Notes Mon 26.3

<u>Def</u>. An object $I \in \underline{M}$ is called <u>injective</u> if for any diagram:

$$\begin{tikzcd} M' \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & M \\ I & \end{tikzcd}$$

with φ a <u>monomorphism</u>, there is a morphism $\tilde{\gamma} : M \to I$ s.t. $\gamma = \tilde{\gamma} \circ \varphi$.

$$\begin{tikzcd} M' \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & M \arrow[dl, dotted, "\tilde{\gamma}"] \\ I & \end{tikzcd}$$

<u>Prop</u>. TFAE for $I \in \underline{M}$:

(i) I is injective

(ii) Any ex. seq. $0 \to I \to M \to M'' \to 0$ splits.

(iii) The functor $\operatorname{Hom}_{\underline{M}}(-, I) : \underline{M}^{op} \to \underline{Ab}$ is exact.

<u>Pf</u>. <u>Exer</u>.

For injective modules over noeth. comm. rings see [RD, pp. 120-122] [RD = Residues & Duality]

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<u>Prop</u>. Let $f : A \to B$ be a ring hom, and I an injective in <u>Mod</u> A (= left A-mods). Then

$$J := \operatorname{Hom}_A(B, I)$$

is injective in <u>Mod</u> B.

(30) <u>proof</u> Note that B is a left A-mod. via f, and a right B-module; this makes J into a left B-module. In a formula: for $\varphi \in J$,

$(b\varphi)(b') = \varphi(b'b)$; $b, b' \in B$. Check that $b\varphi \in J$:

$(b\varphi)(ab') = \varphi(ab'b) = a \cdot \varphi(b'b) = a \cdot (b\varphi)(b')$

it's indeed A-linear!

Now for $N \in \underline{\text{Mod}}\, B$ we have an isom.

$$\psi: \quad \text{Hom}_B(N, J) = \text{Hom}_B(N, \text{Hom}_A(B, I))$$
$$\cong \text{Hom}_A(N, I)$$

which is natural (in N). So $\text{Hom}_B(-, J)$ is exact

$\Rightarrow J$ injective Δ

Thm (Baer criterion) Let A be a ring and I a left A-module. I is injective iff $\forall$ left ~~[illegible]~~ ideal $\mathfrak{a} \subset A$ and any $\gamma : \mathfrak{a} \to I$, γ extends to $\tilde{\gamma} : A \to I$. <u>Idea of pf</u>: clever use of Zorn's Lemma.

<u>Thm....</u> $\underline{\text{Mod}}\, A$ has <u>enough</u> inj<u>ectives</u> ; i.e. any $M \in \underline{\text{Mod}}\, A$ can be embedded (= $\exists$ monom) in an injective A-module.

<u>Proof</u>. <u>Step 1</u>. If $\{I_x\}_{x \in X}$ is a collection of injective A-modules, then $I := \prod_{x \in X} I_x$ is also injective. This is immediate from def.

Step 2. For $A = \mathbb{Z}$ the module $I = \mathbb{Q}/\mathbb{Z}$ is injective. This is almost immediate from the Baer criterion(*): enough to check for

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "n"] \arrow[d, "\gamma"'] & \mathbb{Z} \arrow[dl, dashed, "\tilde{\gamma}"] \\ \mathbb{Q}/\mathbb{Z} & \end{tikzcd}$$

$n \neq 0$.

if $\gamma(1) = r + \mathbb{Z}$
take $\tilde{\gamma}(1) = \frac{r}{n} + \mathbb{Z}$.

(*) Baer: enough to check $M' = K \hookrightarrow M = A$
left ideal

Step 3. For $A = \mathbb{Z}$ take any ab. grp M. For $0 \neq m \in M$ let $M' := \mathbb{Z} \cdot m \subset M$, so $M' \cong \mathbb{Z}/n_m\mathbb{Z}$ for $n_m > 0$. Take $\gamma_m : M' \to \mathbb{Q}/\mathbb{Z}$, $\gamma'(m) := \frac{1}{n_m} + \mathbb{Z}$. This extends to $\tilde{\gamma}_m : M \to \mathbb{Q}/\mathbb{Z}$; $\tilde{\gamma}_m(m) \neq 0$.

Take $I := \prod_m (\mathbb{Q}/\mathbb{Z})$, $\underset{\prod_m \tilde{\gamma}_m}{\overset{\shortparallel}{\varphi}} : M \to I$.

Step 4. A any ring, $M \in \underline{\mathrm{Mod}}\, A$. Choose $\varphi : M \hookrightarrow I$, I inj. $\mathbb{Z}$-module. By φ get $\psi : M \to J := \mathrm{Hom}_{\mathbb{Z}}(A, I)$. Since

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[rr, bend right, "\varphi"'] & J \arrow[r] & I \end{tikzcd}$$

$\Rightarrow \psi$ is injection. $\square$

Exa. Let $\underline{N} := \{\text{torsion ab. grps}\}$
$\underline{M} := \{\text{finite ab. grps}\}$

so $\underline{M} \subset \underline{N} \subset \underline{Ab}$, full abelian subcats.

$\underline{M}$ has no projectives nor injectives (but 0

$\underline{N}$ has no projectives (but 0).

$\underline{N}$ has enough injectives, because:

Prop. If A is left noetherian, then any direct sum of injectives is injective.

proof. Exercise Hint: use Baer criterion

* * *

Prop. Let $(X, \mathcal{A})$ be a ringed scheme. Then $\underline{\text{Mod}}\, \mathcal{A}$ has enough injectives.

Proof. Take a left $\mathcal{A}$-module $\mathcal{M}$. For any $x \in X$ there is an injective $\mathcal{A}_x$-module $\mathcal{I}_x$, and an embedding $\mathcal{M}_x \overset{\varphi_x}{\hookrightarrow} \mathcal{I}_x$. ($\mathcal{M}_x$ is the stalk.) There is an ind. hom. $\varphi_x : \mathcal{M} \to g_{x*}(\mathcal{I}_x)$, where $g_x : \{x\} \to X$ is the inclusion.

Define $\mathcal{I} := \prod_{x \in X} g_{x*}(\mathcal{I}_x)$. Then $\varphi : \mathcal{M} \to \mathcal{I}$ is injective (a monom.), & $\mathcal{I}$ is an injective $\mathcal{A}$-mod. $\square$

Flat Modules

This section relates only to ab. cats. like $\underline{\mathrm{Mod}}\, A$ (or $\underline{\mathrm{Mod}}\, \mathcal{A}$).

<u>Def</u>. A right A-module M is <u>flat</u> if for any injection $N' \hookrightarrow N$ of left A-modules, the hom $M \otimes_A N' \to M \otimes_A N$ is injective.

I.e: the functor $M \otimes_A - : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Ab}}$ is exact.

∠

Since a projective module is flat, we see that $\underline{\mathrm{Mod}}\, A^{op}$ (and also $\underline{\mathrm{Mod}}\, A$) has enough flat modules.

Outline of Derived Cat's

Here is where we are going. [Later I will give full defs & proofs.]

Let $\underline{M}$ be an additive category.

A complex of objects of $\underline{M}$, or a complex in $\underline{M}$, is a diagram

$$M = \left(\cdots \to M^0 \xrightarrow{d^0} M^1 \xrightarrow{d^1} M^2 \to \cdots \right)$$

in $\underline{M}$, s.t. $d^{i+1} \circ d^i = 0$. (*)

For complexes M & N, a morphism $\varphi : M \to N$ is a collection of morphisms $\{\varphi^i : M^i \to N^i\}_{i \in \mathbb{Z}} = \varphi$ in $\underline{M}$ s.t.

$$d^i_N \circ \varphi^i = \varphi^{i+1} \circ d^i_M .$$

In other words there is a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & M^{-1} \arrow[r, "d"] \arrow[d, "\varphi"'] & M^0 \arrow[r] \arrow[d, "\varphi"'] & \cdots \\
\cdots \arrow[r] & N^{-1} \arrow[r, "d"] & N^0 \arrow[r] & \cdots
\end{tikzcd}$$

(*) We sometimes write d^i_M; or just d, if i & M are clear. d is called the differential, or the coboundary op.

We denote by $\underline{C}(\underline{M})$ the category of complexes in $\underline{M}$. This is an additive category: direct sums, kernels & cokernels are degree-wise. E.g.

$$\mathrm{Ker}(\varphi : M \to N) = (\cdots \to K^0 \to K^1 \to \cdots)$$

where $K^i := \mathrm{Ker}(\varphi^i : M^i \to N^i) \in \underline{M}$.

are degree-wise. If $\underline{M}$ is abelian then so is $\underline{C}(\underline{M})$.

Suppose $\underline{P} \subset \underline{M}$ is a full additive subcat. Then $\underline{C}(\underline{P})$, the cat. of complexes in $\underline{P}$, is a full additive subcat. of $\underline{C}(\underline{M})$.

<u>Example</u>. A is a comm. ring, $\underline{M} = \underline{\mathrm{Mod}}\, A$, $\underline{P} :=$ {projective A-modules}. Then $\underline{C}(\underline{M})$ is abelian, and $\underline{C}(\underline{P})$ is additive.

* * *

Consider an additive category $\underline{M}$.
Let $M, N \in \underline{C}(\underline{M})$. We define a complex of abelian groups

$$\mathrm{Hom}_{\underline{M}}(M, N) \in \underline{C}(\underline{Ab})$$

as follows. In degree i we let

$$\boxed{\mathrm{Hom}_{\underline{M}}(M,N)^i := \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\underline{M}}(M^j, N^{i+j}) \in \underline{Ab}.}$$

Thus $\varphi \in \mathrm{Hom}_{\underline{M}}(M,N)^i$ is a collection $\varphi = \{\varphi^j\}_{j\in\mathbb{Z}}$

where

$$\begin{tikzcd}
 & \vdots \\
\vdots & N^i \\
M^0 \arrow[ur, "\varphi^0"] & N^{i-1} \\
M^{-1} \arrow[ur, "\varphi^{-1}"] & N^{i-2} \\
M^{-2} \arrow[ur, "\varphi^{-2}"] & \vdots \\
\vdots &
\end{tikzcd}$$

slope is i.

The differential is "ad(d)", i.e. commutator with d: for $\varphi \in \mathrm{Hom}_{\underline{M}}(M,N)^i$,

$$\boxed{d(\varphi) := d_N \circ \varphi - (-1)^i \varphi \circ d_M.}$$

It is easy to check that $d \circ d = 0$.

A hom. of complexes $\varphi : M \to N$, i.e. φ a morphism in $\underline{C}(\underline{M})$, is precisely $\varphi \in \mathrm{Hom}_{\underline{M}}(M,N)^0$ s.t. $d(\varphi) = 0$.

Thus:

$$\mathrm{Hom}_{\underline{C}(\underline{M})}(M,N) = Z^0\left(\mathrm{Hom}_{\underline{M}}(M,N)\right)$$

the ab. grp. of 0-cocycles

Rem. The cat. $\underline{C}(\underline{M})$ is thus an additive DG category. We might say more on this later

The Homotopy Category.

Let $\underline{M}$ be an additive category

A morphism $\varphi : M \to N$ in $\underline{C}(\underline{M})$ is called <u>null-homotopic</u> if it is a 0-coboundary in $\mathrm{Hom}_{\underline{M}}(M,N)$; ie.

$$\varphi \in B^0(\mathrm{Hom}_{\underline{M}}(M,N));$$

ie. $\varphi = d(\psi)$, for some $\psi \in \mathrm{Hom}_{\underline{M}}(M,N)^{-1}$.

Morphisms $\varphi, \varphi' : M \to N$ are called <u>homotopic</u> if $\varphi - \varphi'$ is null-homotopic. Denote this relation by $\varphi \sim \varphi'$.

$\varphi : M \to N$ is a homotopy equiv. if $\exists\ \psi : N \to M$ s.t. $\psi \circ \varphi \sim 1_M$ & $\varphi \circ \psi \sim 1_N$.

<u>Prop.</u>

The null-homotopic morphisms in $\underline{C}(\underline{M})$ form a 2-sided ideal. I.e. if $\varphi : M \to N$ is null-hom, and $\psi : M' \to M$, $\tau : N \to N'$ are arbitrary morphisms in $\underline{C}(\underline{M})$, then $\varphi \circ \psi$ & $\tau \circ \varphi$ are null-hom.

<u>proof.</u> Say $\varphi = d(\alpha)$, $\alpha \in \mathrm{Hom}_{\underline{M}}(M,N)^{-1}$. Then

$$\begin{aligned} d(\psi \circ \alpha) &= d(\psi) \circ \alpha + \psi \circ d(\alpha) \\ &= \psi \circ \varphi \end{aligned}$$

[Leibniz rule holds !!]

so $\psi \circ \varphi$ null. hom. $[d(\psi) = 0]$

Likewise for $\tau \circ \varphi$.

$\square$

Since the null-homotopic hom's form a 2-sided ideal in the additive category $\underline{C}(\underline{M})$, we can pass to the quotient category; and this quot. cat. is also additive. (Just like in ring theory.)

Def. The quotient of $\underline{C}(\underline{M})$ modulo the null-homotopic homomorphisms is called the homotopy category, and it is denoted by $\underline{K}(\underline{M})$.

* * *

The objects of $\underline{K}(\underline{M})$ are the same as those of $\underline{C}(\underline{M})$; namely the complexes in $\underline{M}$. And

$$\mathrm{Hom}_{\underline{K}(\underline{M})}(M,N) = \frac{\mathrm{Hom}_{\underline{C}(\underline{M})}(M,N)}{\{\text{null-homotopic homs}\}}$$

$$= \frac{Z^0\,\mathrm{Hom}(\;)}{B^0\,\mathrm{Hom}()} = H^0\,\mathrm{Hom}_{\underline{M}}(M,N).$$

The is an additive functor

$$\underline{C}(\underline{M}) \longrightarrow \underline{K}(\underline{M})$$

which is the identity on objects, and surjective on Hom groups.

Note the $\varphi : M \to N$ in $\underline{C}(\underline{M})$ becomes an isom. in $\underline{K}(\underline{M})$ iff it is a hom. equiv.