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The homotopy cat. $\underline{K}(\underline{M})$ is not abelian. It is a triangulated category. A full definition of this will be given next time (?). later! Let me just say a bit. ...

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$\underline{C}(\underline{M})$ has an additive automorphism T, which is called the shift (or translation, or suspension). For an object

$$M = (\cdots \to M^0 \xrightarrow{d} M^1 \to \cdots) \in \underline{C}(\underline{M})$$

it's shift is:

$$T(M) := (\cdots \longrightarrow \underset{\substack{\uparrow \\ i+1}}{M^i} \xrightarrow{-d} \underset{\substack{\uparrow \\ i+2}}{M^{i+1}} \longrightarrow \cdots)$$

I.e. $\boxed{T(M)^i = M^{i+1}}$ and the diff. is $\boxed{d_{T(M)} := -d_M.}$

We often write $M[k] := T^k(M)$, $k \in \mathbb{Z}$.

on morphisms: for $\varphi : M \to N$ in $\underline{C}(\underline{M})$, $T(\varphi) : T(M) \to T(N)$ is the same hom. on the components: $T(\varphi)^i = \varphi^{i+1} : M^{i+1} \to N^{i+1}$.

The autom. $T : \underline{C}(\underline{M}) \circlearrowleft$ induces an auto. $T : \underline{K}(\underline{M}) \circlearrowleft$.

(40) A *triangle* in $\underline{K}(\underline{M})$ is a diagram

$$(*) \quad L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} L[1]$$

in $\underline{K}(\underline{M})$. Among the triangles there is a subset of *distinguished triangles*. These are analogues of short exact sequences; but they are not so obvious.

Def. Suppose $\underline{K}$ & $\underline{L}$ are two triangulated categories (e.g. $\underline{K} = \underline{K}(\underline{M})$ & $\underline{L} = \underline{K}(\underline{N})$ for additive cats. $\underline{M}$ & $\underline{N}$). A *triangulated functor*

$$F : \underline{K} \to \underline{L}$$

is an additive functor, that sends distinguished triangles to distinguished tri.

[together ~~is~~ with a natural isom

$$\xi_F : T_{\underline{L}} \circ F \xrightarrow{\simeq} F \circ T_{\underline{M}}$$
]

Namely given a dist. tri. (*) in $\underline{M}$, the tri.

$$F(L) \xrightarrow{F(\alpha)} F(M) \xrightarrow{F(\beta)} F(N) \xrightarrow{F(\gamma)} \underset{\substack{\xi \; \wr\wr \\ (F(L))[1]}}{F(L[1])}$$

is a dist. tri. in $\underline{L}$.

(41) "triang. functor" is the analogue of exact functor between abelian categories.

Now $\underline{M}$ is abelian.

Def. A morphism $\varphi: M \to N$ in $\underline{C}(M)$ is called a quasi-isomorphism if the induced morphisms $H^i(\varphi): H^i(M) \to H^i(N)$ in $\underline{M}$ are all isomorphisms. We write qisom

Prop. Suppose $\varphi, \psi: M \to N$ in $\underline{C}(\underline{M})$ are homotopic ($\varphi \sim \psi$). Then $H^i(\varphi) = H^i(\psi)$, as morphism $H^i(M) \to H^i(N)$ in $\underline{M}$.

Pf Exercise.

Due to the prop, any morphism $\varphi \in \mathrm{Hom}_{\underline{K}(\underline{M})}(M, N)$

induces well-defined morphisms
$$H^i(\varphi): H^i(M) \to H^i(N) \quad \text{in } \underline{M}.$$
So it makes sense to ask if φ is a q.isom.

Clearly the composition of q-isoms is also a q-isom. Also $1_M: M \to M$ is a q-isom.

(42) Thus we get a subcategory

$$\underline{S} \subset \underline{K}(\underline{M})$$

with same objects (the complexes), where the morphisms are the q-isoms.

We will prove that $\underline{K}(\underline{M})$ can be localized with respect to $\underline{S}$. This is essentially the same proof as the one in ring theory: if A is a (nc) ring, and $S \subset A$ is a denominator set, then the classical ring of fractions A_S exists.

Here the localization is

$$\boxed{\underline{D}(\underline{M}) := \underline{K}(\underline{M})_{\underline{S}}},$$

the derived category. $\underline{D}(\underline{M})$ is a triang. cat; there is a triang. functor

$$Q : \underline{K}(\underline{M}) \to \underline{D}(\underline{M}),$$

which is the identity on objects; any $\varphi : M \to N$ in $\underline{S}$ (i.e. a q-isom in $\underline{K}(\underline{M})$) becomes an isom $Q(\varphi) : M \to N$ in $\underline{D}(\underline{M})$.

The pair $(\underline{D}(\underline{M}), Q)$ is universal for this:

(13) given a tri. cat. $\underline{L}$ and a tri. functor $F : \underline{K}(\underline{M}) \to \underline{L}$ s.t. $F(\varphi)$ is an isom. $\forall \varphi \in \underline{S}$, then $\exists$ tri. funct.

$$F_{\underline{S}} : \underline{D}(\underline{M}) \to \underline{L},$$

and a unique isom. of tri. func's

$$\eta : F \xrightarrow{\sim} F_{\underline{S}} \circ Q$$

from $\underline{K}(\underline{M})$ to $\underline{L}$.

$$\begin{tikzcd}
\underline{S} \subset \underline{K}(\underline{M}) \arrow[rr, "F"] \arrow[dr, "Q"'] \arrow[drr, phantom, "\eta \Downarrow \cong"] & & \underline{L} \\
& \underline{D}(\underline{M}) \arrow[ur, "F_{\underline{S}}"'] &
\end{tikzcd}$$

Def. Let $F, G : \underline{K} \to \underline{L}$ be tri. fun's bet. tri. cat's. Recall that F is equipped with isom $\xi_F : T \circ F \xrightarrow{\sim} F \circ T$ (shift isom) Likewise there is ξ_G. A natural trans. of tri. funs. is a nat. trans. $\eta : F \to G$ s.t. diag. is comm:

$$\begin{tikzcd}
T \circ F \arrow[r, "\xi_F"] \arrow[d, "1_T \circ \eta"'] & F \circ T \arrow[d, "\eta \circ 1_T"] \\
T \circ G \arrow[r, "\xi_G"'] & G \circ T
\end{tikzcd}$$

<u>Example</u>. Let $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ be an ex. seq. in $\underline{M}$. Then $\exists$ morphism $\gamma : N \to L[1]$ in $\underline{D}(\underline{M})$, and the triangle

(*)
$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} L[1]$$

is disting.

Moreover: We will show that the functor $\underline{M} \to \underline{D}(\underline{M})$ sending $M \in \underline{M}$ to the complex $(\dots \to 0 \to M \to 0 \to \cdots)$, is fully faithful. Thus $\underline{M} \subset \underline{D}(\underline{M})$, a full subcat. The ex. seq. in $\underline{M}$ can be recovered as the dist. tri. (*).

Outline of Derived Functors

Let $\underline{M}$ & $\underline{N}$ be abelian categories, and let $F: \underline{M} \to \underline{N}$ be an additive functor. The functor F extends to a functor

$$\underline{C}(F): \underline{C}(\underline{M}) \longrightarrow \underline{C}(\underline{N})$$

like this: given a complex $M = (\to M^i \xrightarrow{d} M^{i+1} \to \cdots)$ we let

$$\underline{C}(F)(M) := \left(\to F(M^i) \xrightarrow{F(d)} F(M^{i+1}) \to \cdots\right)$$

$\in \underline{C}(\underline{N})$. Likewise for morphisms $\varphi: M \to N$ in $\underline{C}(\underline{M})$, get $\underline{C}(F)(\varphi): \underline{C}(F)(M) \to \underline{C}(F)(N)$.

If $\varphi: M \to N$ is null-homotopic then so is $\underline{C}(F)(\varphi): \underline{C}(F)(M) \to \underline{C}(F)(N)$. (Exercise?). Thus we get an induced functor

$$\underline{K}(F): \underline{K}(\underline{M}) \longrightarrow \underline{K}(\underline{N}).$$

This turns out to be a trian. functor (automatically). We want to derive $\underline{K}(F)$.

Changing notation, we consider a trian. funct. $F : \underline{K}(\underline{M}) \to \underline{L}$, where $\underline{L}$ is some tri. cat. (possibly $\underline{L} = \underline{D}(\underline{N})$, $\underline{N}$ abelian).

<u>Def.</u> Let $\underline{M}$ be an abelian cat, $\underline{L}$ a triang. cat., and $F : \underline{K}(\underline{M}) \to \underline{L}$ a triang. funct. A <u>right derived</u> functor of F is a trian. funct.

$$RF : \underline{D}(\underline{M}) \to \underline{L},$$

with a nat. trans. of trian. functors

$$\eta : F \to RF \circ Q$$

(from $\underline{K}(\underline{M})$ to $\underline{L}$),

$$\begin{tikzcd}
\underline{K}(M) \arrow[r, "F"] \arrow[d, "Q"'] \arrow[dr, phantom, "\searrow \eta"] & \underline{L} \\
\underline{D}(\underline{M}) \arrow[ur, "RF"] \arrow[ur, bend right=60, "G"'] \arrow[ur, phantom, bend right=30, "\downarrow \zeta"] &
\end{tikzcd}$$

with the following universal property:

(*) Given any trian. funct. $G : \underline{D}(\underline{M}) \to \underline{L}$ and nat. tr. of tri. fun. $\eta' : F \to G \circ Q$, there's a unique nat. tr. (of --) $\zeta : RF \to G$ s.t. $\eta' = \zeta \circ \eta$.

<u>Prop.</u> A right der. functor RF is unique, up to a unique nat. isom. (of tri. fun.). Pf.: This is trivial!

Similarly:

<u>Def.</u> Let $\underline{M}$ be an abelian cat, $\underline{L}$ a triang. cat, and $F : \underline{K}(\underline{M}) \to \underline{L}$ a tri. fun. A <u>left derived functor</u> of F is a tri. fun.

$$LF : \underline{D}(\underline{M}) \longrightarrow \underline{L}$$

with a nat. trans. of tri. fun.

$$\eta : LF \circ Q \longrightarrow F$$

from $\underline{K}(\underline{M})$ to $\underline{L}$, with this universal property:

(*) given any tri. fun.

$$G : \underline{D}(\underline{M}) \longrightarrow \underline{L}$$

and nat. tr. $\eta' : G \circ Q \to F$, there's a unique nat. tr. $\zeta : G \to LF$ s.t. $\eta' = \eta \circ \zeta$.

$$\begin{tikzcd}
\underline{K}(\underline{M}) \arrow[r, "F"] \arrow[d, "Q"'] \arrow[dr, phantom, "\Uparrow \eta \quad \Uparrow \eta'"] & \underline{L} \\
\underline{D}(\underline{M}) \arrow[ur, "LF"] \arrow[ur, bend right=60, "G"'] \arrow[ur, phantom, bend right=30, "\Uparrow \zeta"] &
\end{tikzcd}$$

<u>Prop.</u> A left derived functor LF is unique, up to a unique nat. isom. of tri. func. (same proof).

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