(10)

Products & Coproducts

Let $\underline{M}$ be a category.
Recall the concept of product: for a collection $\{M_i\}_{i\in I}$ of objects of $\underline{M}$, the product $\prod_{i\in I} M_i$, if it exists, is an object $M \in \underline{M}$, with morphisms

$$p_i : M \to M_i, \quad \text{called projections,}$$

s.t. for any $N \in \underline{M}$ and $\{\varphi_i : N \to M_i\}$ there is a unique $\varphi: N \to M$ s.t. $\varphi_i = p_i \circ \varphi$

$$\begin{tikzcd} N \arrow[r, "\exists!\,\varphi"] \arrow[dr, "\varphi_i"'] & M \arrow[d, "p_i"] \\ & M_i \end{tikzcd}$$

(In a nutshell: the function (of sets)

$$\mathrm{Hom}_{\underline{M}}(N, M) \to \prod_{i\in I} \mathrm{Hom}_{\underline{M}}(N, M_i)$$
$$\varphi \mapsto \{p_i \circ \varphi\}_{i\in I}$$

must be bijective, $\forall\ N \in \underline{M}$.)

We know that the pair $(M, \{p_i\})$ is unique up to a unique isom (if it exists).

<u>Exe</u>. In <u>Set</u> and in <u>Mod</u> A all products (indexes by $I \in \underline{U}$) exist: they are the usual cartesian products.

Similarly we have the notion of <u>coproduct</u> in $\underline{M}$. A product $\coprod_{i \in I} M_i$ of a collection $\{M_i\}_{i \in I}$ of objects of $\underline{M}$ is an object M, with morphisms $e_i : M_i \to M$, called the embeddings. The property is that for any $(N, \{\varphi_i : M_i \to N\})$ there is a unique $\varphi : M \to N$ s.t. $\varphi_i = \varphi \circ e_i$.

$$\begin{tikzcd} M_i \arrow[d, "e_i"'] \arrow[dr, "\varphi_i"] & \\ M \arrow[r, "\exists! \varphi"] & N \end{tikzcd}$$

The pair $(M, \{e_i\})$ is unique (up to unique $\cong$).

<u>Exe</u>. In <u>Set</u> the coproduct is the disjoint union. In <u>Mod</u> A the coproduct is the direct sum.

(12) Additive Categories

Def. A category $\underline{M}$ is called additive if it satisfies these conditions:

(i) For any $M, N \in \underline{M}$, the set $\mathrm{Hom}_{\underline{M}}(M, N)$ has a structure of an abelian group.

(ii) The compositions

$$\mathrm{Hom}_{\underline{M}}(M_0, M_1) \times \mathrm{Hom}_{\underline{M}}(M_1, M_2) \xrightarrow{\circ} \mathrm{Hom}_{\underline{M}}(M_0, M_2)$$

are bilinear functions.

(iii) There is a zero object $\overset{0}{\mathrm{v}}$ in $\underline{M}$.

(iv) $\underline{M}$ has finite coproducts.

Observe:

- $\mathrm{Hom}_{\underline{M}}(M, N) \neq \emptyset$, since it's a group
- $\mathrm{Hom}_{\underline{M}}(M, 0) = 0$, the zero ab. grp, since it has only one element. Likewise $\mathrm{Hom}_{\underline{M}}(0, M) = 0$. We denote these arrows by 0.
- The coproduct in $\underline{M}$ is denoted by $\oplus$, and is also called a direct sum.

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<u>Prop</u>. Let $\underline{M}$ be an additive category. Let $\{M_i\}_{i\in I}$ be a finite collection of objects in $\underline{M}$, and let $M := \bigoplus_{i\in I} M_i$ be the coproduct, with embeddings $e_i : M_i \to M$.

1) For any i let $p_i : M \to M_i$ be the unique morphism s.t. $p_i \circ e_i = \mathbb{1}_{M_i}$ (the id. morphism of M_i), and $p_i \circ e_j = 0$ for $j \neq i$.
Then $(M, \{p_i\})$ is a product of $\{M_i\}$.

2) $\sum_{i\in I} e_i \circ p_i = \mathbb{1}_M$.

<u>proof</u>. <u>Exercise</u>.

<u>Exa</u>. Let A be a ring. The category $\underline{\mathrm{Mod}}\, A$ is additive.
Let $\underline{M} \subset \underline{\mathrm{Mod}}\, A$ be the full subcat. consisting of free modules. Then $\underline{M}$ is also additive.

(13.1) About Rings as Linear Categories

(1) Let A be a ring. Define a category

$\underline{M}$ like this: there is one ~~op~~ object

$P \in \underline{M}$, and $\mathrm{Hom}_{\underline{M}}(P) := A$.

This is a linear category. Does it have finite direct sums?

Note that $\underline{M}$ is the full ~~su~~ subcat. of $\underline{\mathrm{Mod}}\,A^{op}$ (right A-mods) on the object $P := A \in \underline{\mathrm{Mod}}\,A^{op}$, the free right module of rank 1:

$$\mathrm{Hom}_{\underline{\mathrm{Mod}}\,A^{op}}(P, P) = A$$

acting by left mult. on $P = A$.

2 13.2

Suppose we have $\underline{N} \cong \underline{P} \oplus \underline{P}$ in $\underline{M}$. Then

$$\underline{N} = \underline{P}$$

(since unique object) so

$$\underline{P} \cong \underline{P} \oplus \underline{P} \quad \text{in } \underline{M}.$$

So $\exists$ $e_i, p_i : \underline{P} \to \underline{P}$, $i = 1,2$

s.t. $$1_{\underline{P}} = e_1 \circ p_1 + e_2 \circ p_2$$

Now $p_i \circ e_i = 1_{\underline{P}}$ and $e_i \circ p_j = 0$ for $i \neq j$. All this in the ring A.

This implies ~~the same~~ that

$$\underline{P} \cong \underline{P} \oplus \underline{P} \quad \text{also in } \mathrm{Mod}\, A.$$

Hence A can't be commutative nor right noetherian

(3) (13.3)

However we can take

$A := \mathrm{End}_K(V)$, where K is a field, & V is a countable rank K-module. Then $A \cong A \oplus A$ as right A-modules. So $\underline{M}$ has finite, and even countable $\oplus$.

(14)

Abelian Categories

Let $\underline{M}$ be an additive category.

<u>Def.</u> Let $f: M \to N$ be a morph. in $\underline{M}$.

i) A <u>Kernel</u> of f is a morphism $g: K \to N$ s.t. $f \circ g = 0$, and for any $g': K' \to M$ in $\underline{M}$ s.t. $f \circ g' = 0$ there is a unique $h: K' \to K$ s.t. $g' = g \circ h$.

$$\begin{tikzcd}
K' \arrow[rr, bend left, "g'"] \arrow[r, "\exists! h"] \arrow[rr, bend right=60] & K \arrow[r, "g"] \arrow[d] & M \arrow[r, "f"] & N \\
& & 0 \arrow[ur] &
\end{tikzcd}$$

<u>Prop.</u> 1) If a kernel $(g: K \to M) = Ker(f)$ exists, then it is unique (up to a unique isom.)

2) The kernel $g: K \to M$ is a monomorphism.

<u>Pf.</u> <u>Exercise.</u>

<u>Exa.</u> In <u>Mod</u> A, kernels exists:

$$Ker(f) = (K \xrightarrow{g} M), \quad K := \{ m \in M \mid f(m) = 0 \}$$

g is the inclusion

<u>notation</u>

We usually write $K = ker(f)$ and leave g implicit.

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Def. Let $f: M \to N$ be a morphism in $\underline{M}$.
A __cokernel__ of f is a morphism $g: N \to C$ s.t. $g \circ f = 0$, and for any $g': N \to C'$ s.t. $g' \circ f = 0$, there exists a unique $h: C \to C'$ s.t. $g' = h \circ g$.

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[dr] & N \arrow[r, "g"] \arrow[rr, bend left, "g'"] & C \arrow[r, dotted, "\exists! h"] & C' \\
& 0 \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

Prop. (1) If a cokernel $(g: N \to C) =: \mathrm{Coker}(f)$ exists, then it's unique (up to ...)
(2) The cokernel $\mathrm{Coker}(f)$ is an epimorphism.

Pf Exercise.

Example In $\underline{\mathrm{Mod}}\, A$ we take $C := N/\bar{M}$, $g: N \to C$ can. proj. where $\bar{M} := \{f(m) \mid m \in M\}$.

Def. Assume that $\underline{M}$ has kernels and cokernels.
(1). For $f: M \to N$ let $\mathrm{Im}(f) := \mathrm{Ker}(\mathrm{Coker}(f))$.
I.e. if $\mathrm{coker}(f) = (c: N \to C)$, then $\mathrm{Im}(f) = \mathrm{Ker}(c)$.
(2) Let $\mathrm{Coim}(f) := \mathrm{Coker}(\mathrm{Ker}(f))$.
I.e. if $(k: K \to M) = \mathrm{Ker}(f)$, then $\mathrm{Coim}(f) = \mathrm{Coker}(k)$.

In a diagram we get:

$$
\begin{tikzcd}
K \arrow[r, "k"] \arrow[dr, "0"'] & M \arrow[r, "f"] \arrow[d, "\alpha"] \arrow[dr, "\gamma"] \arrow[rr, bend left, "0"] & N \arrow[r, "c"] & C \\
& M' \arrow[r, "f'"] & N' \arrow[u, "\beta"] \arrow[ur, "0"'] &
\end{tikzcd}
$$

$\mathrm{Coker}(k) = \mathrm{Coim}(f)$ $\quad$ $\mathrm{Im}(f) = \mathrm{Ker}(c)$

now:

$c \circ f = 0 \Rightarrow \exists!\ \gamma : M \to N'$.

Now $\beta \circ \gamma \circ k = f \circ k = 0$; β mono., $\Rightarrow$
$\gamma \circ k = 0 \Rightarrow \exists!\ f' : M' \to N'$
s.t. diagram comm.

We get induced morphism

$(*)$ $\qquad f' : \mathrm{Coim}(f) \to \mathrm{Im}(f)$.

<u>Def</u>. An <u>abelian</u> <u>category</u> is an additive category $\underline{M}$ with these additional properties:

(i) All morphisms in $\underline{M}$ admit <u>kernels</u> and <u>cokernels</u>.

(ii) For any $f : M \to N$, the morphism
$\mathrm{Coim}(f) \to \mathrm{Im}(f)$ above $(*)$ is an <u>isomorphism</u>

I.e.:

$(\heartsuit)$ $\quad \mathrm{Coker}(\mathrm{Ker}(f)) \cong \mathrm{Ker}(\mathrm{Coker}(f))$.

<u>Rem.</u> From now we forget "coimage".

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Exe. Let A be a ring. The category $\underline{Mod}\, A$ is abelian.

Consider the full subcat. $\underline{M} \subset \underline{Mod}\, A$ consisting of free modules. Is $\underline{M}$ abelian? Usually not.

Exercise. Take $A = \mathbb{Z}$, so $\underline{Mod}\, A = \underline{Ab}$. Prove that the additive category $\underline{M} = \{\text{free } \mathbb{Z}\text{-modules}\}$ is not abelian. (Hint: it has kernels & cokernels, but (♡) fails.)

Here is a delicate issue:

Def. Let $\underline{M}$ be an abelian category and let $\underline{N}$ be a full subcategory. We say that $\underline{N}$ is a full abelian subcat. of $\underline{M}$ if $\underline{N}$ is closed under direct sums, kernels and cokernels.

Rem. This means that $\underline{N}$ is an ab. cat, and the inclusion $\underline{N} \to \underline{M}$ is an exact functor (later).

Exa. The cat. of finite abelian groups is a full abelian subcat. of Ab.

Exa. A is left noetherian ring iff the full cat. $\underline{\mathrm{Mod}}_f A$ of fin. gen left. modules is a full abelian subcat. of $\underline{\mathrm{Mod}}\, A$.

Exa. Let $(X, \mathcal{A})$ be a ringed space. So X is a topological space, and $\mathcal{A}$ is a shef of rings on X.

Let $\underline{\mathrm{PMod}}\, \mathcal{A}$ be the category of presheaves of $\mathcal{A}$-modules on $\mathcal{A}$. This is an abelian category. For $f: \mathcal{M} \to \mathcal{N}$ the p. shef $\mathcal{K} = \mathrm{Ker}_{\mathrm{pr}}(f)$ is:

$$\Gamma(U, \mathcal{K}) = \mathrm{Ker}\big(f: \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})\big), \quad U \subset X \text{ open.}$$

And $\mathcal{C} = \mathrm{Coker}(f)$ is

$$\Gamma(U, \mathcal{C}) = \mathrm{Coker}(\quad " \quad).$$

Let $\underline{\mathrm{Mod}}\, \mathcal{A}$ be the full subcat of $\underline{\mathrm{PMod}}\, \mathcal{A}$ consisting of sheaves. This is also an abelian category, but with different cokernels.
So it's not a full abelian subcat.

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$$\mathrm{Coker}_{\underline{\mathrm{Mod}}\,A}(f) = \left(\mathrm{Coker}_{\underline{\mathrm{PMod}}\,A}(f)\right)^{\sim}$$ ¶?

the associated sheaf.

more precisely:
- $\underline{\mathrm{Mod}}\,A$ is closed under kernels
- not closed under cokernels
- has its own intrinsic cokernels ¶.

For educational purposes we state:

<u>Thm</u> (Freyd & Mitchell). Let $\underline{M}$ be a small abelian category (i.e. $\mathrm{Ob}(\underline{M}) \in \underline{U}$). Then $\underline{M}$ is equivalent to a full abelian subcat. of $\underline{\mathrm{Mod}}\,A$, for some ring A.

What does this help? It means that for most of what we do, we can ! pretend that $\underline{M} \subset \underline{\mathrm{Mod}}\,A$, a full abelian subcat.

<u>Prop.</u> Let $\underline{M}$ be an additive category. Then $\underline{M}^{op}$, the opposite category, is also additive (with same ab. grp. str. on

$$\mathrm{Hom}_{\underline{M}^{op}}(M, N) = \mathrm{Hom}_{\underline{M}}(N, M).$$ zero = zero.

<u>pf.</u> Bilinearity of comp. in $\underline{M}^{op}$ is clear. Existence of finite coproducts in $\underline{M}^{op}$ is because $\underline{M}$ has finite products. $\triangle$

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Prop Let $\underline{M}$ be an abelian category. Then $\underline{M}^{op}$ is also abelian.

pf. We know that $\underline{M}^{op}$ is additive. It has kernels & cokernels:

$$\mathrm{Ker}_{\underline{M}^{op}}(f) = \mathrm{Coker}_{\underline{M}}(f) \quad \text{etc.}$$

Also the symmetric axiom (♡) holds. □

Prop Let $\underline{M}$ be an abelian category, and let $f : M \to N$ be a morphism in $\underline{M}$.

(1) f is a monomorphism iff $\mathrm{Ker}(f) \cong 0$

(2) f is an epimorphism iff $\mathrm{Coker}(f) \cong 0$

(3) f is an isom. iff it's both an epim. & a monom.

Pf. Exercise.

Additive Functors

<u>Def</u>. Let $\underline{M}$ and $\underline{N}$ be additive categories. A functor $F : \underline{M} \to \underline{N}$ is called an <u>additive</u> functor if for any $M_1, M_2 \in \underline{M}$ the function

$$F : \quad Hom_{\underline{M}}(M_1, M_2) \to Hom_{\underline{N}}(F(M_1), F(M_2))$$

is a hom. of abelian groups.

<u>Exe</u>. A a ring, The forgetful functor
$F : \underline{Mod}\, A \to \underline{Ab}$ is additive

go to 21.1

<u>Def</u>. Let $\underline{M}$ and $\underline{N}$ be abelian categories. An additive functor $F : \underline{M} \to \underline{N}$ is:

(1) <u>Left exact</u> if it commutes with <u>kernels</u>.

I.e. $\quad Ker_{\underline{N}}(F(\varphi)) = F(Ker_{\underline{M}}(\varphi))$

for any $\varphi : M_1 \to M_2$ in $\underline{M}$.

to P. 22

21.1

__Prop__ Let $\underline{M}$ & $\underline{N}$ be additive categories, and let $F: \underline{M} \to \underline{N}$ be additive functors. Then $F(0_{\underline{M}}) \cong 0_{\underline{N}}$.

proof. The zero object $0_{\underline{N}} \in \underline{N}$ is characterized by $1 = 0$ in $\mathrm{Hom}_{\underline{N}}(0_{\underline{N}}, 0_{\underline{N}})$. Now let $N := F(0_{\underline{M}}) \in \underline{N}$.

So

$$F: \mathrm{Hom}_{\underline{M}}(0_{\underline{M}}, 0_{\underline{M}}) \to \mathrm{Hom}_{\underline{N}}(N, N)$$

sends $F(0) = 0$; since additive

and $F(1) = 1$; any functor.

Since $1 = 0$ in $\mathrm{Hom}_{\underline{M}}(0_{\underline{M}}, 0_{\underline{M}})$, the same holds for N.

So $N \cong 0_{\underline{N}}$. $\square$

another proof: for any $M \in \underline{M}$, $\mathrm{End}_{\underline{M}}(M)$ is a ring, and $\mathrm{Hom}_{\underline{M}}(M_0, M_1)$ is a bimodule over $\mathrm{End}(M_1)$-$\mathrm{End}(M_0)$.

Thus $M \cong 0$ iff $\mathrm{End}_{\underline{M}}(M)$ is the <u>zero ring</u>.

Now $F: \mathrm{End}_{\underline{M}}(M) \to \mathrm{End}_{\underline{N}}(F(M))$ is a ring hom.

21.2

<u>Prop.</u> Let $\underline{M}$ & $\underline{N}$ be additive categories, and let $F: \underline{M} \to \underline{N}$ be an additive functor. Then F commutes with finite coproducts. Namely if $\{M_i\}_{i \in I}$ is a finite collection of objects of $\underline{M}$, and $(M, \{e_i\})$ is its coproduct, then $(F(M), \{F(e_i)\})$ is a coproduct of $\{F(M_i)\}$.

Pf Exercise.

to p. 21

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(2) <u>Right exact</u> if it commutes with <u>cokernels</u>. I.e.

$$\mathrm{Coker}_{\underline{N}}(F(\varphi)) = F(\mathrm{Coker}_{\underline{M}}(\varphi)).$$

(3) <u>Exact</u> if it is both left & right exact.

$$\begin{tikzcd}
\overset{K}{\mathrm{Ker}_{\underline{M}}(\varphi)} \arrow[r, "k"] & M \arrow[r, "\varphi"] & N \arrow[r, "c"] & \overset{C}{\mathrm{Coker}_{\underline{M}}(\varphi)}
\end{tikzcd}$$

$F \Big\downarrow$

$$\begin{tikzcd}
F(K) \arrow[r, "F(k)"] \arrow[dr, dashed, "\cong"'] & F(M) \arrow[r, "F(\varphi)"] & F(N) \arrow[r, "F(c)"] \arrow[dr] & F(C) \\
& \mathrm{Ker}_{\underline{N}}(F(\varphi)) \arrow[u] & & \mathrm{Coker}_{\underline{N}}(F(\varphi)) \arrow[u, dashed, "\cong"']
\end{tikzcd}$$

<u>Def</u> An <u>exact</u> <u>sequence</u> in an ab. cat. $\underline{M}$ is a diagram

$$\cdots M_0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} M_2 \rightarrow \cdots$$

(finite or infinite on either side)

s.t. $\mathrm{Ker}(\varphi_i) = \mathrm{Im}(\varphi_{i-1})$ for all i (s.t. φ_i & φ_{i-1} are defined).

A <u>short</u> ex. seq. is an ex. seq.

(*) $$0 \to M_0 \to M_1 \to M_2 \to 0$$

(23)

<u>Prop</u> Let $F : \underline{M} \to \underline{N}$ be an additive functor between abelian categories. The functor F is left exact ; resp. right exact; resp. exact iff for every exact seq.

$$0 \to M_0 \to M_1 \to M_2 \to 0$$

in $\underline{M}$, the seq.

$$0 \to F(M_0) \to F(M_1) \to F(M_2) \quad ;$$

resp.

$$F(M_0) \to F(M_1) \to F(M_2) \to 0 \quad ;$$

resp

$$0 \to F(M_0) \to F(M_1) \to F(M_2) \to 0$$

is exact.

<u>proof</u>. <u>Exercise</u>.
(Hint: $M_0 \cong \mathrm{Ker}(M_1 \to M_2)$, etc.)

<u>Example</u>. Let A be a comm. ring, and $M \in \underline{\mathrm{Mod}}\, A$.

- Define $F : \underline{\mathrm{Mod}}\, A \circlearrowleft$, $F(N) := M \otimes_A N$. This is right exact.
- $G : (\underline{\mathrm{Mod}}\, A)^{op} \to \underline{\mathrm{Mod}}\, A$, $G(N) := \mathrm{Hom}_A(N, M)$
- $H : \underline{\mathrm{Mod}}\, A \circlearrowleft$, $H(N) := \mathrm{Hom}_A(M, N)$.

These are left exact.

(24)

Recall that a functor $F : \underline{C} \to \underline{D}$ is an <u>equivalence</u> if $\exists$ $G : \underline{D} \to \underline{C}$, and natural isom's $G \circ F \cong 1\!\!1_{\underline{C}}$ and $F \circ G \cong 1\!\!1_{\underline{D}}$.
The functor G is called a <u>quasi-inverse</u> of F. ok

<u>Prop</u>. Let $\underline{M}$ and $\underline{N}$ be abelian categories, (* p25) and let $F : \underline{M} \to \underline{N}$ be an additive functor.
If F is an equivalence, then it is exact.

<u>Proof</u> We will prove that F respects kernels. The proof for cokernels is similar.
Take $f : M_0 \to M_1$ with $K = \operatorname{Ker}(f)$.

$$\begin{tikzcd}
 & M \arrow[dl, "\exists!\, h"'] \arrow[d, "g"] \arrow[dr, bend left, "0"] & \\
K \arrow[r, "k"] & M_0 \arrow[r, "f"] & M_1
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{M} \arrow[d, "F"] \\
\underline{N}
\end{tikzcd}$$

$$\begin{tikzcd}
 & F(M) = N \arrow[dl, "\exists!\, F(h)"'] \arrow[d, "\bar{g} = F(g)"] \arrow[dr, bend left, "0"] & \\
F(K) \arrow[r, "F(k)"] & F(M_0) \arrow[r, "F(f)"] & F(M_1)
\end{tikzcd}$$

Say $\bar{g} : N \to F(M_0)$ in $\underline{N}$ s.t. $F(f) \circ \bar{g} = 0$. Since F is essentially surjective on objects, $N \cong F(M)$ for some M.
So we may assume $N = F(M)$. (replace $\bar{g} : N \to F(M_0)$ with $F(M) \xrightarrow{\cong} N \xrightarrow{\bar{g}} F($
Next $F : \operatorname{Hom}_{\underline{M}}(M, M_0) \to \operatorname{Hom}_{\underline{N}}(F(M), F(M_0))$ is an isom. of ab. grps ;

(25) So $\exists!$ $g: M \to M_0$ s.t. $F(g) = \bar{g}$.

And $f \circ g = 0$. Since $k = \ker(f)$,

$\exists!$ $h: M \to K$ s.t. $k \circ h = g$.

So $\bar{h} := F(h): N \to F(K)$ is unique

s.t. $F(k) \circ \bar{h} = \bar{g}$. $\square$

(*) We know that $F: \underline{C} \to \underline{D}$ is an equivalence iff these conditions hold:

(i) F is essentially surjective on objects (i.e. surjective on isom. classes of objects).

(ii) F is fully faithful (i.e. bijective on Hom sets)

<u>Prop.</u> Let $F: \underline{M} \to \underline{N}$ be an additive functor between additive categories. TFAE:

(i) F has a quasi-inverse.

(ii) F has an <u>additive</u> quasi-inverse.

<u>pf.</u> <u>Exercise</u>.

Splitting, Projectives & Injectives

Let $\underline{M}$ be an abelian category.
An exact sequence

$$(*) \qquad \begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "\varphi"] & M \arrow[r, "\psi"] \arrow[l, bend right, "\alpha"'] & M'' \arrow[r] \arrow[l, bend right, "\beta"'] & 0 \end{tikzcd}$$

in $\underline{M}$ is called <u>split</u> if it's isomorphic to the exact sequence

$$0 \to M' \xrightarrow{e'} M' \oplus M'' \xrightarrow{p''} M'' \to 0 ,$$

where e' is the embedding, and p'' is the projection (recall that $\oplus M_i \cong \prod M_i$).
Namely there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "\varphi"] \arrow[d, "="'] & M \arrow[r, "\psi"] \arrow[d, "\cong"', "\gamma"] & M'' \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & M' \arrow[r, "e'"] & M' \oplus M'' \arrow[r, "p''"] & M'' \arrow[r] & 0 \end{tikzcd}$$

We know that finding γ is equivalent to finding $\beta : M'' \to M$ s.t. $\psi \circ \beta = 1_{M''}$; and to finding $\alpha : M \to M'$ s.t. $\alpha \circ \varphi = 1_{M'}$ (splitting ψ & φ resp.).

<u>Exe</u> The exact seq. (in $\underline{Ab}$)

$$0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \to \mathbb{Z}/(2) \to 0$$

is not split

<u>Def.</u> An object $\underline{P} \in \underline{M}$ is called <u>projective</u> if for any diagram $M \xrightarrow{\Psi} M'' \xleftarrow{\gamma} \underline{P}$ s.t. Ψ is an <u>epimorphism</u> there exists $\tilde{\gamma}: \underline{P} \to M$ s.t. $\gamma = \Psi \circ \tilde{\gamma}$.

$$\begin{tikzcd} & P \arrow[d, "\gamma"] \arrow[dl, dashed, "\exists \tilde{\gamma}"'] \\ M \arrow[r, "\Psi"] & M'' \end{tikzcd}$$

<u>Prop</u> TFAE for $\underline{P} \in \underline{M}$;

(i) $\underline{P}$ is projective.

(ii) any exact seq. $0 \to M' \to M \to \underline{P} \to 0$ splits.

(iii) The functor $\mathrm{Hom}_{\underline{M}}(\underline{P}, -) : \underline{M} \to \underline{Ab}$ is exact.

<u>proof</u> Exer.

<u>Example</u> Let A be a ring. An A-module $\underline{P}$ (left) is projective iff it is a direct summand of a free module. Any (left) A-module is a quotient of a projective A-module.

<u>Def.</u> We say that $\underline{M}$ <u>has enough projectives</u> if any $M \in \underline{M}$ admits an epimorphism $\underline{P} \to M$, with $\underline{P}$ projective (i.e. say "M is a quotient of $\underline{P}$")

(28)

Exercise. Let $\underline{M} \subset \underline{Ab}$ be the category of finite abelian groups. We know it is a full ab. subcat of $\underline{Ab} = \underline{Mod}\ \mathbb{Z}$. Show that the only proj. object in $\underline{M}$ is 0.

$\Rightarrow$ not enough proj's.

Example. Take $X := \mathbb{P}^1_{\mathbb{k}}$, the projective line over a field $\mathbb{k}$ (can assume $\mathbb{k}$ alg. closed). Consider the abelian category $\underline{Coh}\ \mathcal{O}_X$ of coherent $\mathcal{O}_X$-modules. It's abelian ✓.
I claim that the only projective object in $\underline{Coh}\ \mathcal{O}_X$ is 0. This is quite involved! Let me just show why $\mathcal{O}_X$ is not projective ...
Let t_0, t_1 be the homogeneous coordinates. They generate $\mathcal{O}_X(1)$, and there is an exact seq.

$$0 \to \mathcal{O}_X(-1) \xrightarrow{[t_0\ -t_1]} \mathcal{O}_X \oplus \mathcal{O}_X \xrightarrow{\begin{bmatrix} t_0 \\ t_1 \end{bmatrix}} \mathcal{O}_X(1) \to 0$$

in $\underline{Coh}\ \mathcal{O}_X$. Twisting we get

$$0 \to \mathcal{O}_X(-2) \xrightarrow{\varphi} \mathcal{O}_X(-1)^2 \xrightarrow{\psi} \mathcal{O}_X \to 0$$

which is exact, but not split!

$\Rightarrow$ not enough proj's!

Def. An object $I \in \underline{M}$ is called <u>injective</u> if for any diagram:

$$\begin{tikzcd} M' \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & M \\ I & \end{tikzcd}$$

with φ a monomorphism, there is a morphism $\tilde{\gamma} : M \to I$ s.t. $\gamma = \tilde{\gamma} \circ \varphi$.

$$\begin{tikzcd} M' \arrow[r, "\varphi"] \arrow[d, "\gamma"'] & M \arrow[dl, dotted, "\tilde{\gamma}"] \\ I & \end{tikzcd}$$

Prop. TFAE for $I \in \underline{M}$:

(i) I is injective

(ii) Any ex. seq. $0 \to I \to M \to M'' \to 0$ splits

(iii) The functor $\mathrm{Hom}_{\underline{M}}(-, I) : \underline{M}^{op} \to \underline{Ab}$ is exact.

Pf. Exer.

$\boxed{\text{For injective modules over noeth. comm. rings see [RD, pp. 120-122] [RD = Residues \& Duality]}}$

< to here 21.3

Prop. Let $f : A \to B$ be a ring hom, and I an injective in $\underline{\mathrm{Mod}}\, A$ (= left A-mods). Then

$$J := \mathrm{Hom}_A(B, I) \text{ is injective}$$

in $\underline{\mathrm{Mod}}\, B$.