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The homotopy cat. $\underline{K}(\underline{M})$ is not abelian.
 It is a triangulated category. A full definition of this will be given next time (?).

Let me just say a bit. ... later!

$\underline{C}(\underline{M})$ has an additive automorphism T , which is called the shift (or translation, or suspension). For an object

$$M = (\dots \rightarrow M^0 \xrightarrow{d} M^1 \rightarrow \dots) \in \underline{C}(\underline{M})$$

it's shift is:

$$T(M) := (\dots \rightarrow M^i \xrightarrow{-d} M^{i+1} \rightarrow \dots)$$

\uparrow
 $i+1$

\uparrow
 $i+2$

I.e. $T(M)^i = M^{i+1}$ and the diff. is $d_{T(M)} := -d_M$.

We often write $M[k] := T^k(M)$, $k \in \mathbb{Z}$.

on morphisms: For $\varphi: M \rightarrow N$ in $\underline{C}(\underline{M})$, $T(\varphi): T(M) \rightarrow T(N)$ is the same hom. on the components: $T(\varphi)^i = \varphi^{i+1}: M^{i+1} \rightarrow N^{i+1}$.

The autom. $T: \underline{C}(\underline{M}) \rightarrow \underline{C}(\underline{M})$ induces an auto. $T: \underline{K}(\underline{M}) \rightarrow \underline{K}(\underline{M})$.

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A triangle in $\underline{K}(\underline{M})$ is a diagram

$$(*) \quad \underline{L} \xrightarrow{\alpha} \underline{M} \xrightarrow{\beta} \underline{N} \xrightarrow{\gamma} \underline{L}[1]$$

in $\underline{K}(\underline{M})$. Among the triangles there is a subset of distinguished triangles.

These are analogues of short exact sequences; but they are not so obvious.

Def

Suppose \underline{K} & \underline{L} are two triangulated categories (e.g. $\underline{K} = \underline{K}(\underline{M})$ & $\underline{L} = \underline{K}(\underline{N})$ for additive cets. \underline{M} & \underline{N}). A triangulated functor

$$F: \underline{K} \rightarrow \underline{L}$$

is an additive functor that sends distinguished triangles to distinguished tri.

together with a natural isom

$$\xi_F: T_{\underline{L}} \circ F \xrightarrow{\cong} F \circ T_{\underline{M}}$$

Namely given a dist. tri. $(*)$ in \underline{M} , the tri.

$$F(\underline{L}) \xrightarrow{F(\alpha)} F(\underline{M}) \xrightarrow{F(\beta)} F(\underline{N}) \xrightarrow{F(\gamma)} F(\underline{L}[1])$$

is a dist. tri. in \underline{L} .

\cong
 $(F(\underline{L}))[1]$

(41) "triang. functor" is the analogue of exact functor between abelian categories.

Now \underline{M} is abelian. ∞

Def. A morphism $\varphi: M \rightarrow N$ in $\underline{C}(\underline{M})$ is called a quasi-isomorphism if the induced morphisms $H^i(\varphi): H^i(M) \rightarrow H^i(N)$ in \underline{M} are all isomorphisms. We write q-isom

Prop. Suppose $\varphi, \psi: M \rightarrow N$ in $\underline{C}(\underline{M})$ are homotopic ($\varphi \sim \psi$). Then $H^i(\varphi) = H^i(\psi)$, as morphism $H^i(M) \rightarrow H^i(N)$ in \underline{M} .

Ex Exercise.

Due to the prop, any morphism $\varphi \in \text{Hom}_{\underline{C}(\underline{M})}(M, N)$

induces well-defined morphisms $H^i(\varphi): H^i(M) \rightarrow H^i(N)$ in \underline{M} .

So it makes sense to ask if φ is a q-isom.

Clearly the composition of q-isoms is also a q-isom. Also $1_M: M \rightarrow M$ is a q-isom.

(42) Thus we get a subcategory

$$\underline{S} \subset \underline{K}(M)$$

with same objects (the complexes), where the morphisms are the q -isoms.

We will prove that $\underline{K}(M)$ can be localized with respect to \underline{S} . This is essentially the same proof as the one in ring theory:

if A is a (nc) ring, and $S \subset A$ is a denominator set, then the classical ring of fractions A_S exists.

Hence the localization is

$$\underline{D}(M) := \underline{K}(M)_{\underline{S}}$$

the derived category. $\underline{D}(M)$ is a triang. cat; there is a triang. functor

$$Q: \underline{K}(M) \rightarrow \underline{D}(M)$$

which is the identity on objects;

any $\varphi: M \rightarrow N$ in \underline{S} (i.e. a q -isom in $\underline{K}(M)$)

becomes an isom

$$Q(\varphi): M \rightarrow N \text{ in } \underline{D}(M).$$

The pair $(\underline{D}(M), Q)$ is universal for this:

(13) given a tri. cat. \underline{L} and a

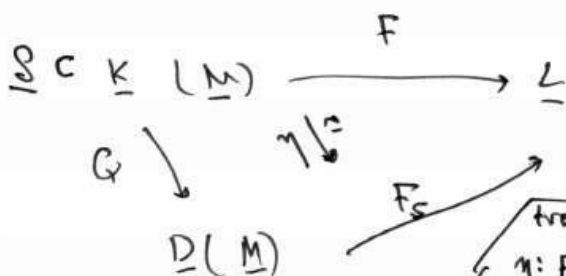
tri. functor $F: \underline{K}(M) \rightarrow \underline{L}$ s.t.

$F(\varphi)$ is an isom. $\forall \varphi \in \underline{K}(M)$, then \exists tri. functor

$F_{\underline{L}}: \underline{D}(M) \rightarrow \underline{L}$,
and a unique isom. of tri. functors

$$\eta: F \xrightarrow{\cong} F_{\underline{L}} \circ Q$$

from $\underline{K}(M)$ to \underline{L} .



Def.
Let $F, G: \underline{K} \rightarrow \underline{L}$ be tri. functors bet. tri. cats. Recall that F is equipped with isom. $\xi_F: T \circ F \xrightarrow{\cong} F \circ T$ (shift isom.) Likewise there is ξ_G . A natural trans. of tri. functors is a nat. trans. $\eta: F \rightarrow G$ s.t. diag. is comm.:

$$\begin{array}{ccc} T \circ F & \xrightarrow{\xi_F} & F \circ T \\ \downarrow \eta \circ T & & \downarrow \eta \circ T \\ T \circ G & \xrightarrow{\xi_G} & G \circ T \end{array}$$

Example. Let $0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0$

be an ex. seq. in \underline{M} . Then \exists morphism $\gamma: N \rightarrow L[1]$ in $\underline{D}(M)$, and the triangle

$$(*) \quad L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} L[1]$$

is dist. tri.

Moreover:

We will show that the functor $\underline{M} \rightarrow \underline{D}(M)$ sending $M \in \underline{M}$ to the complex $(\dots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \dots)$, is fully faithful. Thus $\underline{M} \subset \underline{D}(M)$, a full subcat. The ex. seq. in \underline{M} can be recovered as the dist. tri. (*).

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Outline of Derived Functors

Let \underline{M} & \underline{N} be abelian categories, and
let $F: \underline{M} \rightarrow \underline{N}$ be an additive functor.
The functor F extends to a functor

$$\underline{C}(F): \underline{C}(\underline{M}) \rightarrow \underline{C}(\underline{N})$$

like this: given a complex $M = (\cdots \rightarrow M^i \xrightarrow{d^i} M^{i+1} \rightarrow \cdots)$
we let

$$\underline{C}(F)(M) := (\cdots \rightarrow F(M^i) \xrightarrow{F(d^i)} F(M^{i+1}) \rightarrow \cdots)$$

$\in \underline{C}(\underline{N})$. Likewise for morphisms $\varphi: M \rightarrow N$
in $\underline{C}(\underline{M})$, get $\underline{C}(F)(\varphi): \underline{C}(F)(M) \rightarrow \underline{C}(F)(N)$.

If $\varphi: M \rightarrow N$ is null-homotopic then so
is $\underline{C}(F)(\varphi): \underline{C}(F)(M) \rightarrow \underline{C}(F)(N)$.

(Exercise?). Thus we get an induced functor

$$\underline{K}(F): \underline{K}(\underline{M}) \rightarrow \underline{K}(\underline{N}).$$

This turns out to be a trian. functor (automatically).

We want to derive $\underline{K}(F)$.



(45) Changing notation, we consider a triang. funct.

$$F: \underline{K}(\underline{M}) \rightarrow \underline{L},$$

where \underline{L} is some tri. cat. (possibly $\underline{L} = \underline{D}(\underline{N})$, \underline{N} abelian).

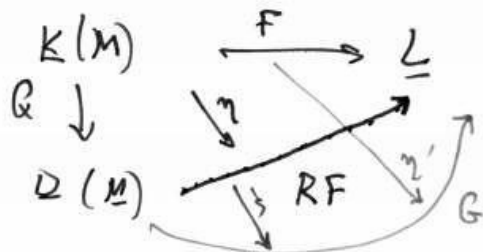
Def. Let \underline{M} be an abelian cat, \underline{L} a triang. cat, and $F: \underline{K}(\underline{M}) \rightarrow \underline{L}$ a triang. funct. A right derived functor of F is a triang. funct.

$$RF: \underline{D}(\underline{M}) \rightarrow \underline{L},$$

with a nat. trans. of triang. functors

$$\eta: F \rightarrow RF \circ Q$$

(from $\underline{K}(\underline{M})$ to \underline{L}),



With the following universal property:

(*) Given any triang. funct. $G: \underline{D}(\underline{M}) \rightarrow \underline{L}$ and nat. tr. of tri. fun. $\gamma': F \rightarrow G \circ Q$, there's a

unique nat. tr. $\gamma: RF \rightarrow G$ st. $\gamma' = \gamma \circ \eta$.

Prop. A right der. functor RF is unique, up to a unique nat. isom. (of tri. fun.). \square : This is trivial!

(46) Similarly:

Def Let \underline{M} be an abelian cat, \underline{L} a triang. cat, and $F: \underline{K}(\underline{M}) \rightarrow \underline{L}$ a tri. fun. A left derived functor of F is a tri. fun.

$$LF: \underline{D}(\underline{M}) \rightarrow \underline{L}$$

with a nat. trans. of tri. fun.

$$\eta: LF \circ Q \rightarrow F$$

from $\underline{K}(\underline{M})$ to \underline{L} , with this universal property:
 (*) given any tri. fun.

$$G: \underline{D}(\underline{M}) \rightarrow \underline{L}$$

and nat. tr. $\eta': G \circ Q \rightarrow F$, there's a unique nat. tr. $\zeta: G \rightarrow LF$ st. $\eta' = \eta \circ \zeta$.

$$\begin{array}{ccc}
 \underline{K}(\underline{M}) & \xrightarrow{F} & \underline{L} \\
 \downarrow Q & \nearrow \eta & \nearrow \eta' \\
 \underline{D}(\underline{M}) & \xrightarrow{LF} & \underline{L} \\
 & \nearrow \zeta & \nearrow G
 \end{array}$$

Prop. A left derived functor LF is unique, up to a unique nat. isom. of tri. fun.

(same proof)

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