RHom}_A\big(\mathrm{RHom}_A(N, R), R\big)$$

are isomorphisms. Now by Corollary 7.3 we know that $R \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. A combination of Proposition 2.8 and Theorem 2.13(1) tells us that $M \otimes^{\mathrm{L}}_A R \in \mathsf{D}^{-}_{\mathrm{f}}(A)$. Thus in both cases $N \in \mathsf{D}^{-}_{\mathrm{f}}(A)$, and according to Proposition 7.2(2) the morphism in question is an isomorphism. Condition (4) is part of condition (3) in the commutative situation. □

Definition 7.18. A cohomologically noetherian cohomologically bounded DG ring A is called *Gorenstein* if the DG module A has finite injective dimension relative to $\mathsf{D}(A)$.

Proposition 7.19. *Let A be a cohomologically noetherian cohomologically bounded DG ring. The following conditions are equivalent*:

(i) *A is Gorenstein.*
(ii) *The DG A-module A is dualizing.*

Proof. Since conditions (i) and (iii) of Definition 7.1 are automatic for $R := A$, this is clear. □

Remark 7.20. For DG rings that are not cohomologically bounded, a comparison like in Proposition 7.17 does not seem to work nicely. Corollary 7.16 is very similar to [FIJ, Theorem III]; but of course the assumptions are not the same.

We do not know a reasonable definition of Gorenstein DG rings that are not cohomologically bounded.

Here is a rather surprising result, that was pointed out to us by Jørgensen.

Theorem 7.21. *Let A be a DG ring, which is cohomologically bounded and cohomologically essentially finite type over some noetherian ring $\mathbb{K}$. If $\bar{A}$ is a perfect DG A-module, then the canonical homomorphism $A \to \bar{A}$ is a quasi-isomorphism.*

Example 7.26 shows that the assumption that A is cohomologically bounded is really needed.

Proof. We will prove that $\bar{A}_{\mathfrak{p}} \otimes_{\bar{A}} \mathrm{H}^i(A) = 0$ for every $i < 0$ and every $\mathfrak{p} \in \operatorname{Spec} \bar{A}$.

Fix such i and $\mathfrak{p}$. Because the assertion is invariant under DG ring quasi-isomorphisms, we may assume, by Lemma 7.8 (replacing A with A_{eft}), that A^0 is a noetherian ring. Consider the ring $A^0_{\mathfrak{p}} := \tilde{S}^{-1} \cdot A^0$, where $\pi : A \to \bar{A}$ is the canonical homomorphism, $S := \bar{A} - \mathfrak{p}$, and $\tilde{S} := \pi^{-1}(S) \cap A^0$. Next define the DG ring $A_{\mathfrak{p}} := A^0_{\mathfrak{p}} \otimes_{A^0} A$. Then $A^0_{\mathfrak{p}} \to \bar{A}_{\mathfrak{p}}$ is surjective, and $A^0_{\mathfrak{p}}$ is a noetherian local ring. By Proposition 5.3(2) the DG $A_{\mathfrak{p}}$-module $\bar{A}_{\mathfrak{p}} \cong A_{\mathfrak{p}} \otimes_A \bar{A}$ is perfect; and by Theorem 5.20 this is a compact object of $\mathsf{D}(A_{\mathfrak{p}})$. Also $\bar{A}_{\mathfrak{p}}$ is nonzero. According to [Jo, Theorem 0.2] we have $\operatorname{amp}(\mathrm{H}(A_{\mathfrak{p}})) \leq \operatorname{amp}(\mathrm{H}(\bar{A}_{\mathfrak{p}})) = 0$. Therefore $\mathrm{H}^i(A_{\mathfrak{p}}) = 0$ for all $i < 0$. But $\mathrm{H}^i(A_{\mathfrak{p}}) \cong \bar{A}_{\mathfrak{p}} \otimes_{\bar{A}} \mathrm{H}^i(A)$. □

We conclude this section with several examples and remarks.

Example 7.22. Suppose A is a Gorenstein noetherian ring, and $\boldsymbol{a} = (a_1, \ldots, a_n)$ is a sequence of elements in A. Let $B := \mathrm{K}(A; \boldsymbol{a})$, the Koszul complex, which is cohomologically noetherian and cohomologically bounded. The DG ring homomorphism $A \to B$ is cohomologically pseudo-finite, $R_A := A[n]$ is a dualizing DG A-module, and hence $R_B := \operatorname{RHom}_A(B, R_A)$ is a dualizing DG B-module. But B is semi-free as DG A-module, and therefore

$$R_B = \operatorname{RHom}_A(B, A[n]) \cong \operatorname{Hom}_A(B, A[n]) \cong B$$

in $\mathsf{D}(B)$. We see that B is Gorenstein, in the sense of Definition 7.18.

Example 7.23. Here is a comparison to Hinich's notion of dualizing DG module from [Hi1]. Let A be a noetherian local ring, with maximal ideal $\mathfrak{m}$. For the sake of simplicity, let us assume that A contains a field $\mathbb{K}$, such that $\mathbb{K} \to A/\mathfrak{m}$ is finite. Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a sequence of elements in A that generates an $\mathfrak{m}$-primary ideal, and let $B := \mathrm{K}(A; \boldsymbol{a})$ be the associated Koszul complex. Thus $\mathbb{K} \to B$ is a cohomologically pseudo-finite homomorphism of DG rings, and according to Proposition 7.5 the DG B-module $R_B := \operatorname{Hom}_{\mathbb{K}}(B, \mathbb{K})$ is dualizing. Now $\operatorname{Hom}_{\mathbb{K}}(\mathrm{H}(R_B), \mathbb{K}) \cong \mathrm{H}(B)$ as graded $\mathrm{H}(B)$-modules, so R_B is a dualizing DG module in the sense of Hinich. Taking Corollary 7.16 into consideration, we see that any dualizing DG B-module R (in our sense) satisfies the condition of Hinich.

Remark 7.24. The results in our paper so far suggest an analogy between the following two scenarios:

(dg) The *DG scenario*: a cohomologically pseudo-noetherian DG ring A, with reduction $\bar{A} := \mathrm{H}^0(A)$.

(ad) The *adic scenario*: a noetherian ring A, $\mathfrak{a}$-adically complete w.r.t. an ideal $\mathfrak{a}$, with reduction $\bar{A} := A/\mathfrak{a}$.

We refer to this as the *DG vs. adic analogy*. This analogy restricts to the "degenerate cases" in these scenarios:

(dg) The cohomology $\mathrm{H}(A)$ is bounded.

(ad) The defining ideal $\mathfrak{a}$ is nilpotent.

Of course, this observation is not new (cf. [Lu1], [Lu2], [TV] and [AG]).

The DG vs. adic analogy holds also for "finite homomorphisms":

(dg) A cohomologically pseudo-finite homomorphism $f : A \to B$ between cohomologically pseudo-noetherian DG rings (Definition 7.4).

(ad) A *formally finite* or *pseudo-finite* homomorphism $f : A \to B$ between adically complete noetherian rings, as in [Ye1] and [AJL2] respectively.

There is a further analogy between "dualizing objects" in the two scenarios:

(dg) A dualizing DG module R over a cohomologically pseudo-noetherian DG ring A (Definition 7.1).

(ad) A *t-dualizing complex* R over an adically complete noetherian ring A, as in [Ye1] and [AJL1].

The analogies above raise two questions:

(1) Is there a DG analogue of the *c-dualizing complex* of [AJL1]?
(2) Is there a DG analogue of the *GM Duality* of [AJL1] and the *MGM Equivalence* of [PSY]?

Remark 7.25. Recall that a noetherian ring A of finite Krull dimension is regular (i.e. all its local rings $A_{\mathfrak{p}}$ are regular) iff it has finite global cohomological dimension.

Now suppose A is a cohomologically pseudo-noetherian DG ring. By "Krull dimension" we could mean that of $\bar{A}$, but "regular local ring" has no apparent meaning here. Hence we propose this definition: A is called *regular* if it has *finite global cohomological dimension*. By this we mean that there is a natural number d, such that if $M \in \mathsf{C}(A)$ is generated in the integer interval $[i_0, i_1]$, then M has projective dimension at most $d + i_1 - i_0$; cf. Definitions 1.16 and 2.4, and Examples 2.5 and 2.6. According to Theorem 5.11, we see that any $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$, including $M = \bar{A}$, is perfect.

Now assume that A is a regular DG ring, but with *bounded cohomology*. Then, taking $M = \bar{A}$, Theorem 7.21 says that $A \to \bar{A}$ is a quasi-isomorphism. The conclusion is that *the only regular DG rings with bounded cohomology are the regular rings* (up to quasi-isomorphism).

Under the DG vs. adic analogy of Remark 7.24, this corresponds to an adic ring A with a nilpotent defining ideal $\mathfrak{a}$. If A is regular, then it cannot have nonzero nilpotent elements. Therefore $\mathfrak{a} = 0$ here, and $A \to \bar{A}$ is bijective.

Example 7.26. Take a field $\mathbb{K}$, and let $A := \mathbb{K}[t]$, the polynomial ring in a variable t of degree -2. We view A as a DG ring with zero differential, so $\mathrm{H}(A) \cong A$, and it is cohomologically noetherian, but not cohomologically bounded below. The DG ring homomorphism $\mathbb{K} \to A$ is cohomologically pseudo-finite. Hence the DG A-module $R := \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ is dualizing. This DG module is not bounded above.

Note that here $\bar{A} \cong \mathbb{K}$ is a perfect DG A-module. To show this, we shall produce a finite semi-free resolution of $\bar{A}$. The DG module $A^{\leq -2}$, which is both the stupid and the smart truncation of A at -2, is free, since $A^{\leq -2} \cong A[2]$ as DG A-modules. Let $\phi : A^{\leq -2} \to A$ be the inclusion, and let $P := \mathrm{cone}(\phi)$. There is an obvious quasi-isomorphism $P \to \bar{A}$.

The adic analogue is the ring of formal power series $A := \mathbb{K}[[t]]$, with ideal of definition $\mathfrak{a} := (t)$. The corresponding t-dualizing complex is $R := \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{K}}(A, \mathbb{K})$, which is an artinian A-module of infinite length.

Remark 7.27. Our definition of dualizing DG modules, Definition 7.1, might seem an almost straightforward generalization of Grothendieck's original definition in [RD]. However there are at least two subtle points: (a) Finding the correct notion of injective dimension of a DG module (condition (ii) of Definition 7.1). (b) Allowing a dualizing DG module to have unbounded above cohomology (condition (i) of Definition 7.1; cf. Corollary 7.3 and Remark 7.24).

All results in this section, up to and including Corollary 7.11, might also seem to be straightforward generalizations of Grothendieck's corresponding results in [RD]. But the technical difficulties (mainly when A is cohomologically unbounded) cannot be neglected.

We should mention that Theorem 7.9 can be made a bit stronger, by replacing the condition that A is tractable with the weaker condition that there is a cohomologically essentially finite type homomorphism $\mathbb{K} \to A$, where $\mathbb{K}$ is a noetherian ring with a dualizing complex. Theorem 7.21 can be similarly strengthened.

Some earlier papers, notably [Hi1] and [FIJ], had adopted other definitions of dualizing DG modules; see Example 7.23 and Proposition 7.17 respectively. These definitions are not consistent with our definition in general, and there does not appear to be a well-developed theory for them.

In [Lu2, Definition 4.2.5], Lurie gives a definition of a dualizing E_∞ module over an E_∞ ring. Now any DG ring A can be viewed as an E_∞ ring, and DG A-modules can be viewed E_∞ A-modules. Under this correspondence, it seems that a dualizing DG A-module, in the sense of Definition 7.1 above, becomes a dualizing module in the sense of [Lu2]. (We are being careful, because a precise comparison of the definitions is not so easy.) Thus our results in this section, up to and including Corollary 7.11, might be viewed as special instances of Lurie's statements. Still, an attempt to produce a full proof of our results based on the corresponding results in [Lu2] (e.g. deducing our Theorem 7.9 from [Lu2, Theorem 4.3.14], or deducing our Theorem 7.10 from [Lu2, Proposition 4.2.9]) might be nontrivial, and most likely it would be longer than our own direct proofs. This is because, as far as we know, there do not exist full comparison results for the monoidal operations between the E_∞ and the DG setups.

Our Corollary 7.12 appears to be totally new. We could not find anything resembling it in Lurie's papers, nor elsewhere in the literature. Likewise for Theorem 7.21 (except for Jørgensen's original local result).

Remark 7.28. A result that is noticeably missing from our paper is a DG analogue of [Lu2, Theorem 4.3.5]. Translated to the DG terminology, it states that if the ring $\bar{A}$ admits a dualizing DG module, then the DG ring A admits a dualizing DG module. We do not know whether this result can be proved within the DG framework; this is a question that we find interesting.

Note however that the corresponding result in the adic case, namely when A is a complete $\mathfrak{a}$-adic ring extension of $\bar{A}$ (cf. Remark 7.24), was proved a long time ago by Faltings [Fa]. The proof of the nilpotent case in [Fa] is quite easy; but the passage to the complete adic case is somewhat involved there. The proof can be greatly simplified by first proving the existence of a t-dualizing complex R'_A over A, and then applying derived completion to obtain a c-dualizing complex $R_A := \mathrm{L}\Lambda_{\mathfrak{a}}(R'_A)$. See [Ye1] [AJL1], [AJL2] and [PSY] for information on derived completion, and on dualizing complexes over adic rings.

8. Cohen-Macaulay DG Modules

In this section we work with cohomologically pseudo-noetherian commutative DG rings (see Convention 4.2 and Definition 1.18).

Let A be such a DG ring. Recall that $\bar{A} = \mathrm{H}^0(A)$, and $\mathsf{D}^0(A)$ is the full subcategory of $\mathsf{D}(A)$ consisting of the DG modules M such that $\mathrm{H}^i(M) = 0$ for all $i \neq 0$. Inside $\mathsf{D}^0(A)$ we have $\mathsf{D}^0_{\mathrm{f}}(A) = \mathsf{D}_{\mathrm{f}}(A) \cap \mathsf{D}^0(A)$.

Lemma 8.1. *Consider the canonical DG ring homomorphism $\pi : A \to \bar{A}$. The functor*

$$\mathrm{Q} \circ \mathrm{rest}_\pi : \mathsf{Mod}\, \bar{A} \to \mathsf{D}^0(A)$$

is an equivalence. It restricts to an equivalence

$$\mathrm{Q} \circ \mathrm{rest}_{\pi} : \mathsf{Mod}_{\mathrm{f}} \bar{A} \to \mathsf{D}^{0}_{\mathrm{f}}(A).$$

Proof. Smart truncation shows that any object of $\mathsf{D}^{0}(A)$ is isomorphic to an object of $\mathsf{Mod}\,\bar{A}$. Finiteness of $\bar{A}$-modules is preserved. It remains to show that $\mathrm{Q} \circ \mathrm{rest}_{\pi}$ is a fully faithful functor.

So take $M, N \in \mathsf{Mod}\,\bar{A}$, and let $\tilde{M} \to M$ be a semi-free resolution over A with $\sup(\tilde{M}) \leq 0$. Then

$$\mathrm{Hom}_{\mathsf{D}(A)}(M, N) \cong \mathrm{H}^{0}\big(\mathrm{Hom}_{A}(\tilde{M}, N)\big) \cong \mathrm{Hom}_{\bar{A}}(\mathrm{H}^{0}(\tilde{M}), N) \cong \mathrm{Hom}_{\bar{A}}(M, N).$$

□

Definition 8.2. Let let R be a dualizing DG A-module. A DG module $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ is called *Cohen-Macaulay with respect to R* if $\mathrm{RHom}_{A}(M, R) \in \mathsf{D}^{0}_{\mathrm{f}}(A)$.

In other words, the condition is that $\mathrm{RHom}_{A}(M, R)$ is isomorphic, in $\mathsf{D}(A)$, to an object of $\mathsf{Mod}_{\mathrm{f}} \bar{A}$. As usual "Cohen-Macaulay" is abbreviated to "CM". Let us denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)_{\mathrm{CM}:R}$ the full subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ consisting of DG modules that are CM w.r.t. R.

Remark 8.3. Observe that the functor $\mathrm{RHom}_{A}(-, R)$ gives rise to a duality between $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)_{\mathrm{CM}:R}$ and $\mathsf{D}^{0}_{\mathrm{f}}(A)$. And the latter is equivalent to $\mathsf{Mod}_{\mathrm{f}} \bar{A}$. Therefore $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)_{\mathrm{CM}:R}$ is an artinian abelian category.

If $A \to \bar{A}$ is not a quasi-isomorphism, then A does not belong to $\mathsf{D}^{0}(A)$, and therefore R is not a CM DG module w.r.t. itself.

We do not know any definition of Cohen-Macaulay DG rings; except when $A \to \bar{A}$ is a quasi-isomorphism, in which case the condition is that the ring $\bar{A}$ should be CM.

For a comparison to Cohen-Macaulay modules and Grothendieck's notion of Cohen-Macaulay complexes, see [YZ3, Theorem 6.2] and [YZ4, Section 7].

The groups $\mathrm{DPic}^{0}(A) \subseteq \mathrm{DPic}(A)$ were introduced in Definitions 6.2 and 6.16.

Lemma 8.4. *Let P be a tilting DG A-module. The following are equivalent*:

(i) *The auto-equivalence* $P \otimes^{\mathrm{L}}_{A} -$ *of* $\mathsf{D}(A)$ *preserves the subcategory* $\mathsf{D}^{0}_{\mathrm{f}}(A)$.

(ii) *The class of P is in* $\mathrm{DPic}^{0}(A)$.

Proof. (i) ⇒ (ii): Let $\bar{P} := \bar{A} \otimes^{\mathrm{L}}_{A} P$. By assumption it belongs to $\mathsf{D}^{0}_{\mathrm{f}}(A)$. But then the class of $\bar{P}$ is in $\mathrm{Pic}(\bar{A}) = \mathrm{DPic}^{0}(\bar{A})$, so by definition the class of P is in $\mathrm{DPic}^{0}(A)$.

(ii) ⇒ (i): Here $\bar{P} := \bar{A} \otimes^{\mathrm{L}}_{A} P$ is isomorphic to an invertible $\bar{A}$-module, so $\bar{P} \otimes^{\mathrm{L}}_{\bar{A}} -$ preserves $\mathsf{D}^{0}_{\mathrm{f}}(\bar{A})$. Now take any $M \in \mathsf{D}^{0}_{\mathrm{f}}(A)$. By Lemma 8.1 we can assume that $M \in \mathsf{Mod}_{\mathrm{f}}(\bar{A})$. Then

$$P \otimes^{\mathrm{L}}_{A} M \cong P \otimes^{\mathrm{L}}_{A} \bar{A} \otimes^{\mathrm{L}}_{\bar{A}} M \cong \bar{P} \otimes^{\mathrm{L}}_{\bar{A}} M \in \mathsf{D}^{0}_{\mathrm{f}}(\bar{A}).$$

We see that $\mathrm{H}^{i}(P \otimes^{\mathrm{L}}_{A} M) = 0$ for all $i \neq 0$. □

Recall the connected component idempotent functors of a DG ring from Definition 4.25.

Theorem 8.5. *Let $f : A \to B$ be a cohomologically pseudo-finite homomorphism between cohomologically pseudo-noetherian DG rings. Assume that A and B have dualizing DG modules R_A and R_B respectively, and that $\bar{B}$ is nonzero. Let $E_1, \ldots, E_n$ be the connected component decomposition functors of B.*

(1) *There are unique integers $k_1, \ldots, k_n$ such that, letting*

$$R'_B := \bigoplus_{i=1}^{n} E_i(R_B)[k_i] \in \mathsf{D}(B),$$

the class of the tilting DG B-module $\mathrm{RHom}_A(R'_B, R_A)$ *is inside* $\mathrm{DPic}^0(B)$.

(2) *Let $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(B)$. The following conditions are equivalent*:
  (i) *M is CM w.r.t. to R'_B.*
  (ii) $\mathrm{rest}_f(M)$ *is CM w.r.t. to R_A.*

Note that R'_B and $R''_B := \mathrm{RHom}_A(B, R_A)$ are dualizing DG B-modules, so

$$\mathrm{RHom}_A(R'_B, R_A) \cong \mathrm{RHom}_B(R'_B, R''_B)$$

is a tilting DG B-module (by Theorem 7.10(2)), and hence item (1) above makes sense.

Proof. (1) Let R''_B be as above. By the classifications in Corollaries 7.11 and 6.15, there are unique $k_1, \ldots, k_n \in \mathbb{Z}$, and a tilting DG B-module Q whose class in $\mathrm{DPic}^0(B)$ is unique, such that $R''_B \cong Q \otimes^{\mathrm{L}}_B R'_B$. Let $P := \mathrm{RHom}_B(Q, B)$, which by Corollary 6.6 is the quasi-inverse of Q. Then $R'_B \cong P \otimes^{\mathrm{L}}_B R''_B$. Using adjunction we get isomorphisms

$$\begin{aligned}\mathrm{RHom}_A(R'_B, R_A) &\cong \mathrm{RHom}_B(R'_B, R''_B) \cong \mathrm{RHom}_B(P \otimes^{\mathrm{L}}_B R''_B, R''_B) \\ &\cong \mathrm{RHom}_B\big(P, \mathrm{RHom}_B(R''_B, R''_B)\big) \cong \mathrm{RHom}_B(P, B) \cong Q\end{aligned}$$

in $\mathsf{D}(B)$. This proves (1).

(2) By Lemma 8.4 the functor $Q \otimes^{\mathrm{L}}_B -$ preserves $\mathsf{D}^0_{\mathrm{f}}(B)$. Take any $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(B)$. Then, using Lemma 7.13, we get isomorphisms

$$\begin{aligned}\mathrm{RHom}_A(M, R_A) &\cong \mathrm{RHom}_B(M, R''_B) \\ &\cong \mathrm{RHom}_B(M, Q \otimes^{\mathrm{L}}_B R'_B) \cong \mathrm{RHom}_B(M, R'_B) \otimes^{\mathrm{L}}_B Q.\end{aligned}$$

This gives (i) $\Rightarrow$ (ii). The converse is very similar. □

Corollary 8.6. *Let $f : A \to B$ be a cohomologically finite homomorphism between tractable DG rings. Assume* $\operatorname{Spec} \bar{B}$ *is connected. The following are equivalent for $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(B)$:*
  (i) *M is CM w.r.t. some dualizing DG B-module.*
  (ii) $\mathrm{rest}_f(M)$ *is CM w.r.t. some dualizing DG A-module.*

Proof. The implication (ii) $\Rightarrow$ (i) comes from Theorem 8.5(2) – and does not require $\operatorname{Spec} \bar{B}$ to be connected.

For the implication (i) $\Rightarrow$ (ii), let R_B be a dualizing DG B-module such that M is CM with respect to it. Take any dualizing DG A-module R_A, and let $k \in \mathbb{Z}$ be such that $\mathrm{RHom}_A(R_B[k], R_A)$ is inside $\mathrm{DPic}^0(B)$. See Theorem 8.5(1). Then by Theorem 8.5(2) we know that $\mathrm{rest}_f(M)$ is CM w.r.t. $R_A[-k]$. □

Theorem 8.7. *Let $f : A \to B$ be a homomorphism between cohomologically pseudo-noetherian DG rings, such that $\bar{f} : \bar{A} \to \bar{B}$ is surjective. Assume A and B have dualizing DG modules. Let R_B be a dualizing DG B-module and let $M, N \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(B)$.*

(1) *If M is CM w.r.t. R_B, and there is an isomorphism* $\mathrm{rest}_f(M) \cong \mathrm{rest}_f(N)$ *in* $\mathsf{D}(A)$, *then N is also CM w.r.t. R_B.*

(2) *If M and N are CM w.r.t. R_B, then the homomorphism*

$$\mathrm{rest}_f : \mathrm{Hom}_{\mathsf{D}(B)}(M, N) \to \mathrm{Hom}_{\mathsf{D}(A)}\big(\mathrm{rest}_f(M), \mathrm{rest}_f(N)\big)$$

is bijective.

Proof. (1) We may assume that $\bar{B} \neq 0$. Let $E_1, \ldots, E_n$ be the connected component decomposition functors of B. Write $F := \mathrm{rest}_f$, $M_i := E_i(M)$ and $N_i := E_i(N)$. Choose some dualizing DG A-module R_A, and let $k_1, \ldots, k_n$ be the integers from Theorem 8.5(1). Thus the dualizing DG B-module $R'_B := \bigoplus_{i=1}^{n} E_i(R_B)[k_i]$ satisfies this: the tilting DG B-module $Q := \mathrm{RHom}_A(R'_B, R_A)$ is inside $\mathrm{DPic}^0(B)$. Define $M' := \bigoplus_i M_i[k_i]$ and $N' := \bigoplus_i N_i[k_i]$.

We are given that M is CM w.r.t. R_B. Using the equivalence of Corollary 4.23 we see that M' is CM w.r.t. R'_B. Thus by Theorem 8.5(2) the DG A-module $F(M')$ is CM w.r.t. R_A. Proposition 4.27 implies that $F(M') \cong F(N')$ in $\mathsf{D}(A)$, and hence $F(N')$ is CM w.r.t. R_A. Using Theorem 8.5(2) once more we conclude that N' is CM w.r.t. R'_B; and hence N is CM w.r.t. R_B.

(2) Let us write $R''_B := \mathrm{RHom}_A(B, R_A)$, $D''_B := \mathrm{RHom}_B(-, R''_B)$ and $D_A := \mathrm{RHom}_A(-, R_A)$. Define $M' := \bigoplus_i M_i[k_i]$ and $N' := \bigoplus_i N_i[k_i]$ as above, so these are CM w.r.t. R''_B. There are isomorphisms

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{D}(B)}(M, N) \cong^1 \mathrm{Hom}_{\mathsf{D}(B)}(M', N') \cong^2 \mathrm{Hom}_{\mathsf{D}(B)}\big(D''_B(N'), D''_B(M')\big) \\
&\quad \cong^3 \mathrm{Hom}_{\mathsf{Mod}\,\bar{B}}\big(D''_B(N'), D''_B(M')\big) \cong^4 \mathrm{Hom}_{\mathsf{Mod}\,\bar{A}}\big(F(D''_B(N')), F(D''_B(M'))\big) \\
&\quad \cong^5 \mathrm{Hom}_{\mathsf{Mod}\,\bar{A}}\big(D_A(F(N')), D_A(F(M'))\big) \cong^{2,3} \mathrm{Hom}_{\mathsf{D}(A)}\big(F(M'), F(N')\big) \\
&\quad \cong^6 \mathrm{Hom}_{\mathsf{D}(A)}\big(F(M), F(N)\big) .
\end{aligned}$$

They are gotten as follows: the isomorphism $\cong^1$ is by Corollary 4.23 and Proposition 4.26; the isomorphism $\cong^2$ is by Proposition 7.2(3); the isomorphism $\cong^3$ is by Lemma 8.1; the isomorphism $\cong^4$ is because $\mathrm{H}^0(f) : \bar{A} \to \bar{B}$ is surjective; the isomorphism $\cong^5$ is because $F \circ D''_B \cong D_A \circ F$ as functors; and isomorphism $\cong^6$ is due to Proposition 4.27. The composition of all these isomorphisms is F. □

Corollary 8.8. *Let $f : A \to B$ be a homomorphism between cohomologically pseudo-noetherian DG rings, such that $\bar{f} : \bar{A} \to \bar{B}$ is bijective. Let R_A be a dualizing DG A-module, and define $R_B := \mathrm{RHom}_A(B, R_A)$. Then the functor*

$$\mathrm{rest}_f : \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(B)_{\mathrm{CM}:R_B} \to \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)_{\mathrm{CM}:R_A}$$

is an equivalence.

Proof. In view of the last theorem, it suffices to show that rest_f is essentially surjective on objects. Take any $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)_{\mathrm{CM}:R_A}$. Then $N := \mathrm{RHom}_A(M, R_A)$ is (isomorphic to) a module in $\mathsf{Mod}_{\mathrm{f}}\,\bar{A}$. Because $\mathrm{rest}_{\bar{f}} : \mathsf{Mod}_{\mathrm{f}}\,\bar{B} \to \mathsf{Mod}_{\mathrm{f}}\,\bar{A}$ is an equivalence, there is $N' \in \mathsf{Mod}_{\mathrm{f}}\,\bar{B}$ that is sent to N. Therefore the DG module

$$M' := \mathrm{RHom}_A(N', R_B) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(B)_{\mathrm{CM}:R_B}$$

satisfies $\mathrm{rest}_f(M') \cong M$. □

Remark 8.9. Here is a quick explanation of the role of CM DG modules in [Ye5]. Suppose $\mathbb{K} \to A \to B$ are ring homomorphisms, $M \in \mathsf{D}(A)$ and $N \in \mathsf{D}(B)$. Under suitable assumptions we want to have a canonical isomorphism

$$\smile : \mathrm{Sq}_{A/\mathbb{K}}(M) \otimes^{\mathrm{L}}_A \mathrm{Sq}_{B/A}(N) \xrightarrow{\simeq} \mathrm{Sq}_{B/\mathbb{K}}(M \otimes^{\mathrm{L}}_A N)$$

in $\mathsf{D}(B)$, that we call the *cup product*. This isomorphism was already constructed in [YZ1, Theorem 4.11]; but unfortunately this part of [YZ1] also contained a serious mistake.

The construction in [Ye5] goes like this. We choose a semi-free DG ring resolution $\mathbb{K} \to \tilde{A}$ of $\mathbb{K} \to A$, and then a semi-free DG ring resolution $\tilde{A} \to \tilde{B}$ of $\tilde{A} \to B$.

So

$$\mathrm{Sq}_{A/\mathbb{K}}(M) = \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(A, M \otimes^{\mathrm{L}}_{\mathbb{K}} M)$$

etc. The construction goes through a few "standard moves" (adjunction formulas mostly), until we arrive at the following situation. Consider the surjective DG ring homomorphism $f : \tilde{B} \otimes_{\mathbb{K}} \tilde{B} \to \tilde{B} \otimes_{\tilde{A}} \tilde{B}$, and the DG modules

$$K := N \otimes^{\mathrm{L}}_{A} N \otimes^{\mathrm{L}}_{A} \mathrm{Sq}_{A/\mathbb{K}}(M)$$

and

$$L := \mathrm{RHom}_{\tilde{B} \otimes_{\mathbb{K}} \tilde{B}}\big(\tilde{B} \otimes_{\tilde{A}} \tilde{B}, (M \otimes^{\mathrm{L}}_{A} N) \otimes^{\mathrm{L}}_{\mathbb{K}} (M \otimes^{\mathrm{L}}_{A} N)\big)$$

in $\mathsf{D}(\tilde{B} \otimes_{\tilde{A}} \tilde{B})$. The "standard moves" give us a canonical isomorphism $\chi : \mathrm{rest}_f(K) \xrightarrow{\simeq} \mathrm{rest}_f(L)$ in $\mathsf{D}(\tilde{B} \otimes_{\mathbb{K}} \tilde{B})$; but what we need to continue the construction is a canonical isomorphism $\bar{\chi} : K \xrightarrow{\simeq} L$ in $\mathsf{D}(\tilde{B} \otimes_{\tilde{A}} \tilde{B})$ such that $\mathrm{rest}_f(\bar{\chi}) = \chi$. The only conceivable hope was that something like Theorem 8.7 should appear.

Fortunately, in the situation where we require the cup product, the ring $\mathbb{K}$ is a regular noetherian ring; $\mathbb{K} \to A$ is essentially finite type; $A \to B$ is essentially Gorenstein (i.e. essentially finite type, flat, and the fibers are Gorenstein rings); M is a rigid dualizing complex over A relative to $\mathbb{K}$; and N is a tilting complex over B (and hence it is a relative dualizing complex for $A \to B$). These assumptions imply that K is a dualizing DG module over the DG ring $\tilde{B} \otimes_{\tilde{A}} \tilde{B}$, and therefore it is a CM DG module w.r.t. itself. Now Theorem 8.7 says that there exists a unique isomorphism $\bar{\chi} : K \xrightarrow{\simeq} L$ satisfying $\mathrm{rest}_f(\bar{\chi}) = \chi$.

9. Rigid DG Modules

Recall that all our DG rings are commutative (Convention 4.2); namely we work inside the category $\mathsf{DGR}^{\leq 0}_{\mathrm{sc}}$. At the beginning of this section we do not make any finiteness assumptions on DG rings.

In our new paper [Ye5] we introduced the *squaring operation* for commutative DG rings. It is summarized in Theorem 9.1 below, which is a combination of [Ye5, Theorems 0.3.4, 0.3.5 and 7.16].

Given a homomorphism $u : A \to B$ of DG rings, a *K-flat resolution* of u is a commutative diagram

$$\begin{tikzcd}
 & & & \boxed{\text{K-flat}} \\
\boxed{\text{qu-isom}} & \tilde{A} \arrow[r, "\tilde{u}"] \arrow[d, "v"'] & \tilde{B} \arrow[d, "w"] & \boxed{\text{surj qu-isom}} \\
 & A \arrow[r, "u"'] & B &
\end{tikzcd}$$

in $\mathsf{DGR}^{\leq 0}_{\mathrm{sc}}$, where v is a quasi-isomorphism, w is a surjective quasi-isomorphism, and $\tilde{B}$ is K-flat as a DG $\tilde{A}$-module. Such resolutions always exist.

A K-flat resolution $\tilde{A} \to \tilde{B}$ of $A \to B$ gives rise to a functor $\mathrm{Sq}^{\tilde{B}/\tilde{A}}_{B/A} : \mathsf{D}(B) \to \mathsf{D}(B)$,

$$\mathrm{Sq}^{\tilde{B}/\tilde{A}}_{B/A}(M) := \mathrm{RHom}_{\tilde{B} \otimes_{\tilde{A}} \tilde{B}}(B, M \otimes^{\mathrm{L}}_{\tilde{A}} M).$$

Theorem 9.1 ([Ye5]). *Let $A \to B$ be a homomorphism of DG rings.*

(1) *There is a functor*

$$\mathrm{Sq}_{B/A} : \mathsf{D}(B) \to \mathsf{D}(B)\,,$$

called the squaring operation, together with a compatible system of isomorphisms of functors $\mathrm{Sq}_{B/A} \cong \mathrm{Sq}^{\tilde{B}/\tilde{A}}_{B/A}$, where $\tilde{A} \to \tilde{B}$ runs over all K-flat resolutions of $A \to B$ in $\mathsf{DGR}^{\leq 0}_{\mathrm{sc}}$.

(2) *The functor* $\mathrm{Sq}_{B/A}$ *is quadratic, in the following sense: for any morphism* $\phi : M \to N$ *in* $\mathsf{D}(B)$*, and any element* $b \in \bar{B}$*, there is equality*

$$\mathrm{Sq}_{B/A}(b \cdot \phi) = b^2 \cdot \phi \, .$$

The compatible system of isomorphisms of functors in item (1) of the theorem is stated in detail in [Ye5, Theorem 0.3.4]. These isomorphisms make the functor $\mathrm{Sq}_{B/A}$ unique up to a unique isomorphism. In part (2) we use the fact that $\mathsf{D}(B)$ is a $\bar{B}$-linear category (see Proposition 4.4).

Definition 9.2. Let $A \to B$ be a homomorphism of DG rings.

(1) Let M be a DG B-module. A *rigidifying isomorphism* for M relative to A is an isomorphism

$$\rho : M \xrightarrow{\simeq} \mathrm{Sq}_{B/A}(M)$$

in $\mathsf{D}(B)$.

(2) A *rigid DG module over* B *relative to* A is a pair (M, ρ), consisting of a DG B-module M, and a rigidifying isomorphism $\rho : M \xrightarrow{\simeq} \mathrm{Sq}_{B/A}(M)$ in $\mathsf{D}(B)$.

Unlike in [YZ1], here we do not impose any finiteness conditions on M as part of the definition of rigidity.

Definition 9.3. Let $A \to B$ be a homomorphism of DG rings.

(1) Let (M, ρ) and (M', ρ') be rigid DG modules over B relative to A. A *rigid morphism*

$$\phi : (M, \rho) \to (M', \rho')$$

is a morphism $\phi : M \to M'$ in $\mathsf{D}(B)$, such that the diagram

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\phi"'] & \mathrm{Sq}_{B/A}(M) \arrow[d, "\mathrm{Sq}_{B/A}(\phi)"] \\ M' \arrow[r, "\rho'"] & \mathrm{Sq}_{B/A}(M') \end{tikzcd}$$

in $\mathsf{D}(B)$ is commutative.

(2) The category of rigid DG modules over B relative to A is denoted by $\mathsf{D}(B)_{\mathrm{rig}/A}$.

Theorem 9.4 (Uniqueness of Rigid Automorphisms)**.** *Let* $A \to B$ *be a homomorphism of DG rings, and let* (M, ρ) *be a rigid DG module over* B *relative to* A*. Assume that the adjunction morphism* $B \to \mathrm{RHom}_B(M, M)$ *is an isomorphism. Then the only automorphism of* (M, ρ) *in* $\mathsf{D}(B)_{\mathrm{rig}/A}$ *is the identity.*

Proof. The idea of the proof is already in [Ye2, Theorem 5.2]. The adjunction condition implies that the ring homomorphism $\bar{B} \to \mathrm{End}_{\mathsf{D}(B)}(M)$ is bijective. Take any automorphism ϕ of (M, ρ) in $\mathsf{D}(B)_{\mathrm{rig}/A}$. Then $\phi = b \cdot \mathrm{id}_M$, for a unique invertible element $b \in \bar{B}$. By item (2) of Theorem 9.1 we know that $\mathrm{Sq}_{B/A}(\phi) = b^2 \cdot \mathrm{Sq}_{B/A}(\mathrm{id}_M)$. On the other hand, because ϕ and id_M are both rigid, we get

$$\begin{aligned} \mathrm{Sq}_{B/A}(\phi) &= \rho \circ \phi \circ \rho^{-1} = \rho \circ (b \cdot \mathrm{id}_M) \circ \rho^{-1} \\ &= b \cdot (\rho \circ \mathrm{id}_M \circ \rho^{-1}) = b \cdot \mathrm{Sq}_{B/A}(\mathrm{id}_M). \end{aligned}$$

This shows that $b^2 \cdot \mathrm{id}_M = b \cdot \mathrm{id}_M$, so $b^2 = b$, and hence $b = 1$. □

Definition 9.5. Let A be a tractable cohomologically pseudo-noetherian DG ring, with traction $\mathbb{K} \to A$ (see Definition 7.7). A *rigid dualizing DG module over* A *relative to* $\mathbb{K}$ is a rigid DG module (R, ρ) over A relative to $\mathbb{K}$ (Definition 9.2), such that R is a dualizing DG module over A (Definition 7.1).

Lemma 9.6. *Suppose $\mathbb{K}$ is a noetherian ring, A is a cohomologically pseudo-noetherian DG ring, and $\mathbb{K} \to A$ is a cohomologically essentially finite type homomorphism. Let $\mathbb{K} \to \tilde{A}$ be a K-flat resolution of $\mathbb{K} \to A$, and define $\tilde{A}^{\mathrm{en}} := \tilde{A} \otimes_{\mathbb{K}} \tilde{A}$.*

(1) *The DG ring $\tilde{A}^{\mathrm{en}}$ is cohomologically pseudo-noetherian, and the homomorphism $\mathbb{K} \to \tilde{A}^{\mathrm{en}}$ is a cohomologically essentially finite type.*
(2) *Let $P \in \mathsf{D}(A)$ be a tilting DG A-module. Then $P \otimes^{\mathrm{L}}_{\mathbb{K}} P \in \mathsf{D}(\tilde{A}^{\mathrm{en}})$ is a tilting DG $\tilde{A}^{\mathrm{en}}$-module.*

The derived tensor product $P \otimes^{\mathrm{L}}_{\mathbb{K}} P$ in item (2) is calculated as follows: we take any K-flat resolution $\tilde{P} \to P$ in $\mathsf{C}(\tilde{A})$, and define $P \otimes^{\mathrm{L}}_{\mathbb{K}} P := \tilde{P} \otimes_{\mathbb{K}} \tilde{P}$. Because $\tilde{P}$ is K-flat over $\mathbb{K}$, this is independent of the resolution, up to a canonical isomorphism in $\mathsf{D}(\tilde{A}^{\mathrm{en}})$.

Proof. (1) As in the proof of Lemma 7.8, we can find a commutative diagram in $\mathsf{DGR}^{\leq 0}_{\mathrm{sc}}$

$$\begin{tikzcd}
\mathbb{K} \arrow[r] \arrow[d] & \tilde{A} \arrow[r] \arrow[d, "\tilde{f}"] & A \arrow[r] \arrow[d] & \bar{A} \\
\tilde{A}_{\mathrm{eft}} \arrow[r, "\tilde{g}"] & \tilde{A}_{\mathrm{loc}} \arrow[r] & A_{\mathrm{loc}} \arrow[ur] &
\end{tikzcd}$$

such that $\tilde{f}$ is a K-flat quasi-isomorphism, $\tilde{g}$ is a quasi-isomorphism, the DG ring $\tilde{A}_{\mathrm{eft}}$ is K-flat over $\mathbb{K}$, the ring $\tilde{A}^0_{\mathrm{eft}}$ is essentially finite type over $\mathbb{K}$, and each module $\tilde{A}^i_{\mathrm{eft}}$ is finite over $\tilde{A}^0_{\mathrm{eft}}$. Now $\tilde{A}$, $\tilde{A}_{\mathrm{eft}}$ and $\tilde{A}_{\mathrm{loc}}$ are all K-flat over $\mathbb{K}$. Hence, letting $\tilde{A}^{\mathrm{en}}_{\mathrm{loc}} := \tilde{A}_{\mathrm{loc}} \otimes_{\mathbb{K}} \tilde{A}_{\mathrm{loc}}$ and $\tilde{A}^{\mathrm{en}}_{\mathrm{eft}} := \tilde{A}_{\mathrm{eft}} \otimes_{\mathbb{K}} \tilde{A}_{\mathrm{left}}$, we get DG ring quasi-isomorphisms $\tilde{A}^{\mathrm{en}}_{\mathrm{eft}} \to \tilde{A}^{\mathrm{en}}_{\mathrm{loc}}$ and $\tilde{A}^{\mathrm{en}} \to \tilde{A}^{\mathrm{en}}_{\mathrm{loc}}$. Therefore it suffices to prove the required assertions for $\tilde{A}^{\mathrm{en}}_{\mathrm{eft}}$.

Consider the ring $(\tilde{A}^{\mathrm{en}}_{\mathrm{eft}})^0 = \tilde{A}^0_{\mathrm{eft}} \otimes_{\mathbb{K}} \tilde{A}^0_{\mathrm{eft}}$. It is essentially finite type over $\mathbb{K}$, and thus it is also noetherian. For any $i \leq 0$ we have

$$(\tilde{A}^{\mathrm{en}}_{\mathrm{eft}})^i = \bigoplus_{i \leq p \leq 0} \tilde{A}^p_{\mathrm{eft}} \otimes_{\mathbb{K}} \tilde{A}^{i-p}_{\mathrm{eft}},$$

and this is a finite module over $(\tilde{A}^{\mathrm{en}}_{\mathrm{eft}})^0$. Therefore the ring $\mathrm{H}^0(\tilde{A}^{\mathrm{en}}_{\mathrm{eft}})$, being a quotient of $(\tilde{A}^{\mathrm{en}}_{\mathrm{eft}})^0$, is essentially finite type over $\mathbb{K}$. Each $\mathrm{H}^i(\tilde{A}^{\mathrm{en}}_{\mathrm{eft}})$, being a subquotient of $(\tilde{A}^{\mathrm{en}}_{\mathrm{eft}})^i$, is a finite module over $\mathrm{H}^0(\tilde{A}^{\mathrm{en}}_{\mathrm{eft}})$.

(2) Let $Q \in \mathsf{D}(A)$ be a quasi-inverse of P. Then

$$(Q \otimes^{\mathrm{L}}_{\mathbb{K}} Q) \otimes^{\mathrm{L}}_{A} (P \otimes^{\mathrm{L}}_{\mathbb{K}} P) \cong (Q \otimes^{\mathrm{L}}_{A} P) \otimes^{\mathrm{L}}_{\mathbb{K}} (Q \otimes^{\mathrm{L}}_{A} P) \cong A \otimes^{\mathrm{L}}_{\mathbb{K}} A \cong \tilde{A}^{\mathrm{en}}$$

in $\mathsf{D}(\tilde{A}^{\mathrm{en}})$. □

Theorem 9.7 (Uniqueness of Rigid Dualizing DG Modules)**.** *In the situation of Definition 9.5, suppose that (R, ρ) and (R', ρ') are rigid dualizing DG modules over A relative to $\mathbb{K}$. Then there is a unique isomorphism $(R, \rho) \cong (R', \rho')$ in $\mathsf{D}(A)_{\mathrm{rig}/\mathbb{K}}$.*

Proof. The idea of first half of the proof goes back to the original work of Van den Bergh [VdB]. According to Theorem 7.10(2), there is a tilting DG module P such that $R' \cong P \otimes^{\mathrm{L}}_{A} R$ in $\mathsf{D}(A)$. Choose a K-flat DG ring resolution $\mathbb{K} \to \tilde{A}$ of $\mathbb{K} \to A$, and let $\tilde{A}^{\mathrm{en}} := \tilde{A} \otimes_{\mathbb{K}} \tilde{A}$. Then we have isomorphisms

$$\begin{aligned} P \otimes^{\mathrm{L}}_{A} R &\cong^1 R' \cong^2 \mathrm{RHom}_{\tilde{A}^{\mathrm{en}}}(A, R' \otimes^{\mathrm{L}}_{\mathbb{K}} R') \\ &\cong^1 \mathrm{RHom}_{\tilde{A}^{\mathrm{en}}}\big(A, (P \otimes^{\mathrm{L}}_{A} R) \otimes^{\mathrm{L}}_{\mathbb{K}} (P \otimes^{\mathrm{L}}_{A} R)\big) \\ &\cong^3 \mathrm{RHom}_{\tilde{A}^{\mathrm{en}}}\big(A, (R \otimes^{\mathrm{L}}_{\mathbb{K}} R) \otimes^{\mathrm{L}}_{\tilde{A}^{\mathrm{en}}} (P \otimes^{\mathrm{L}}_{\mathbb{K}} P)\big) \\ &\cong^4 \mathrm{RHom}_{\tilde{A}^{\mathrm{en}}}(A, R \otimes^{\mathrm{L}}_{\mathbb{K}} R) \otimes^{\mathrm{L}}_{\tilde{A}^{\mathrm{en}}} (P \otimes^{\mathrm{L}}_{\mathbb{K}} P) \\ &\cong^2 R \otimes^{\mathrm{L}}_{\tilde{A}^{\mathrm{en}}} (P \otimes^{\mathrm{L}}_{\mathbb{K}} P) \cong^3 R \otimes^{\mathrm{L}}_{A} P \otimes^{\mathrm{L}}_{A} P \end{aligned}$$

in $\mathsf{D}(A)$. The isomorphisms $\cong^1$ come from the given isomorphism $R' \cong P \otimes^{\mathrm{L}}_{A} R$ in $\mathsf{D}(A)$; the isomorphisms $\cong^2$ come from the rigidifying isomorphisms ρ' and ρ, together with the isomorphisms from Theorem 9.1(1); the isomorphism $\cong^3$ is by a standard tensor product identity (that is valid also in this derived setting, because $\tilde{A}$ is K-flat over $\mathbb{K}$); and the the isomorphism $\cong^4$ is by Lemma 7.13, that applies because $\tilde{A}^{\mathrm{en}}$ is cohomologically pseudo-noetherian, $R \otimes^{\mathrm{L}}_{\mathbb{K}} R$ is in $\mathsf{D}^{+}(\tilde{A}^{\mathrm{en}})$, and $P \otimes^{\mathrm{L}}_{\mathbb{K}} P$ is perfect over $\tilde{A}^{\mathrm{en}}$ – thanks to Lemma 9.6. Now Theorem 7.10(3) tells us that $P \otimes^{\mathrm{L}}_{A} P \cong P$ in $\mathsf{D}(A)$. This implies that $P \cong A$. We conclude that $R' \cong R$ in $\mathsf{D}(A)$.

The remainder of the proof is the same as in the proof of [Ye2, Theorem 5.2]. Let $\phi : R \xrightarrow{\simeq} R'$ be any isomorphism in $\mathsf{D}(A)$. The calculation in the proof of Theorem 9.4 shows that $\mathrm{Sq}_{A/\mathbb{K}}(\phi) = a \cdot (\rho' \circ \phi \circ \rho^{-1})$ for a unique invertible element $a \in \bar{A}$. Then $a^{-1} \cdot \phi : R \to R$ is a rigid isomorphism; and it is unique by Theorem 9.4. □

Existence of rigid dualizing DG modules in this generality is not known, but we believe it it true – this is Conjecture 0.12 in the Introduction. Conjecture 9.8 below implies Conjecture 0.12.

Suppose $u : A \to B$ is a homomorphism between cohomologically pseudo-noetherian DG rings. We say that u is *cohomologically essentially smooth of relative dimension n* if $\bar{u} : \bar{A} \to \bar{B}$ is essentially smooth of relative dimension n, and the induced homomorphism $\mathrm{H}(A) \otimes_{\bar{A}} \bar{B} \to \mathrm{H}(B)$ is bijective. In this case, the module of differentials $\Omega^1_{\bar{B}/\bar{A}}$ is projective of rank n, and hence $\Omega^n_{\bar{B}/\bar{A}}$ is projective of rank 1. We define $\Omega^n_{B/A}[n] \in \mathsf{D}(B)$ to be the lift of $\Omega^n_{\bar{B}/\bar{A}}[n] \in \mathsf{D}(\bar{B})$, under the group isomorphism $\mathrm{DPic}(B) \cong \mathrm{DPic}(\bar{B})$ of Theorem 6.14. There is an alternative description of $\Omega^n_{B/A}[n]$ in terms of the *cotangent complex* of B/A.

Conjecture 9.8. In the situation of Definition 9.5, suppose (R_A, ρ_A) is a rigid dualizing DG module over A relative to $\mathbb{K}$. Let B be another DG ring, and let $A \to B$ be a cohomologically essentially finite type homomorphism.

(1) If $A \to B$ is cohomologically pseudo-finite, then the DG module
$$R_B := \mathrm{RHom}_A(B, R_A) \in \mathsf{D}(B)$$
has an induced rigidifying isomorphism ρ_B relative to $\mathbb{K}$.

(2) If $A \to B$ is cohomologically essentially smooth of relative dimension n, then the DG module
$$R_B := \Omega^n_{B/A}[n] \otimes^{\mathrm{L}}_{A} R_A \in \mathsf{D}(B)$$
has an induced rigidifying isomorphism ρ_B relative to $\mathbb{K}$.

In the case when A and B are rings, Conjectures 0.12 and 9.8 were already proved in [YZ1]. The proofs will be repeated, with improvements, in [Ye7]. In the case when the cohomology of A is bounded, Conjectures 0.12 and 9.8 were very recently proved by Shaul [Sh2]. What remains to prove is the case when $\mathrm{H}(A)$ is unbounded.

Remark 9.9. Finally, a few words on Conjecture 0.13 from the Introduction. We can see three possible routes to try to prove it. The first route is by generalizing the original proof of [YZ2, Theorem 1.2], that talked about a regular tractable ring A, to the DG ring case. The second route is by generalizing the proof of [AIL, Theorem 4], which applied to any tractable ring A, to the DG ring case.

There is also a more conceptual route. In [Ye5] we introduced the *rectangle operation*

$$\operatorname{Rect}_{A/\mathbb{K}}(M,N) := \operatorname{RHom}_{\tilde{A}\otimes_{\mathbb{K}}\tilde{A}}(A, M \otimes^{\mathrm{L}}_{\mathbb{K}} N),$$

for $M, N \in \mathsf{D}(A)$. Here $\mathbb{K} \to \tilde{A}$ is a K-flat resolution of $\mathbb{K} \to A$, but the rectangle operation is independent of this resolution. Notice that the square is a special instance of the rectangle: $\operatorname{Sq}_{A/\mathbb{K}}(M) = \operatorname{Rect}_{A/\mathbb{K}}(M,M)$.

Recently Shaul [Sh1] proved that when A is a tractable ring, and under certain finiteness conditions on M and N, there is a canonical isomorphism

$$\operatorname{Rect}_{A/\mathbb{K}}(M,N) \cong D\big(D(M) \otimes^{\mathrm{L}}_{A} D(N)\big),$$

where $D := \operatorname{RHom}_A(-, R_A)$, and R_A is the rigid dualizing complex of A relative to $\mathbb{K}$. Thus, writing $M \otimes^{!}_{A/\mathbb{K}} N := \operatorname{Rect}_{A/\mathbb{K}}(M,N)$, this operation becomes a symmetric monoidal structure on (a suitable subcategory of) $\mathsf{D}^{+}_{\mathrm{f}}(A)$.

Suppose now that $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ is rigid relative to $\mathbb{K}$. Define $L := D(M) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. Then L satisfies $L \otimes^{\mathrm{L}}_{A} L \cong L$. If M is nonzero on each connected component of $\operatorname{Spec} \bar{A}$, then the same holds for L. It is not hard to show that in this case $L \cong A$ in $\mathsf{D}(A)$. Therefore $M \cong R_A$, so it is a rigid dualizing complex.

If this work of Shaul could be extended to cover tractable DG rings, we would have a proof of Conjecture 0.13, at least when A is cohomologically bounded.

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Department of Mathematics, Ben Gurion University, Be'er Sheva 84105, Israel

E-mail address: `amyekut@math.bgu.ac.il`