Noether Normalization

All rings are still commutative.

<u>Def</u> Let $A \to B$ be a hom. of commutative rings, and let $(b_1, \dots, b_n)$ be a sequence of elements in B. We say that this sequence is <u>algebraically independent over</u> A if the ring homomorphism

$$A[t_1, \dots, t_n] \to B, \quad t_i \mapsto b_i$$

from the polynomial ring is <u>injective</u>.

A ring hom. $A[t_1, \dots, t_n] \xrightarrow{f} B$ from a polynomial ring can be understood as <u>substitution</u>. A polynomial $p(\underline{t}) = p(t_1, \dots, t_n)$ is sent to the element

$$p(b_1, \dots, b_n) := f(p) \in B,$$

where $b_i := f(t_i)$. The image of f is the subring $A[b_1, \dots, b_n] := \mathrm{Im}(f) \subset B$.

It is called the A-subring of B generated by $b_1, \dots, b_n$.

A polynomial $p(t)$ belongs to $\mathrm{Ker}(f)$ iff $p(b_1, \dots, b_n) = 0$ in the substitution notation. We then say that the elts. $b_1 \dots b_n$ <u>satisfy the polynomial equation</u> $p(\underline{t}) = 0$.

Thus the seq. $(b_1, \dots, b_n)$ is algebraically independent over A iff it does not satisfy any nontrivial polynomial equation over A.

Def A ring homomorphism $A \to B$ is called <u>finite</u> if it makes B into a finitely generated A-module.

<u>Lemma</u> ① If $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are finite ring hom's, then $g \circ f : A \to C$ is finite.

pf. <u>Exercise.</u>

<u>Lemma</u> ② Let $\mathbb{K}$ be a field, let $B := \mathbb{K}[t_1, \dots, t_n]$ be the polynomial ring, and let $b \in B$ be a nonconstant polynomial. Then there exist polynomials $b_1, \dots, b_{n-1} \in B$ s.t. B is a finitely generated module over the subring $\mathbb{K}[b_1, \dots, b_{n-1}, b]$.

The lemma asserts that the ring hom.

$$\mathbb{K}[s_1, \dots, s_n] \longrightarrow B = \mathbb{K}[t_1, \dots, t_n], \quad s_i \mapsto b_i$$

is finite. Here $s_1, \dots, s_n$ are new variables, and $b_n := b$.

(109) (Nagata)

<u>proof</u>. Write $b(t_1,\dots,t_n) = \sum_{\underline{i}} \lambda_{\underline{i}} \cdot \underline{t}^{\underline{i}}$

where $\underline{i} = (i_1,\dots,i_n) \in \mathbb{N}^n$, $\lambda_{\underline{i}} \in \mathbb{K}$ and $\underline{t}^{\underline{i}} := t_1^{i_1} \cdots t_n^{i_n}$. Let

$$I := \{\underline{i} \in \mathbb{N}^n \mid \lambda_{\underline{i}} \neq 0\},$$ which is a finite set. Choose $e \in \mathbb{N}$ large enough s.t.

$$e > i_1,\dots,i_n \quad \forall\ (i_1,\dots,i_n) \in I.$$

Consider the function

$$\mathcal{E} : I \to \mathbb{N},$$

$$\mathcal{E}(\underline{i}) = \mathcal{E}(i_1 \dots i_n) := i_n + i_1 e + i_2 e^2 + \dots + i_{n-1} e^{n-1}$$

Thus $(i_{n-1},\dots,i_1,i_n)$ is the base e expansion of the number $\mathcal{E}(\underline{i})$. We see that $\mathcal{E}$ is injective. Let $\underline{i}_{max} \in I$ be the multi-index where $\mathcal{E}$ is maximal. And let $\mathcal{E}_{max} := \mathcal{E}(\underline{i}_{max})$, $\lambda_{max} := \lambda_{\underline{i}_{max}}$. Note that $\underline{i}_{max} \neq (0,\dots,0)$, $\mathcal{E}_{max} > 0$ and $\lambda_{max} \neq 0$.

Define $b_i \in \mathbb{K}[\underline{t}]$, $1 \le i \le n-1$, to be

$$b_i := t_i - t_n^{e^i}.$$

So $$\boxed{t_i = b_i + t_n^{e^i}. \quad (+)}$$

For any $\underline{i}$ we have

$$\underline{t}^{\underline{i}} = t_1^{i_1} \cdots t_n^{i_n} = (b_1 + t_n^{e})^{i_1} \cdots (b_{n-1} + t_n^{e^{n-1}})^{i_{n-1}} \cdot t_n^{i_n}$$

Expanding this by powers of t_n we get

$$\underline{t}^{\underline{i}} = t_n^{i_n + i_1 e + \cdots + i_{n-1} e^{n-1}} + (\text{lower order terms in } t_n)$$

$$= t_n^{\varepsilon(\underline{i})} + \sum_{j=0}^{\varepsilon(\underline{i})-1} g_{\underline{i},j}(b_1,\dots,b_{n-1}) \cdot t_n^j \,, \quad g_{\underline{i},j} \in \mathbb{K}[s_1,\dots,s_{n-1}]$$

Therefore

$$b(\underline{t}) = \sum_{\underline{i} \in I} \lambda_{\underline{i}} \underline{t}^{\underline{i}} =$$

$$= \lambda_{max} \cdot t_n^{\varepsilon_{max}} + \underbrace{\sum_{j=0}^{\varepsilon_{max}-1} g_j(b_1,\dots,b_{n-1}) \cdot t_n^j}_{\substack{\text{contribs of } \underline{i} \neq \underline{i}_{max} \\ \text{and the lower order terms of } \underline{i}_{max}}}$$

$g_j \in \mathbb{K}[s_1,\dots,s_{n-1}]$

Let $A := \mathbb{K}[b_1,\dots,b_{n-1}, b] \subset \mathbb{K}[\underline{t}]$. Let s be a new variable. Define

$$h(s) := s^{\varepsilon_{max}} + \sum_{j=0}^{\varepsilon_{max}-1} \frac{1}{\lambda_{max}} \cdot g_j(b_1,\dots,b_{n-1}) \cdot s^j - \frac{1}{\lambda_{max}} \cdot b \,,$$

which is a <u>monic</u> polynomial in $A[s]$, and $h(t_n) = 0$.

Therefore t_n is integral over A, and $A[t_n]$ is finite over A. But

$$A[t_n] = B = \mathbb{K}[\underline{t}].$$

$\square$

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