from: Stenstrom, "Rings of Quotients"

Chapter II. Rings of Fractions

In §10 (Example 3) of the preceding chapter it was briefly indicated how the well-known construction of the field of fractions of a commutative integral domain could be generalized to arbitrary commutative rings. When trying to extend this construction to non-commutative rings, one finds that this is not always possible, but that one can give necessary and sufficient conditions for the existence of a ring of fractions. Such a condition was first found by Ø. Ore [1] around 1930 for the case of a skew-field of fractions of a domain. The existence of a total ring of fractions of an arbitrary ring was first considered by K. Asano [1]. General rings of fractions were studied by Elizarov [1], and a systematic theory of rings and modules of fractions was developed by P. Gabriel [1, 2] in connection with his theory of general rings of quotients.

§1. The Ring of Fractions

Let A be a ring and let S be a multiplicatively closed subset of A, i.e. $s, t \in S$ implies $st \in S$ and $1 \in S$. We define a *right ring of fractions* of A with respect to S as a ring $A[S^{-1}]$ together with a ring homomorphism $\varphi: A \to A[S^{-1}]$ satisfying:

$F1$. $\varphi(s)$ is invertible for every $s \in S$.

$F2$. Every element in $A[S^{-1}]$ has the form $\varphi(a)\varphi(s)^{-1}$ with $s \in S$.

$F3$. $\varphi(a)=0$ if and only if $as=0$ for some $s \in S$.

Similarly one defines a left ring of fractions $[S^{-1}]A$ of A with respect to S. It is not immediately clear that the axioms $F1$–3 determine $A[S^{-1}]$ uniquely, but that so is the case follows from the fact that $A[S^{-1}]$ is a solution of a universal problem. (We will use the term "universal problem" in an informal sense, because attempts towards a formalization of it tend to become a bit awkward. The unicity of solutions of universal problems can therefore not be proved formally here, but it can be done in each specific case by the same kind of argument as was employed in the typical case of tensor products (Prop. I.8.1).)

Proposition 1.1. *When $A[S^{-1}]$ exists, it has the following universal property: for every ring homomorphism $\psi: A \to B$ such that $\psi(s)$ is invertible in B for every $s \in S$, there exists a unique homomorphism $\sigma: A[S^{-1}] \to B$ such that $\sigma\varphi = \psi$.*

Proof. σ is defined as $\sigma(\varphi(a)\varphi(s)^{-1}) = \psi(a)\psi(s)^{-1}$. We have to verify that this is well-defined. So suppose $\varphi(a)\varphi(s)^{-1} = \varphi(b)\varphi(t)^{-1}$. Then $\varphi(a) = \varphi(b)\varphi(t)^{-1}\varphi(s)$ $= \varphi(b)\varphi(c)\varphi(u)^{-1}$ for some $c \in A$ and $u \in S$, by $F2$. This gives

$$\varphi(a)\varphi(u) = \varphi(b)\varphi(c)$$

and

$$\varphi(s)\varphi(u) = \varphi(t)\varphi(c),$$

from which we obtain with the use of $F3$ that $auv = bcv$ and $suv' = tcv'$ for some v and v' in S. Since $\psi(v)$ and $\psi(v')$ are invertible, this implies

$$\psi(a)\psi(u) = \psi(b)\psi(c)$$

and

$$\psi(s)\psi(u) = \psi(t)\psi(c),$$

and we can go backwards to find that $\psi(a)\psi(s)^{-1} = \psi(b)\psi(t)^{-1}$.

We leave to the reader to verify that σ is a homomorphism. It is clear that $\sigma\varphi = \psi$ and that σ is unique. □

Because of the unicity of the solution of a universal problem we may draw the following two conclusions:

Corollary 1.2. *$A[S^{-1}]$ is unique up to isomorphism.*

Corollary 1.3. *If both $A[S^{-1}]$ and $[S^{-1}]A$ exist, then they are naturally isomorphic.*

The existence of a solution of the universal problem stated in Prop. 1.1 does not imply that $A[S^{-1}]$ exists. This is clear since it may happen that $[S^{-1}]A$ exists but $A[S^{-1}]$ does not exist (see Example 5 below).

Note that it follows from $F3$ that the canonical homomorphism $\varphi: A \to A[S^{-1}]$ is a monomorphism if and only if S is contained in the set of non-zero-divisors. When so is the case, A is usually identified with $\varphi(A)$, and elements in $A[S^{-1}]$ may be written as as^{-1}.

We now turn to the question of the existence of $A[S^{-1}]$. It is rather easy to show that the universal problem in Prop. 1.1 always has a solution (in a presentation of A by generators and relations, one adjoins a new generator $\bar{s}$ for each $s \in S$, and adds the new relations $s\bar{s} = \bar{s}s = 1$). However, this solution is practically useless because in general it is difficult to tell what its elements look like, e.g. it is hard to determine the kernel of the canonical homomorphism from A into it.

Proposition 1.4. *Let S be a multiplicatively closed subset of A. $A[S^{-1}]$ exists if and only if S satisfies:*

$S1$. If $s \in S$ and $a \in A$, there exist $t \in S$ and $b \in A$ such that $sb = at$.

$S2$. If $sa = 0$ with $s \in S$, then $at = 0$ for some $t \in S$.

When $A[S^{-1}]$ exists, it has the form

$$A[S^{-1}] = A \times S/\sim$$

where $\sim$ is the equivalence relation defined as $(a,s) \sim (b,t)$ if there exist $c, d \in A$ such that $ac = bd$ and $sc = td \in S$.

Proof. Suppose $A[S^{-1}]$ exists and let $\varphi: A \to A[S^{-1}]$ be the canonical map. We first verify $S1$. If $a \in A$ and $s \in S$, then $F2$ gives $\varphi(s)^{-1}\varphi(a) = \varphi(b)\varphi(t)^{-1}$ for some $b \in A$ and $t \in S$. Hence $\varphi(at) = \varphi(sb)$, and by $F3$ this means that $atu = sbu$

for some $u \in S$. Since $t\,u \in S$, this gives $S1$. If $s\,a=0$ with $s \in S$, then $\varphi(a)=0$ by $F1$, and $F3$ implies that $a\,t=0$ for some $t \in S$. Thus we also have $S2$.

Now suppose instead that $S1$ and $S2$ are satisfied. It is routine matter to verify that the stated relation $\sim$ is an equivalence relation. We form $A \times S/\sim$ and define addition and multiplication in the obvious way:

$$(a,s)+(b,t)=(a\,c+b\,d,u) \quad \text{where} \quad u=s\,c=t\,d \in S,$$

$$(a,s)\cdot(b,t)=(a\,c,t\,u) \quad \text{where} \quad s\,c=b\,u \quad \text{and} \quad u \in S.$$

One then has to verify that these definitions are independent of the choice of representing couples. This we leave to the industrious reader, as well as the verification that $A \times S/\sim$ is a ring and that $a \mapsto (a,1)$ is a ring homomorphism $\varphi: A \to A \times S/\sim$. It is easy to see that we have arrived at a right ring of fractions: the inverse of $\varphi(s)$ is $(1,s)$ when $s \in S$, and $(a,s)=\varphi(a)\,\varphi(s)^{-1}$; finally, the kernel of φ is $\{a \in A \mid a\,s=0 \text{ for some } s \in S\}$. ▯

S is a *right denominator set* when it is multiplicatively closed and satisfies $S1$ and $S2$. Similarly one defines left and two-sided denominator sets. Sometimes S is called *right permutable* if it satisfies $S1$ and *right reversible* if it satisfies $S2$. We will now show that $S1$ implies $S2$ under certain circumstances. A right ideal of the form $r(S)=\{a \in A \mid S\,a=0\}$, for some subset S of A, is called a right *annihilator*.

Proposition 1.5. *Assume A satisfies* ACC *on right annihilators. If S is multiplicatively closed and satisfies $S1$, then S is a right denominator set.*

Proof. Suppose $s\,a=0$ for some $s \in S$. There exists an integer n such that $r(s^k)=r(s^n)$ for all $k \geq n$. By $S1$ one can find $t \in S$ and $b \in A$ such that $s^n\,b=a\,t$. Then $s^{n+1}\,b=s\,a\,t=0$, so $b \in r(s^{n+1})=r(s^n)$. Hence $a\,t=s^n\,b=0$, which establishes $S2$. ▯

Examples

1. Commutative rings. When A is a commutative ring, both $S1$ and $S2$ are automatically satisfied, so every multiplicatively closed set S is a denominator set.

2. Ore rings. The most important example of a multiplicatively closed set is the set S_{reg} of all *regular* elements (i.e. non-zero-divisors) of A. The ring of fractions $A[S_{reg}^{-1}]$ is usually called the *classical right ring of quotients* (or sometimes the *total right ring of fractions*) of A. We will denote it by $Q_{cl}^r(A)$, or more often simply by Q_{cl}. The corresponding left-hand notation is $Q_{cl}^l(A)$. Recall that when both $Q_{cl}^r(A)$ and $Q_{cl}^l(A)$ exist, they coincide. Since condition $S2$ is automatically satisfied, we have:

Proposition 1.6 (Ore [1]). *$Q_{cl}^r(A)$ exists if and only if A satisfies the right Ore condition, i.e. for a and s in A with s regular, there exist b and t in A with t regular, such that $a\,t=s\,b$.*

3. Ore domains. If A has no zero-divisors, the right Ore condition states that $a\,A \cap s\,A \neq 0$ for all non-zero elements a, s of A, and this is the same as saying that $\mathfrak{a} \cap \mathfrak{b} \neq 0$ for all non-zero right ideals $\mathfrak{a}$ and $\mathfrak{b}$. A ring without zero-divisors and satisfying the right Ore condition is called a *right Ore domain*, and its classical right ring of quotients is a skew-field. Numerous examples of Ore domains are obtained from:

Proposition 1.7. *Every right noetherian ring without zero-divisors is a right Ore domain.*

Proof. Let a and b be non-zero elements of A. Consider the right ideals $\mathfrak{a}_n=b\,A+a\,b\,A+\cdots+a^n\,b\,A$. Since A is right noetherian, there exists a smallest n for which $\mathfrak{a}_n=\mathfrak{a}_{n+1}$, and then $a^{n+1}\,b=b\,c_0+a\,b\,c_1+\cdots+a^n\,b\,c_n$. This gives

$$b\,c_0=a(a^n\,b-b\,c_1-\cdots-a^{n-1}\,b\,c_n)\neq 0$$

where the minimality of n implies that the bracket is $\neq 0$. Thus $b\,A \cap a\,A \neq 0$, which is the right Ore condition. ▯

This result will be considerably improved in the next section.

4. Bezout domains. A *right Bezout domain* is a ring without zero-divisors in which every finitely generated right ideal is principal.

Proposition 1.8. *Every right Bezout domain is a right Ore domain.*

Proof. If A does not satisfy the Ore condition, then there exist non-zero elements a and b of A such that $a\,A \cap b\,A=0$. Put $a\,A+b\,A=c\,A$ and write $b=c\,d$. Then $a\,A=a\,A/(a\,A \cap b\,A) \cong (a\,A+b\,A)/b\,A=c\,A/b\,A \cong A/d\,A$. Since $d \neq 0$, this is a contradiction. ▯

5. Skew polynomial rings. Our aim is to construct examples of rings which are Ore domains to the right but not to the left. Let K be a field and $\varphi: K \to K$ an endomorphism of K. The elements of the *skew polynomial ring* $K_\varphi[X]$ are formal polynomials

$$a_0+X\,a_1+\cdots+X^n\,a_n$$

with addition defined as usual, but with multiplication given by $a\,X=X\,\varphi(a)$, i.e.

$$X^m\,a \cdot X^n\,b=X^{m+n}\,\varphi^n(a)\,b.$$

This gives rise to a ring which contains K as a subring. Each element f of $K_\varphi[X]$ has a well-defined degree $\deg(f)$, and one has $\deg(f\,g)=\deg(f)+\deg(g)$. It follows in particular that $K_\varphi[X]$ has no zero-divisors. It is easily verified that $K_\varphi[X]$ is a right euclidean domain, i.e. for any f and g there exist q and r with $\deg r<\deg q$ such that $f=g\,q+r$. This implies that $K_\varphi[X]$ is a right principal domain and therefore a right Ore domain by Prop. 1.7 (or 1.8).

If φ is an automorphism of K, one can perform the euclidean division algorithm also to the left, and $K_\varphi[X]$ is a left Ore domain. But if there exists $a \in K$ which is not in the image of φ, then the left Ore condition fails for the couple $(X, X\,a)$, as one easily verifies. Thus $K_\varphi[X]$ is a left Ore domain if and only if φ is an automorphism of K.

6. Rings of quotients. A ring A is called a *ring of quotients* if every non-zero-divisor of A is invertible, i.e. A is its own classical right and left ring of quotients. E.g. every regular ring is a ring of quotients, as one easily sees. Also rings with minimum condition on right ideals are rings of quotients, in fact one has more generally:

Proposition 1.9. *If A satisfies* DCC *on principal right ideals, then A is a ring of quotients.*