DERIVED EQUIVALENCES BETWEEN ASSOCIATIVE DEFORMATIONS

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Abstract. We prove that if two associative deformations (parameterized by the same complete local ring) are derived Morita equivalent, then they are Morita equivalent (in the classical sense).

0. Introduction

Let $\mathbb{K}$ be a commutative ring, and let A and B be associative unital $\mathbb{K}$-algebras. We denote by $\mathsf{Mod}\,A$ and $\mathsf{Mod}\,B$ the corresponding categories of left modules. One says that A and B are *Morita equivalent relative to* $\mathbb{K}$ (in the classical sense) if there is a $\mathbb{K}$-linear equivalence of categories $\mathsf{Mod}\,A \to \mathsf{Mod}\,B$.

Let $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A)$ denote the bounded derived category of complexes of left A-modules. This is a $\mathbb{K}$-linear triangulated category. If there is a $\mathbb{K}$-linear equivalence of triangulated categories $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B)$, then one says that A and B are *derived Morita equivalent relative to* $\mathbb{K}$.

There are plenty of examples of pairs of algebras that are derived Morita equivalent, but are not Morita equivalent in the classical sense.

Now suppose $\mathbb{K}$ is a complete noetherian local ring, with maximal ideal $\mathfrak{m}$ and residue field $\Bbbk$. Let A be a flat $\mathfrak{m}$-adically complete $\mathbb{K}$-algebra, such that the $\Bbbk$-algebra $\bar{A} := \Bbbk \otimes_{\mathbb{K}} A$ is commutative. We then say that A is an *associative* $\mathbb{K}$-*deformation of* $\bar{A}$; see [Ye3].

The most important example of an associative deformation is when $\Bbbk = \mathbb{R}$; $\bar{A} = \mathrm{C}^{\infty}(X)$, the $\mathbb{R}$-algebra of smooth functions on a differentiable manifold X; $\mathbb{K} = \mathbb{R}[[\hbar]]$, the ring of formal power series in the variable $\hbar$; and $A = \bar{A}[[\hbar]]$. In this case the multiplication in A is called a *star product*.

Let us assume that A and B are associative $\mathbb{K}$-deformations, and moreover the commutative rings $\bar{A}$ and $\bar{B}$ have connected prime spectra (i.e. they have no non-trivial idempotents). The main result of the paper (Theorem 2.7) says that if T is a *two-sided tilting complex over* B-A *relative to* $\mathbb{K}$, then $T \cong P[n]$ for some invertible bimodule P and integer n. (Tilting complexes and their properties are recalled in Section 1.) A direct consequence (Corollary 2.8) is that if A and B are derived Morita equivalent, then they are Morita equivalent in the classical sense.

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1. Base Change for Tilting Complexes

In this section we recall some facts about two-sided tilting complexes, and also prove one new theorem. Throughout this section $\mathbb{K}$ is a commutative ring. By "$\mathbb{K}$-algebra" we mean an associative unital algebra; i.e. a ring A, with center $\mathrm{Z}(A)$, together with a ring homomorphism $\mathbb{K} \to \mathrm{Z}(A)$.

For a $\mathbb{K}$-algebra A we denote by A^{op} the opposite algebra, namely with reverse multiplication. We view right A-modules as left A^{op}-modules. Let B be some other $\mathbb{K}$-algebra. By *B-A-bimodule relative to* $\mathbb{K}$ we mean a $\mathbb{K}$-central B-A-bimodule. We view B-A-bimodules relative to $\mathbb{K}$ as left $B \otimes_{\mathbb{K}} A^{\mathrm{op}}$ -modules.

The category of left A-modules is denoted by $\mathsf{Mod}\, A$. This is a $\mathbb{K}$-linear abelian category. Classical Morita theory says that any $\mathbb{K}$-linear equivalence $\mathsf{Mod}\, A \to \mathsf{Mod}\, B$ is of the form $P \otimes_A -$, where P is some invertible B-A-bimodule relative to $\mathbb{K}$.

The derived category of $\mathsf{Mod}\, A$ is $\mathsf{D}(\mathsf{Mod}\, A)$. This is a $\mathbb{K}$-linear triangulated category. We follow the conventions of [RD] on derived categories. For instance, $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ is the full subcategory of $\mathsf{D}(\mathsf{Mod}\, A)$ consisting of bounded complexes.

Here is a definition from Rickard's paper [Ri1].

Definition 1.1. Let A and B be $\mathbb{K}$-algebras. If there exists a $\mathbb{K}$-linear equivalence of triangulated categories $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B)$ then we say that A and B are *derived Morita equivalent relative to* $\mathbb{K}$.

Now assume that A is *flat* over $\mathbb{K}$. Since $A \otimes_{\mathbb{K}} B$ is flat over B, it follows that the forgetful functor $\mathsf{Mod}\, A \otimes_{\mathbb{K}} B \to \mathsf{Mod}\, B$ sends flat modules to flat modules.

Given three $\mathbb{K}$-algebras A, B, C, and complexes $M \in \mathsf{D}^{-}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$ and $N \in \mathsf{D}^{-}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} C^{\mathrm{op}})$, and assuming A is flat over $\mathbb{K}$, the derived tensor product

$$M \otimes^{\mathrm{L}}_{B} N \in \mathsf{D}^{-}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} C^{\mathrm{op}})$$

can be defined as follows: choose a quasi-isomorphism $P \to M$ with P a bounded above complex of projective $A \otimes_{\mathbb{K}} B^{\mathrm{op}}$ -modules. Then P is a bounded above complex of flat B^{op}-modules, and we take

$$M \otimes^{\mathrm{L}}_{B} N := P \otimes_{B} N.$$

This operation is functorial in M and N. As usual the requirements can be relaxed: it is enough to resolve M by a bounded above complex P of bimodules that are flat over B^{op}. If C is flat over $\mathbb{K}$ then we can resolve N instead of M. The derived tensor product $M \otimes^{\mathrm{L}}_{B} N$ is "indifferent" to the algebras A and C: we can forget them before or after calculating $M \otimes^{\mathrm{L}}_{B} N$, and get the same answer in $\mathsf{D}^{-}(\mathsf{Mod}\, \mathbb{K})$.

We record the following useful technical results.

Lemma 1.2 (Projective truncation trick)**.** *Let $M \in \mathsf{D}(\mathsf{Mod}\, A)$ and let i_0 be an integer. Suppose that $\mathrm{H}^i M = 0$ for all $i > i_0$, and $P := \mathrm{H}^{i_0} M$ is a projective A-module. Then there is an isomorphism $M \cong P[-i_0] \oplus N$ in $\mathsf{D}(\mathsf{Mod}\, A)$, where N is a complex satisfying $N^i = 0$ for all $i \geq i_0$.*

Proof. By the usual truncation trick (cf. [RD, Section I.7]) we can assume that $M^i = 0$ for all $i > i_0$. Hence we get an exact sequence $M^{i_0 - 1} \xrightarrow{\mathrm{d}} M^{i_0} \to P \to 0$. But P is projective, and therefore $M^{i_0} \cong P \oplus \mathrm{d}(M^{i_0-1})$. Define $N^{i_0-1} := \mathrm{Ker}(\mathrm{d}) \subset M^{i_0-1}$ and $N^i := M^i$ for $i < i_0 - 1$. □

Recall that a complex $M \in \mathsf{D}(\mathsf{Mod}\, A)$ is called *perfect* if it is isomorphic to bounded complex of finitely generated projective modules. We denote by $\mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{perf}}$ the full subcategory of perfect complexes.

Lemma 1.3. *Let $M \in \mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{perf}}$ and let i_0 be an integer. If $\mathrm{H}^i M = 0$ for all $i > i_0$, then the A-module $N := \mathrm{H}^{i_0} M$ is finitely presented.*

Proof. This is a bit stronger then [Ye1, Lemma 1.1(2)]. By truncation reasons we can assume that $M \cong P$, where P is a bounded complex of finitely generated projective A-modules, and $P^i = 0$ for $i > i_0$. So we get an exact sequence $P^{i_0-1} \to P^{i_0} \to N \to 0$. Suppose P^{i_0} is a direct summand of A^r (the free module of rank r), and P^{i_0-1} is a direct summand of A^s. Then be rearranging terms we get an exact sequence $A^{r+s} \to A^r \to N \to 0$. □

Lemma 1.4 (Künneth trick)**.** *Let A be a $\mathbb{K}$-algebra, let $M \in \mathsf{D}^-(\mathsf{Mod}\, A^{\mathrm{op}})$ and let $N \in \mathsf{D}^-(\mathsf{Mod}\, A)$. Let $i_0, j_0 \in \mathbb{Z}$ be such that $\mathrm{H}^i M = 0$ and $\mathrm{H}^j N = 0$ for all $i > i_0$ and $j > j_0$. Then*

$$(\mathrm{H}^{i_0} M) \otimes_A (\mathrm{H}^{j_0} N) \cong \mathrm{H}^{i_0+j_0}(M \otimes^{\mathrm{L}}_A N)$$

as $\mathbb{K}$-modules.

Proof. See [Ye1, Lemma 2.1]. □

The next definition is from [Ri2].

Definition 1.5. Let A and B be flat $\mathbb{K}$-algebras. A *two-sided tilting complex over B-A relative to* $\mathbb{K}$ is a complex $T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$ with the following property:

(∗) there exists a complex $S \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$, and isomorphisms $S \otimes^{\mathrm{L}}_B T \cong A$ and $T \otimes^{\mathrm{L}}_A S \cong B$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} A^{\mathrm{op}})$ and $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} B^{\mathrm{op}})$ respectively.

The complex S is called an *inverse of* T.

In case $B = A$ we say that T is a *two-sided tilting complex over A relative to* $\mathbb{K}$.

The inverse S in the definition is unique up to isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$. Of course S is a two-sided tilting complex over A-B relative to $\mathbb{K}$.

A two-sided tilting complex T induces a $\mathbb{K}$-linear equivalence of triangulated categories

$$T \otimes^{\mathrm{L}}_A - : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B).$$

This functor restricts to equivalences

$$\mathsf{D}^{\star}(\mathsf{Mod}\, A) \to \mathsf{D}^{\star}(\mathsf{Mod}\, B),$$

where $\star$ is either $+, -$ or b; and also to an equivalence

$$\mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{perf}} \to \mathsf{D}(\mathsf{Mod}\, B)_{\mathrm{perf}}.$$

See [Ri2] or [Ye1, Corollary 1.6(4)].

Conversely we have the next important result, due to Rickard [Ri2]. For alternative proofs see [Ke1] or [Ye1, Corollary 1.9].

Theorem 1.6 (Rickard)**.** *Let A and B be flat $\mathbb{K}$-algebras that are derived Morita equivalent relative to $\mathbb{K}$. Then there exists a two-sided tilting complex over B-A relative to $\mathbb{K}$.*

Remark 1.7. Suppose $F : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B)$ is a $\mathbb{K}$-linear equivalence of triangulated categories. Then F restricts to an equivalence between the subcategories of perfect complexes (cf. [Ke2]). This implies that F has finite cohomological dimension (bounded by the amplitude of $\mathrm{H}\, F(A)$). Hence F restricts to an equivalence between the bounded derived categories – i.e. a derived Morita equivalence.

Remark 1.8. In our paper [Ye1] the base ring $\mathbb{K}$ is taken to be a field. However the results in Sections 1-3 of that paper hold for any commutative base ring $\mathbb{K}$, as long as the $\mathbb{K}$-algebras in question are *flat*.

It is possible to remove even the flatness condition, at the price of working with DG algebras. Here is how to do it: choose a DG $\mathbb{K}$-algebra $\tilde{A}$ such that $\tilde{A}^i = 0$ for $i > 0$ and every $\tilde{A}^i$ flat as $\mathbb{K}$-module, with a DG algebra quasi-isomorphism $\tilde{A} \to A$. We call $\tilde{A} \to A$ a flat DG algebra resolution of A relative to $\mathbb{K}$. This can be done (cf. [YZ, Section 1] for commutative $\mathbb{K}$-algebras). Likewise choose a flat DG algebra resolution $\tilde{B} \to B$.

Let $\tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{A})^{\mathrm{b}}$ be the derived category of DG $\tilde{A}$-modules with bounded cohomologies. It is known (cf. [YZ, Proposition 1.4]) that the restriction of scalars functor $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A) \to \tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{A})^{\mathrm{b}}$ is an equivalence. Therefore a $\mathbb{K}$-linear equivalence $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,A) \to \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B)$ is the same as a $\mathbb{K}$-linear equivalence $\tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{A})^{\mathrm{b}} \to \tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{B})^{\mathrm{b}}$. Now the proof of [Ye1, Theorem 1.8] shows that there is a complex $T \in \tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{B} \otimes_{\mathbb{K}} \tilde{A}^{\mathrm{op}})^{\mathrm{b}}$ which is two-sided tilting.

A different choice of flat DG algebra resolutions $\tilde{A} \to A$ and $\tilde{B} \to B$ will give rise to an equivalent triangulated category $\tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{B} \otimes_{\mathbb{K}} \tilde{A}^{\mathrm{op}})^{\mathrm{b}}$. In this sense two-sided tilting complexes are independent of the resolutions.

See Remark 1.11 for the history of the next theorem.

Theorem 1.9. *Let A and B be flat $\mathbb{K}$-algebras. Assume A is commutative with connected spectrum. Let T be a two-sided tilting complex over B-A relative to $\mathbb{K}$. Then there is an isomorphism*

$$T \cong P[n]$$

in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B \otimes_{\mathbb{K}} A^{\mathrm{op}})$ for some invertible B-A-bimodule P and integer n.

Proof. We may assume that $A \neq 0$, so that $T \neq 0$. The complex T is perfect over B and over A^{op} (cf. [Ye1, Theorem 1.6]). As in [Ye1, Proposition 2.4] the complex T induces a $\mathbb{K}$-algebra isomorphism $A \cong \mathrm{Z}(B)$.

Let

$$n := -\sup\{i \mid \mathrm{H}^i T \neq 0\},$$

and let $P := \mathrm{H}^{-n} T$. This is a B-A-bimodule. By Lemma 1.3, P is finitely presented as right A-module.

For a prime $\mathfrak{p} \in \operatorname{Spec} A$, with corresponding local ring $A_{\mathfrak{p}}$, we write $P_{\mathfrak{p}} := P \otimes_A A_{\mathfrak{p}}$. Define $Y \subset \operatorname{Spec} A$ to be the support of P, i.e.

$$Y := \{\mathfrak{p} \in \operatorname{Spec} A \mid P_{\mathfrak{p}} \neq 0\}.$$

Since P is finitely generated it follows that Y is a closed subset of $\operatorname{Spec} A$.

Take any prime $\mathfrak{p} \in Y$, and let $B_{\mathfrak{p}} := B \otimes_A A_{\mathfrak{p}}$. Then, by [Ye1, Lemma 2.5], the complex

$$T_{\mathfrak{p}} := B_{\mathfrak{p}} \otimes_B T \otimes_A A_{\mathfrak{p}} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B_{\mathfrak{p}} \otimes_{\mathbb{K}} A_{\mathfrak{p}}^{\mathrm{op}})$$

is a two-sided tilting complex over $B_{\mathfrak{p}}$-$A_{\mathfrak{p}}$. Since

$$\mathrm{H}^{-n} T_{\mathfrak{p}} \cong P_{\mathfrak{p}} \neq 0,$$

[Ye1, Theorem 2.3] implies that

$$T_{\mathfrak{p}} \cong P_{\mathfrak{p}}[n] \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B_{\mathfrak{p}} \otimes_{\mathbb{K}} A_{\mathfrak{p}}^{\mathrm{op}}). \tag{1.10}$$

Thus $P_{\mathfrak{p}}$ is an invertible $B_{\mathfrak{p}}$-$A_{\mathfrak{p}}$-bimodule. This implies that $P_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module, of rank $r > 0$. According to [CA, Section II.5.1, Corollary] there is an open neighborhood U of $\mathfrak{p}$ in $\operatorname{Spec} A$ on which P is free of rank r. In particular $P_{\mathfrak{q}} \neq 0$ for all $\mathfrak{q} \in U$. Therefore $U \subset Y$.

The conclusion is that Y is also open in $\operatorname{Spec} A$. Since $\operatorname{Spec} A$ is connected it follows that $Y = \operatorname{Spec} A$. Another conclusion is that P is projective as A-module – see [CA, Section II.5.2, Theorem 1].

Going back to equation (1.10) we see that $(\mathrm{H}^i T)_{\mathfrak{p}} \cong \mathrm{H}^i T_{\mathfrak{p}} = 0$ for all $i \neq -n$. Therefore $\mathrm{H}^i T = 0$ for $i \neq -n$. By truncation we get an isomorphism $T \cong P[n]$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,B \otimes_{\mathbb{K}} A^{\mathrm{op}})$. Finally by [Ye1, Proposition 2.2] the B-A-bimodule P is invertible. ∎

Remark 1.11. Theorem 1.9 (for a field $\mathbb{K}$) is [Ye1, Theorem 2.6]. However the proof there is only correct when A is noetherian (the hidden assumption is that $\operatorname{Spec} A$ is a noetherian topological space).

The same result was proved independently (and pretty much simultaneously, i.e. circa 1997) by Rouquier and Zimmermann [RZ].

Corollary 1.12. *Let A and B be flat $\mathbb{K}$-algebras with A commutative. If A and B are derived Morita equivalent relative to $\mathbb{K}$, then they are Morita equivalent relative to $\mathbb{K}$.*

Proof. Use the first paragraph in the proof of [Ye1, Theorem 2.6] to pass to the case when $\operatorname{Spec} A$ is connected, and then apply Theorem 1.9. □

We denote by $\operatorname{Pic}_{\mathbb{K}}(A)$ the noncommutative Picard group of A, consisting of isomorphism classes of invertible A-A-bimodules relative to $\mathbb{K}$. The operation is $- \otimes_A -$. Here is a definition from [Ye1] extending this notion to the derived setting:

Definition 1.13. Let A be a flat $\mathbb{K}$-algebra. The *derived Picard group of A relative to $\mathbb{K}$* is

$$\operatorname{DPic}_{\mathbb{K}}(A) := \frac{\{\text{two-sided tilting complexes over } A \text{ relative to } \mathbb{K}\}}{\text{isomorphism}},$$

where isomorphism is in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} A^{\mathrm{op}})$. The operation is $- \otimes^{\mathrm{L}}_A -$, and the unit element is the bimodule A.

There is a canonical injective group homomorphism

$$\operatorname{Pic}_{\mathbb{K}}(A) \times \mathbb{Z} \to \operatorname{DPic}_{\mathbb{K}}(A).$$

It formula is $(P, n) \mapsto P[n]$.

Remark 1.14. When A is either local, or commutative with connected spectrum, the homomorphism above is in fact bijective. On the other hand, if A is the algebra of upper triangular $n \times n$ matrices over $\mathbb{K}$ ($n > 0$, $\mathbb{K}$ a field), then the bimodule $A^* := \operatorname{Hom}_{\mathbb{K}}(A, \mathbb{K})$ is a two-sided tilting complex that does not belong to $\operatorname{Pic}_{\mathbb{K}}(A) \times \mathbb{Z}$. This is a sort of “Calabi-Yau” phenomenon. See [Ye1] for details.

Let A and B be $\mathbb{K}$-algebras, and let P be an invertible B-A-bimodule relative to $\mathbb{K}$. Let $\mathbb{K}'$ be any commutative $\mathbb{K}$-algebra, and define $A' := \mathbb{K}' \otimes_{\mathbb{K}} A$, $B' := \mathbb{K}' \otimes_{\mathbb{K}} B$. and $P' := \mathbb{K}' \otimes_{\mathbb{K}} P$. Then P' is an invertible B'-A'-bimodule relative to $\mathbb{K}'$. When we take $B = A$ this fact gives rise to a group homomorphism

$$\operatorname{Pic}_{\mathbb{K}}(A) \to \operatorname{Pic}_{\mathbb{K}'}(A').$$

For the derived version we need flatness. The next theorem is the only new result in this section of the paper.

Theorem 1.15. *Let A, B, C be flat $\mathbb{K}$-algebras, and let $\mathbb{K}'$ be a commutative $\mathbb{K}$-algebra. Define $A' := \mathbb{K}' \otimes_{\mathbb{K}} A$, $B' := \mathbb{K}' \otimes_{\mathbb{K}} B$ and $C' := \mathbb{K}' \otimes_{\mathbb{K}} C$. Given complexes*

$$M \in \mathsf{D}^{-}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$$

and

$$N \in \mathsf{D}^{-}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} C^{\mathrm{op}}),$$

let us define

$$M' := \mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} M \in \mathsf{D}^{-}(\mathsf{Mod}\, A' \otimes_{\mathbb{K}'} B'^{\,\mathrm{op}})$$

and

$$N' := \mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} N \in \mathsf{D}^{-}(\mathsf{Mod}\, B' \otimes_{\mathbb{K}'} C'^{\,\mathrm{op}}).$$

Then there is an isomorphism

$$M' \otimes^{\mathrm{L}}_{B'} N' \cong \mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} (M \otimes^{\mathrm{L}}_B N)$$

in $\mathsf{D}^{-}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} C^{\mathrm{op}})$, functorial in M and N.

Proof. First let us observe that $A \otimes_{\mathbb{K}} B^{\mathrm{op}}$ is a flat $\mathbb{K}$-algebra, and

$$A' \otimes_{\mathbb{K}'} B'^{\,\mathrm{op}} \cong \mathbb{K}' \otimes_{\mathbb{K}} (A \otimes_{\mathbb{K}} B^{\mathrm{op}})$$

as $\mathbb{K}'$-algebras.

Choose an isomorphism $M \cong P$ in $\mathsf{D}^{-}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$, where each P^i is projective over $A \otimes_{\mathbb{K}} B^{\mathrm{op}}$. Then

$$M' \cong \mathbb{K}' \otimes_{\mathbb{K}} P \in \mathsf{D}^{-}(\mathsf{Mod}\, A' \otimes_{\mathbb{K}'} B'^{\,\mathrm{op}}),$$

and each $\mathbb{K}' \otimes_{\mathbb{K}} P^i$ is flat over A' and over $B'^{\,\mathrm{op}}$.

Similarly let us choose an isomorphism $N \cong Q$ in $\mathsf{D}^{-}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} C^{\mathrm{op}})$; so $N' \cong \mathbb{K}' \otimes_{\mathbb{K}} Q$.

Now

$$M' \otimes^{\mathrm{L}}_{B'} N' \cong (\mathbb{K}' \otimes_{\mathbb{K}} P) \otimes_{B'} (\mathbb{K}' \otimes_{\mathbb{K}} Q)$$

in $\mathsf{D}^{-}(\mathsf{Mod}\, A' \otimes_{\mathbb{K}} C'^{\,\mathrm{op}})$. There is a canonical isomorphism

$$(\mathbb{K}' \otimes_{\mathbb{K}} P) \otimes_{B'} (\mathbb{K}' \otimes_{\mathbb{K}} Q) \cong \mathbb{K}' \otimes_{\mathbb{K}} (P \otimes_B Q)$$

as complexes of $A' \otimes_{\mathbb{K}} C'^{\,\mathrm{op}}$ -modules; and therefore this is also an isomorphism also in $\mathsf{D}^{-}(\mathsf{Mod}\, A' \otimes_{\mathbb{K}} C'^{\,\mathrm{op}})$.

Next we have

$$M \otimes^{\mathrm{L}}_{B} N \cong P \otimes_B Q$$

in $\mathsf{D}^{-}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} C^{\mathrm{op}})$. But since $P \otimes_B Q$ is a complex of flat $\mathbb{K}$-modules, we also have

$$\mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} (M \otimes^{\mathrm{L}}_{B} N) \cong \mathbb{K}' \otimes_{\mathbb{K}} (P \otimes_B Q)$$

in $\mathsf{D}^{-}(\mathsf{Mod}\, A' \otimes_{\mathbb{K}} C'^{\,\mathrm{op}})$. □

Corollary 1.16. *Let A and B be flat $\mathbb{K}$-algebras, and let $\mathbb{K}'$ be a commutative $\mathbb{K}$-algebra. Define $A' := \mathbb{K}' \otimes_{\mathbb{K}} A$ and $B' := \mathbb{K}' \otimes_{\mathbb{K}} B$. Suppose T is a two-sided tilting complex over B-A relative to $\mathbb{K}$, with inverse S. Define*

$$T' := \mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B' \otimes_{\mathbb{K}'} A'^{\,\mathrm{op}})$$

and

$$S' := \mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} S \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A' \otimes_{\mathbb{K}'} B'^{\,\mathrm{op}}).$$

Then T' is a is a two-sided tilting complex over B'-A' relative to $\mathbb{K}'$, with inverse S'.

Proof. By the theorem we have

$$T' \otimes^{\mathrm{L}}_{A'} S' \cong \mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} (T \otimes^{\mathrm{L}}_{A} S) \cong \mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} B \cong B'$$

in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B' \otimes_{\mathbb{K}'} B'^{\mathrm{op}})$; and similarly $S' \otimes^{\mathrm{L}}_{B'} T' \cong A'$. □

Corollary 1.17. *Let A be a flat $\mathbb{K}$-algebra, and let $\mathbb{K}'$ be a commutative $\mathbb{K}$-algebra. Define $A' := \mathbb{K}' \otimes_{\mathbb{K}} A$. Then the formula $T \mapsto \mathbb{K}' \otimes^{\mathrm{L}}_{\mathbb{K}} T$ defines a group homomorphism*

$$\mathrm{DPic}_{\mathbb{K}}(A) \to \mathrm{DPic}_{\mathbb{K}'}(A').$$

Proof. Immediate from the previous corollary. □

2. Associative Deformations

In this section we keep the following setup:

Setup 2.1. $\mathbb{K}$ is a complete local noetherian commutative ring, with maximal ideal $\mathfrak{m}$ and residue field $\Bbbk = \mathbb{K}/\mathfrak{m}$.

Let M be a $\mathbb{K}$-module. Its $\mathfrak{m}$-adic completion is the $\mathbb{K}$-module

$$\hat{M} := \lim_{\leftarrow i} M/\mathfrak{m}^i M.$$

Recall that M is called $\mathfrak{m}$-*adically complete* (some texts, e.g. [CA], use the term "separated and complete") if the canonical homomorphism $M \to \hat{M}$ is bijective. Every finitely generated $\mathbb{K}$-module is complete; but this is not true for infinitely generated modules. For instance, if N is a free $\mathbb{K}$-module of infinite rank, and if the ideal $\mathfrak{m}$ is not nilpotent, then the canonical homomorphism $N \to \hat{N}$ in injective but not surjective. Still in this instance the induced homomorphism $\Bbbk \otimes_{\mathbb{K}} N \to \Bbbk \otimes_{\mathbb{K}} \hat{N}$ is bijective. See [Ye2, Theorem 1.12].

In [Ye2, Corollary 2.12] we prove that a $\mathbb{K}$-module M is flat and $\mathfrak{m}$-adically complete if and only if $M \cong \hat{N}$ for some free $\mathbb{K}$-module N.

Sometimes one is given a ring homomorphism $\Bbbk \to \mathbb{K}$ lifting the canonical surjection $\mathbb{K} \to \Bbbk$; and then $\mathbb{K}$ becomes a $\Bbbk$-algebra. In this case the free $\mathbb{K}$-module N can be expresses as $N = \mathbb{K} \otimes_{\Bbbk} V$ for some $\Bbbk$-module V; and its completion is $M = \hat{N} = \mathbb{K} \hat{\otimes}_{\Bbbk} V$. Moreover $V \cong \Bbbk \otimes_{\mathbb{K}} N \cong \Bbbk \otimes_{\mathbb{K}} M$ as $\Bbbk$-modules.

Example 2.2. Take $\mathbb{K} := \Bbbk[[\hbar]]$, the power series ring in the variable $\hbar$ over the field $\Bbbk$. The maximal ideal $\mathfrak{m}$ is generated by $\hbar$. For a $\Bbbk$-module V we have a canonical isomorphism $\Bbbk[[\hbar]] \hat{\otimes}_{\Bbbk} V \cong V[[\hbar]]$, the latter being set of formal power series with coefficients in V.

The next definition is used in [Ye3]:

Definition 2.3. Let A be a flat $\mathfrak{m}$-adically complete $\mathbb{K}$-algebra, such that the $\Bbbk$-algebra $\bar{A} := \Bbbk \otimes_{\mathbb{K}} A$ is commutative. Then we call A an *associative* $\mathbb{K}$-*deformation of* $\bar{A}$.

If $\mathbb{K}$ is a $\Bbbk$-algebra then we can find a (noncanonical) isomorphism of $\mathbb{K}$-modules $A \cong \mathbb{K} \hat{\otimes}_{\Bbbk} \bar{A}$. The multiplication induced on $\mathbb{K} \hat{\otimes}_{\Bbbk} \bar{A}$ by such an isomorphism is called a *star product*.

Example 2.4. Suppose $\bar{A}$ is some commutative $\Bbbk$-algebra, and $\mathbb{K} = \Bbbk[[\hbar]]$. Then a star product $\star$ on the $\Bbbk[[\hbar]]$-module $A := \bar{A}[[\hbar]]$ is expressed by a series $\{\beta_i\}_{i\geq 1}$ of $\Bbbk$-bilinear functions $\beta_i : \bar{A} \times \bar{A} \to \bar{A}$, as follows:

$$c_1 \star c_2 = c_1 c_2 + \sum_{i\geq 1} \beta_i(c_1, c_2)\hbar^i$$

for $c_1, c_2 \in \bar{A}$.

We shall need this version of the Nakayama Lemma:

Lemma 2.5. *Let $\mathbb{K}$ be as in Setup 2.1, let A be an $\mathfrak{m}$-adically complete $\mathbb{K}$-algebra, and let M be a finitely generated left A-module. If $\Bbbk \otimes_{\mathbb{K}} M = 0$ then $M = 0$.*

Proof. Let $\mathfrak{a} := \mathfrak{m}A$, which is a two-sided ideal of A, and $\mathfrak{m}^i A = \mathfrak{a}^i$ for every i. It follows that A is $\mathfrak{a}$-adically complete. According to [CA, Section III.3.1, Lemma 3] the ideal $\mathfrak{a}$ is inside the Jacobson radical of A. By the usual Nakayama Lemma (which holds also for noncommutative rings, cf. [CA, Section II.3.2, Proposition 4]) we see that $M/\mathfrak{a}M = 0$ implies $M = 0$. □

Note that there is no commutativity or finiteness assumption on the algebra A; only its structure as $\mathbb{K}$-module is important.

The next proposition might be of interest.

Proposition 2.6. *Let $\mathbb{K}$ be as in Setup 2.1, let A be an $\mathfrak{m}$-adically complete $\mathbb{K}$-algebra, and let M be a perfect complex in $\mathsf{D}(\mathsf{Mod}\, A)$. If $\Bbbk \otimes^{\mathrm{L}}_{\mathbb{K}} M = 0$ then $M = 0$.*

Proof. Assume $M \neq 0$, and let $\mathrm{H}^{i_0} M$ be its highest nonzero cohomology module. By Lemmas 1.3 and 2.5 we see that $\Bbbk \otimes_{\mathbb{K}} \mathrm{H}^{i_0} M \neq 0$. On the other hand by the Künneth trick (Lemma 1.4) we have

$$\Bbbk \otimes_{\mathbb{K}} \mathrm{H}^{i_0} M \cong \mathrm{H}^{i_0}(\Bbbk \otimes^{\mathrm{L}}_{\mathbb{K}} M).$$

Hence $\Bbbk \otimes^{\mathrm{L}}_{\mathbb{K}} M \neq 0$. □

Here is the main result of our paper:

Theorem 2.7. *Let $\mathbb{K}$ be as in Setup 2.1, and let A and B be a flat $\mathfrak{m}$-adically complete $\mathbb{K}$-algebras, such that the $\Bbbk$-algebras $\bar{A} := \Bbbk \otimes_{\mathbb{K}} A$ and $\bar{B} := \Bbbk \otimes_{\mathbb{K}} B$ are commutative with connected spectra. Suppose T is a two-sided tilting complex over B-A relative to $\mathbb{K}$. Then there is an isomorphism*

$$T \cong P[n]$$

in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$, for some invertible B-A-bimodule P and integer n.

Proof. This is very similar to the proof of Theorem 1.9. We may assume that $A \neq 0$. Define

$$n := -\sup\{i \mid \mathrm{H}^i T \neq 0\},$$

and let $P := \mathrm{H}^{-n} T$. This is a B-A-bimodule. By Lemma 1.3, P is a nonzero finitely generated right A-module. So according to Lemma 2.5 the right $\bar{A}$-module $\bar{P} := \Bbbk \otimes_{\mathbb{K}} P$ is nonzero. By the Künneth trick (Lemma 1.4) there is an isomorphism

$$\bar{P} = \Bbbk \otimes_{\mathbb{K}} \mathrm{H}^{-n} T \cong \mathrm{H}^{-n}(\Bbbk \otimes^{\mathrm{L}}_{\mathbb{K}} T).$$

According to Corollary 1.16 the complex $\bar{T} := \Bbbk \otimes^{\mathrm{L}}_{\mathbb{K}} T$ is a two-sided tilting complex over $\bar{B}$-$\bar{A}$ relative to $\Bbbk$. Since $\bar{A}$ is commutative and $\operatorname{Spec} \bar{A}$ is connected, we can apply Theorem 1.9. The conclusion is that $\bar{T}$ has exactly one nonzero cohomology module. But by the calculation above this must be $\mathrm{H}^{-n} \bar{T} \cong \bar{P}$. Therefore we get an isomorphism $\bar{T} \cong \bar{P}[n]$ in $\mathsf{D}(\mathsf{Mod}\, \bar{B} \otimes_{\Bbbk} \bar{A}^{\mathrm{op}})$, and $\bar{P}$ is an invertible $\bar{B}$-$\bar{A}$ bimodule relative to $\Bbbk$.

Let $S \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\mathrm{op}})$ be an inverse of T. Define

$$m := -\sup\{i \mid \mathrm{H}^i S \neq 0\},$$

$Q := \mathrm{H}^{-m} S$, $\bar{S} := \Bbbk \otimes^{\mathrm{L}}_{\mathbb{K}} S$ and $\bar{Q} := \Bbbk \otimes_{\mathbb{K}} Q$. By the same considerations as above we see that $\bar{S} \cong \bar{Q}[m]$ in $\mathsf{D}(\mathsf{Mod}\, \bar{A} \otimes_{\Bbbk} \bar{B}^{\mathrm{op}})$, and $\bar{Q}$ is an invertible $\bar{A}$-$\bar{B}$ bimodule relative to $\Bbbk$.

From Corollary 1.16 it follows that

$$\bar{P}[n] \otimes_{\bar{A}} \bar{Q}[m] \cong \bar{T} \otimes^{\mathrm{L}}_{\bar{A}} \bar{S} \cong \bar{B}.$$

Therefore $n = -m$. Using the Künneth trick we see that

$$B \cong \mathrm{H}^0(T \otimes^{\mathrm{L}}_{A} S) \cong (\mathrm{H}^{-n} T) \otimes_A (\mathrm{H}^n Q) = P \otimes_A Q.$$

Similarly we get

$$A \cong Q \otimes_B P.$$

So P is an invertible B-A-bimodule relative to $\mathbb{K}$.

Since P is a projective A^{op}-module, and it is the highest nonzero cohomology of T, by Lemma 1.3 we have an isomorphism $T \cong M \oplus P[n]$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{op}})$ for some complex M. Suppose, for the sake of contradiction, that $M \neq 0$; and let

$$l := \sup\{i \mid \mathrm{H}^i M \neq 0\}.$$

Then $l < -n$, so $l + n < 0$. By the Künneth trick we get

$$(\mathrm{H}^l M) \otimes_A Q \cong (\mathrm{H}^l M) \otimes_A (\mathrm{H}^n S) \cong \mathrm{H}^{l+n}(M \otimes^{\mathrm{L}}_A S),$$

which is a direct summand of the B^{op}-module

$$\mathrm{H}^{l+n}(T \otimes^{\mathrm{L}}_A S) \cong \mathrm{H}^{l+n} B = 0.$$

But Q is an invertible bimodule, and therefore $\mathrm{H}^l M = 0$. This is a contradiction. Hence $T \cong P[n]$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{op}})$.

Finally, the last isomorphism implies that $\mathrm{H}^i T = 0$ for all $i \neq -n$. By truncation we obtain the isomorphism $T \cong P[n]$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\mathrm{op}})$. □

The upshot is that associative deformations behave like commutative algebras, as far as derived Morita theory is concerned. Specifically:

Corollary 2.8. *Let $\mathbb{K}$ be as in Setup* 2.1*, and let A and B be a flat $\mathfrak{m}$-adically complete $\mathbb{K}$-algebras, such that the $\Bbbk$-algebras $\bar{A} := \Bbbk \otimes_{\mathbb{K}} A$ and $\bar{B} := \Bbbk \otimes_{\mathbb{K}} B$ are commutative with connected spectra. Assume that A and B are derived Morita equivalent relative to $\mathbb{K}$. Then A and B are Morita equivalent relative to $\mathbb{K}$. Moreover the $\Bbbk$-algebras $\bar{A}$ and $\bar{B}$ are isomorphic.*

Proof. By Theorem 1.6 there is a two-sided tilting complex T over B-A-relative to $\mathbb{K}$. Therefore by Theorem 2.7 there is an invertible B-A-bimodule P relative to $\mathbb{K}$. So we have classical Morita equivalence between A and B.

Now the bimodule $\bar{P} := \Bbbk \otimes_{\mathbb{K}} P$ is an invertible $\bar{B}$-$\bar{A}$-bimodule relative to $\Bbbk$. Since these are commutative $\Bbbk$-algebras they must be isomorphic. □

Corollary 2.9. *Let $\mathbb{K}$ be as in Setup* 2.1*, and let A be a flat $\mathfrak{m}$-adically complete $\mathbb{K}$-algebra, such that the $\Bbbk$-algebra $\bar{A} := \Bbbk \otimes_{\mathbb{K}} A$ is commutative with connected spectrum. Then*

$$\mathrm{DPic}_{\mathbb{K}}(A) = \mathrm{Pic}_{\mathbb{K}}(A) \times \mathbb{Z}.$$

Proof. As mentioned earlier, there is a canonical inclusion of $\mathrm{Pic}_{\mathbb{K}}(A) \times \mathbb{Z}$ into $\mathrm{DPic}_{\mathbb{K}}(A)$. By Theorem 2.7 this is a bijection. □

Remark 2.10. Let $\mathbb{K}$ be any commutative ring, and let A be a flat noetherian $\mathbb{K}$-algebra. A *dualizing complex* over A relative to $\mathbb{K}$ is a complex $R \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} A^{\mathrm{op}})$ satisfying a list of conditions; see [Ye1, Definition 4.1]. Presumably [Ye1, Theorem 4.5] holds in this case (it was only proved when $\mathbb{K}$ is a field). Then the group $\mathrm{DPic}_{\mathbb{K}}(A)$ classifies isomorphism classes of dualizing complexes (if at least one dualizing complex exists).

Now assume we are in the situation of Corollary 2.9, and that $\bar{A}$ is a finitely generated $\Bbbk$-algebra. Then A is noetherian. It is reasonable to suppose that A will have some dualizing complex R relative to $\mathbb{K}$. What Corollary 2.9 tells us is that any other dualizing complex R' must be isomorphic to $P[n] \otimes_A R$ for some invertible bimodule P and integer n.

Remark 2.11. In the paper [BW] Bursztyn and Waldmann consider the local ring $\mathbb{K} = \Bbbk[[\hbar]]$, and a fixed commutative $\Bbbk$-algebra $\bar{A}$ with connected spectrum. They prove that the Picard group $\mathrm{Pic}_{\Bbbk}(\bar{A})$ acts on the set of gauge equivalence classes of associative $\mathbb{K}$-deformations A of $\bar{A}$. The orbit of a deformation A under this action is the set of deformations that Morita equivalent to A. The stabilizer of A in $\mathrm{Pic}_{\Bbbk}(\bar{A})$ is the image of $\mathrm{Pic}_{\mathbb{K}}(A)$. And the kernel of the homomorphism $\mathrm{Pic}_{\mathbb{K}}(A) \to \mathrm{Pic}_{\Bbbk}(\bar{A})$ is the group of outer gauge equivalences of A.

Presumably these results remain true for any complete ring $\mathbb{K}$ as in Setup 2.1, not just for $\mathbb{K} = \Bbbk[[\hbar]]$.

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