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Tensor Products of Rings

Thm. Let A, B, C be (comm.) rings, and let $A \to B$ and $A \to C$ be ring homomorphisms (commut.). The abelian group $B \otimes_A C$ has a unique ring structure, whose unit is $1_B \otimes 1_C$, and whose multiplication satisfies

$$(\diamond) \qquad (b_1 \otimes c_1) \cdot (b_2 \otimes c_2) = (b_1 b_2) \otimes (c_1 c_2)$$

for all $b_1, b_2 \in B$ and $c_1, c_2 \in C$.

Proof Step 1. For $(b_1, c_1) \in B \times C$ define a function

$$\begin{cases} \beta^1_{(b_1,c_1)} : B \times C \to B \otimes_A C, \\ \beta^1_{(b_1,c_1)}(b,c) := bb_1 \otimes cc_1 . \end{cases}$$

This is an A-bilin. function, so it induces a hom.

$$\mu^1_{(b_1,c_1)} : B \otimes_A C \to B \otimes_A C, \text{ s.t. } \mu^1_{(b_1,c_1)}(b_2 \otimes c_2) = (b_1 b_2) \otimes (c_1 c_2).$$

Step 2. For $x \in B \otimes_A C$ define

$$\begin{cases} \beta^2_x : B \times C \to B \otimes_A C \\ \beta^2_x(b,c) := \mu^1_{(b,c)}(x) . \end{cases}$$

We claim that β^2_x is A-bilinear. To see why, write $x = \sum_i b_i \otimes c_i$. (finite sum) Then

$$\beta^2_x(b,c) = \mu^1_{(b,c)}(x) = \mu^1_{(b,c)}\left(\sum_i b_i \otimes c_i\right) = \sum_i \mu^1_{(b,c)}(b_i \otimes c_i)$$

$$= \sum_i \beta^2_{b_i \otimes c_i}(b,c).$$

(6.7) It suffices to show that $\beta^2_{b_i \otimes c_i}(b,c)$ is bilinear in (b,c).
But for every i:

$$\beta^2_{b_i \otimes c_i}(b,c) = \mu'_{(b,c)}(b_i \otimes c_i) = (bb_i) \otimes (cc_i),$$

and this is bilinear in (b,c).

So we get a homom.

$$\mu^2_x : B \otimes_A C \to B \otimes_A C \qquad \text{satisfying}$$

$$\mu^2_x(b_1 \otimes c_1) = \mu'_{(b_1,c_1)}(x).$$

Thus

$$\mu^2_{b_2 \otimes c_2}(b_1 \otimes c_1) = \mu'_{(b_1,c_1)}(b_2 \otimes c_2) = (b_1 b_2) \otimes (c_1 c_2).$$

Step 3. Define the multiplication

$$\mu : (B \otimes_A C) \times (B \otimes_A C) \to B \otimes_A C$$

to be

$$\mu(x,y) := \mu^2_y(x).$$

Then $\mu(b_1 \otimes c_1, b_2 \otimes c_2) = (b_1 b_2) \otimes (c_1 c_2)$.

It is now easy to see that μ is commutative, associative, distributive w.r.t. addition, and $1_B \otimes 1_C$ is a unit.

Step 4 (uniqueness). Suppose μ' is another multip. on $B \otimes_A C$, that makes it a comm. ring, and satisfies $(\diamondsuit)$.

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We need to show $\mu' = \mu$. Take $x, y \in B \otimes_A C$.
Write $x = \sum_i b_i \otimes c_i$ and $y = \sum_j d_j \otimes e_j$.
Then

$$\mu'(x,y) = \mu'\Big(\sum_i b_i \otimes c_i, \sum_j d_j \otimes e_j\Big) \overset{[\text{distrib of } \mu']}{=}$$

$$= \sum_{i,j} \mu'(b_i \otimes c_i, d_j \otimes e_j) \underset{[\text{cond. }(\Delta)]}{=} \sum_{i,j} (b_i d_j) \otimes (c_i e_j)$$

$$= \mu(x,y).$$

$\square$

Exercise

Let A be a ring and let s, t be variables. Show that

$$A[s] \otimes_A A[t] \cong A[s,t].$$

(59) Let us denote by <u>CRing</u> the category of commutative rings.

<u>Thm.</u> Let $A \xrightarrow{f} B$ and $A \xrightarrow{g} C$ be homomorphisms in <u>CRing</u>.

(1). The functions

$$u : B \longrightarrow B \otimes_A C, \quad b \mapsto b \otimes 1_C$$

and

$$v : C \longrightarrow B \otimes_A C, \quad c \mapsto 1_B \otimes c$$

are ring hom's, and $u \circ f = v \circ g$.

(2) Suppose $u' : B \to D$ and $v' : C \to D$ are hom's in <u>CRing</u> s.t.

$$u' \circ f = v' \circ g.$$

Then there is a unique hom.

$$w : B \otimes_A C \to D$$

s.t.

$$w(b \otimes c) = u'(b) \cdot v'(c) \qquad (♡)$$

for all $b \in B$ and $c \in C$.

The situation is this: a comm. diagram in <u>CRing</u>

$$\begin{tikzcd}
 & B \arrow[dr, "u"] \arrow[drr, bend left, "u'"] & & \\
A \arrow[ur, "f"] \arrow[rr] \arrow[dr, "g"'] & & B \otimes_A C \arrow[r, "w"] & D \\
 & C \arrow[ur, "v"'] \arrow[urr, bend right, "v'"'] & &
\end{tikzcd}$$

Proof (1) u is additive since $b \otimes 1_C$ is additive in b. u respects multip. by cond. (☆), and $u(1_B) = 1_B \otimes 1_C$. So u is a ring hom. Same for v.
For $a \in A$ we have

$$(u \circ f)(a) = (a \cdot 1_B) \otimes 1_C = 1_B \otimes (a \cdot 1_C) = (v \circ g)(a).$$

(2) Define a func. $\beta : B \times C \to D$ by $\beta(b, c) := u'(b) \cdot v'(c)$. This is A-bilin, so get A-lin. hom. $w : B \otimes_A C \longrightarrow D$ which satisfies (♡).

That w respects multiplication can be checked on pure tensors, and this is clear from formulas (◇) and (♡).
Uniqueness of w is verified similarly.

$\square$