Princeton, le 23 février 2016

Dear Kebutiali,

A few comments on your "Non abelian ... surfaces".

_ Why use the exponential? I expect you can just use any C^∞ map φ defined near $0 \in \mathrm{Lie}(G)$ such that $\varphi(0) = e$ and that at 0, $d\varphi$ is the identity [For instance, for $G = GL(n)$, $\mathfrak{g} = M_n$, it can be convenient to use $\varphi(X) = 1 + X$] One ~~then~~ will have $\varphi(X+Y)$ $= \varphi(X)\varphi(Y) + O(|X|.|Y|)$ for X, Y in $\mathfrak{g}$ near 0, and I don't think you need more.

_ Your choice of regularity condition is strange to me. To define the multiplicative integral of $f(t)\,dt$ on $[0,1]$, f with values in $\mathfrak{g}$, I expect that f in L^1 suffices (for $\varepsilon \to 0$, décompose $[0,1]$ in intervals I_i such that $\int_{I_i} |f(t)|\,dt < \varepsilon$ and, for φ as above, take the product of the $\varphi(\int_{I_i} f(t)\,dt)$)

_ I expect that using G-torsors can simplify life, and gives additional useful formulé. In one dimension, on $[0,1]$, if you have $f(t)\,dt$ as above and also $g(t)$ with values in G, the change of gauge formula is the limit of: divide $[0,1]$ by $x_0 = 0, x_1, \ldots, x_n = 1$ ($I_i = [x_{i-1}, x_i]$). Let $\gamma_i = \int_{I_i} f(t)\,dt$. One has $g(0)\varphi(\gamma_1)\ldots\varphi(\gamma_n)g(1)^{-1} =$

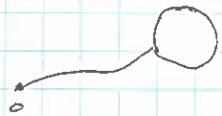
$$g(0)\varphi(\gamma_1)\underbrace{g(x_1)^{-1}g(x_1)}\varphi(\gamma_1)\underbrace{g(x_2)^{-1}g(x_2)} \ldots \quad g(x_{n-1})\varphi(\gamma_n)g(1) =$$

$$\underbrace{g(0)\varphi(\gamma_1)g(0)^{-1} \cdot g(0)g(x_1)^{-1}} \cdot \ \ldots$$

cf change of gauge for a connection.

Instead of $\left\{ \begin{array}{ccc} \text{cross module} & H \xrightarrow{\Phi} G \\ \text{forms} & \beta \quad\quad \alpha \end{array} \right.$,

it would be natural to consider instead of α a G-torsor P with a connection ∇. If g^P is the bundle of Lie algebras g twisted by P, the curvature is in $\Omega^2(g^P)$. As G acts on H, h^P is defined, and, for connection-curvature pairs, β becomes a lifting of the curvature in $\Omega^2(h^P)$. In this setting, your multiplicative integral would be a ~~lift~~, for a disk D, a



lifting to $(H^P)_o$ of the monodromy of (P, ∇) in $(G^P)_o$.
That things are well defined in that setting include a behaviour by change of trivialisation of P.

A gain is that in the connection-curvature pair case so formulated, one can simplify the story. Over a disk D, the curvature being in $d\Phi(h)$, one can reduce the structural group of (P, ∇) to $H/\mathrm{Ker}(\Phi)$. One can then lift (P, ∇) to a torsor with connection (P', ∇') over H; if β' is its curvature, $\beta' - \beta$ will be with values in $\mathrm{Lie}\,\mathrm{Ker}(\Phi)$, and I expect that the multiplicative integral will simply be the monodromy of (P', ∇'), corrected by the $\exp(\int \beta' - \beta)$. *

Best

P.

* and this reduces Stokes to the usual Stokes formula