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Derived equivalences between associative deformations☆

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ABSTRACT

We prove that if two associative deformations (parameterized by the same complete local ring) are derived Morita equivalent, then they are Morita equivalent (in the classical sense).

0. Introduction

Let $\mathbb{K}$ be a commutative ring, and let A and B be associative unital $\mathbb{K}$-algebras. We denote by $\mathsf{Mod}\, A$ and $\mathsf{Mod}\, B$ the corresponding categories of left modules. One says that A and B are *Morita equivalent relative to* $\mathbb{K}$ (in the classical sense) if there is a $\mathbb{K}$-linear equivalence of categories $\mathsf{Mod}\, A \to \mathsf{Mod}\, B$.

Let $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ denote the bounded derived category of complexes of left A-modules. This is a $\mathbb{K}$-linear triangulated category. If there is a $\mathbb{K}$-linear equivalence of triangulated categories $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B)$, then one says that A and B are *derived Morita equivalent relative to* $\mathbb{K}$.

There are plenty of examples of pairs of algebras that are derived Morita equivalent, but are not Morita equivalent in the classical sense.

Now suppose $\mathbb{K}$ is a complete noetherian local ring, with maximal ideal $\mathfrak{m}$ and residue field $\Bbbk$. Let A be a flat $\mathfrak{m}$-adically complete $\mathbb{K}$-algebra, such that the $\Bbbk$-algebra $\bar{A} := \Bbbk \otimes_{\mathbb{K}} A$ is commutative. We then say that A is an *associative* $\mathbb{K}$*-deformation of* $\bar{A}$; see [11].

The most important example of an associative deformation is when $\Bbbk = \mathbb{R}$; $\bar{A} = \mathrm{C}^{\infty}(X)$, the $\mathbb{R}$-algebra of smooth functions on a differentiable manifold X; $\mathbb{K} = \mathbb{R}[[\hbar]]$, the ring of formal power series in the variable $\hbar$; and $A = \bar{A}[[\hbar]]$. In this case the multiplication in A is called a *star product*.

Let us assume that A and B are associative $\mathbb{K}$-deformations, and moreover the commutative rings $\bar{A}$ and $\bar{B}$ have connected prime spectra (i.e. they have no nontrivial idempotents). The main result of the paper (Theorem 2.7) says that if T is a *two-sided tilting complex over B–A relative to* $\mathbb{K}$, then $T \cong P[n]$ for some invertible bimodule P and integer n. (Tilting complexes and their properties are recalled in Section 1.) A direct consequence (Corollary 2.8) is that if A and B are derived Morita equivalent, then they are Morita equivalent in the classical sense.

1. Base change for tilting complexes

In this section we recall some facts about two-sided tilting complexes, and also prove one new theorem. Throughout this section $\mathbb{K}$ is a commutative ring. By "$\mathbb{K}$-algebra" we mean an associative unital algebra; i.e. a ring A, with center $\mathrm{Z}(A)$, together with a ring homomorphism $\mathbb{K} \to \mathrm{Z}(A)$.

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For a $\mathbb{K}$-algebra A we denote by A^{op} the opposite algebra, namely with reverse multiplication. We view right A-modules as left A^{op}-modules. Let B be some other $\mathbb{K}$-algebra. By *B–A-bimodule relative to* $\mathbb{K}$ we mean a $\mathbb{K}$-central B–A-bimodule. We view B–A-bimodules relative to $\mathbb{K}$ as left $B \otimes_{\mathbb{K}} A^{\mathrm{op}}$-modules.

The category of left A-modules is denoted by $\mathsf{Mod}\ A$. This is a $\mathbb{K}$-linear abelian category. Classical Morita theory says that any $\mathbb{K}$-linear equivalence $\mathsf{Mod}\ A \to \mathsf{Mod}\ B$ is of the form $P \otimes_A -$, where P is some invertible B–A-bimodule relative to $\mathbb{K}$.

The derived category of $\mathsf{Mod}\ A$ is $\mathsf{D}(\mathsf{Mod}\ A)$. This is a $\mathbb{K}$-linear triangulated category. We follow the conventions of [7] on derived categories. For instance, $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\ A)$ is the full subcategory of $\mathsf{D}(\mathsf{Mod}\ A)$ consisting of bounded complexes.

Here is a definition from Rickard's paper [5].

Definition 1.1. Let A and B be $\mathbb{K}$-algebras. If there exists a $\mathbb{K}$-linear equivalence of triangulated categories $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\ A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\ B)$ then we say that A and B are *derived Morita equivalent relative to* $\mathbb{K}$.

Now assume that A is *flat* over $\mathbb{K}$. Since $A \otimes_{\mathbb{K}} B$ is flat over B, it follows that the forgetful functor $\mathsf{Mod}\ A \otimes_{\mathbb{K}} B \to \mathsf{Mod}\ B$ sends flat modules to flat modules.

Given three $\mathbb{K}$-algebras A, B, C, and complexes $M \in \mathsf{D}^{-}(\mathsf{Mod}\ A \otimes_{\mathbb{K}} B^{\mathrm{op}})$ and $N \in \mathsf{D}^{-}(\mathsf{Mod}\ B \otimes_{\mathbb{K}} C^{\mathrm{op}})$, and assuming A is flat over $\mathbb{K}$, the derived tensor product

$$M \otimes^{\mathrm{L}}_{B} N \in \mathsf{D}^{-}(\mathsf{Mod}\ A \otimes_{\mathbb{K}} C^{\mathrm{op}})$$

can be defined as follows: choose a quasi-isomorphism $P \to M$ with P a bounded above complex of projective $A \otimes_{\mathbb{K}} B^{\mathrm{op}}$-modules. Then P is a bounded above complex of flat B^{op}-modules, and we take

$$M \otimes^{\mathrm{L}}_{B} N := P \otimes_B N.$$

This operation is functorial in M and N. As usual the requirements can be relaxed: it is enough to resolve M by a bounded above complex P of bimodules that are flat over B^{op}. If C is flat over $\mathbb{K}$ then we can resolve N instead of M. The derived tensor product $M \otimes^{\mathrm{L}}_{B} N$ is "indifferent" to the algebras A and C: we can forget them before or after calculating $M \otimes^{\mathrm{L}}_{B} N$, and get the same answer in $\mathsf{D}^{-}(\mathsf{Mod}\ \mathbb{K})$.

We record the following useful technical results.

Lemma 1.2 (*Projective Truncation Trick*). *Let $M \in \mathsf{D}(\mathsf{Mod}\ A)$ and let i_0 be an integer. Suppose that $\mathrm{H}^i M = 0$ for all $i > i_0$, and $P := \mathrm{H}^{i_0} M$ is a projective A-module. Then there is an isomorphism $M \cong P[-i_0] \oplus N$ in $\mathsf{D}(\mathsf{Mod}\ A)$, where N is a complex satisfying $N^i = 0$ for all $i \geq i_0$.*

Proof. By the usual truncation trick (cf. [7, Section I.7]) we can assume that $M^i = 0$ for all $i > i_0$. Hence we get an exact sequence $M^{i_0-1} \xrightarrow{\mathrm{d}} M^{i_0} \to P \to 0$. But P is projective, and therefore $M^{i_0} \cong P \oplus \mathrm{d}(M^{i_0-1})$. Define $N^{i_0-1} := \mathrm{Ker}(\mathrm{d}) \subset M^{i_0-1}$ and $N^i := M^i$ for $i < i_0 - 1$. $\square$

Recall that a complex $M \in \mathsf{D}(\mathsf{Mod}\ A)$ is called *perfect* if it is isomorphic to bounded complex of finitely generated projective modules. We denote by $\mathsf{D}(\mathsf{Mod}\ A)_{\mathrm{perf}}$ the full subcategory of perfect complexes.

Lemma 1.3. *Let $M \in \mathsf{D}(\mathsf{Mod}\ A)_{\mathrm{perf}}$ and let i_0 be an integer. If $\mathrm{H}^i M = 0$ for all $i > i_0$, then the A-module $N := \mathrm{H}^{i_0} M$ is finitely presented.*

Proof. This is a bit stronger then [9, Lemma 1.1(2)]. By truncation reasons we can assume that $M \cong P$, where P is a bounded complex of finitely generated projective A-modules, and $P^i = 0$ for $i > i_0$. So we get an exact sequence $P^{i_0-1} \to P^{i_0} \to N \to 0$. Suppose P^{i_0} is a direct summand of A^r (the free module of rank r), and P^{i_0-1} is a direct summand of A^s. Then be rearranging terms we get an exact sequence $A^{r+s} \to A^r \to N \to 0$. $\square$

Lemma 1.4 (*Künneth Trick*). *Let A be a $\mathbb{K}$-algebra, let $M \in \mathsf{D}^{-}(\mathsf{Mod}\ A^{\mathrm{op}})$ and let $N \in \mathsf{D}^{-}(\mathsf{Mod}\ A)$. Let $i_0, j_0 \in \mathbb{Z}$ be such that $\mathrm{H}^i M = 0$ and $\mathrm{H}^j N = 0$ for all $i > i_0$ and $j > j_0$. Then*

$$(\mathrm{H}^{i_0} M) \otimes_A (\mathrm{H}^{j_0} N) \cong \mathrm{H}^{i_0+j_0}(M \otimes^{\mathrm{L}}_{A} N)$$

as $\mathbb{K}$-modules.

Proof. See [9, Lemma 2.1]. $\square$

The next definition is from [6].

Definition 1.5. Let A and B be flat $\mathbb{K}$-algebras. A *two-sided tilting complex over B–A relative to* $\mathbb{K}$ is a complex $T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\ B \otimes_{\mathbb{K}} A^{\mathrm{op}})$ with the following property:

(∗) there exists a complex $S \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\ A \otimes_{\mathbb{K}} B^{\mathrm{op}})$, and isomorphisms $S \otimes^{\mathrm{L}}_{B} T \cong A$ and $T \otimes^{\mathrm{L}}_{A} S \cong B$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\ A \otimes_{\mathbb{K}} A^{\mathrm{op}})$ and $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\ B \otimes_{\mathbb{K}} B^{\mathrm{op}})$ respectively.

The complex S is called an *inverse of T*.

In case $B = A$ we say that T is a *two-sided tilting complex over A relative to* $\mathbb{K}$.

The inverse S in the definition is unique up to isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$. Of course S is a two-sided tilting complex over A–B relative to $\mathbb{K}$.

A two-sided tilting complex T induces a $\mathbb{K}$-linear equivalence of triangulated categories

$$T \otimes^{\mathrm{L}}_{A} - : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B).$$

This functor restricts to equivalences

$$\mathsf{D}^{\star}(\mathsf{Mod}\, A) \to \mathsf{D}^{\star}(\mathsf{Mod}\, B),$$

where $\star$ is either $+$, $-$ or b; and also to an equivalence

$$\mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{perf}} \to \mathsf{D}(\mathsf{Mod}\, B)_{\mathrm{perf}}.$$

See [6] or [9, Corollary 1.6(4)].

Conversely we have the next important result, due to Rickard [6]. For alternative proofs see [3] or [9, Corollary 1.9].

Theorem 1.6 (*Rickard*). *Let A and B be flat $\mathbb{K}$-algebras that are derived Morita equivalent relative to $\mathbb{K}$. Then there exists a two-sided tilting complex over B–A relative to $\mathbb{K}$.*

Remark 1.7. Suppose $F : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B)$ is a $\mathbb{K}$-linear equivalence of triangulated categories. Then F restricts to an equivalence between the subcategories of perfect complexes (cf. [4]). This implies that F has finite cohomological dimension (bounded by the amplitude of $\mathrm{H}\, F(A)$). Hence F restricts to an equivalence between the bounded derived categories — i.e. a derived Morita equivalence.

Remark 1.8. In our paper [9] the base ring $\mathbb{K}$ is taken to be a field. However the results in Sections 1-3 of that paper hold for any commutative base ring $\mathbb{K}$, as long as the $\mathbb{K}$-algebras in question are *flat*.

It is possible to remove even the flatness condition, at the price of working with DG algebras. Here is how to do it: choose a DG $\mathbb{K}$-algebra $\tilde{A}$ such that $\tilde{A}^i = 0$ for $i > 0$ and every $\tilde{A}^i$ flat as $\mathbb{K}$-module, with a DG algebra quasi-isomorphism $\tilde{A} \to A$. We call $\tilde{A} \to A$ a flat DG algebra resolution of A relative to $\mathbb{K}$. This can be done (cf. [12, Section 1] for commutative $\mathbb{K}$-algebras). Likewise choose a flat DG algebra resolution $\tilde{B} \to B$.

Let $\tilde{\mathsf{D}}(\mathsf{DGMod}\, \tilde{A})^{\mathrm{b}}$ be the derived category of DG $\tilde{A}$-modules with bounded cohomologies. It is known (cf. [12, Proposition 1.4]) that the restriction of scalars functor $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A) \to \tilde{\mathsf{D}}(\mathsf{DGMod}\, \tilde{A})^{\mathrm{b}}$ is an equivalence. Therefore a $\mathbb{K}$-linear equivalence $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B)$ is the same as a $\mathbb{K}$-linear equivalence $\tilde{\mathsf{D}}(\mathsf{DGMod}\, \tilde{A})^{\mathrm{b}} \to \tilde{\mathsf{D}}(\mathsf{DGMod}\, \tilde{B})^{\mathrm{b}}$. Now the proof of [9, Theorem 1.8] shows that there is a complex $T \in \tilde{\mathsf{D}}(\mathsf{DGMod}\, \tilde{B} \otimes_{\mathbb{K}} \tilde{A}^{\mathrm{op}})^{\mathrm{b}}$ which is two-sided tilting.

A different choice of flat DG algebra resolutions $\tilde{A} \to A$ and $\tilde{B} \to B$ will give rise to an equivalent triangulated category $\tilde{\mathsf{D}}(\mathsf{DGMod}\, \tilde{B} \otimes_{\mathbb{K}} \tilde{A}^{\mathrm{op}})^{\mathrm{b}}$. In this sense two-sided tilting complexes are independent of the resolutions.

See Remark 1.11 for the history of the next theorem.

Theorem 1.9. *Let A and B be flat $\mathbb{K}$-algebras. Assume A is commutative with connected spectrum. Let T be a two-sided tilting complex over B–A relative to $\mathbb{K}$. Then there is an isomorphism*

$$T \cong P[n]$$

in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$ for some invertible B–A-bimodule P and integer n.

Proof. We may assume that $A \neq 0$, so that $T \neq 0$. The complex T is perfect over B and over A^{op} (cf. [9, Theorem 1.6]). As in [9, Proposition 2.4] the complex T induces a $\mathbb{K}$-algebra isomorphism $A \cong \mathrm{Z}(B)$.

Let

$$n := -\sup\{i \mid \mathrm{H}^i T \neq 0\},$$

and let $P := \mathrm{H}^{-n} T$. This is a B–A-bimodule. By Lemma 1.3, P is finitely presented as right A-module.

For a prime $\mathfrak{p} \in \operatorname{Spec} A$, with corresponding local ring $A_{\mathfrak{p}}$, we write $P_{\mathfrak{p}} := P \otimes_A A_{\mathfrak{p}}$. Define $Y \subset \operatorname{Spec} A$ to be the support of P, i.e.

$$Y := \{\mathfrak{p} \in \operatorname{Spec} A \mid P_{\mathfrak{p}} \neq 0\}.$$

Since P is finitely generated it follows that Y is a closed subset of $\operatorname{Spec} A$.

Take any prime $\mathfrak{p} \in Y$, and let $B_{\mathfrak{p}} := B \otimes_A A_{\mathfrak{p}}$. Then, by [9, Lemma 2.5], the complex

$$T_{\mathfrak{p}} := B_{\mathfrak{p}} \otimes_B T \otimes_A A_{\mathfrak{p}} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B_{\mathfrak{p}} \otimes_{\mathbb{K}} A_{\mathfrak{p}}^{\mathrm{op}})$$

is a two-sided tilting complex over $B_{\mathfrak{p}}$–$A_{\mathfrak{p}}$. Since

$$\mathrm{H}^{-n} T_{\mathfrak{p}} \cong P_{\mathfrak{p}} \neq 0,$$

[9, Theorem 2.3] implies that

$$T_{\mathfrak{p}} \cong P_{\mathfrak{p}}[n] \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B_{\mathfrak{p}} \otimes_{\mathbb{K}} A_{\mathfrak{p}}^{\mathrm{op}}). \tag{1.10}$$

Thus $P_{\mathfrak{p}}$ is an invertible $B_{\mathfrak{p}}$–$A_{\mathfrak{p}}$-bimodule. This implies that $P_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module, of rank $r > 0$. According to [2, Section II.5.1, Corollary] there is an open neighborhood U of $\mathfrak{p}$ in $\operatorname{Spec} A$ on which P is free of rank r. In particular $P_{\mathfrak{q}} \neq 0$ for all $\mathfrak{q} \in U$. Therefore $U \subset Y$.

The conclusion is that Y is also open in $\operatorname{Spec} A$. Since $\operatorname{Spec} A$ is connected it follows that $Y = \operatorname{Spec} A$. Another conclusion is that P is projective as A-module — see [2, Section II.5.2, Theorem 1].

Going back to Eq. (1.10) we see that $(\mathrm{H}^i T)_{\mathfrak{p}} \cong \mathrm{H}^i T_{\mathfrak{p}} = 0$ for all $i \neq -n$. Therefore $\mathrm{H}^i T = 0$ for $i \neq -n$. By truncation we get an isomorphism $T \cong P[n]$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$. Finally by [9, Proposition 2.2] the B–A-bimodule P is invertible. $\square$

Remark 1.11. Theorem 1.9 (for a field $\mathbb{K}$) is [9, Theorem 2.6]. However the proof there is only correct when A is noetherian (the hidden assumption is that $\operatorname{Spec} A$ is a noetherian topological space).

The same result was proved independently (and pretty much simultaneously, i.e. circa 1997) by Rouquier and Zimmermann [8].

Corollary 1.12. *Let A and B be flat $\mathbb{K}$-algebras with A commutative. If A and B are derived Morita equivalent relative to $\mathbb{K}$, then they are Morita equivalent relative to $\mathbb{K}$.*

Proof. Use the first paragraph in the proof of [9, Theorem 2.6] to pass to the case when $\operatorname{Spec} A$ is connected, and then apply Theorem 1.9. $\square$

We denote by $\operatorname{Pic}_{\mathbb{K}}(A)$ the noncommutative Picard group of A, consisting of isomorphism classes of invertible A–A-bimodules relative to $\mathbb{K}$. The operation is $- \otimes_A -$. Here is a definition from [9] extending this notion to the derived setting:

Definition 1.13. Let A be a flat $\mathbb{K}$-algebra. The *derived Picard group of A relative to* $\mathbb{K}$ is

$$\operatorname{DPic}_{\mathbb{K}}(A) := \frac{\{\text{two-sided tilting complexes over } A \text{ relative to } \mathbb{K}\}}{\text{isomorphism}},$$

where isomorphism is in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} A^{\mathrm{op}})$. The operation is $- \otimes^{\mathrm{L}}_A -$, and the unit element is the bimodule A.

There is a canonical injective group homomorphism

$$\operatorname{Pic}_{\mathbb{K}}(A) \times \mathbb{Z} \to \operatorname{DPic}_{\mathbb{K}}(A).$$

It formula is $(P, n) \mapsto P[n]$.

Remark 1.14. When A is either local, or commutative with connected spectrum, the homomorphism above is in fact bijective. On the other hand, if A is the algebra of upper triangular $n \times n$ matrices over $\mathbb{K}$ ($n \geq 2$, $\mathbb{K}$ a field), then the bimodule $A^* := \operatorname{Hom}_{\mathbb{K}}(A, \mathbb{K})$ is a two-sided tilting complex that does not belong to $\operatorname{Pic}_{\mathbb{K}}(A) \times \mathbb{Z}$. This is a sort of "Calabi–Yau" phenomenon. See [9] for details.

Let A and B be $\mathbb{K}$-algebras, and let P be an invertible B–A-bimodule relative to $\mathbb{K}$. Let $\mathbb{K}'$ be any commutative $\mathbb{K}$-algebra, and define $A' := \mathbb{K}' \otimes_{\mathbb{K}} A$, $B' := \mathbb{K}' \otimes_{\mathbb{K}} B$. and $P' := \mathbb{K}' \otimes_{\mathbb{K}} P$. Then P' is an invertible B'–A'-bimodule relative to $\mathbb{K}'$. When we take $B = A$ this fact gives rise to a group homomorphism

$$\operatorname{Pic}_{\mathbb{K}}(A) \to \operatorname{Pic}_{\mathbb{K}'}(A').$$

For the derived version we need flatness. The next theorem is the only new result in this section of the paper.

Theorem 1.15. *Let A, B, C be flat $\mathbb{K}$-algebras, and let $\mathbb{K}'$ be a commutative $\mathbb{K}$-algebra. Define $A' := \mathbb{K}' \otimes_{\mathbb{K}} A$, $B' := \mathbb{K}' \otimes_{\mathbb{K}} B$ and $C' := \mathbb{K}' \otimes_{\mathbb{K}} C$. Given complexes*

$$M \in \mathsf{D}^{-}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$$

and

$$N \in \mathsf{D}^{-}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} C^{\mathrm{op}}),$$

let us define

$$M' := \mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} M \in \mathsf{D}^{-}(\mathsf{Mod}\, A' \otimes_{\mathbb{K}'} B'^{\,\mathrm{op}})$$

and

$$N' := \mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} N \in \mathsf{D}^{-}(\mathsf{Mod}\, B' \otimes_{\mathbb{K}'} C'^{\,\mathrm{op}}).$$

Then there is an isomorphism

$$M' \otimes^{\mathrm{L}}_{B'} N' \cong \mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} (M \otimes^{\mathrm{L}}_{B} N)$$

in $\mathsf{D}^{-}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} C^{\mathrm{op}})$, functorial in M and N.

Proof. First let us observe that $A \otimes_{\mathbb{K}} B^{\mathrm{op}}$ is a flat $\mathbb{K}$-algebra, and

$$A' \otimes_{\mathbb{K}'} B'^{\,\mathrm{op}} \cong \mathbb{K}' \otimes_{\mathbb{K}} (A \otimes_{\mathbb{K}} B^{\mathrm{op}})$$

as $\mathbb{K}'$-algebras.

Choose an isomorphism $M \cong P$ in $\mathsf{D}^{-}(\mathsf{Mod}\; A \otimes_{\mathbb{K}} B^{\mathrm{op}})$, where each P^i is projective over $A \otimes_{\mathbb{K}} B^{\mathrm{op}}$. Then

$$M' \cong \mathbb{K}' \otimes_{\mathbb{K}} P \in \mathsf{D}^{-}(\mathsf{Mod}\; A' \otimes_{\mathbb{K}'} B'^{\,\mathrm{op}}),$$

and each $\mathbb{K}' \otimes_{\mathbb{K}} P^i$ is flat over A' and over $B'^{\,\mathrm{op}}$.

Similarly let us choose an isomorphism $N \cong Q$ in $\mathsf{D}^{-}(\mathsf{Mod}\; B \otimes_{\mathbb{K}} C^{\mathrm{op}})$; so $N' \cong \mathbb{K}' \otimes_{\mathbb{K}} Q$.

Now

$$M' \otimes^{\mathrm{L}}_{B'} N' \cong (\mathbb{K}' \otimes_{\mathbb{K}} P) \otimes_{B'} (\mathbb{K}' \otimes_{\mathbb{K}} Q)$$

in $\mathsf{D}^{-}(\mathsf{Mod}\; A' \otimes_{\mathbb{K}} C'^{\,\mathrm{op}})$. There is a canonical isomorphism

$$(\mathbb{K}' \otimes_{\mathbb{K}} P) \otimes_{B'} (\mathbb{K}' \otimes_{\mathbb{K}} Q) \cong \mathbb{K}' \otimes_{\mathbb{K}} (P \otimes_B Q)$$

as complexes of $A' \otimes_{\mathbb{K}} C'^{\,\mathrm{op}}$-modules; and therefore this is also an isomorphism also in $\mathsf{D}^{-}(\mathsf{Mod}\; A' \otimes_{\mathbb{K}} C'^{\,\mathrm{op}})$.

Next we have

$$M \otimes^{\mathrm{L}}_{B} N \cong P \otimes_B Q$$

in $\mathsf{D}^{-}(\mathsf{Mod}\; A \otimes_{\mathbb{K}} C^{\mathrm{op}})$. But since $P \otimes_B Q$ is a complex of flat $\mathbb{K}$-modules, we also have

$$\mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} (M \otimes^{\mathrm{L}}_{B} N) \cong \mathbb{K}' \otimes_{\mathbb{K}} (P \otimes_B Q)$$

in $\mathsf{D}^{-}(\mathsf{Mod}\; A' \otimes_{\mathbb{K}} C'^{\,\mathrm{op}})$. $\square$

Corollary 1.16. *Let A and B be flat $\mathbb{K}$-algebras, and let $\mathbb{K}'$ be a commutative $\mathbb{K}$-algebra. Define $A' := \mathbb{K}' \otimes_{\mathbb{K}} A$ and $B' := \mathbb{K}' \otimes_{\mathbb{K}} B$. Suppose T is a two-sided tilting complex over B–A relative to $\mathbb{K}$, with inverse S. Define*

$$T' := \mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\; B' \otimes_{\mathbb{K}'} A'^{\,\mathrm{op}})$$

and

$$S' := \mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} S \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\; A' \otimes_{\mathbb{K}'} B'^{\,\mathrm{op}}).$$

Then T' is a is a two-sided tilting complex over B'–A' relative to $\mathbb{K}'$, with inverse S'.

Proof. By the theorem we have

$$T' \otimes^{\mathrm{L}}_{A'} S' \cong \mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} (T \otimes^{\mathrm{L}}_{A} S) \cong \mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} B \cong B'$$

in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\; B' \otimes_{\mathbb{K}'} B'^{\mathrm{op}})$; and similarly $S' \otimes^{\mathrm{L}}_{B'} T' \cong A'$. $\square$

Corollary 1.17. *Let A be a flat $\mathbb{K}$-algebra, and let $\mathbb{K}'$ be a commutative $\mathbb{K}$-algebra. Define $A' := \mathbb{K}' \otimes_{\mathbb{K}} A$. Then the formula $T \mapsto \mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} T$ defines a group homomorphism*

$$\mathrm{DPic}_{\mathbb{K}}(A) \to \mathrm{DPic}_{\mathbb{K}'}(A').$$

Proof. Immediate from the previous corollary. $\square$

2. Associative deformations

In this section we keep the following setup:

Setup 2.1. $\mathbb{K}$ is a complete local noetherian commutative ring, with maximal ideal $\mathfrak{m}$ and residue field $\Bbbk = \mathbb{K}/\mathfrak{m}$.

Let M be a $\mathbb{K}$-module. Its $\mathfrak{m}$-adic completion is the $\mathbb{K}$-module

$$\hat{M} := \lim_{\leftarrow i} M/\mathfrak{m}^i M.$$

Recall that M is called $\mathfrak{m}$-*adically complete* (some texts, e.g. [2], use the term "separated and complete") if the canonical homomorphism $M \to \hat{M}$ is bijective. Every finitely generated $\mathbb{K}$-module is complete; but this is not true for infinitely generated modules. For instance, if N is a free $\mathbb{K}$-module of infinite rank, and if the ideal $\mathfrak{m}$ is not nilpotent, then the canonical homomorphism $N \to \hat{N}$ in injective but not surjective. Still in this instance the induced homomorphism $\Bbbk \otimes_{\mathbb{K}} N \to \Bbbk \otimes_{\mathbb{K}} \hat{N}$ is bijective. See [10, Theorem 1.12].

In [10, Corollary 2.12] we prove that a $\mathbb{K}$-module M is flat and $\mathfrak{m}$-adically complete if and only if $M \cong \hat{N}$ for some free $\mathbb{K}$-module N.

Sometimes one is given a ring homomorphism $\Bbbk \to \mathbb{K}$ lifting the canonical surjection $\mathbb{K} \to \Bbbk$; and then $\mathbb{K}$ becomes a $\Bbbk$-algebra. In this case the free $\mathbb{K}$-module N can be expresses as $N = \mathbb{K} \otimes_{\Bbbk} V$ for some $\Bbbk$-module V; and its completion is $M = \hat{N} = \mathbb{K} \hat{\otimes}_{\Bbbk} V$. Moreover $V \cong \Bbbk \otimes_{\mathbb{K}} N \cong \Bbbk \otimes_{\mathbb{K}} M$ as $\Bbbk$-modules.

Example 2.2. Take $\mathbb{K} := \Bbbk[[\hbar]]$, the power series ring in the variable $\hbar$ over the field $\Bbbk$. The maximal ideal $\mathfrak{m}$ is generated by $\hbar$. For a $\Bbbk$-module V we have a canonical isomorphism $\Bbbk[[\hbar]] \hat{\otimes}_{\Bbbk} V \cong V[[\hbar]]$, the latter being set of formal power series with coefficients in V.

The next definition is used in [11]:

Definition 2.3. Let A be a flat $\mathfrak{m}$-adically complete $\mathbb{K}$-algebra, such that the $\Bbbk$-algebra $\bar{A} := \Bbbk \otimes_{\mathbb{K}} A$ is commutative. Then we call A an *associative* $\mathbb{K}$*-deformation of* $\bar{A}$.

If $\mathbb{K}$ is a $\Bbbk$-algebra then we can find a (noncanonical) isomorphism of $\mathbb{K}$-modules $A \cong \mathbb{K} \hat{\otimes}_{\Bbbk} \bar{A}$. The multiplication induced on $\mathbb{K} \hat{\otimes}_{\Bbbk} \bar{A}$ by such an isomorphism is called a *star product*.

Example 2.4. Suppose $\bar{A}$ is some commutative $\Bbbk$-algebra, and $\mathbb{K} = \Bbbk[[\hbar]]$. Then a star product $\star$ on the $\Bbbk[[\hbar]]$-module $A := \bar{A}[[\hbar]]$ is expressed by a series $\{\beta_i\}_{i\geq 1}$ of $\Bbbk$-bilinear functions $\beta_i : \bar{A} \times \bar{A} \to \bar{A}$, as follows:

$$c_1 \star c_2 = c_1 c_2 + \sum_{i\geq 1} \beta_i(c_1, c_2)\hbar^i$$

for $c_1, c_2 \in \bar{A}$.

We shall need this version of the Nakayama Lemma:

Lemma 2.5. *Let $\mathbb{K}$ be as in Setup 2.1, let A be an $\mathfrak{m}$-adically complete $\mathbb{K}$-algebra, and let M be a finitely generated left A-module. If $\Bbbk \otimes_{\mathbb{K}} M = 0$ then $M = 0$.*

Proof. Let $\mathfrak{a} := \mathfrak{m}A$, which is a two-sided ideal of A, and $\mathfrak{m}^i A = \mathfrak{a}^i$ for every i. It follows that A is $\mathfrak{a}$-adically complete. According to [2, Section III.3.1, Lemma 3] the ideal $\mathfrak{a}$ is inside the Jacobson radical of A. By the usual Nakayama Lemma (which holds also for noncommutative rings, cf. [2, Section II.3.2, Proposition 4]) we see that $M/\mathfrak{a}M = 0$ implies $M = 0$. □

Note that there is no commutativity or finiteness assumption on the algebra A; only its structure as $\mathbb{K}$-module is important.

The next proposition might be of interest.

Proposition 2.6. *Let $\mathbb{K}$ be as in Setup 2.1, let A be an $\mathfrak{m}$-adically complete $\mathbb{K}$-algebra, and let M be a perfect complex in $\mathsf{D}(\mathsf{Mod}\ A)$. If $\Bbbk \otimes^{\mathrm{L}}_{\mathbb{K}} M = 0$ then $M = 0$.*

Proof. Assume $M \neq 0$, and let $\mathrm{H}^{i_0}M$ be its highest nonzero cohomology module. By Lemmas 1.3 and 2.5 we see that $\Bbbk \otimes_{\mathbb{K}} \mathrm{H}^{i_0}M \neq 0$. On the other hand by the Künneth trick (Lemma 1.4) we have

$$\Bbbk \otimes_{\mathbb{K}} \mathrm{H}^{i_0}M \cong \mathrm{H}^{i_0}(\Bbbk \otimes^{\mathrm{L}}_{\mathbb{K}} M).$$

Hence $\Bbbk \otimes^{\mathrm{L}}_{\mathbb{K}} M \neq 0$. □

Here is the main result of our paper:

Theorem 2.7. *Let $\mathbb{K}$ be as in Setup 2.1, and let A and B be a flat $\mathfrak{m}$-adically complete $\mathbb{K}$-algebras, such that the $\Bbbk$-algebras $\bar{A} := \Bbbk \otimes_{\mathbb{K}} A$ and $\bar{B} := \Bbbk \otimes_{\mathbb{K}} B$ are commutative with connected spectra. Suppose T is a two-sided tilting complex over B–A relative to $\mathbb{K}$. Then there is an isomorphism*

$$T \cong P[n]$$

in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\ B \otimes_{\mathbb{K}} A^{\mathrm{op}})$, for some invertible B–A-bimodule P and integer n.

Proof. This is very similar to the proof of Theorem 1.9. We may assume that $A \neq 0$. Define

$$n := -\sup\{i \mid \mathrm{H}^i T \neq 0\},$$

and let $P := \mathrm{H}^{-n}T$. This is a B–A-bimodule. By Lemma 1.3, P is a nonzero finitely generated right A-module. So according to Lemma 2.5 the right $\bar{A}$-module $\bar{P} := \Bbbk \otimes_{\mathbb{K}} P$ is nonzero. By the Künneth trick (Lemma 1.4) there is an isomorphism

$$\bar{P} = \Bbbk \otimes_{\mathbb{K}} \mathrm{H}^{-n}T \cong \mathrm{H}^{-n}(\Bbbk \otimes^{\mathrm{L}}_{\mathbb{K}} T).$$

According to Corollary 1.16 the complex $\bar{T} := \Bbbk \otimes^{\mathrm{L}}_{\mathbb{K}} T$ is a two-sided tilting complex over $\bar{B}$–$\bar{A}$ relative to $\Bbbk$. Since $\bar{A}$ is commutative and $\operatorname{Spec} \bar{A}$ is connected, we can apply Theorem 1.9. The conclusion is that $\bar{T}$ has exactly one nonzero cohomology module. But by the calculation above this must be $\mathrm{H}^{-n}\bar{T} \cong \bar{P}$. Therefore we get an isomorphism $\bar{T} \cong \bar{P}[n]$ in $\mathsf{D}(\mathsf{Mod}\ \bar{B} \otimes_{\Bbbk} \bar{A}^{\mathrm{op}})$, and $\bar{P}$ is an invertible $\bar{B}$–$\bar{A}$-bimodule relative to $\Bbbk$.

Let $S \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\ A \otimes_{\mathbb{K}} B^{\mathrm{op}})$ be an inverse of T. Define

$$m := -\sup\{i \mid \mathrm{H}^i S \neq 0\},$$

$Q := \mathrm{H}^{-m}S$, $\bar{S} := \Bbbk \otimes^{\mathrm{L}}_{\mathbb{K}} S$ and $\bar{Q} := \Bbbk \otimes_{\mathbb{K}} Q$. By the same considerations as above we see that $\bar{S} \cong \bar{Q}[m]$ in $\mathsf{D}(\mathsf{Mod}\ \bar{A} \otimes_{\Bbbk} \bar{B}^{\mathrm{op}})$, and $\bar{Q}$ is an invertible $\bar{A}$–$\bar{B}$-bimodule relative to $\Bbbk$.

From Corollary 1.16 it follows that

$$\bar{P}[n] \otimes_{\bar{A}} \bar{Q}[m] \cong \bar{T} \otimes^{\mathrm{L}}_{\bar{A}} \bar{S} \cong \bar{B}.$$

Therefore $n = -m$. Using the Künneth trick we see that

$$B \cong \mathrm{H}^0(T \otimes^{\mathrm{L}}_A S) \cong (\mathrm{H}^{-n}T) \otimes_A (\mathrm{H}^n Q) = P \otimes_A Q.$$

Similarly we get

$$A \cong Q \otimes_B P.$$

So P is an invertible B–A-bimodule relative to $\mathbb{K}$.

Since P is a projective A^{op}-module, and it is the highest nonzero cohomology of T, by Lemma 1.3 we have an isomorphism $T \cong M \oplus P[n]$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\ A^{\mathrm{op}})$ for some complex M. Suppose, for the sake of contradiction, that $M \neq 0$; and let

$$l := \sup\{i \mid \mathrm{H}^i M \neq 0\}.$$

Then $l < -n$, so $l + n < 0$. By the Künneth trick we get

$$(\mathrm{H}^l M) \otimes_A Q \cong (\mathrm{H}^l M) \otimes_A (\mathrm{H}^n S) \cong \mathrm{H}^{l+n}(M \otimes^{\mathrm{L}}_A S),$$

which is a direct summand of the B^{op}-module

$$\mathrm{H}^{l+n}(T \otimes^{\mathrm{L}}_A S) \cong \mathrm{H}^{l+n} B = 0.$$

But Q is an invertible bimodule, and therefore $\mathrm{H}^l M = 0$. This is a contradiction. Hence $T \cong P[n]$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\ A^{\mathrm{op}})$.

Finally, the last isomorphism implies that $\mathrm{H}^i T = 0$ for all $i \neq -n$. By truncation we obtain the isomorphism $T \cong P[n]$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\ B \otimes_{\mathbb{K}} A^{\mathrm{op}})$. □

The upshot is that associative deformations behave like commutative algebras, as far as derived Morita theory is concerned. Specifically:

Corollary 2.8. *Let $\mathbb{K}$ be as in Setup 2.1, and let A and B be a flat $\mathfrak{m}$-adically complete $\mathbb{K}$-algebras, such that the $\Bbbk$-algebras $\bar{A} := \Bbbk \otimes_{\mathbb{K}} A$ and $\bar{B} := \Bbbk \otimes_{\mathbb{K}} B$ are commutative with connected spectra. Assume that A and B are derived Morita equivalent relative to $\mathbb{K}$. Then A and B are Morita equivalent relative to $\mathbb{K}$. Moreover the $\Bbbk$-algebras $\bar{A}$ and $\bar{B}$ are isomorphic.*

Proof. By Theorem 1.6 there is a two-sided tilting complex T over B–A-relative to $\mathbb{K}$. Therefore by Theorem 2.7 there is an invertible B–A-bimodule P relative to $\mathbb{K}$. So we have classical Morita equivalence between A and B.

Now the bimodule $\bar{P} := \Bbbk \otimes_{\mathbb{K}} P$ is an invertible $\bar{B}$–$\bar{A}$-bimodule relative to $\Bbbk$. Since these are commutative $\Bbbk$-algebras they must be isomorphic. □

Corollary 2.9. *Let $\mathbb{K}$ be as in Setup 2.1, and let A be a flat $\mathfrak{m}$-adically complete $\mathbb{K}$-algebra, such that the $\Bbbk$-algebra $\bar{A} := \Bbbk \otimes_{\mathbb{K}} A$ is commutative with connected spectrum. Then*

$$\mathrm{DPic}_{\mathbb{K}}(A) = \mathrm{Pic}_{\mathbb{K}}(A) \times \mathbb{Z}.$$

Proof. As mentioned earlier, there is a canonical inclusion of $\mathrm{Pic}_{\mathbb{K}}(A) \times \mathbb{Z}$ into $\mathrm{DPic}_{\mathbb{K}}(A)$. By Theorem 2.7 this is a bijection. □

Remark 2.10. Let $\mathbb{K}$ be any commutative ring, and let A be a flat noetherian $\mathbb{K}$-algebra. A *dualizing complex* over A relative to $\mathbb{K}$ is a complex $R \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\ A \otimes_{\mathbb{K}} A^{\mathrm{op}})$ satisfying a list of conditions; see [9, Definition 4.1]. Presumably [9, Theorem 4.5] holds in this case (it was only proved when $\mathbb{K}$ is a field). Then the group $\mathrm{DPic}_{\mathbb{K}}(A)$ classifies isomorphism classes of dualizing complexes (if at least one dualizing complex exists).

Now assume we are in the situation of Corollary 2.9, and that $\bar{A}$ is a finitely generated $\Bbbk$-algebra. Then A is noetherian. It is reasonable to suppose that A will have some dualizing complex R relative to $\mathbb{K}$. What Corollary 2.9 tells us is that any other dualizing complex R' must be isomorphic to $P[n] \otimes_A R$ for some invertible bimodule P and integer n.

Remark 2.11. In the paper [1] Bursztyn and Waldmann consider the local ring $\mathbb{K} = \Bbbk[[\hbar]]$, and a fixed commutative $\Bbbk$-algebra $\bar{A}$ with connected spectrum. They prove that the Picard group $\mathrm{Pic}_{\Bbbk}(\bar{A})$ acts on the set of gauge equivalence classes of associative $\mathbb{K}$-deformations A of $\bar{A}$. The orbit of a deformation A under this action is the set of deformations that Morita equivalent to A. The stabilizer of A in $\mathrm{Pic}_{\Bbbk}(\bar{A})$ is the image of $\mathrm{Pic}_{\mathbb{K}}(A)$. And the kernel of the homomorphism $\mathrm{Pic}_{\mathbb{K}}(A) \to \mathrm{Pic}_{\Bbbk}(\bar{A})$ is the group of outer gauge equivalences of A.

Presumably these results remain true for any complete ring $\mathbb{K}$ as in Setup 2.1, not just for $\mathbb{K} = \Bbbk[[\hbar]]$.

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References

[1] H. Bursztyn, S. Waldmann, Bimodule Deformations, Picard Groups and Contravariant Connections, K-Theory 31 (1) (2004).
[2] N. Bourbaki, Commutative Algebra, Hermann, Paris, 1972.
[3] B. Keller, A remark on tilting theory and DG algebras, Manuscripta Math. 79 (1993) 247–252.
[4] B. Keller, Deriving DG categories, Ann. Sci. Ecole Norm. Sup. 27 (1994) 63–102.
[5] J. Rickard, Morita theory for derived categories, J. London Math. Soc. 39 (1989) 436–456.
[6] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. 43 (1991) 37–48.
[7] R. Hartshorne, Residues and Duality, in: Lecture Notes in Math., vol. 20, Springer-Verlag, Berlin, 1966.
[8] R. Rouquier, A. Zimmermann, Picard groups for derived module categories, Proc. London Math. Soc. 87 (2003) 197–225.
[9] A. Yekutieli, Dualizing complexes, Morita equivalence and the derived Picard group of a ring, J. London Math. Soc. 60 (1999) 723–746.
[10] A. Yekutieli, On flatness and completion for infinitely generated modules over noetherian rings, eprint arXiv:0902.4378v2.
[11] A. Yekutieli, Twisted deformation quantization of algebraic varieties, eprint arXiv:0905.0488v1.
[12] A. Yekutieli, J.J. Zhang, Rigid complexes via DG algebras, Algebr. Represent. Theory 12 (1) (2009) 19–52.