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Products & Coproducts

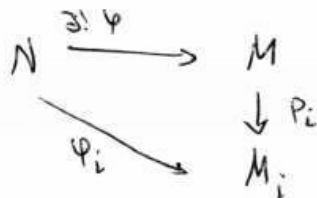
Let  $\underline{M}$  be a category.

Recall the concept of product: for a collection  $\{M_i\}_{i \in I}$  of objects of  $\underline{M}$ , the product  $\prod_{i \in I} M_i$ , if it exists, is an object

$M \in \underline{M}$ , with morphisms

$p_i : M \rightarrow M_i$ , called projections,

s.t. for any  $N \in \underline{M}$  and  $\{\varphi_i : N \rightarrow M_i\}$  there is a unique  $\varphi : N \rightarrow M$  s.t.  $\varphi_i = p_i \circ \varphi$



(In a nutshell: the function (of sets)

$$\text{Hom}_{\underline{M}}(N, M) \rightarrow \prod_{i \in I} \text{Hom}_{\underline{M}}(N, M_i)$$

$$\varphi \mapsto \{p_i \circ \varphi\}_{i \in I}$$

must be bijective,  $\forall N \in \underline{M}$ .)

We know that  $\forall$  <sup>the pair</sup>  $(M, \{p_i\})$  is unique up to a unique isom (if it exists).

①

Exe. In Set and in Mod A all products (indexed by  $I \in \mathbb{U}$ ) exist: they are the usual cartesian products.

∞

Similarly we have the notion of coproduct in M. A product  $\coprod_{i \in I} M_i$  of a collection

$\{M_i\}_{i \in I}$  of objects of M is an object

$M$ , with morphisms  $e_i: M_i \rightarrow M$ ,

called the embeddings. The property is that

for any  $(N, \{\varphi_i: M_i \rightarrow N\})$  there is a unique  $\varphi: M \rightarrow N$  s.t.  $\varphi_i = \varphi \circ e_i$ .

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi_i} & \\ e_i \downarrow & & \\ M & \xrightarrow{\exists! \varphi} & N \end{array}$$

The pair  $(M, \{e_i\})$  is unique (up to unique  $\cong$ ).

Exe. In Set the coproduct is the disjoint union. In Mod A the coproduct is the direct sum.

∞

## ⑫ Additive Categories

Def. A category  $\underline{M}$  is called additive if it satisfies these conditions:

(i) For any  $M, N \in \underline{M}$ , the set  $\text{Hom}_{\underline{M}}(M, N)$  has a structure of an abelian group.

(ii) The compositions

$$\text{Hom}_{\underline{M}}(M_0, M_1) \times \text{Hom}_{\underline{M}}(M_1, M_2) \xrightarrow{\circ} \text{Hom}_{\underline{M}}(M_0, M_2)$$

are bilinear functions.

(iii) There is a zero object  $0$  in  $\underline{M}$ .

(iv)  $\underline{M}$  has finite coproducts.

Observe:

- $\text{Hom}_{\underline{M}}(M, N) \neq \emptyset$ , since it's a group
- $\text{Hom}_{\underline{M}}(M, 0) = 0$ , ... the zero ab. grp, since it has only one element. Likewise  $\text{Hom}_{\underline{M}}(0, M) = 0$ . We denote these arrows by  $0$ .
- The coproduct in  $\underline{M}$  is denoted by  $\oplus$ , and is also called a direct sum.

(13)

Proof. Let  $\underline{M}$  be an additive category.

Let  $\{M_i\}_{i \in I}$  be a finite collection of objects in  $\underline{M}$ , and let  $M := \bigoplus_{i \in I} M_i$  be the coproduct, with embeddings  $e_i: M_i \rightarrow M$ .

1) For any  $i$  let  $p_i: M \rightarrow M_i$  be the unique morphism s.t.  $p_i \circ e_i = \mathbf{1}_{M_i}$  (the id. morphism of  $M_i$ ), and  $p_i \circ e_j = 0$  for  $j \neq i$ .

Then  $(M, \{p_i\})$  is a product of  $\{M_i\}$ .

$$2) \sum_{i \in I} e_i \circ p_i = \mathbf{1}_M.$$

Proof. Exercise.

Exe. Let  $A$  be a ring. The category  $\underline{\text{Mod}} A$  is additive

Let  $\underline{M} \subset \underline{\text{Mod}} A$  be the full subcat. consisting of free modules. Then  $\underline{M}$  is also additive

(13.1) About Rings as Linear Categories  
Let  $A$  be a ring. Define a category

$\mathcal{M}$  like this: there is one object

$P \in \mathcal{M}$ , and  $\text{Hom}_{\mathcal{M}}(P) = A$ .

This is a linear category. Does it have finite direct sums?

Note that  $\mathcal{M}$  is the full subcategory

of  $\text{Mod } A^{\text{op}}$  (right  $A$ -mods) on the object  $P := A \in \text{Mod } A^{\text{op}}$ , the free right

module of rank 1:

$$\text{Hom}_{\text{Mod } A^{\text{op}}}(P, P) = A$$

acting by left mult. on  $P = A$ .

13.2

Suppose we have  $N \cong P \oplus P$   
in  $\underline{M}$ . Then

$$N = P$$

(since unique object.)  $\otimes$

$$P \cong P \otimes P \text{ in } \underline{M}.$$

so  $\exists e_i \rightarrow p_i : P \rightarrow P, i=1,2$

$$\text{st. } 1_P = e_1 \circ p_1 + e_2 \circ p_2$$

Now  $p_i \circ e_i = 1_P$  and  $e_i \circ p_j = 0$

for  $i \neq j$ . All this in the ring  $A$ .

This implies ~~the same~~ that

$$P \cong P \otimes P \text{ also in } \text{Mod } A.$$

Hence  $A$  can't be commutative nor  
right noetherian

~~13.2~~ (13.3)

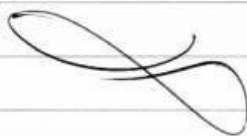
However we can take

$A := \text{End}_K(V)$  where  $K$  is  
a field, &  $V$  is a countable rank

$K$ -module. Then  $A \cong A \oplus A$  as

right  $A$ -modules so  $\underline{M}$  has

finite, and even countable  $\oplus$ .



(14)

## Abelian Categories

Let  $\underline{M}$  be an additive category.

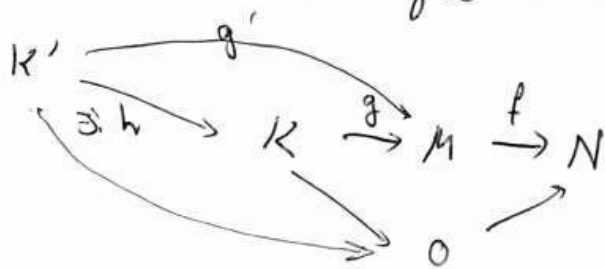
Def. Let  $f: M \rightarrow N$  be a morph. in  $\underline{M}$ .

$\therefore$  A Kernel of  $f$  is a monomorphism

$g: K \rightarrow M$  s.t.  $fg = 0$ , and for

any  $g': K' \rightarrow M$  in  $\underline{M}$  s.t.  $fg' = 0$

there is a unique  $h: K' \rightarrow K$  s.t.  $g' = gh$ .



Prop. 1) If a kernel  $(g: K \rightarrow M) = \text{Ker}(f)$  exists, then it is unique (up to a unique isom.)

2) The kernel  $g: K \rightarrow M$  is a monomorphism.

Ex. Exercise.

Exo. In Mod  $A$ , kernels exist:

$$\text{Ker}(f) = (K \xrightarrow{g} M), \quad K := \{m \in M \mid f(m) = 0\}$$

$g$  is the inclusion

notation

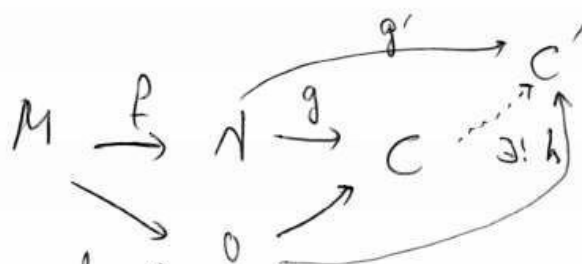
We usually write  $K = \text{Ker}(f)$  and leave  $g$  implicit.



(15)

Def. Let  $f: M \rightarrow N$  be a morphism in  $\underline{M}$ .

A cokernel of  $f$  is a morphism  $g: N \rightarrow C$  s.t.  $g \circ f = 0$ , and for any  $g': N \rightarrow C'$  s.t.  $g' \circ f = 0$ , there exists a unique  $h: C \rightarrow C'$  s.t.  $g' = h \circ g$ .



Prop. (1) If a cokernel  $(g: N \rightarrow C) =: \text{Coker}(f)$  exists, then it's unique (up to ...)

(2) The cokernel  $\text{Coker}(f)$  is an epimorphism.

Ex. Exercise.

Example In  $\text{Mod } A$  we take  $C := N/\bar{M}$ ,  
 $g: N \rightarrow C$  can. proj. where  $\bar{M} := \{f(m) \mid m \in M\}$ .



Def. Assume that  $\underline{M}$  has kernels and cokernels.

(1) For  $f: M \rightarrow N$  let  $\text{Im}(f) := \text{Ker}(\text{Coker}(f))$ .

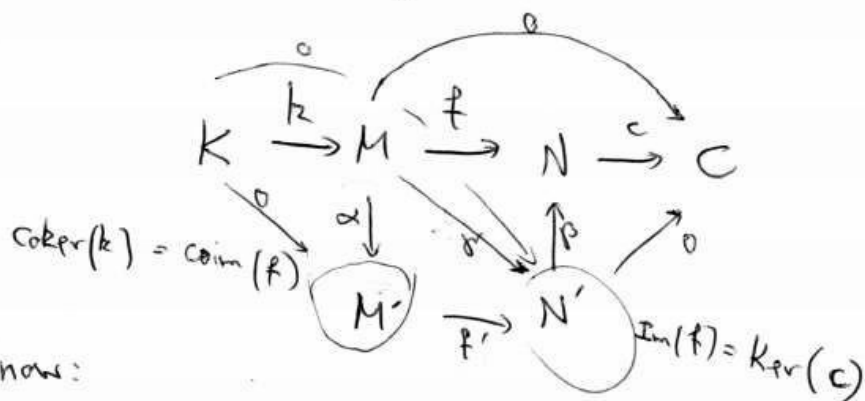
I.e. if  $\text{Coker}(f) = (c: N \rightarrow C)$ , then  $\text{Im}(f) = \text{Ker}(c)$ .

(2) Let  $\text{Coim}(f) := \text{Coker}(\text{Ker}(f))$ .

I.e. if  $(k: K \rightarrow M) = \text{Ker}(f)$ , then  $\text{Coim}(f) = \text{Coker}(k)$ .

(10)

In a diagram we get:



now:

$$\text{cof} = 0 \Rightarrow \exists! \gamma: M \rightarrow N'$$

$$\text{Now } \beta \circ \delta \circ k = f \circ k = 0 \text{ ; } \beta \text{ mono, } \Rightarrow$$

$$\gamma \circ k = 0 \Rightarrow \exists! f': M' \rightarrow N'$$

s.t. diagram comm.

We get induced morphism

$$(*) \quad f': \text{Coim}(f) \rightarrow \text{Im}(f)$$

Def. An abelian category is an additive category  $\underline{M}$  with these additional properties:

(i) All morphisms in  $\underline{M}$  admit kernels and cokernels.

(ii) For any  $f: M \rightarrow N$ , the morphism

$$\text{Coim}(f) \rightarrow \text{Im}(f) \text{ above } \gamma \text{ is an } \underline{\text{isomorphism}}$$

I.e.:

$$(10) \quad \text{Coker}(\text{Ker}(f)) = \text{Ker}(\text{Coker}(f))$$

Rem. From now we forget "coimage".

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Exe. Let  $A$  be a ring. The category  $\text{Mod } A$  is abelian.

Consider the full subcat  $\underline{M} \subset \text{Mod } A$  consisting of free modules. Is  $\underline{M}$  abelian? ...

Usually, not.

Exercise. Take  $A = \mathbb{Z}$ , so  $\text{Mod } A = \underline{Ab}$ .

Prove that the <sup>additive</sup> category  $\underline{M} = \{\text{free } \mathbb{Z}\text{-modules}\}$  is not abelian. (Hint: it has kernels & cokernels, but  $(\heartsuit)$  fails.)

Here is a delicate issue:

Def. Let  $\underline{M}$  be an abelian category, and let  $\underline{N}$  be a full subcategory. We say that  $\underline{N}$  is a full abelian subcat. of  $\underline{M}$  if  $\underline{N}$  is closed under direct sums, kernels and cokernels.

Prop. This means that  $\underline{N}$  is an ab. cat, and the inclusion  $\underline{N} \rightarrow \underline{M}$  is an exact functor (later).

(18)

Exe. The cat. of finite abelian groups is a full abelian subcat. of  $\underline{Ab}$ .

Exe.  $A$  is left noetherian ring iff the full cat.  $\underline{Mod}_f A$  of fin. gen. left. modules is a full abelian subcat. of  $\underline{Mod} A$ .

Exe Let  $(X, \mathcal{A})$  be a ringed space. So  $X$  is a topological space, and  $\mathcal{A}$  is a sheaf of rings on  $X$ .

Let  $\underline{PMod} \mathcal{A}$  be the category of presheaves of  $\mathcal{A}$ -modules on  $\mathcal{A}$ . This is an abelian category. For  $f: \mathcal{K} \rightarrow \mathcal{A}$  the p. sheaf  $\mathcal{K} = \text{Ker}(f)$

is:

$$\Gamma(U, \mathcal{K}) = \text{Ker}(f: \Gamma(U, \mathcal{K}) \rightarrow \Gamma(U, \mathcal{A})), \quad U \in \mathcal{X} \text{ open.}$$

And  $\mathcal{C} = \text{Coker}(f)$  is

$$\Gamma(U, \mathcal{C}) = \text{Coker} \left( \Gamma(U, \mathcal{K}) \rightarrow \Gamma(U, \mathcal{A}) \right)$$

Let  $\underline{Mod} \mathcal{A}$  be the full subcat of  $\underline{PMod} \mathcal{A}$  consisting of sheaves. This is also an abelian category, but with different colimits:  
So it's not a full abelian subcat.

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$$\text{Coker}_{\text{Mod } A}(f) = \left( \text{Coker}_{\text{BMod } A}(f) \right)^{\sim}$$

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more precisely:  
 •  $\text{Mod } A$  is closed under kernels  
 •  $\text{Mod } A$  is not closed under cokernels  
 • has its own intrinsic cokernels  $\varphi$ .

↑  
 the associated shf.

For educational purposes we state:

Thm (Freyd & Mitchell). Let  $\underline{M}$  be a small abelian category (i.e.  $\text{Ob}(\underline{M}) \in \mathcal{U}$ ).

Then  $\underline{M}$  is equivalent to a full abelian subcat. of  $\text{Mod } A$ , for some ring  $A$ .

What does this help? It means that for most of what we do, we can pretend that  $\underline{M} \in \text{Mod } A$ , a full abelian subcat.

Prop. Let  $\underline{M}$  be an additive category.

Then  $\underline{M}^{\text{op}}$ , the opposite category, is also additive (with same ab. grp. str. on

$$\text{Hom}_{\underline{M}^{\text{op}}}(M, N) = \text{Hom}_{\underline{M}}(N, M). \quad \left. \vphantom{\text{Hom}_{\underline{M}^{\text{op}}}(M, N)} \right\} \text{zero} = \text{zero.}$$

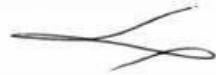
pt. Bilinearity of comp. in  $\underline{M}^{\text{op}}$  is clear. Existence of finite coproducts in  $\underline{M}^{\text{op}}$  is because  $\underline{M}$  has finite products.  $\square$

(20)

Prop Let  $\underline{M}$  be an abelian category.  
Then  $\underline{M}^{\text{op}}$  is also abelian.

prf. We know that  $\underline{M}^{\text{op}}$  is additive.  
It has kernels & cokernels:

$\text{Ker}_{\underline{M}^{\text{op}}}(f) = \text{Coker}_{\underline{M}}(f)$  etc.  
Also the symmetric axiom (ii) holds.  $\square$



Prop Let  $\underline{M}$  be an abelian category,  
and let  $f: M \rightarrow N$  be a morphism in  $\underline{M}$ .

- (a)  $f$  is a monomorphism iff  $\text{Ker}(f) \cong 0$
- (b)  $f$  is an epimorphism iff  $\text{Coker}(f) \cong 0$
- (c)  $f$  is an isom. iff it's both an  
epim. & a monom.

Ex. Exercise.

(21)

## Additive Functors

Def. Let  $\underline{M}$  and  $\underline{N}$  be additive categories. A functor  $F: \underline{M} \rightarrow \underline{N}$  is called an additive functor if for any  $M_1, M_2 \in \underline{M}$  the function

$F: \text{Hom}_{\underline{M}}(M_1, M_2) \rightarrow \text{Hom}_{\underline{N}}(F(M_1), F(M_2))$   
is a hom. of abelian groups.

Exe.  $A$  a ring, The forgetful functor  
 $F: \text{Mod } A \rightarrow \underline{Ab}$  is additive

← go to  
21.1

Def. Let  $\underline{M}$  and  $\underline{N}$  be abelian categories. An additive functor  $F: \underline{M} \rightarrow \underline{N}$  is:

(1) Left exact if it commutes with kernels.

$$\text{I.e. } \text{Ker}_{\underline{N}}(F(\varphi)) = F(\text{Ker}_{\underline{M}}(\varphi))$$

for any  $\varphi: M_1 \rightarrow M_2$  in  $\underline{M}$ .

→  
to P. 22

21.1

Prop Let  $\underline{M}$  &  $\underline{N}$  be additive categories,  
and let  $F: \underline{M} \rightarrow \underline{N}$  be additive functors.  
Then  $F(O_M) \cong O_N$ .

proof. The zero object  $O_N \in \underline{N}$  is  
characterized by  $1=0$  in

$\text{Hom}_{\underline{N}}(O_N, O_N)$ . Now let  $N := F(O_M) \in \underline{N}$ .

so

$$F: \text{Hom}_{\underline{M}}(O_M, O_M) \rightarrow \text{Hom}_{\underline{N}}(N, N)$$

sends  $F(0) = 0$  ; since additive

and  $F(1) = 1$  ; any functor.

Since  $1=0$  in  $\text{Hom}_{\underline{M}}(O_M, O_M)$ , the  
same holds for  $N$ .

$$\text{So } N \cong O_N.$$

□

another proof: For any  
 $M \in \underline{M}$ ,  $\text{End}_{\underline{M}}(M)$  is a ring, and  
 $\text{Hom}_{\underline{M}}(M, M)$  is a bimodule over  $\text{End}_{\underline{M}}(M)$  and  $\text{End}_{\underline{M}}(M)$  is the zero ring.  
Thus  $M \cong 0$  iff  $\text{End}_{\underline{M}}(M)$  is the zero ring.  
Now  $F: \text{End}_{\underline{M}}(M) \rightarrow \text{End}_{\underline{N}}(F(M))$  is a ring hom.



Q1.2)

Prop. Let  $\underline{M}$  &  $\underline{N}$  be additive categories, and let  $F: \underline{M} \rightarrow \underline{N}$  be an additive functor.

Then  $F$  commutes with finite coproducts.

Namely if  $\{M_i\}_{i \in I}$  is a finite collection of objects of  $\underline{M}$ , and  $(M, \{e_i\})$  is its coproduct, then  $(F(M), \{F(e_i)\})$  is a coproduct of  $\{F(M_i)\}$ .

Exercise.

←  
to p. 21

(22)

(2) Right exact if it commutes with cokernels. I.e.

$$\text{Coker}_N(F(\varphi)) = F(\text{Coker}_M(\varphi)).$$

(3) Exact if it is both left & right exact.

$$\begin{array}{c}
 \text{Ker}_M^k(\varphi) \xrightarrow{k} M \xrightarrow{\varphi} N \xrightarrow{c} \text{Coker}_M^c(\varphi) \\
 \downarrow F \\
 F(\text{Ker}_M^k(\varphi)) \xrightarrow{F(k)} F(M) \xrightarrow{F(\varphi)} F(N) \xrightarrow{F(c)} F(\text{Coker}_M^c(\varphi))
 \end{array}$$

$$\begin{array}{ccccccc}
 F(K) & \xrightarrow{F(k)} & F(M) & \xrightarrow{F(\varphi)} & F(N) & \xrightarrow{F(c)} & F(C) \\
 \downarrow \cong & & \uparrow & & \downarrow & & \downarrow \cong \\
 & & \text{Ker}_N(F(\varphi)) & & & & \text{Coker}_N(F(\varphi))
 \end{array}$$



Def An exact sequence in an ab. cat.  $\mathcal{M}$  is a diagram

$$\dots M_0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} M_2 \rightarrow \dots$$

(finite or infinite  $\dots$  on either side)

s.t.  $\text{Ker}(\varphi_i) = \text{Im}(\varphi_{i-1})$  for all  $i$  (s.t.  $\varphi_i$  &  $\varphi_{i-1}$  are defined).

A short ex. seq. is an ex. seq.

$$(*) \quad 0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0$$

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Prop Let  $F: \underline{M} \rightarrow \underline{N}$  be an additive functor between abelian categories. The functor  $F$  is left exact; resp. right exact; resp. exact iff for every exact seq.

$$0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0$$

in  $\underline{M}$ , the seq.

$$0 \rightarrow F(M_0) \rightarrow F(M_1) \rightarrow F(M_2) ;$$

resp.

$$F(M_0) \rightarrow F(M_1) \rightarrow F(M_2) \rightarrow 0 ;$$

resp

$$0 \rightarrow F(M_0) \rightarrow F(M_1) \rightarrow F(M_2) \rightarrow 0$$

is exact.

proof. Exercise

(Hint:  $M_0 \cong \text{Ker}(M_1 \rightarrow M_2)$ , etc.)

✂

Example. Let  $A$  be a comm. ring, and  $M \in \underline{\text{Mod}} A$ .

• Define  $F: \underline{\text{Mod}} A \rightarrow \underline{\text{Mod}} A$ ,  $F(N) := M \otimes_A N$ . This is right exact.

•  $G: (\underline{\text{Mod}} A)^{\text{op}} \rightarrow \underline{\text{Mod}} A$ ,  $G(N) := \text{Hom}_A(N, M)$ .

•  $H: \underline{\text{Mod}} A \rightarrow \underline{\text{Mod}} A$ ,  $H(N) := \text{Hom}_A(M, N)$ .

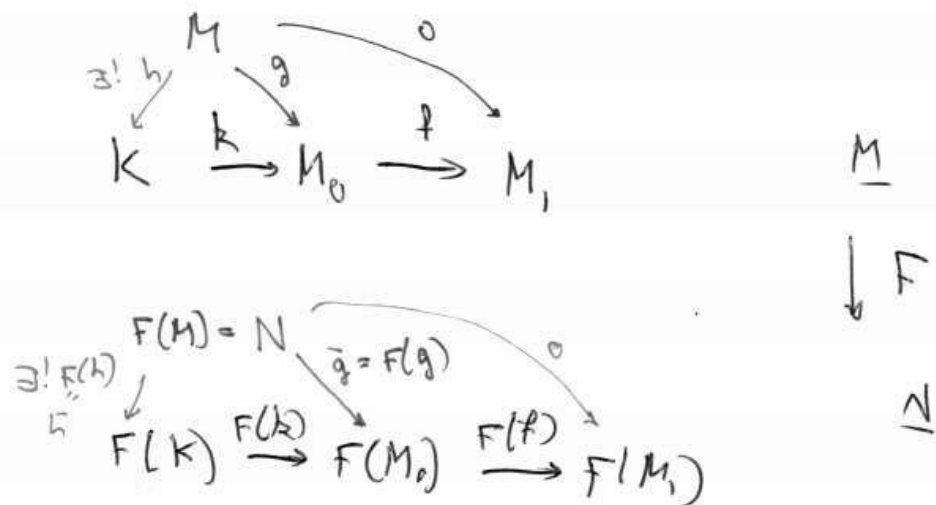
These are left exact.

(24)

Recall that a functor  $F: \underline{C} \rightarrow \underline{D}$  is an equivalence if  $\exists G: \underline{D} \rightarrow \underline{C}$ , and natural isom's  $G \circ F \cong \mathbb{1}_{\underline{C}}$  and  $F \circ G \cong \mathbb{1}_{\underline{D}}$ .  
 The functor  $G$  is called a quasi-inverse of  $F$ .

Prop. Let  $\underline{M}$  and  $\underline{N}$  be abelian categories, and let  $F: \underline{M} \rightarrow \underline{N}$  be an additive functor. If  $F$  is an equivalence, then it is exact.

Proof We will prove that  $F$  respects kernels. The proof for cokernels is similar. Take  $f: M_0 \rightarrow M_1$  with  $K = \text{Ker}(f)$ .



Say  $\bar{g}: N \rightarrow F(M_0)$  in  $\underline{N}$ . Since  $F$  is essentially surjective on objects,  $N \cong F(M)$  for some  $M$ .

So we may assume  $N = F(M)$ . (replace  $\bar{g}: N \rightarrow F(M_0)$  with  $F(M) \xrightarrow{\cong} N \xrightarrow{\bar{g}} F(M_0)$ )  
 Next  $F: \text{Hom}_{\underline{M}}(M, M_0) \rightarrow \text{Hom}_{\underline{N}}(F(M), F(M_0))$  is an isom. of ab. grps.

(25)  $\Rightarrow \exists! g: M \rightarrow M_0$  s.t.  $F(g) = \hat{g}$ .

And  $f \circ g = 0$ . Since  $K = \ker(F)$ ,

$\exists! h: M \rightarrow K$  s.t.  $kh = g$ .

So  $\bar{h} := F(h): N \rightarrow F(K)$  is unique

s.t.  $F(k) \circ \bar{h} = \hat{g}$ .  $\square$

(\*) We know that  $F: \underline{C} \rightarrow \underline{D}$  is an equivalence iff these conditions hold:

- (i)  $F$  is essentially surjective on objects (i.e. surjective on iso. classes of objects).
- (ii)  $F$  is fully faithful (i.e. bijective on Hom sets).

Prop Let  $F: \underline{M} \rightarrow \underline{N}$  be an additive functor between additive categories. TFAE:

- (i)  $F$  has a quasi-inverse.
- (ii)  $F$  has an additive quasi-inverse.

Ex Exercise.

(26)

## Splitting, Projectives & Injectives

Let  $\mathcal{M}$  be an abelian category.

An exact sequence

$$(*) \quad 0 \rightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \rightarrow 0$$

in  $\mathcal{M}$  is called split if it's isomorphic to the exact sequence

$$0 \rightarrow M' \xrightarrow{e'} M' \oplus M'' \xrightarrow{p''} M'' \rightarrow 0,$$

where  $e'$  is the embedding, and  $p''$  is the projection (recall that  $\oplus M_i \cong \prod M_i$ ).

Namely there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & M' & \xrightarrow{\varphi} & M & \xrightarrow{\psi} & M'' \rightarrow 0 \\ & & \cong \downarrow & & \cong \downarrow & & \cong \downarrow \\ 0 & \rightarrow & M' & \xrightarrow{e'} & M' \oplus M'' & \xrightarrow{p''} & M'' \rightarrow 0 \end{array}$$

We know that finding  $\beta$  is equivalent to finding  $p: M'' \rightarrow M$  s.t.  $\psi \circ \beta = 1_{M''}$ ; and to finding  $\alpha: M \rightarrow M'$  s.t.  $\alpha \circ \varphi = 1_{M'}$  (splitting  $\psi$  &  $\varphi$  resp.).

(JF)

Exe The exact seq. (in  $\underline{Ab}$ )

$$0 \rightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}/(2) \rightarrow 0$$

is not split



Def. An object  $P \in \underline{M}$  is called projective

if for any diagram

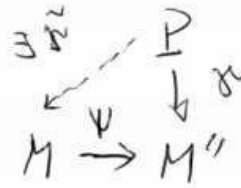
$$M \xrightarrow{\psi} M'' \xleftarrow{\varphi} P \xrightarrow{\alpha} Y$$

is an epimorphism

there exists

$$\tilde{\varphi}: P \rightarrow M \text{ s.t.}$$

$$\varphi = \psi \circ \tilde{\varphi}$$



Prop TFAE for  $P \in \underline{M}$ :

(i)  $P$  is projective.

(ii) any exact seq.  $0 \rightarrow M' \rightarrow M \rightarrow P \rightarrow 0$  splits.

(iii) The functor  $\text{Hom}_M(P, -) : \underline{M} \rightarrow \underline{Ab}$  is exact.

Proof Exer.

Example. Let  $A$  be a ring. An  $A$ -module  $P$  is projective iff it is a direct summand of a free module. Any (left)  $A$ -module is a quotient of a projective  $A$ -module.

Def. We say that  $\underline{M}$  has enough projectives if any  $M \in \underline{M}$  admits an epimorphism  $P \rightarrow M$ , with  $P$  projective (i.e. say " $M$  is a quotient of  $P$ ")

(28)

Exercise. Let  $\underline{M} \subset \underline{Ab}$  be the category of finite abelian groups. We know it is a full ab. subset of  $\underline{Ab} = \text{Mod } \mathbb{Z}$ . Show that the only proj. object in  $\underline{M}$  is 0.  $\Rightarrow$  not enough proj's.

Example. Take  $X := \mathbb{P}^1_{\mathbb{K}}$ , the projective line over a field  $\mathbb{K}$  (can assume  $\mathbb{K}$  alg. closed). Consider the abelian category  $\underline{\text{Coh}} \mathcal{O}_X$  of coherent  $\mathcal{O}_X$ -modules. I claim that the only projective object in  $\underline{\text{Coh}} \mathcal{O}_X$  is 0. This is quite involved! Let me just show why  $\mathcal{O}_X$  is not projective.

Let  $t_0, t_1$  be the homogeneous coordinates. They generate  $\mathcal{O}_X(1)$ , and there is an exact seq.

$$0 \rightarrow \mathcal{O}_X(-1) \xrightarrow{[t_0 \ t_1]} \mathcal{O}_X \oplus \mathcal{O}_X \xrightarrow{\begin{bmatrix} t_0 \\ t_1 \end{bmatrix}} \mathcal{O}_X(1) \rightarrow 0$$

in  $\underline{\text{Coh}} \mathcal{O}_X$ . Twisting we get

$$0 \rightarrow \mathcal{O}_X(-2) \xrightarrow{\psi} \mathcal{O}_X(-1)^2 \xrightarrow{\Psi} \mathcal{O}_X \rightarrow 0$$

which is exact, but not split!

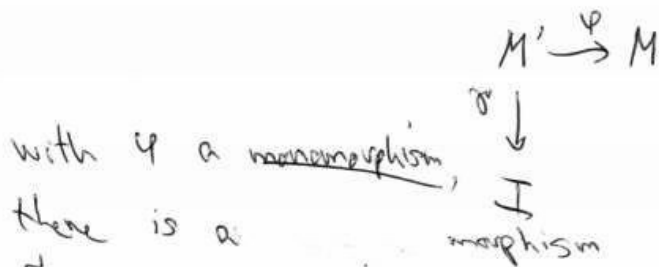
$\Rightarrow$  not enough proj's!

$\infty$



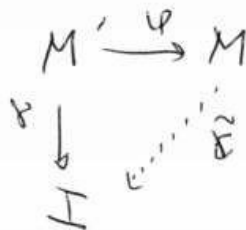
(29)

Def. An object  $I \in \underline{M}$  is called injective if for any diagram:



$$\tilde{\alpha}: M \rightarrow I \text{ s.t.}$$

$$\tilde{\alpha} \circ \varphi = \alpha.$$



Prop. TFAE for  $I \in \underline{M}$ :

(i)  $I$  is injective

(ii) Any ex. seq.  $0 \rightarrow I \rightarrow M \rightarrow M'' \rightarrow 0$  splits

(iii) The functor  $\text{Hom}_M(-, I): \underline{M}^{\text{op}} \rightarrow \underline{A}$  is exact.

Exer.

For injective modules over noeth. comm. rings see [RD, pp. 120-122] [RD = Residues & Duality]

← to here 21.3

Prop. The  $f: A \rightarrow B$  be a ring hom, and  $I$  an injective in  $\text{Mod } A$  (= left  $A$ -mods) then

$J := \text{Hom}_A(B, I)$  is injective in  $\text{Mod } B$ .