lecture 14.3.12

(1)

Derived Categories

Spring 2011/12 A. Yekutieli

Introduction

By way of introduction to the subject, let's consider duality.

Let K be a field. Given a K-module M (i.e. a vector space), let

$$D(M) := \mathrm{Hom}_K(M, K)$$

the dual module. There is a canonical homomorphism

$$\eta_M : M \to D(D(M)),$$

namely $\eta(m)(\varphi) := \varphi(m)$ for $m \in M$ and $\varphi \in D(M)$. If M is finitely generated, then η_M is an isomorphism.
(Actually it's "iff".)

②

To formalize the above, let $\underline{\mathrm{Mod}}\,K$ denote the category of K-modules. Then

$$D : \underline{\mathrm{Mod}}\,K \to \underline{\mathrm{Mod}}\,K$$

is a contravariant functor, and

$$\eta : \mathbb{1} \to D \circ D$$

is a natural transformation. Here $\mathbb{1}$ denotes the identity functor of $\underline{\mathrm{Mod}}\,K$.

Now let's replace K by any commutative ring A. Again we can define a $\underset{\text{cont.var.}}{\widehat{\text{functor}}}$

$$D : \underline{\mathrm{Mod}}\,A \to \underline{\mathrm{Mod}}\,A$$

$$D(M) := \mathrm{Hom}_A(M, A)$$

and a natural trans.

$$\eta : \mathbb{1} \to D \circ D.$$

It is easy to see that $\eta_M : M \to (D \circ D)(M)$ is an isom. if M is a <u>fin. gen.</u> <u>free</u> A-module.

We can't expect to have <u>reflexivity</u> (i.e. η_M an isom.) if M is not fin. gen.

(3) But there are many fin. gen. mods that are not free ...

Let's restrict attention to the case $A = \mathbb{Z}$, that we know well. A fin. gen. $\mathbb{Z}$-module, i.e. a fin. gen. abelian group M, is of this form:

$$M \cong G \oplus H \quad , G \text{ free}, H \text{ finite}.$$

(This is not canonical; there is a canonical ex. seq $0 \to H \to M \to G \to 0$, and we choose a splitting.)

For the free ab. group G there is reflexivity: $\eta_G : G \to D(D(G))$ is $\cong$.

But $D(H) = \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Z}) = 0$.

So this duality is <u>useless</u> for <u>finite ab. grps.</u>!

However, for such H we can define

$$D'(H) := \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{Q}/\mathbb{Z}).$$

(We can think of $\mathbb{Q}/\mathbb{Z}$ as the group of roots of unity in $\mathbb{C}$, via the exp map.)

There is a duality here too: the can. hom.

$$\eta'_H : H \to D'(D'(H))$$

is an isom.

If G is a free fin. gen. ab. grp., then $D'(D'(G)) \cong \hat{G}$, the profinite completion of G, so the duality D' is useless for free groups!

Later in the course we will introduce the derived category $\underline{D}(\underline{\mathrm{Mod}}\,\mathbb{Z})$, and the contravariant triangulated functor

$$D : \underline{D}(\underline{\mathrm{Mod}}\,\mathbb{Z}) \to \underline{D}(\underline{\mathrm{Mod}}\,\mathbb{Z})$$

$$D(M) := R\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$$

"right derived Hom". There will be a nat. trans. of triangulated functors

$$\eta : \mathbb{1} \to D \circ D.$$

The objects of $\underline{D}(\underline{\mathrm{Mod}}\,\mathbb{Z})$ are the complexes of $\mathbb{Z}$-modules.

If M is a bounded complex with fin. gen. cohomology modules, then

$$\eta_M : M \to D(D(M))$$

is an isom. in $\underline{D}(\mathrm{Mod}\, \mathbb{Z})$.

In particular, if M is a fin. gen. $\mathbb{Z}$-module, then we can think of it as a complex as follows:

$$(*) \qquad (\cdots \to 0 \to \underset{\substack{\uparrow \\ \text{deg. } 0}}{M} \to 0 \to \cdots)$$

So η_M is an isom.

(Formally, there is an embedding

$$\mathrm{Mod}\, \mathbb{Z} \hookrightarrow \underline{D}(\mathrm{Mod}\, \mathbb{Z})$$

by $(*)$.)

This sort of duality holds for many noetherian comm. rings A. But instead of $R\mathrm{Hom}_A(-, A)$, the duality is this:

$$D(M) = R\mathrm{Hom}_A(M, R)$$

where R is a dualizing complex.

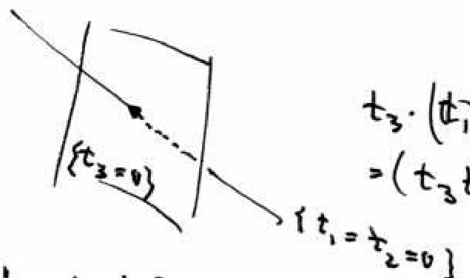
The du. cplx. R is unique (up to known variations).

If A is a regular ring then $R = A$ is dualizing; if A is a Cohen-Macaulay ring then R is a single A-module; but if A is more singular, then R must live in several degrees.

Example.

$X \subset \mathbb{A}^3_{\mathbb{R}}$

line $\cup$ plane



$t_3 \cdot ((t_1) + (t_2)) = (t_3 t_1, t_3 t_2)$

$X = \operatorname{Spec} A$

$A = \mathbb{R}[t_1, t_2, t_3] / (t_3 t_1, t_3 t_2)$

The dualizing cplx. here lives in two adjacent degrees.

One can also talk about dualizing complexes over noncommutative rings. We may get to learn about this.

7

Categories

In this course we will not try to be precise about set theory. The blanket assumption is that we are given a universe $\underline{U}$ (in the sense of Grothendieck). This is an infinite set, closed under many set theoretical operations.

A small set (or $\underline{U}$-small set) a set $S \in \underline{U}$.

A category $\underline{C}$ is a $\underline{U}$-category if the object set $Ob(\underline{C}) \subset \underline{U}$, and the morphism sets

$$\mathrm{Hom}_{\underline{C}}(C, D) \in \underline{U} \quad \text{for all } C, D \in Ob(\underline{C}).$$

(see [KS, §1.1], or [Neeman]. for another approach.)

We denote by $\underline{Set}$ the category of all small sets. So $Ob(\underline{Set}) = \underline{U}$, and $\underline{Set}$ is a $\underline{U}$-category.

For a (small) ring A, we denote by $\underline{Mod}\, A$ the category of all small left A-modules M. An abelian group M is small if its underlying set is a small set. etc.

And so on... By default we only work with $\underline{U}$-categories

(8)

Let $\underline{C}$ be a category. A morphism $f: C \to D$ in $\underline{C}$ is an *epimorphism* if for any $g, g': D \to E$, $g \circ f = g' \circ f$ implies $g = g'$.

$$C \xrightarrow{f} D \overset{g}{\underset{g'}{\rightrightarrows}} E$$

$f: C \to D$ is a *monomorphism* if for any $g, g': E \to C$, $f \circ g = f \circ g'$ implies $g = g'$.

$$E \overset{g}{\underset{g'}{\rightrightarrows}} C \xrightarrow{f} D$$

Exa. In *Set*, epimorphisms are surjections, and monomorphisms are injections. Same for *Mod A*.

Rem. Let $\underline{C}' \subset \underline{C}$ be a full subcategory, and let $f: C \to D$ be a morphism in $\underline{C}'$. The property of f being an "epimorphism" or a "monomorphism" could change if we check it in $\underline{C}$ or in $\underline{C}'$!
(Find an example!)

An *initial* object $C_0 \in Ob(\underline{C})$ is such that for any $C \in Ob(\underline{C})$ there is exactly one morphism $C_0 \to C$; i.e. $Hom_{\underline{C}}(C_0, C)$ is a singleton.

(9) An object $C_\infty \in \underline{C}$ is a <u>terminal</u> object if $\mathrm{Hom}_{\underline{C}}(C, C_\infty)$ is a singleton $\forall\, C \in \underline{C}$.

Initial & terminal objects are unique, up to unique isoms. (They may not exist!)

A <u>zero</u> object $0 \in \underline{C}$ is an object that is both initial & terminal.

<u>Exs</u> In <u>Set</u>, $\emptyset$ is initial, and any singleton is terminal. There is no zero object.

<u>Exs</u>. Let $\underline{Ab} := \underline{Mod}\ \mathbb{Z}$, i.e. the category of small abelian groups. The group $0 := \{0\}$ is a zero object in $\underline{Ab}$. Same for $\underline{Mod}\ A$.

< 14.3