BEN GURION UNIVERSITY אוניברסיטת בן גוריון בנגב

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Course Announcement:

Derived Categories

2nd Semester 2011/12
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Audience & Prerequisites. This is a very advanced course (in BGU standards), aimed at M.Sc. and Ph.D. students, post-docs and researchers. Participants from outside the BGU community are welcome. The lectures will most likely be in English (certainly so if we have non-Israeli participants).

The prerequisites for this course are homological algebra, commutative algebra, algebraic topology and algebraic geometry. Ambitious students who have not studied these topics can ask my permission to attend; they will be expected to fill the gaps on their own.

The course will meet twice a week, each time for a 2 hour lecture. The semester begins on 11 March 2012. Potential participants are urged to get in touch with me for information about registration, and for setting up a schedule for the course.

About the Subject. *Derived categories* were invented by Grothendieck and Verdier around 1960, not very long after the "old" homological algebra (of derived functors between abelian categories) was established. This "new" homological algebra, of derived categories and derived functors between them, provides a significantly richer and more flexible machinery than the old homological algebra.

Here is a sketch of what a derived category looks like (for those who know old homological algebra). Let A be a commutative ring, and let $\mathsf{Mod}\, A$ be the category of A-modules. The objects of the derived category $\mathsf{D}(\mathsf{Mod}\, A)$ are the *complexes* $M = (\cdots \to M^0 \to M^1 \to \cdots)$ of A-modules. The morphisms in $\mathsf{D}(\mathsf{Mod}\, A)$ are the homomorphisms of complexes $\phi : M \to N$, composed with inverses of *quasi-isomorphisms*; this is a construction analogous to localization of rings. (A quasi-isomorphism is a homomorphism of complexes $\phi : M \to N$ such that the induces homomorphisms $\mathrm{H}^i(\phi) : \mathrm{H}^i(M) \to \mathrm{H}^i(N)$ are bijective.) The category $\mathsf{Mod}\, A$ is embedded in $\mathsf{D}(\mathsf{Mod}\, A)$ as the complexes concentrated in degree 0.

Given an additive functor $F : \mathsf{Mod}\, A \to \mathsf{Mod}\, B$, where B is another ring, there is a *left derived functor* $\mathrm{L}F : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B)$. The old derived functors $\mathrm{L}^i F$ can be recovered as $\mathrm{L}^i F(M) \cong \mathrm{H}^i(\mathrm{L}F(M))$ for $M \in \mathsf{Mod}\, A$. The way it works is this: any complex $M \in \mathsf{D}(\mathsf{Mod}\, A)$ admits an isomorphism $P \xrightarrow{\simeq} M$, where P is a "generalized projective complex". The complex $\mathrm{L}F(M) \in \mathsf{D}(\mathsf{Mod}\, B)$ is defined to be $\mathrm{L}F(M) := F(P)$, where $F(P)$ is the complex $\cdots \to F(P^0) \to F(P^1) \to \cdots$. If $P' \xrightarrow{\simeq} M$ is another such resolution, then there is a canonical isomorphism $F(P) \cong F(P')$, and hence $\mathrm{L}F(M)$ is well defined. Similarly there is a *right derived functor* $\mathrm{R}F$, constructed using "generalized injective resolutions".

The first spectacular consequence of this construction is that *dualizing complexes* appear. In the setup discussed above, a dualizing complex over the ring A is a complex M such that the right derived functor $\mathrm{RHom}_A(-, M)$ is a duality (i.e. a contravariant equivalence) of $\mathsf{D}(\mathsf{Mod}\, A)$.

(I am supressing certain finiteness conditions.) These dualizing complexes are at the heart of *Grothendieck duality theory* for schemes, which generalizes the earlier *Serre duality* in three ways: it works for singular schemes, it works for affine schemes, and it is relative (i.e. the setup is a morphism of schemes $f : X \to Y$, rather than just a smooth projective variety X over an algebraically closed field K, as in Serre's formulation). Grothendieck's duality theory appeared in the book *Residues and Duality* from 1966. Since then there were several extensions, and alternative approaches, to Grothendieck Duality, due to Deligne, Lipman, Yekutieli, Neeman and others.

Since their introduction, derived categories slowly made their way into adjacent areas of mathematics. Around 1980 there was a lot of progress surrounding the *Kazhdan-Lusztig conjecture* in representation theory, which was solved using the methods of *perverse sheaves*, *intersection cohomology*, *algebraic $\mathcal{D}$-modules* and the *Riemann-Hilbert correspondence*. All these methods involve derived categories in an essential way. Among the names to mention here are Bernstein, Beilinson, Deligne and Kashiwara. Intersection cohomology also plays a role in several combinatorial methods.

In the 1990's derived categories entered noncommutative algebra from a few directions. One direction was the study of *finite dimensional algebras*, including group rings in positive characteristics. In this context I'll mention the *Auslander-Reiten quiver* of the derived category of an algebra (due to Happel), and *derived morita theory* (due to Rickard). Another direction was in the M. Artin school of *noncommutative algebraic geometry*, where noetherian noncommutative rings were studied with the tools of algebraic geometry. A central tool here was the theory of *noncommutative dualizing complexes*, that was developed by Yekutieli, Zhang and Van den Bergh.

Also in the 1990's we saw a very interesting influence of algebraic topology (mainly *Quillen's theory of model structures*, and the *Brown Representability Theorem*) on the foundations of derived categories. One of the outcomes was the introduction of *unbounded resolutions*. In this context I'll mention the work of Keller and Neeman. These influences of algebraic topology have evolved in the last decade into *derived algebraic geometry*, which is still in its first steps, and comes in several "flavors". Let me mention the names of a few of the innovators: Kontsevich, Lurie and Toën.

Around 2000 it was noticed that derived categories can explain (and sometimes help to classify) phenomena in *birational algebraic geometry*, such as *blow-ups*. Here I will mention the *McKay Correspondence* for surfaces, and *noncommutative resolutions*. Among the contributors were Kapranov, Bondal, Orlov and Bridgeland.

The last (and most recent) aspect of derived categories I want to mention is the interaction with *theoretical physics*. One notable idea is that *D-branes*, that are high dimensional variants of strings, should be considered as objects in suitable derived categories (of sheaves on certain 3-dimensional complex projective varieties). A related phenomenon is *mirror symmetry*, that relates complex algebraic varieties to real symplectic manifolds. It is believed that this relation is expressed in terms of derived categories. Here again the main innovator is Kontsevich.

Topics. Here is a tentative list of topics for the course. Participants are encouraged to suggest other topics, based on their particular interests.

(1) Abelian categories and additive functors.
(2) Triangulated categories. The derived category of an abelian category.
(3) Derived functors: definition and uniqueness. Resolutions, and existence of derived functors.
(4) Commutative algebra via derived categories (dualizing complexes; regular, Gorenstein and CM rings; Grothendieck's Local Duality Theorem).
(5) Geometric derived categories (of sheaves on spaces). Direct and inverse image functors, Grothendieck Duality, Poicaré-Verdier Duality, perverse sheaves.
(6) Derived categories associated to nocommutative rings (including dualizing complexes, tilting complexes and derived Morita theory).
(7) DG algebras, DG categories, enhancements of triangulated categories. Derived algebraic geometry (one flavor).
(8) Derived categories in modern algebraic geometry and modern string theory (a survey).

Bibliography.

(1) Hartshorne, "Residues and Duality", Springer.
(2) Kashiwara and Schapira, "Sheaves on Manifolds", Springer.
(3) Lipman, in: "Foundations of Grothendieck duality for diagrams of schemes", Springer.
(4) Weibel, "An introduction to homological algebra", Cambridge.
(5) Gelfand and Manin, "Methods of Homological Algebra", Springer.
(6) Various lecture notes (TBA)