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Symplectic and Poisson derived geometry and deformation quantization

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Abstract. We review recent results and ongoing investigations of the symplectic and Poisson geometry of derived moduli spaces, and describe applications to deformation quantization of such spaces. This paper has been written for the proceedings of the "Algebraic Geometry" AMS summer institute, University of Utah, Salt Lake City 2015.

Contents

Introduction

From the vantage point of the timeline of the AMS Summer Institutes, this contribution is a continuation and an update of B. Toën's 2005 overview [**To1**]. Our goal here is to highlight some of the remarkable developments in derived geometry that we witnessed in the past ten years.

One of the very first important results on the symplectic geometry of moduli spaces in algebraic geometry is undoubtedly Mukai's proof of the existence of a symplectic form on the moduli of simple sheaves on Calabi-Yau surfaces S, [**Mu**]. This was later generalized by Inaba to the moduli space of simple perfect complexes, [**In**]. By looking at these proofs one realizes that two ingredients plays different roles in establishing the result: one hand, the fact that we are considering moduli of *sheaves* and, on the other hand, the fact that we are working on a *Calabi-Yau* variety. Derived algebraic geometry gives a somehow more conceptual and unified explanation of Mukai's and Inaba's results, as follows. First of all, these moduli

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Artin stack locally of finite presentation over k. Let $f_i : X_i \to Y$, $i = 1, 2$ be maps of derived Artin stacks each endowed with coisotropic structures relative to π. Then, Melani and Safronov prove that the derived fiber product $X_1 \times_Y X_2$ has a natural, induced $(n-1)$-shifted Poisson structure such that the natural map $X_1 \times_Y X_2 \to X_1 \times X_2$ is a morphism of $(n-1)$-shifted Poisson stacks, where in the target X_2 is endowed with the $(n-1)$-shifted Poisson structure from point (i) above, and X_1 with the corresponding opposite $(n-1)$-shifted Poisson structure (see [**Me-Sa**] for details). A classical, i.e. 0-shifted, and purely cohomological precursor of this result was proved in [**Gi-Ba**]. Aside from its conceptual significance, the coisotropic intersection theorem of [**Me-Sa**] has many purely utilitarian corollaries. It allows us to extend the list of examples at the end of Section 2.3, by providing many more examples of shifted Poisson structures on moduli stacks, hence of moduli stacks admitting natural deformation quantizations (see Section 3). For instance, recently Spaide [**Spa**] applied coisotropic inersections to construct and characterize shifted Poisson structures on moduli spaces of framed sheaves in arbitrary dimension as well as on the moduli of monopoles.

3. Deformation quantization

Recall that for an ordinary smooth scheme X over k, a classical (unshifted) Poisson structure π can be viewed as the infinitesimal to the deformation of $\mathcal{O}_X$ as a sheaf of associative algebras on X. According to the algebraic deformation quantization results of Kontsevich [**Ko1**] and Yekutieli [**Ye**] every ordinary Poisson scheme (X, π) admits a quantization as a stack of algebroids. That is we can always find a stack of algebroids $\mathcal{X}$ defined over $k[[\hbar]]$ with $(\mathcal{X} \bmod \hbar) = X$ and with infinitesimal π. Moreover [**Ko1, Ye**] all possible quantizations with a given infinitesimal depend on a choice of a *formality quasi-isomorphism* (Drinfeld associator) and are classified by deformation of (X, π) as a Poisson scheme over $k[[\hbar]]$. In particular the *trivial* Poisson deformation corresponding to the $k[[\hbar]]$-linear Poisson bivector $\hbar \cdot \pi$ gives rise to a preferred quantization of (X, π). This preferred quantization is Kontsevich's canonical quantization, or in the case of a non-degenerate π is the algebraic Fedosov canonical quantization of Bezrukavnikov-Kaledin [**Bez-Ka**].

In this section we discuss the extension of the deformation quantization problem to shifted Poisson structures on derived Artin stacks. We argue that the canonical n-shifted quantization always exists as long as $n \neq 0$ and again depends on the choice of a Drinfeld associator. Ineterestingly enough the special case when $n = 0$ remains the hardest case and the best existing quantization results are still those of [**Ko1, Ye**]. The natural question of extending the [**Ko1, Ye**] quantization of smooth Poisson schemes to 0-shifted Poisson derived Artin stacks requires new ideas and will not be treated here.

3.1. Weak and strong quantization. Informally, shifted Poisson structures arise when we study deformations of X in which we allow only partial non commutativity in the deformed product structure. More precisely, an n-shifted Poisson structure can be viewed as the infinitesimal for deforming the commutative $(= \mathbb{E}_\infty)$ algebra structure on $\mathcal{P}_X(\infty)$ to an $\mathbb{E}_{n+1}$-algebra structure.

[Ko1] M. Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys. 66(2003), 157-216.

[Ye] A. Yekutieli, Twisted deformation quantization of algebraic varieties. Adv. Math. 268 (2015),241-305.