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Goldie's theorem leads naturally to several generalizations. On the one hand, it is of interest to know which rings can be embedded into Artinian rings, since many properties of Artinian rings pass to their subrings. The remainder of §3.2 is spent on questions of this sort, including Small's theorem.

On the other hand, one may wish to find non-Ore localization procedures which generalize Goldie's theorem and apply to wider classes of rings. One such theory is Johnson's theory of nonsingular rings, which is presented in some detail in §3.3, including Goodearl's theorems; however, this topic is not pursued in later chapters. Johnson's theory has been generalized further to an abstract localization theory, sketched in §3.4; one important application is Martindale's ring of quotients and, in particular, his central closure, which is an indispensible tool in parts of the structure theory.

§3.1 Classical Rings of Fractions

In this section we consider general conditions under which a given submonoid S of elements of R can be made invertible, cf., definition 1.10.1. Often we shall require S to be *regular*, i.e., every element of S is regular in R.

Recall $s \in R$ is *regular* if $rs \neq 0$ and $sr \neq 0$ for all nonzero r in R. Hopefully there will be no confusion with our other usage of "regular"—"regular ring" means von Neumann regular, cf., definition 2.11.18.

Definition 3.1.1: Suppose S is a (multiplicative) submonoid of R. Q is a (left) *ring of fractions of* R with respect to S if the following two conditions are satisfied: (1) There is a ring homomorphism $v: R \to Q$ such that vs is invertible for all s in S, with $\ker v = \{r \in R: sr = 0 \text{ for some } s \text{ in } S\}$, and (2) every element of Q has the form $s^{-1}r$ where $s \in S$ and $r \in R$. (Technically, this should be written $(vs)^{-1}vr$, but we shall write $s^{-1}r$ abusing the notation slightly.)

Important Special Case. If $S = \{\text{all regular elements of } R\}$ and if $\varphi: R \to Q$ is an injection then Q is called the *classical ring of fractions of* R, and R is called an *order* in Q.

Classical rings of fractions need not exist, and our first objective will be to find necessary and sufficient conditions for constructing rings of fractions.

Definition 3.1.2: A (left) *denominator set* (also called *Ore set*) is a submonoid S of R satisfying the following two conditions:

(i) For any s_1 and S and r_1 in R there exist s_2 in S and r_2 in R such that $s_2 r_1 = r_2 s_1$.

(ii) If $rs = 0$ for r in R, s in S, then there is s' in S with $s'r = 0$.

Proposition 3.1.3: *If R has a ring of fractions with respect to S then S is a denominator set.*

Proof: We check the conditions of definition 3.1.2.

(i) Given s_1, r_1 we have $(1^{-1}r_1)(s_1^{-1}1) = s^{-1}r$ for suitable s, r; then $1^{-1}sr_1 = 1^{-1}rs_1$ implies $sr_1 - rs_1 \in \ker v$ so for some s' in S we have $0 = s'(sr_1 - rs_1) = s'sr_1 - s'rs_1$; let $s_2 = s's$ and $r_2 = s'r$.

(ii) If $rs = 0$ then $0 = (1^{-1}rs)(s^{-1}1) = 1^{-1}r$, implying $s'r = 0$ for some s' in S. Q.E.D.

Before considering the converse, let us discuss two important special cases. First, any submonoid S of $Z(R)$ is obviously a denominator set, and the corresponding ring of fractions $S^{-1}R$ exists and was already constructed three different ways (construction 1.10.2, construction 1.10.3, and exercise 1.10.1).

On the other hand, suppose S is regular. Then condition (ii) is automatic, so we need only check (i). In case $S = \{\text{all regular elements of } R\}$ we call (i) the *Ore condition*, and R is called (left) Ore; for example, any PLID is Ore by remark 1.6.19. The proof that Ore rings have classical rings of fraction follows the lines of the general proof presented below, but with the simplification that one does not worry about zero divisors.

We are ready now to proceed to the construction of a ring of fractions with respect to an arbitrary denominator set. As usual there are two approaches—the "brute force" construction which follows one's naive intuition but involves verifications at every step of the way, and a slick construction using direct limits. The slick construction yields a much more encompassing result (cf., theorem 3.4.25ff) but leaves us without a "feel" for computing in the ring of fractions. Consequently, we shall wade through the brute force construction, and the reader who is so inclined can view it as a corollary of theorem 3.4.25 below.

Theorem 3.1.4: *Suppose S is a denominator set.*

(i) *$S \times R$ has an equivalence $\sim$ defined as follows: $(s_1, r_1) \sim (s_2, r_2)$ if there exist r, r' in R such that $rr_1 = r'r_2$ and $rs_1 = r's_2 \in S$. Write $S^{-1}R$ for the set of equivalence classes, and write $s^{-1}r$ for the equivalence class of (s, r).*

(ii) *If $s \in S$ and $as \in S$ then $(as)^{-1}(ar) = s^{-1}r$ in $S^{-1}R$.*

(iii) *To check a function f on $S^{-1}R$ is well-defined it suffices to prove f has the same value on $s_1^{-1}r_1$ and $(rs_1)^{-1}(rr_1)$ where $rs_1 \in S$.*

(iv) *$S^{-1}R$ has a natural ring structure (given in the proof), which is a ring of fractions for R with respect of S.*

Proof:

(i) Clearly $\sim$ is reflexive and symmetric; to prove transitivity we need the following technical lemma, which also cleans up the other sticky points in proving the theorem:

Lemma: *Suppose $(s_1, r_1) \sim (s_2, r_2)$ and $as_1 = a's_2 \in S$ for a, a' in R. Then there is b in R with $bar_1 = ba'r_2$ and $bas_1 = ba's_2 \in S$.*

Proof of lemma: There are r, r' in R satisfying $rr_1 = r'r_2$ and $rs_1 = r's_2 \in S$, by hypothesis. Take r'', s'' such that

$$s''(rs_1) = r''(as_1).$$

Then $(s''r - r''a)s_1 = 0$ so there is s'_1 in S satisfying $s'_1(s''r - r''a) = 0$, i.e.,

$$s'_1 s''r = s'_1 r''a.$$

Now $s'_1 r''(a's_2) = s'_1 r''as_1 = s'_1 s''rs_1 = s'_1 s''r's_2$ so $(s'_1 r''a' - s'_1 s''r')s_2 = 0$. Hence for some s'_2 we have

$$s'_2 s'_1 r''a' = s'_2 s'_1 s''r'.$$

Let $b = s'_2 s'_1 r''$ and $s = s'_2 s'_1 s''$. Then $ba' = sr'$ and

$$ba = s'_2(s'_1 r''a) = s'_2(s'_1 s''r) = sr.$$

Hence $bar_1 = srr_1 = sr'r_2 = ba'r_2$. Also, by hypothesis, $bas_1 = ba's_2$ and is in S since $bas_1 = s'_2 s'_1 r''(as_1) = s'_2 s'_1 s''(rs_1) \in S$. Q.E.D. for lemma.

For transitivity of $\sim$ suppose $(s_1, r_1) \sim (s_2, r_2)$ and $(s_2, r_2) \sim (s_3, r_3)$. For $i = 1, 2$ we take a_i, a'_i in R with $a_i r_i = a'_i r_{i+1}$ and $a_i s_i = a'_i s_{i+1} \in S$. Since S is a denominator set we have s in S and a in R with $sa_2 = aa'_1$; then $(aa_1)s_1 = (aa'_1)s_2 = (sa_2)s_2$ and $(sa_2)s_2 = s(a_2 s_2) \in S$. Applying the lemma to aa_1 and sa_2 we get b in R with $b(aa_1)r_1 = b(sa_2)r_2$ and $b(aa_1)s_1 = b(sa_2)s_2 \in S$. Then

$$(baa_1)r_1 = bsa_2 r_2 = (bsa'_2)r_3 \quad \text{and} \quad (baa_1)s_1 = bsa_2 s_2 = (bsa'_2)s_3 \in S.$$

Thus $\sim$ is transitive, and so is an equivalence.

(ii) follows at once from definition of $\sim$.

(iii) is immediate for if $rr_1 = r'r_2$ and $rs_1 = r's_2 \in S$ then, by hypothesis,

$$f(s_1^{-1}r_1) = f((rs_1)^{-1}rr_1) = f((r's_2)^{-1}r'r_2) = f(s_2^{-1}r_2).$$

(iv) We shall define, respectively, the product and sum of $s_1^{-1}r_1$ and $s_2^{-1}r_2$. Although not needed formally the case $s_1 = r_2 = 1$ is very instructive, for if we take r in R, s in S with $sr_1 = rs_2$ then "intuitively" one would want $(1^{-1}r_1)(s_2^{-1}1) = r_1 s_2^{-1} = s^{-1}r$. This trick for switching s_2 past r_1 motivates the formal definitions:

$(s_1^{-1}r_1)(s_2^{-1}r_2) = (as_1)^{-1}rr_2$ where $r, a \in R$ are chosen such that $as_1 \in S$ and $ar_1 = rs_2$

$(s_1^{-1}r_1) + (s_2^{-1}r_2) = (as_1)^{-1}(ar_1 + rr_2)$ where r, a are chosen such that $as_1 \in S$ and $as_1 = rs_2$.

We shall show multiplication is well-defined in three steps.

Step I. Independent of choice of r, a. By definition 3.1.2(i) there are r_0 in R and a_0 in S for which $a_0 r_1 = r_0 s_2$, and we shall prove $(as_1)^{-1}rr_2 = (a_0 s_1)^{-1}r_0 r_2$ for each choice of r, a. Taking r' in R and s' in S satisfying $r'a_0 = s'a$ we have

$$s'(rs_2) = s'(ar_1) = (s'a)r_1 = r'a_0 r_1 = r'r_0 s_2,$$

so $(s'r - r'r_0)s_2 = 0$. Hence $s(s'r - r'r_0) = 0$ for some s in S, so $ss'r = sr'r_0$. Moreover, $sr'a_0 s_1 = ss'as_1 \in S$ so by (ii) we see indeed

$$(a_0 s_1)^{-1}r_0 r_2 = (sr'a_0 s_1)^{-1}sr'r_0 r_2 = (ss'as_1)^{-1}ss'rr_2 = (as_1)^{-1}rr_2.$$

Step II. Well-defined in first argument. By (iii) it suffices to show $((bs_1)^{-1}br_1) \cdot (s_2^{-1}r_2) = (s_1^{-1}r_1)(s_2^{-1}r_2)$ whenever $bs_1 \in S$. But taking r, a such that $a(br_1) = rs_2$ and $a(bs_1) \in S$ we have

$$((bs_1)^{-1}br_1)(s_2^{-1}r_2) = (abs_1)^{-1}rr_2 = (s_1^{-1}r_1)(s_2^{-1}r_2)$$

by definition since $(ab)s_1 \in S$ and $(ab)r_1 = rs_2$.

Step III. Well-defined in second argument. As in Step II, if $bs_2 \in S$ then taking r, a such that $as_1 \in S$ and $ar_1 = rbs_2$ we have

$$(s_1^{-1}r_1)((bs_2)^{-1}br_2) = (as_1)^{-1}rbr_2 = (s_1^{-1}r_1)(s_2^{-1}r_2)$$

viewing $ar_1 = (rb)s_2$.

This proves multiplication is well-defined; addition is well-defined in the same way, and the ring verifications are now routine, cf., exercise 2. Finally note $r_1 \in \ker \nu$ iff $1^{-1}r_1 = 1^{-1}0$, which occurs when there are r, r' in R such that $rr_1 = r'0 = 0$ and $r = r' \in S$, proving $S^{-1}R$ is indeed a ring of fractions of R with respect to S. Q.E.D.

Properties of Fractions

Remark 3.1.5: Suppose S is a regular denominator set. Since the canonical homomorphism $\nu: R \to S^{-1}R$ has kernel 0, we can identify $R \subseteq S^{-1}R$ in which all elements of S become invertible.

Now we shall show the ring of fractions of R with respect to S is unique, by characterizing $S^{-1}R$ as a universal in terms of remark 3.1.5. Given $A \subset R$ write $s^{-1}A$ for $\{s^{-1}a: s \in S \text{ and } a \in A\}$.

Theorem 3.1.6: *If S is a denominator set then the ring of fractions $S^{-1}R$ (together with $\nu: R \to S^{-1}R$) is the universal of definition 1.10.1. In particular, if $f: R \to T$ is a homomorphism with fs invertible for all s in S then f extends uniquely to a homomorphism $\bar{f}: S^{-1}R \to T$; moreover, $\bar{f}$ is given by*

$$\bar{f}(s^{-1}r) = (fs)^{-1}fr \tag{1}$$

and $\ker \bar{f} = S^{-1} \ker f$.

Proof: Any homomorphism $\bar{f}$ extending f must satisfy (1), so we shall show (1) defines a homomorphism. First note for $s_1 \in S$ and $rs_1 \in S$ that $f(rs_1)^{-1}frfs_1 = 1$ so

$$(fs_1)^{-1} = f(rs_1)^{-1}fr.$$

Now $\bar{f}$ is well-defined by theorem 3.1.4(ii) since

$$f(rs_1)^{-1}f(rr_1) = f(rs_1)^{-1}frfr_1 = (fs_1)^{-1}fr_1.$$

To show $^{-}$ is a homomorphism take r, a as in the definition of sum and product in the proof of theorem 3.1.4, where a is chosen in S (which we can do by the dominator set condition).

$$\bar{f}((s_1^{-1}r_1)(s_2^{-1}r_2)) = f(as_1)^{-1}f(rr_2) = (fs_1)^{-1}(fa)^{-1}frfr_2$$
$$= (fs_1)^{-1}fr_1(fs_2)^{-1}fr_2 = \bar{f}(s_1^{-1}r_1)\bar{f}(s_2^{-1}r_2).$$
$$\bar{f}(s_1^{-1}r_1 + s_2^{-1}r_2) = f(as_1)^{-1}f(ar_1 + rr_2) = (fs_1)^{-1}(fa)^{-1}(fafr_1 + frfr_2)$$
$$= (fs_1)^{-1}fr + (fs_1)^{-1}(fa)^{-1}frfr_2$$
$$= \bar{f}(s_1^{-1}r_1) + (fs_1)^{-1}fs_1(fs_2)^{-1}fr_2$$
$$= \bar{f}(s_1^{-1}r_1) + \bar{f}(s_2^{-1}r_2).$$

To prove the last assertion first note $S^{-1}\ker f \subseteq \ker \bar{f}$, so it suffices to show if $0 = \bar{f}(s^{-1}r) = (fs)^{-1}fr$ then $s^{-1}r \in S^{-1}\ker f$. But this is obvious since $fr = 0$. Q.E.D.

Corollary 3.1.7: *Suppose $S_1 \subseteq S_2$ are denominator sets of R. Then there is a canonical homomorphism $S_1^{-1}R \to S_2^{-1}R$ under which we may view $S_2^{-1}R$ as $(S_1^{-1}S_2)^{-1}(S_1^{-1}R)$. Moreover, if $1^{-1}s$ is invertible in $S_1^{-1}R$ for every s in S_2 then $S_1^{-1}R \approx S_2^{-1}R$ canonically. In particular, if every regular element of R is invertible then R is its own classical ring of fractions.*

Since every regular element of a left Artinian ring is invertible, we see that every left Artinian ring is its own classical ring of fractions. If we want to study rings in terms of the classical ring of fractions, the natural procedure is to start with a given kind of left Artinian ring and to characterize orders in this kind of ring. For example, those rings which are orders in fields are the commutative domains. Likewise, we have the following basic result due to Ore:

Proposition 3.1.8: *R is an order in a division ring* iff *R is an Ore domain.*

Proof: ($\Rightarrow$) by proposition 3.1.3, noting that every nonzero element of R is invertible in a larger ring and hence is regular in R; ($\Leftarrow$) is theorem 3.1.4, taking $S = R - \{0\}$ (since $(s^{-1}r)^{-1}$ is then $r^{-1}s$). Q.E.D.

Thus any PLID is an order in a division ring but need *not* be a right order in a division ring, in view of example 1.6.26. Nevertheless, proposition 3.1.8 is one of the cornerstones of ring theory, along with its generalization by Goldie to orders in semisimple Artinian rings, to be discussed in §3.2. We conclude this section with some useful observations concerning any denominator set S of R.

Remark 3.1.9: For any s in S and r_i in R we have $s^{-1}r_1 + s^{-1}r_2 = s^{-1}(r_1 + r_2)$, by an easy calculation.

Lemma 3.1.10: (*Common denominator*) *Given $q_1, \ldots, q_n$ in $S^{-1}R$ one has $r_1, \ldots, r_n$ in R and s in S with $q_i = s^{-1}r_i$ for $1 \le i \le n$.*

Proof: Inductively write $q_i = s^{-1}r_i$ for $1 \le i \le n-1$ and write $q_n = s_n^{-1}r_n$. Then picking r' in R and s' in S with $s's_n = r's$ we have $(s's_n)^{-1}(s'r_n) = q_n$ and $(s's_n)^{-1}(r'r_i) = (r's)^{-1}r'r_i = s^{-1}r_i = q_i$ for $1 \le i \le n-1$. Q.E.D.