A COURSE ON DERIVED CATEGORIES

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0. Introduction

These are notes for an advanced course given a Ben Gurion University in Spring 2012. In this course I am following various sources, mostly [Sc] and [KS2], but going in a sufficiently different route to make written notes desirable. More resources are available on the course web page [CWP].

By way of introduction to the subject, let us consider *duality*. Take a field K. Given a K-module M (i.e. a vector space), let

$$D(M) := \mathrm{Hom}_K(M, K),$$

the dual module. There is a canonical homomorphism

$$\eta_M : M \to D(D(M)),$$

$\eta_M(m)(\phi) := \phi(m)$ for $m \in M$ and $\phi \in D(M)$. If M is finitely generated then η_M is an isomorphism (actually this is "if and only if").

To formalize this situation, let $\mathsf{Mod}\, K$ denote the category of K-modules. Then

$$D : \mathsf{Mod}\, K \to \mathsf{Mod}\, K$$

is a contravariant functor, and

$$\eta : \mathbf{1} \to D \circ D$$

is a natural transformation. Here $\mathbf{1}$ is the identity functor of $\mathsf{Mod}\, K$.

Now let us replace K by any (nonzero) commutative ring A. Again we can define a contravariant functor

$$D : \mathsf{Mod}\, A \to \mathsf{Mod}\, A, \quad D(M) := \mathrm{Hom}_A(M, A),$$

and a natural transformation $\eta : \mathbf{1} \to D \circ D$. It is easy to see that $\eta_M : M \to D(D(M))$ is an isomorphism if M is a finitely generated free module. Of course we can't expect reflexivity (i.e. η_M being an isomorphism) if M is not finitely generated; but what about a finitely generated module that is not free?

In order to understand this better, let us concentrate on the ring $A = \mathbb{Z}$. A finitely generated $\mathbb{Z}$-module M, namely a finitely generated abelian group, is of the form $M \cong G \oplus H$, with G free and H finite. It is important to note that this is not a canonical isomorphism: there is a canonical short exact sequence

$$0 \to H \to M \to G \to 0,$$

and the decomposition $M \cong G \oplus H$ comes from choosing a splitting of this sequence.

We know that for the free abelian group G there is reflexivity. But for the finite abelian group H we have

$$D(H) = \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z}) = 0.$$

Thus, whenever $H \neq 0$, reflexivity fails: $\eta_M : M \to D(D(M))$ is not an isomorphism.

On the other hand, for an abelian group M we can define another sort of dual:

$$D'(M) := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}).$$

(We may view the abelian group $\mathbb{Q}/\mathbb{Z}$ as the group of roots of 1 in $\mathbb{C}$, via the exponential map.) There is a natural transformation $\eta' : \mathbf{1} \to D' \circ D'$, and if H is a finite abelian group then η'_H is an isomorphism. So D' is a duality for finite abelian groups. Yet for a finitely generated free abelian group G we get $D'(D'(G)) = \widehat{G}$, the profinite completion of G. So once more this is not a good duality for all finitely generated abelian groups.

We could try to be more clever and "patch" the two dualities D and D', into something that we will call $D \oplus D'$. This looks pleasing at first – but then we recall that the decomposition $M \cong G \oplus H$ of a finitely generated group is not functorial, so that $D \oplus D'$ can't be a functor.

Later in the course we will introduce the *derived category* $\mathsf{D}(\mathsf{Mod}\,\mathbb{Z})$. The objects of $\mathsf{D}(\mathsf{Mod}\,\mathbb{Z})$ are the complexes of $\mathbb{Z}$-modules. There is a contravariant triangulated functor

$$\mathrm{R}D : \mathsf{D}(\mathsf{Mod}\,\mathbb{Z}) \to \mathsf{D}(\mathsf{Mod}\,\mathbb{Z}),$$
$$\mathrm{R}D(M) := \mathrm{RHom}_{\mathbb{Z}}(M, \mathbb{Z}).$$

This is the *right derived Hom functor*. And there is a natural transformation of triangulated functors

$$\eta : \mathbf{1} \to \mathrm{R}D \circ \mathrm{R}D.$$

If M is a *bounded complex* with *finitely generated cohomology modules* then $\eta_M : M \to \mathrm{R}D(\mathrm{R}D(M))$ is an isomorphism in $\mathsf{D}(\mathsf{Mod}\,\mathbb{Z})$.

We can take a $\mathbb{Z}$-module M and view it as a complex as follows:

$$\cdots \to 0 \to M \to 0 \to \cdots \tag{0.1}$$

where M is in degree 0. This is a fully faithful embedding of $\mathsf{Mod}\,\mathbb{Z}$ in $\mathsf{D}(\mathsf{Mod}\,\mathbb{Z})$. If M is a finitely generated module then η_M is an isomorphism. Thus we have a duality $\mathrm{R}D$ that holds for all finitely generated $\mathbb{Z}$-modules!

Here is the connection between $\mathrm{R}D$ and the "classical" dualities D and D': for $M \in \mathsf{Mod}\,\mathbb{Z}$ there are functorial isomorphisms

$$\mathrm{H}^0(\mathrm{R}D(M)) \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) = D(M)$$

and

$$\mathrm{H}^1(\mathrm{R}D(M)) \cong \mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z}) = D'(M).$$

This sort of duality holds for many noetherian commutative rings A. But the formula for the duality functor

$$\mathrm{R}D : \mathsf{D}(\mathsf{Mod}\,A) \to \mathsf{D}(\mathsf{Mod}\,A)$$

is somewhat different – it is

$$\mathrm{R}D(M) := \mathrm{RHom}_A(M, R),$$

where $R \in \mathsf{D}(\mathsf{Mod}\,A)$ is a *dualizing complex*. Such a dualizing complex is unique (up to shift and tensoring with an invertible module).

Interestingly, the structure of the dualizing complex R depends on the geometry of the ring A (i.e. of the scheme $\mathrm{Spec}\,A$). If A is a regular ring (like $\mathbb{Z}$) then $R = A$ is dualizing. If A is Cohen-Macaulay then R is a single A-module. But if A is a more complicated ring then R must live in several degrees. For example, missing picture[scale=0.35]figure-1.jpg

Figure 1.

consider an affine algebraic variety $X \subset \mathbf{A}^3_{\mathbb{R}}$ consisting of a plane and a line, with coordinate ring

$$A = \mathbb{R}[t_1, t_2, t_3]/(t_3 t_1 + t_3 t_2).$$

See figure 1. The dualizing complex R must live in two adjacent degrees; namely there is some i s.t. $\mathrm{H}^i(R)$ and $\mathrm{H}^{i+1}(R)$ are nonzero.

One can also talk about dualizing complexes over *noncommutative rings*. I am not sure if we will have time to do that in the course. (But this is a favorite topic for me!)

1. Basics Facts on Categories

1.1. **Set Theory.** In this course we will not try to be precise about issues of set theory. The blanket assumption is that we are given a *Grothendieck universe* U. This is an infinite set, closed under most set theoretical operations. A *small set* (or a U-small set) is a set $S \in \mathsf{U}$. A category C is a U-*category* if the set of objects $\operatorname{Ob}(\mathsf{C})$ is a subset of U, and for every $C, D \in \operatorname{Ob}(\mathsf{C})$ the set of morphisms $\operatorname{Hom}_{\mathsf{C}}(C, D)$ is small. See [KS2, Section 1.1]; or see [Ne] for another approach.

We denote by Set the category of all small sets. So $\operatorname{Ob}(\mathsf{Set}) = \mathsf{U}$, and Set is a U-category. An abelian group (or a ring, etc.) is called small if its underlying set is small. For a small ring A we denote by $\mathsf{Mod}\, A$ the category of all small left A-modules.

By default we work with U-categories, and from now on U will remain implicit. The one exception is when we deal with localization of categories, where we shall briefly encounter a set theoretical issue; but for most interesting cases this issue has an easy solution.

1.2. **Zero objects.** Let C be a category. A morphism $f : C \to D$ in C is called an *epimorphism* if it has the right cancelation property: for any $g, g' : D \to E$, $g \circ f = g' \circ f$ implies $g = g'$. The morphism $f : C \to D$ is called a *monomorphism* if it has the left cancelation property: for any $g, g' : E \to C$, $f \circ g = f \circ g'$ implies $g = g'$.

Example 1.1. In Set the monomorphism are the injections, and the epimorphisms are the surjections. A morphism $f : C \to D$ in Set that is both a monomorphism and an epimorphism is an isomorphism. The same holds in $\mathsf{Mod}\, A$.

Remark 1.2. The property of being a monomorphism or an epimorphism is sensitive to the category in question. For instance, consider the category of groups Grp. The forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ respects monomorphisms, but does not respect epimorphisms. (It is easy to write down an epimorphism in Grp that is not a surjection.)

An *initial object* in a category C is an object $C_0 \in \mathsf{C}$, such that for every object $C \in \mathsf{C}$ there is exactly one morphism $C_0 \to C$. Thus the set $\operatorname{Hom}_{\mathsf{C}}(C_0, C)$ is a singleton. An *terminal object* in C is an object $C_\infty \in \mathsf{C}$, such that for every object $C \in \mathsf{C}$ there is exactly one morphism $C \to C_\infty$.

Definition 1.3. A *zero object* in a category C is an object which is both initial and terminal.

Initial, terminal and zero objects are unique up to unique isomorphisms (but they need not exist).

Example 1.4. In Set, $\varnothing$ is an initial object, and any singleton is a terminal object. There is no zero object.

Example 1.5. In $\mathsf{Mod}\, A$, any trivial module (with only the zero element) is a zero object, and we denote this module by 0. This is allowed, since any other zero module is uniquely isomorphic to it.

1.3. **Products and Coproducts.** Let C be a category. For a collection $\{C_i\}_{i \in I}$ of objects of C, indexed by a set I, their *product* is a pair $(C, \{p_i\}_{i \in I})$ consisting of an object C and morphisms $p_i : C \to C_i$. The morphisms $p_i : C \to C_i$ are called projections. The pair $(C, \{p_i\}_{i \in I})$ must have this universal property: given any object D and morphisms $f_i : D \to C_i$, there is a unique morphism $f : D \to C$ s.t. $f_i = p_i \circ f$. Of course if a product $(C, \{p_i\}_{i \in I})$ exists then it is unique up to a unique isomorphism; and we write $\prod_{i \in I} C_i := C$.

Example 1.6. In Set and $\mathsf{Mod}\, A$ all products (indexed by small sets) exist, and they are the usual cartesian products.

For a collection $\{C_i\}_{i \in I}$ of objects of C, their *coproduct* is a pair $(C, \{e_i\}_{i \in I})$ consisting of an object C and morphisms $e_i : C_i \to C$. The morphisms $e_i : C_i \to C$ are called embeddings. The pair $(C, \{e_i\}_{i \in I})$ must have this universal property: given any object D and morphisms $f_i : C_i \to D$, there is a unique morphism $f : C \to D$ s.t. $f_i = f \circ e_i$. If a product $(C, \{e_i\}_{i \in I})$ exists then it is unique up to a unique isomorphism; and we write $\coprod_{i \in I} C_i := C$.

Example 1.7. In Set the coproduct is the disjoint union. In $\mathsf{Mod}\, A$ the coproduct is the direct sum.

1.4. **Equivalence.** Recall that a functor $F : \mathsf{C} \to \mathsf{D}$ is an *equivalence* if there exists a functor $G : \mathsf{D} \to \mathsf{C}$, and natural isomorphisms $G \circ F \cong \mathbf{1}_{\mathsf{C}}$ and $F \circ G \cong \mathbf{1}_{\mathsf{D}}$. Such a functor G is called a *quasi-inverse* of F.

We know that $F : \mathsf{C} \to \mathsf{D}$ is an equivalence iff these two conditions hold:

(i) F is essentially surjective on objects. This means that for every $D \in \mathsf{D}$ there is some $C \in \mathsf{C}$ and an isomorphism $F(C) \xrightarrow{\simeq} D$.

(ii) F is fully faithful. This means that for every $C_0, C_1 \in \mathsf{C}$ the function

$$F : \mathrm{Hom}_{\mathsf{C}}(C_0, C_1) \to \mathrm{Hom}_{\mathsf{D}}(F(C_0), F(C_1))$$

is bijective.

2. Abelian Categories

Definition 2.1. Let $\mathbb{K}$ be a commutative ring. A $\mathbb{K}$-*linear category* is a category A, endowed with a $\mathbb{K}$-module structure on each of the morphism sets $\mathrm{Hom}_{\mathsf{A}}(M_0, M_1)$ for all $M_0, M_1 \in \mathsf{A}$. The condition is this:

- For all $M_0, M_1, M_2 \in \mathsf{A}$ the composition function
$$\mathrm{Hom}_{\mathsf{A}}(M_0, M_1) \times \mathrm{Hom}_{\mathsf{A}}(M_1, M_2) \xrightarrow{\circ} \mathrm{Hom}_{\mathsf{A}}(M_0, M_2)$$
is $\mathbb{K}$-bilinear.

If $\mathbb{K} = \mathbb{Z}$ we say that A is a *linear category*.

Observe that for any object M of a $\mathbb{K}$-linear category A, the set
$$\mathrm{End}_{\mathsf{A}}(M) := \mathrm{Hom}_{\mathsf{A}}(M, M)$$
is a $\mathbb{K}$-algebra. In these notes a $\mathbb{K}$-algebra A is (by default) unital and associative; so in fact A is a ring, together with a ring homomorphism from $\mathbb{K}$ to the center of A.

This observation can be reversed:

Example 2.2. Let A be a $\mathbb{K}$-algebra. Define a category A like this: there is a single object M, and its set of morphisms is
$$\mathrm{Hom}_{\mathsf{A}}(M, M) := A.$$
Composition in A is the multiplication of A. Then A is a $\mathbb{K}$-linear category.

Definition 2.3. An *additive category* is a linear category M satisfying these conditions:

(i) M has a zero object 0.
(ii) M has finite coproducts.

Observe that $\mathrm{Hom}_{\mathsf{M}}(M, N) \neq \emptyset$, since this is an abelian group. Also
$$\mathrm{Hom}_{\mathsf{M}}(M, 0) = \mathrm{Hom}_{\mathsf{M}}(0, M) = 0,$$
the zero abelian group. We denote the unique arrows $0 \to M$ and $M \to 0$ also by 0. So the numeral 0 has a lot of meanings; but they are clear from the contexts. The coproduct in the additive category M is denoted by $\bigoplus$; cf. Example 1.7.

Example 2.4. Let A be a ring. The category $\mathsf{Mod}\,A$ is additive. The full subcategory $\mathsf{M} \subset \mathsf{Mod}\,A$ on the free modules is also additive.

for an object M we denote by $1_M : M \to M$ the identity morphism.

Proposition 2.5. *Let M be an additive category. Let $\{M_i\}_{i \in I}$ be a finite collection of objects of M, and let $M := \bigoplus_{i \in I} M_i$ be the coproduct, with embeddings $e_i : M_i \to M$.*

(1) *For any i let $p_i : M \to M_i$ be the unique morphism s.t. $p_i \circ e_i = 1_{M_i}$, and $p_i \circ e_j = 0$ for $j \neq i$. Then $(M, \{p_i\}_{i \in I})$ is a product of the collection $\{M_i\}_{i \in I}$.*
(2) $\sum_{i \in I} e_i \circ p_i = 1_M$.

Proof. Exercise. □

Example 2.6. One could ask if the linear category A from Example 2.2, built from a ring A, is additive, i.e. does it have finite direct sums? It appears that this depends on whether or not $A \cong A \oplus A$ as left A-modules. Thus if A is nonzero and commutative, or nonzero and noetherian, then this is false. On the other hand if we take a field $\mathbb{K}$, and a countable rank $\mathbb{K}$-module M, then $A := \mathrm{End}_{\mathbb{K}}(M)$ will satisfy $A \cong A \oplus A$.

Definition 2.7. Let M be an additive category, and let $f : M \to N$ be a morphism in M. A *kernel* of f is a pair (K, k), consisting of an object $K \in \mathsf{M}$ and a morphism $k : K \to M$, with these properties:

(i) $f \circ k = 0$.
(ii) If $k' : K' \to M$ is a morphism in M such that $f \circ k' = 0$, then there is a unique morphism $g : K' \to K$ such that $k' = k \circ g$.

In other words, the object K represents the functor $\mathsf{M}^{\mathrm{op}} \to \mathsf{Ab}$,

$$K' \mapsto \{k' \in \mathrm{Hom}_{\mathsf{M}}(K', M) \mid f \circ k' = 0\}.$$

The kernel of f is of course unique up to a unique isomorphism (if it exists), and we denote if by $\mathrm{Ker}(f)$. Sometimes $\mathrm{Ker}(f)$ refers only to the object K, and other times it refers only to the morphism k.

Definition 2.8. Let M be an additive category, and let $f : M \to N$ be a morphism in M. A *cokernel* of f is a pair (C, c), consisting of an object $C \in \mathsf{M}$ and a morphism $c : N \to C$, with these properties:

(i) $c \circ f = 0$.
(ii) If $c' : N \to C'$ is a morphism in M such that $c' \circ f = 0$, then there is a unique morphism $g : C \to C'$ such that $c' = g \circ c$.

The cokernel $\mathrm{Coker}(f)$ is unique up to a unique isomorphism.

Example 2.9. In $\mathsf{Mod}\, A$ all kernels and cokernels exist. Given $f : M \to N$, the kernel is $k : K \to M$, where

$$K := \{m \in M \mid f(m) = 0\},$$

and the k is the inclusion. The cokernel is $c : N \to C$, where $C := N/f(M)$, and c is the canonical projection.

Proposition 2.10. *Let $f : M \to N$ be a morphism, let $k : K \to M$ be a kernel of f, and let $c : N \to C$ be a cokernel of f. Then k is a monomorphism, and c is an epimorphism.*

Proof. Exercise. □

Definition 2.11. Assume the additive category M has kernels and cokernels. Let $f : M \to N$ be a morphism in M.

(1) Define the *image* of f to be
$$\mathrm{Im}(f) := \mathrm{Ker}(\mathrm{Coker}(f)).$$
(2) Define the *coimage* of f to be
$$\mathrm{Coim}(f) := \mathrm{Coker}(\mathrm{Ker}(f)).$$

Consider the following commutative diagram (solid arrows):

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[dr, "0"'] & M \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, dashed, "\gamma"] & N \arrow[r, "c"] & C \\
& M' \arrow[r, dashed, "f'"'] & N' \arrow[u, "\beta"'] \arrow[ur, "0"'] &
\end{tikzcd}$$

where $\alpha = \mathrm{Coker}(k) = \mathrm{Coim}(f)$ and $\beta = \mathrm{Ker}(c) = \mathrm{Im}(f)$. Since $c \circ f = 0$ there is a unique morphism γ making the diagram commutative. Now $\beta \circ \gamma \circ k = f \circ k = 0$; and β is a monomorphism; so $\gamma \circ k = 0$. Hence there is a unique morphism

$f' : M' \to N'$ making the diagram commutative. We conclude that $f : M \to N$ induces a morphism

$$f' : \mathrm{Coim}(f) \to \mathrm{Im}(f). \tag{2.12}$$

Definition 2.13. An *abelian category* is an additive category M with these extra properties:

(i) All morphisms in M admit kernels snd cokernels.
(ii) For any $f : M \to N$ in M the induced morphism f' of equation (2.12) is an isomorphism.

A less precise but (maybe) easier to remember way to state property (ii) is:

$$\mathrm{Ker}(\mathrm{Coker}(f)) = \mathrm{Coker}(\mathrm{Ker}(f)).$$

From now on we forget all about the coimage.

Example 2.14. The category $\mathsf{Mod}\, A$ is abelian.

Definition 2.15. Lert M be an abelian category, and let N be a full subcategory of M. We say that N is a *full abelian subcategory* of M if N is closed under direct sums, kernels and cokernels.

Example 2.16. Let M_1 be the category of finitely generated abelian groups, and let M_0 be the category of finite abelian groups. Then M_0 is a full abelian subcategory of M_1, and M_1 is a full abelian subcategory of Ab.

Example 2.17. Let N be the full subcategory of Ab whose objects are the finitely generated free abelian groups. It is an additive subcategory of Ab (since it is closed under direct sums), but clearly it is not a full abelian subcategory, since it is not closed under cokernels.

What is more interesting is that the additive category N does have its own intrinsic cokernels, but still it fails to be an abelian category.

Example 2.18. A ring A is *left noetherian* iff the category $\mathsf{Mod}_{\mathrm{f}}\, A$ of finitely generated modules is a full abelian subcategory of $\mathsf{Mod}\, A$. Here the issue if kernels.

Example 2.19. Let $(X, \mathcal{A})$ be a ringed space; namely X is a topological space and $\mathcal{A}$ is a sheaf of rings on X. We denote by $\mathsf{PMod}\, \mathcal{A}$ the category of presheaves of left $\mathcal{A}$-modules on X. This is an abelian category. Given a morphism $f : \mathcal{M} \to \mathcal{N}$ in $\mathsf{PMod}\, \mathcal{A}$, its kernel is the presheaf $\mathcal{K}$ defined by

$$\Gamma(U, \mathcal{K}) := \mathrm{Ker}\,\big(f : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N}\big).$$

The cokernel is the presheaf $\mathcal{C}$ defined by

$$\Gamma(U, \mathcal{C}) := \mathrm{Coker}\,\big(f : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N}\big).$$

Now let $\mathsf{Mod}\, \mathcal{A}$ be the full subcategory of $\mathsf{PMod}\, \mathcal{A}$ consisting of sheaves. We know that $\mathsf{Mod}\, \mathcal{A}$ is not closed under cokernels inside $\mathsf{PMod}\, \mathcal{A}$, and hence it is not a full abelian subcategory.

However $\mathsf{Mod}\, \mathcal{A}$ is itself an abelian category, but with different cokernels. Indeed, for a morphism $f : \mathcal{M} \to \mathcal{N}$ in $\mathsf{Mod}\, \mathcal{A}$, its cokernel $\mathrm{Coker}_{\mathsf{Mod}\, \mathcal{A}}(f)$ is the sheafification of the presheaf $\mathrm{Coker}_{\mathsf{PMod}\, \mathcal{A}}(f)$.

For educational purposes we state:

Theorem 2.20 (Freyd & Mitchell)**.** *Let* M *be a small abelian category. Then* M *is equivalent to a full abelian subcategory of* $\mathsf{Mod}\, A$*, for a suitable ring* A*.*

This means that most of the time we can pretend that $\mathsf{M} \subset \mathsf{Mod}\, A$; this could be a helpful heuristic.

Proposition 2.21. (1) *Let be an additive category. Then the opposite category* M^{op} *is also additive.*

(2) *Let* M *be an abelian category. Then the opposite category* M^{op} *is also abelian.*

Proof. (1) First note that

$$\mathrm{Hom}_{\mathsf{M}^{\mathrm{op}}}(M, N) = \mathrm{Hom}_{\mathsf{M}}(N, M),$$

so this is an abelian group. The bilinearity of the composition in M^{op} is clear, and the zero objects are the same. Existence of finite coproducts in M^{op} is because of existence of finite products in M; see Proposition 2.5(1).

(2) M^{op} has kernels and cokernels, since $\mathrm{Ker}_{\mathsf{M}^{\mathrm{op}}}(f) = \mathrm{Coker}_{\mathsf{M}}(f)$ and vice versa. Also the symmetric condition (ii) of Definition 2.13 holds. □

Proposition 2.22. *Let* $f : M \to N$ *be a morphism in an abelian category* M.

(1) f *is a monomorphism iff* $\mathrm{Ker}(f) = 0$.
(2) f *is an epimorphism iff* $\mathrm{Coker}(f) = 0$.
(3) f *is an isomorphism iff it is both a monomorphism and an epimorphism.*

Proof. Exercise. □

3. Additive Functors

Definition 3.1. Let M and N be $\mathbb{K}$-linear categories. A functor $F : \mathsf{M} \to \mathsf{N}$ is called a $\mathbb{K}$-*linear functor* if for every $M_0, M_1 \in \mathsf{M}$ the function

$$F : \mathrm{Hom}_{\mathsf{M}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{N}}(F(M_0), F(M_1))$$

is a $\mathbb{K}$-linear homomorphism.

A $\mathbb{Z}$-linear functor is also called an *additive functor*.

Additive functors commute with finite direct sums. More precisely:

Proposition 3.2. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between linear categories, let $\{M_i\}_{i \in I}$ be a finite collection of objects of M, and assume that the direct sum $(M, \{e_i\}_{i \in I})$ of the collection $\{M_i\}_{i \in I}$ exists in M. Then $\big(F(M), \{F(e_i)\}_{i \in I}\big)$ is a direct sum of the collection $\{F(M_i)\}_{i \in I}$ in N.*

Proof. Exercise. (Hint: use Proposition 2.5.) □

Example 3.3. Let $A \to B$ be a ring homomorphism. The corresponding forgetful functor

$$F : \mathsf{Mod}\, B \to \mathsf{Mod}\, A$$

(also called restriction of scalars) is additive. The functor

$$G : \mathsf{Mod}\, A \to \mathsf{Mod}\, B$$

defined by $G(M) := B \otimes_A M$, called extension of scalars, is also additive.

Proposition 3.4. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between additive categories. Then $F(0_{\mathsf{M}}) = 0_{\mathsf{N}}$.*

Proof. For any object $M \in \mathsf{M}$ we have a ring $\mathrm{End}_{\mathsf{M}}(M)$; and $\mathrm{Hom}_{\mathsf{A}}(M_0, M_1)$ is an $\mathrm{End}_{\mathsf{M}}(M_1)$-$\mathrm{End}_{\mathsf{M}}(M_0)$-bimodule. An object $M \in \mathsf{M}$ is a zero object iff $\mathrm{End}_{\mathsf{M}}(M)$ is the zero ring, i.e. $1 = 0$ in $\mathrm{End}_{\mathsf{M}}(M)$.

Now $F : \mathrm{End}_{\mathsf{M}}(M) \to \mathrm{End}_{\mathsf{N}}(F(M))$ is a ring homomorphism, so it sends the zero ring to the zero ring. □

Definition 3.5. Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between abelian categories.

(1) F is called *left exact* if it commutes with kernels. Namely if $F(\mathrm{Ker}_{\mathsf{M}}(\phi))$ is a kernel of $F(\phi))$ for any $\phi : M_0 \to M_1$ in M.
(2) F is called *right exact* if it commutes with cokernels. Namely if $F(\mathrm{Coker}_{\mathsf{M}}(\phi))$ is a cokernel of $F(\phi)$ for any $\phi : M_0 \to M_1$ in M.
(3) F is called *exact* if it both left exact and right exact.

This is illustrated in the following diagrams. Suppose $\phi : M_0 \to M_1$ is a morphism in M, with kernel K and cokernel C. Applying F to the diagram

$$K \xrightarrow{k} M_0 \xrightarrow{\phi} M_1 \xrightarrow{c} C$$

we get the solid arrows in

$$\begin{tikzcd}
F(K) \arrow[r, "F(k)"] \arrow[dr, dashed, "\psi"'] & F(M_0) \arrow[r, "F(\phi)"] & F(M_1) \arrow[r, "F(c)"] \arrow[d, dashed] & F(C) \\
& \mathrm{Ker}_{\mathsf{N}}(F(\phi)) \arrow[u, dashed] & \mathrm{Coker}_{\mathsf{N}}(F(\phi)) \arrow[ur, dashed, "\chi"'] &
\end{tikzcd}$$

The dashed arrows are from the structure of N. Left exactness requires ψ to be an isomorphism, and right exactness requires χ to be an isomorphism.

Definition 3.6. Let M be an abelian category. An *exact sequence* in M is a diagram

$$\cdots M_0 \xrightarrow{\phi_0} M_1 \xrightarrow{\phi_1} M_2 \cdots$$

(finite or infinite on either side) s.t. $\mathrm{Ker}(\phi_i) = \mathrm{Im}(\phi_{i-1})$ for all i (for which ϕ_i and ϕ_{i-1} are defined).

As usual, a *short exact sequence* is one of the form

$$0 \to M_0 \to M_1 \to M_2 \to 0. \tag{3.7}$$

Proposition 3.8. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between abelian categories.*

(1) *The functor F is left exact iff for every short exact sequence* (3.7) *in* M*, the sequence*
$$0 \to F(M_0) \to F(M_1) \to F(M_2)$$
is exact in N.

(2) *The functor F is right exact iff for every short exact sequence* (3.7) *in* M*, the sequence*
$$F(M_0) \to F(M_1) \to F(M_2) \to 0$$
is exact in N.

Proof. Exercise. (Hint: $M_0 \cong \mathrm{Ker}(M_1 \to M_2)$ etc.) □

Example 3.9. Let A be a commutative ring, and let M be a fixed A-module. Define functors $F, G : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$ and $H : (\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathsf{Mod}\, A$ like this: $F(N) := M \otimes_A N$, $G(N) := \mathrm{Hom}_A(M, N)$ and $H(N) := \mathrm{Hom}_A(N, M)$. Then F is right exact, and G and H are left exact.

Proposition 3.10. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between abelian categories. If F is an equivalence then it is exact.*

Proof. We will prove that F respects kernels; the proof for cokernels is similar. Take a morphism $\phi : M_0 \to M_1$ in M, with kernel K. We have this diagram (solid arrows):

$$\begin{tikzcd} M \arrow[d, dashed, "\psi"'] \arrow[dr, dashed, "\theta"] & & \\ K \arrow[r, "k"] & M_0 \arrow[r, "\phi"] & M_1 \end{tikzcd}$$

Applying F we obtain this diagram (solid arrows):

$$\begin{tikzcd} N = F(M) \arrow[d, dashed, "F(\psi)"'] \arrow[dr, dashed, "\bar{\theta}"] & & \\ F(K) \arrow[r, "F(k)"] & F(M_0) \arrow[r, "F(\phi)"] & F(M_1) \end{tikzcd}$$

in N. Suppose $\bar{\theta} : N \to F(M_0)$ is a morphism in N s.t. $F(\phi) \circ \bar{\theta} = 0$. Since F is essentially surjective on objects, there is some $M \in \mathsf{M}$ with an isomorphism $\alpha : F(M) \xrightarrow{\simeq} N$. After replacing N with $F(M)$ and $\bar{\theta}$ with $\bar{\theta} \circ \alpha$, we can assume that $N = F(M)$.

Now since F is fully faithful, there is a unique $\theta : M \to M_0$ s.t. $F(\theta) = \bar{\theta}$; and $\phi \circ \theta = 0$. So there is a unique $\psi : M \to K$ s.t. $\theta = k \circ \psi$. It follows that $F(\psi) : F(M) \to F(M_0)$ is the unique morphism s.t. $\bar{\theta} = F(k) \circ F\psi)$. □

Here is a result that could afford another proof of the previous proposition.

Proposition 3.11. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between linear categories. The following conditions are equivalent:*

(i) *The functor F has a quasi-inverse.*
(ii) *The functor F has an additive quasi-inverse.*

Proof. Exercise. $\square$

4. Projective and Injective Objects

Here M be an abelian category.

A *splitting* of an epimorphism $\psi : M \to M''$ in M is a morphism $\alpha : M'' \to M$ s.t. $\psi \circ \alpha = 1_{M''}$. A splitting of a monomorphism $\phi : M' \to M$ is a morphism $\beta : M \to M'$ s.t. $\beta \circ \phi = 1_{M'}$. A splitting of a short exact sequence

$$0 \to M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \to 0$$

is a splitting of the epimorphism ψ, or equivalently a splitting of the monomorphism ϕ. The short exact sequence is said to be *split* if it has some splitting.

Definition 4.1. An object $P \in \mathsf{M}$ is called a *projective object* if any diagram (solid arrows)

$$\begin{tikzcd} & P \arrow[dl, dashed, "\tilde{\gamma}"'] \arrow[d, "\gamma"] \\ M \arrow[r, "\psi"'] & N \end{tikzcd}$$

in which ψ is an epimorphism, can be completed (dashed arrow).

Proposition 4.2. *The following conditions are equivalent for $P \in \mathsf{M}$:*

(i) *P is projective.*
(ii) *The additive functor*

$$\mathrm{Hom}_{\mathsf{M}}(P, -) : \mathsf{M} \to \mathsf{Ab}$$

is exact.

Proof. Exercise. □

Definition 4.3. We say M *has enough projectives* if every $M \in \mathsf{M}$ admits an epimorphism $P \to M$ with P a projective object.

Example 4.4. Let A be a ring. An A-module P is projective iff it is a direct summand of a free module; i.e. $P \oplus P' \cong Q$ for some module P' and free module Q. The category $\mathsf{Mod}\, A$ has enough projectives.

Example 4.5. Let M be the category of finite abelian groups. The only projective object in M is 0. So M does not have enough projectives.

Example 4.6. Consider the scheme $X := \mathbf{P}^1_{\mathbb{K}}$, the projective line over a field $\mathbb{K}$ (we can assume $\mathbb{K}$ is algebraically closed, so this is a classical algebraic variety). The structure sheaf (sheaf of functions) is $\mathcal{O}_X$. The category $\mathsf{Coh}\, \mathcal{O}_X$ of coherent $\mathcal{O}_X$-modules is abelian (it is a full abelian subcategory of $\mathsf{Mod}\, \mathcal{O}_X$, cf. Example 2.19). One can show that the only projective object of $\mathsf{Coh}\, \mathcal{O}_X$ is 0, but this is quite involved.

Let us only indicate why $\mathcal{O}_X$ is not projective. Denote by t_0, t_1 the homogenous coordinates of X. These belong to $\Gamma(X, \mathcal{O}_X(1))$, so each determines a homomorphism of sheaves $t_j : \mathcal{O}_X(i) \to \mathcal{O}_X(i+1)$. We get a sequence

$$0 \to \mathcal{O}_X(-2) \xrightarrow{[\,t_0 \ -t_1\,]} \mathcal{O}_X(-1)^2 \xrightarrow{\left[\begin{smallmatrix} t_0 \\ t_1 \end{smallmatrix}\right]} \mathcal{O}_X \to 0$$

in $\mathsf{Coh}\, \mathcal{O}_X$, which is known to exact, and also not split.

Definition 4.7. An object $I \in \mathsf{M}$ is called an *injective object* if any diagram (solid arrows)

$$\begin{tikzcd} I & \\ M \arrow[u, "\gamma"] \arrow[r, "\psi"'] & N \arrow[ul, dashed, "\tilde{\gamma}"'] \end{tikzcd}$$

in which ψ is a monomorphism, can be completed (dashed arrow).

Proposition 4.8. *The following conditions are equivalent for* $I \in \mathsf{M}$:

(i) *I is injective.*
(ii) *The additive functor*

$$\operatorname{Hom}_{\mathsf{M}}(-, I) : \mathsf{M}^{\mathrm{op}} \to \mathsf{Ab}$$

is exact.

Proof. Exercise. □

Here are a few results about injective objects.

Proposition 4.9. *Let $f : A \to B$ be a ring homomorphism, and let I be an injective left A-module. Then $J := \operatorname{Hom}_A(B, I)$ is an injective left B-module.*

=== [to here typing]

!!!!

Example 4.10. Let A be a ring. Unlike projectives, the structure of injective objects in $\mathsf{Mod}\, A$ is very complicated, and not much is known (except that they exist). However if A is a commutative noetherian ring then we know this: every injective module I is a direct sum of indecomposable injective modules. And these indecomposables are parametrized by $\operatorname{Spec} A$, the set of prime ideals of A. These facts are due to Matlis; see [RD, pages 120-122] for details.

9. The Derived Category

(missing stuff)

Remark 9.1. Why "triangle"? This is because sometimes a triangle

$$M \xrightarrow{\alpha} N \xrightarrow{\beta} L \xrightarrow{\gamma} M[1]$$

is written as a diagram

$$\begin{tikzcd} & L \arrow[dl, "\gamma"'] & \\ M \arrow[rr, "\alpha"] & & N \arrow[ul, "\beta"'] \end{tikzcd}$$

But here γ is a map of degree 1.

10. Outline: Derived Functors

(missing stuff)

11. Outline: Existence of Derived Functors

11.1. **Full triangulated subcategories.**

Definition 11.1. Let K be a triangulated category. A *full triangulated subcategory* of K is a full subcategory $\mathsf{J} \subset \mathsf{K}$, s.t. these conditions hold:

(i) J is closed under shifts, i.e. $I \in \mathsf{J}$ iff $I[1] \in \mathsf{J}$.

(ii) J is closed under distinguished triangles, i.e. if

$$I' \to I \to I'' \to I[1]$$

is a distinguished triangle in K s.t. $I', I \in \mathsf{J}$, then also $I'' \in \mathsf{J}$.

When J is a full triangulated subcategory of K, then J itself is triangulated, and the inclusion $\mathsf{J} \to \mathsf{K}$ is a triangulated functor.

11.2. **Right derived functor.**

Recall that an additive functor $F : \mathsf{M} \to \mathsf{N}$ between abelian categories induces a triangulated functor

$$\mathsf{K}(F) : \mathsf{K}(\mathsf{M}) \to \mathsf{K}(\mathsf{N}).$$

In the next theorem we consider a slightly more general situation.

Theorem 11.2. *Let M be an abelian category, E a triangulated category, and $F : \mathsf{K}(\mathsf{M}) \to \mathsf{E}$ a triangulated functor. Assume there is a full triangulated subcategory $\mathsf{J} \subset \mathsf{K}(\mathsf{M})$ with these two properties:*

(i) *If $\phi : I \to I'$ is a quasi-isomorphism in J, then $F(\phi) : F(I) \to F(I')$ is an isomorphism in E.*

(ii) *Every $M \in \mathsf{K}(\mathsf{M})$ admits a quasi-isomorphism $M \to I$ for some $I \in \mathsf{J}$.*

Then the right derived functor $\mathrm{R}F : \mathsf{D}(\mathsf{M}) \to \mathsf{E}$ exists. Moreover, for any $I \in \mathsf{J}$ the morphism

$$\eta_I : F(I) \to (\mathrm{R}F \circ Q)(I)$$

in E is an isomorphism.

Sketch of Proof. (A complete proof will be given later.) Recall that $\mathsf{S} \subset \mathsf{K}(\mathsf{M})$ is the category (multiplicatively closed set of morphisms) consisting of the quasi-isomorphisms in $\mathsf{K}(\mathsf{M})$, and

$$Q : \mathsf{K}(\mathsf{M}) \to \mathsf{K}(\mathsf{M})_{\mathsf{S}} = \mathsf{D}(\mathsf{M})$$

is the localization functor.

Condition (ii) implies that $\mathsf{S} \cap \mathsf{J}$, the quasi-isomorphisms in J, is a left denominator set in J, and the inclusion

$$\mathsf{J}_{\mathsf{S}\cap\mathsf{J}} \to \mathsf{K}(\mathsf{M})_{\mathsf{S}} = \mathsf{D}(\mathsf{M}) \tag{11.3}$$

is an equivalence (of triangulated categories).

For every $M \in \mathsf{K}(\mathsf{M})$ we choose a quasi-isomorphism $\zeta_M : M \to I(M)$ with $I(M) \in \mathsf{J}$. We take care so that $I(M)$ and ζ_M commute with shifts, and that $I(M) = M$ and $\zeta_M = 1_M$ when $M \in \mathsf{J}$. In this way we obtain a triangulated functor

$$I : \mathsf{D}(\mathsf{M}) \to \mathsf{J}_{\mathsf{S}\cap\mathsf{J}}$$

which splits the inclusion (11.3), with a natural isomorphism $\zeta : \mathbf{1}_{\mathsf{D}(\mathsf{M})} \to I$.

We now invoke condition (ii). By the universal property of localization there is a unique functor

$$F_{\mathsf{S}\cap\mathsf{J}} : \mathsf{J}_{\mathsf{S}\cap\mathsf{J}} \to \mathsf{E}$$

extending $F|_{\mathsf{J}} : \mathsf{J} \to \mathsf{E}$. We define

$$\mathrm{R}F := F_{\mathsf{S}\cap\mathsf{J}} \circ I : \mathsf{D}(\mathsf{M}) \to \mathsf{E}.$$

For any $M \in \mathsf{K}(\mathsf{M})$ we define

$$\eta_M : F(M) \to (\mathrm{R}F \circ Q)(M) = F(I(M))$$

to be $\eta_M := F(\zeta_M)$. This is a natural transformation

$$\eta : F \to \mathrm{R}F \circ Q.$$

In a diagram (commutative via η):

$$\begin{tikzcd}
\mathsf{J} \arrow[r, "Q_{\mathsf{J}}"] \arrow[d, "\mathrm{inc}"'] & \mathsf{J}_{\mathsf{S}\cap\mathsf{J}} \arrow[dr, "F_{\mathsf{S}\cap\mathsf{J}}"] & \\
\mathsf{K}(\mathsf{M}) \arrow[r, "Q"] \arrow[r, phantom, shift right=4ex, "\Uparrow\eta"] \arrow[rr, bend right, "F"'] & \mathsf{D}(\mathsf{M}) \arrow[u, "I"'] \arrow[r, "\mathrm{R}F"] & \mathsf{E}
\end{tikzcd}$$

It remains to check that the pair $(\mathrm{R}F, \eta)$ has the universal property. □

11.3. **K-injectives.**

Definition 11.4. Let M be an abelian category. A complex $N \in \mathsf{K}(\mathsf{M})$ is called *acyclic* if $\mathrm{H}^i(N) = 0$ for all i. In other words, if the morphism $0 \to N$ is a quasi-isomorphism.

Definition 11.5. Let M be an abelian category.

(1) A complex $I \in \mathsf{K}(\mathsf{M})$ is called *K-injective* if for every acyclic $N \in \mathsf{K}(\mathsf{M})$, the complex $\mathrm{Hom}_{\mathsf{M}}(N, I)$ is also acyclic.
(2) Let $M \in \mathsf{K}(\mathsf{M})$. A *K-injective resolution* of M is a quasi-isomorphism $M \to I$ in $\mathsf{K}(\mathsf{M})$, where I is K-injective.
(3) We say that $\mathsf{K}(\mathsf{M})$ *has enough K-injectives* if every $M \in \mathsf{K}(\mathsf{M})$ has some K-injective resolution.

The concept of K-injective complex was introduced by Spaltenstein in 1988. At about the same time other authors (Keller, Bockstedt-Neeman, ...) discvered this concept independently, with other names (such as *homotopically injective complex*).

Example 11.6. Let M be either $\mathsf{Mod}\, A$, for some ring A, or $\mathsf{Mod}\, \mathcal{A}$, for some ringed space $(X, \mathcal{A})$. Then $\mathsf{K}(\mathsf{M})$ has enough K-injectives. We will prove this later. (?)

Example 11.7. Let M be an abelian category. Any bounded below complex of injectives is K-injective.

Now assume that M has enough injectives. Let $\mathsf{K}^+(\mathsf{M})$ be the category of bounded below complexes. Any $M \in \mathsf{K}^+(\mathsf{M})$ admits a quasi-isomorphism $M \to I$, with I a bounded below complex of injectives. This generalizes the "old-fashioned" injective resolution

$$0 \to M \to I^0 \to I^1 \to \cdots$$

for $M \in \mathsf{M}$. Thus $\mathsf{K}^+(\mathsf{M})$ has enough K-injectives.

These facts were already known in [RD], in 1966, but of course without the name "K-injective".

There are two wonderful things when $\mathsf{K}(\mathsf{M})$ has enough K-injectives.

Proposition 11.8. *Any quasi-isomorphism $\phi : I \to I'$ between K-injective complexes is a homotopy equivalence; i.e. it is an isomorphism in* $\mathsf{K}(\mathsf{M})$.

This will be proved later.

Let us denote by $\mathsf{K}(\mathsf{M})_{\mathrm{inj}}$ the full subcategory of $\mathsf{K}(\mathsf{M})$ on the K-injective complexes.

Corollary 11.9. *If* $\mathsf{K}(\mathsf{M})$ *has enough K-injectives, then any triangulated functor* $F : \mathsf{K}(\mathsf{M}) \to \mathsf{E}$ (*cf. Theorem* 11.2) *has a right derived functor.*

Proof. Take $\mathsf{J} := \mathsf{K}(\mathsf{M})_{\mathrm{inj}}$ in the theorem. □

Example 11.10.
[I forgot to do this on 2.4 !!!] ===
Suppose we are in the situation of Example 11.7. Let $\mathsf{D}^{+}(\mathsf{M}) := \mathsf{K}^{+}(\mathsf{M})_{\mathsf{S}^{+}}$, the localization of $\mathsf{K}^{+}(\mathsf{M})$ with respect to $\mathsf{S}^{+} := \mathsf{S} \cap \mathsf{K}^{+}(\mathsf{M})$. If $F : \mathsf{M} \to \mathsf{N}$ is an additive functor to some other abelian category N, then (by a variant of Corollary 11.9) we get a right derived functor

$$\mathrm{R}F : \mathsf{D}^{+}(\mathsf{M}) \to \mathsf{D}^{+}(\mathsf{N}).$$

For $M \in \mathsf{M}$ let $\mathrm{R}^{i}F(M) := \mathrm{H}^{i}(\mathrm{R}^{i}F(M))$. We get an additive functor

$$\mathrm{R}^{i}F : \mathsf{M} \to \mathsf{N},$$

which is the usual right derived functor. If F is left exact then the natural transformation $\eta : F \to \mathrm{R}^{0}F$ is an isomorphism.

Here is the second good thing.

Proposition 11.11. *The functor*

$$Q : \mathsf{K}(\mathsf{M})_{\mathrm{inj}} \to \mathsf{D}(\mathsf{M}) \tag{11.12}$$

is fully faithful.

Hence, if $\mathsf{K}(\mathsf{M})$ *has enough K-injectives, then* (11.12) *is an equivalence of triangulated categories.*

This will be proved later. The benefit here is that we can avoid the localization process (inverting the quasi-isomorphisms).

11.4. **Left derived functor.** This is the dual of Theorem 11.2. The proof is the same.

Theorem 11.13. *Let* M *be an abelian category,* E *a triangulated category, and* $F : \mathsf{K}(\mathsf{M}) \to \mathsf{E}$ *a triangulated functor. Assume there is a full triangulated subcategory* $\mathsf{P} \subset \mathsf{K}(\mathsf{M})$ *with these two properties:*

(i) *If* $\phi : P \to P'$ *is a quasi-isomorphism in* P, *then* $F(\phi) : F(P) \to F(P')$ *is an isomorphism in* E.
(ii) *Every* $M \in \mathsf{K}(\mathsf{M})$ *admits a quasi-isomorphism* $P \to M$ *for some* $P \in \mathsf{P}$.

Then the left derived functor $\mathrm{L}F : \mathsf{D}(\mathsf{M}) \to \mathsf{E}$ *exists. Moreover, for any* $P \in \mathsf{P}$ *the morphism*

$$\eta_{P} : (\mathrm{L}F \circ Q)(P) \to F(P)$$

in E *is an isomorphism.*

11.5. **K-projectives etc.**

Definition 11.14. Let M be an abelian category.

(1) A complex $P \in \mathsf{K}(\mathsf{M})$ is called *K-projective* if for every acyclic $N \in \mathsf{K}(\mathsf{M})$, the complex $\operatorname{Hom}_{\mathsf{M}}(P, N)$ is also acyclic.
(2) Let $M \in \mathsf{K}(\mathsf{M})$. A *K-projective resolution* of M is a quasi-isomorphism $P \to M$ in $\mathsf{K}(\mathsf{M})$, where P is K-projective.
(3) We say that $\mathsf{K}(\mathsf{M})$ *has enough K-projectives* if every $M \in \mathsf{K}(\mathsf{M})$ has some K-projective resolution.

=== [to here Monday 2.4.12]

Example 11.15. Let $\mathsf{M} := \mathsf{Mod}\, A$, for some ring A. Then $\mathsf{K}(\mathsf{M})$ has enough K-projectives. We will prove this later. (?)

Example 11.16. Let M be an abelian category. Any bounded above complex of projectives is K-projective.

Now assume that M has enough projectives. Let $\mathsf{K}^-(\mathsf{M})$ be the category of bounded above complexes. Any $M \in \mathsf{K}^-(\mathsf{M})$ admits a quasi-isomorphism $P \to M$, with M a bounded above complex of projectives. This generalizes the "old-fashioned" projective resolution

$$\cdots \to P^{-1} \to P^0 \to M \to 0$$

for $M \in \mathsf{M}$. Thus $\mathsf{K}^-(\mathsf{M})$ has enough K-projectives.

The dual versions of Proposition 11.8, Corollary 11.9, Proposition 11.11 and Example 11.10 hold.

Later we will also discuss *K-flat complexes* over a ring A. These are very useful for constructing certain left derived functors.

11.6. **Derived bifunctors.**

=== [this needs to be fixed!]

The most important functors for us are these:

$$\operatorname{Hom}_{\mathsf{M}}(-,-) : \mathsf{M}^{\mathrm{op}} \times \mathsf{M} \to \mathsf{Mod}\, \mathbb{K}, \tag{11.17}$$

where M is a $\mathbb{K}$-linear abelian category; and

$$- \otimes_A - : \mathsf{Mod}\, A^{\mathrm{op}} \times \mathsf{Mod}\, A \to \mathsf{Mod}\, \mathbb{K} \tag{11.18}$$

where A is a $\mathbb{K}$-algebra. These are *biadditive bifunctors*.

In general, given abelian categories M', M and N, and a biadditive bifunctor

$$F : \mathsf{M}' \times \mathsf{M} \to \mathsf{N},$$

there is an induced *bitriangulated bifunctor*

$$\mathsf{K}(F) : \mathsf{K}(\mathsf{M}') \times \mathsf{K}(\mathsf{M}) \to \mathsf{K}(\mathsf{N}).$$

A right derived bifunctor of F (or of $\mathsf{K}(F)$) is a bitriangulated bifunctor

$$\mathrm{R}F : \mathsf{D}(\mathsf{M}') \times \mathsf{D}(\mathsf{M}) \to \mathsf{D}(\mathsf{N}),$$

with a natural transformation (of bitriangulated bifunctors)

$$\eta : Q \circ \mathsf{K}(F) \to \mathrm{R}F \circ (Q \times Q),$$

that has the same universal property as in (???).

The left derived bifunctor $\mathrm{L}F$ is defined similarly.

11.7. **Deriving Bifunctors: Existence.**
[this needs to be fixed!] ===

Suppose we are given abelian categories M and N, a triangulated category E, and a bitriangulated bifunctor

$$F : \mathsf{K}(\mathsf{N}) \times \mathsf{K}(\mathsf{M}) \to \mathsf{E}.$$

We want to construct the derived functor

$$\mathrm{R}F : \mathsf{D}(\mathsf{N}) \times \mathsf{D}(\mathsf{M}) \to \mathsf{E}.$$

Theorem 11.19. *Suppose there is a full triangulated subcategory $\mathsf{J} \subset \mathsf{K}(\mathsf{M})$ with these two properties:*

(i) *If $\phi : I \to I'$ is a quasi-isomorphism in J, and $\psi : N \to N'$ is a quasi-isomorphism in $\mathsf{K}(\mathsf{N})$, then*

$$F(\psi, \phi) : F(N, I) \to F(N', I')$$

is an isomorphism in E.

(ii) *Every $M \in \mathsf{K}(\mathsf{M})$ admits a quasi-isomorphism $M \to I$ with $I \in \mathsf{J}$.*

Then the derived functor $\mathrm{R}F$ exists, and moreover

$$\eta_{N,I} : F(N, I) \to \mathrm{R}F(N, I)$$

is an isomorphism for any $I \in \mathsf{J}$.

Note the symmetry between M and N in this theorem.

Example 11.20. M is any abelian category, and $\mathsf{N} := \mathsf{M}^{\mathrm{op}}$. Consider the bifunctor

$$F : \mathsf{N} \times \mathsf{M} \to \mathsf{Ab},$$

$$F(N, M) := \mathrm{Hom}_{\mathsf{M}}(N, M).$$

Using the isomorphisms of triangulated categories $\mathsf{K}(\mathsf{N}) = \mathsf{K}(\mathsf{M}^{\mathrm{op}}) \cong \mathsf{K}(\mathsf{M})^{\mathrm{op}}$ and $\mathsf{D}(\mathsf{N}) \cong \mathsf{D}(\mathsf{M})^{\mathrm{op}}$, the derived functor $\mathrm{R}F$ is usually written as

$$\mathrm{RHom}_{\mathsf{M}} : \mathsf{D}(\mathsf{M})^{\mathrm{op}} \times \mathsf{D}(\mathsf{M}) \to \mathsf{D}(\mathsf{Ab}).$$

(If M happens to be a $\mathbb{K}$-linear category, for some commutative ring $\mathbb{K}$, then $\mathrm{RHom}_{\mathsf{M}}$ takes values in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$.)

In case $\mathsf{K}(\mathsf{M})$ has enough K-injectives, then the full subcategory $\mathsf{J} := \mathsf{K}(\mathsf{M})_{\mathrm{inj}} \subset \mathsf{K}(\mathsf{M})$ on the K-injectives satisfies properties (i-ii) of the theorem. If $\mathsf{K}(\mathsf{M})$ has enough K-projectives, then we can take the full subcategory $\mathsf{P} := \mathsf{K}(\mathsf{M})_{\mathrm{proj}} \subset \mathsf{K}(\mathsf{M})$ on the K-projectives. Then $\mathsf{P}^{\mathrm{op}} \subset \mathsf{K}(\mathsf{N})$ satisfies properties (i-ii) of the theorem.

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