<u>Prop.</u> Let A be a noeth. ring, and let $\mathfrak{a} \triangleleft A$ be an ideal. Then the ring $B := A/\mathfrak{a}$ is noetherian.

<u>pf.</u> <u>Exercise.</u>

Hilbert Basis Theorem

Theorem. Let A be a noetherian ring. Then the polynomial ring $A[t]$ is noetherian.

The following terminology is needed for the proof. Let $f(t) = \sum_{i=0}^{n} a_i t^i$ be a polynomial of deg. $n \geq 0$ (so $a_n \neq 0$).
The element $a_n \in A$ is called the leading coefficient of f.

proof. Take any ideal $\mathfrak{b}$ in the ring $A[t]$. We have to prove it is fin. gen. Let us assume the contrary (and show a contradiction).

Construct a sequence $(f_1, f_2, \dots)$ of elements of $\mathfrak{b}$, as follows:

Let $\mathfrak{b}_{i-1}$ be the ideal gen. by $f_1, \dots, f_{i-1}$ (so $\mathfrak{b}_0 = 0$).

By assumption $\mathfrak{b}_{i-1} \neq \mathfrak{b}$. Choose f_i to be an element of minimal degree in $\mathfrak{b} - \mathfrak{b}_{i-1}$.

We have a seq. of ideals $\mathfrak{b}_1 \subsetneq \mathfrak{b}_2 \subsetneq \dots$ in $\mathfrak{b}$.
For any i let a_i be the leading coefficient of $f_i(t)$. Define $\mathfrak{a} \subset A$ to be the ideal gen. by $a_1, a_2, \dots$.

(9.8) Since A is noetherian, the ideal $\mathfrak{a}$ is fin. gen. Hence $\mathfrak{a} = \sum_{j=1}^{i} A \cdot a_j$ for i.

Therefore we have

$$a_{i+1} = \sum_{j=1}^{i} c_j a_j$$

for some $c_j \in A$.

Define the polynomial

$$g := \sum_{j=1}^{i} c_j f_j(t) \cdot t^{\deg(f_{i+1}) - \deg(f_j)}$$

This is a poly. of $\deg(g) = \deg(f_{i+1})$, with leading coefficient $a_{i+1} \neq 0$. Also $g \in \mathfrak{b}_i$. Now

$$h := f_{i+1} - g$$

has $\deg(h) < \deg(f_{i+1})$, and $h \in \mathfrak{b}_{i+1} - \mathfrak{b}_i \subset \mathfrak{b} - \mathfrak{b}_i$.

This contradicts our choice of f_{i+1}. $\square$

Cor. Suppose A is a noetherian ring and B is a finitely generated A-ring. Then B is noetherian.

pf. We can write B as $A[t_1,\dots,t_n]/\mathfrak{b}$ for some $n \geq 1$, and some ideal $\mathfrak{b}$. By Prop. -?-, it suffices to prove that the poly. ring $A[t_1,\dots,t_n]$ is noetherian. But this is a consequence of the basis thm. & induction. $\square$

An A ring B is called <u>essentially finite type</u> if B is a localization $B'_{S'}$ of some fin. type (i.e. fin. gen.) A-ring B', at some m.c. set $S' \subset B'$.

Cor. If A is noetherian and B is an ess. fin. type A-ring, then B is noetherian.

pf. Use last corollary and Thm. -?-. $\square$

<u>Example</u> Let $\mathbb{K}$ be a field or $\mathbb{Z}$, let A be a fin. gen. $\mathbb{K}$-ring, and let $\mathfrak{p} \in \operatorname{Spec} A$. Then the local ring $A_{\mathfrak{p}}$ is noetherian.

Locally free modules

<u>Def</u>. Let A be a ring. An A-module P is called <u>locally free</u> if for every $\mathfrak{p} \in \operatorname{Spec} A$, the $A_{\mathfrak{p}}$-module $P_{\mathfrak{p}}$ is free.

It is not easy to find fin. gen. mods. that are loc. free but not free. For $A = \mathbb{Z}$, $K[t_1, \dots, t_n]$ there are non. They exist for $A := \mathbb{Z}[\sqrt{-5}]$ (number theory) and
$$A := \mathbb{C}[s,t] / \left(t^2 - s(s-1)(s+1)\right)$$
(alg. geom - elliptic curves).

<u>Example</u> $A := \mathbb{Z}/(12)$, $M := \mathbb{Z}/(4)$. $\operatorname{Spec} A = \{\overset{\mathfrak{p}}{(2)}, \overset{\mathfrak{q}}{(3)}\}$

$M_{\mathfrak{p}} \cong A_{\mathfrak{p}} = \mathbb{Z}/(4)$, $\quad M_{\mathfrak{q}} = 0$ $\quad A \cong \frac{\mathbb{Z}}{(4)} \times \frac{\mathbb{Z}}{(3)}$

rank = 1 $\qquad$ rank = 0

<u>Theorem</u> Let A be a local noetherian ring, and let P be a fin. gen. A-module.

TFAE:

(i) P is free.

(ii) P is projective.

Pf. The implications (i) $\Rightarrow$ (ii) is known (and does not require A to be local.)

(10) So let's assume P is projective. Write $K := A/m$ and $\overline{P} := P/mP \cong K \otimes_A P$. Choose a basis $\overline{p}_1, \dots, \overline{p}_r$ for $\overline{P}$, as K-module. Let $p_1, \dots, p_r \in P$ arbitrary lifts of this basis, i.e. $\pi(p_i) = \overline{p}_i$, where $\pi : P \to \overline{P}$ is the can. surject. By Nakayama the elts. $p_1, \dots, p_r$ generate P as A-mod.

Consider the short exact seq. of A-mods

$$(*) \quad 0 \to N \xrightarrow{\psi} A^{\oplus r} \xrightarrow{\varphi} P \to 0 ,$$

where $\varphi(a_1, \dots) := \sum a_i p_i$, and $N := \mathrm{Ker}(\varphi)$.

Since A is noetherian, the mod. N is fin. gen. Because P is projective the seq. $(*)$ is split: There is $\sigma : P \to A^{\oplus r}$ s.t. $\boxed{\varphi \circ \sigma = 1\!\!1_P}$. Hence there is $\tau : A^{\oplus r} \to N$ s.t. $\boxed{\tau \circ \psi = 1\!\!1_N}$. Also $\boxed{1\!\!1_{A^{\oplus r}} = \sigma \circ \varphi + \psi \circ \tau}$.

Let's apply the additive functor $K \otimes_A -$ to $(*)$. We get a sequence of K-mods

$$(**) \quad 0 \to \overline{N} \xrightarrow{\overline{\psi}} K^{\oplus r} \xrightarrow{\overline{\varphi}} \overline{P} \to 0,$$

$\overline{N} := K \otimes_A N$, and $\overline{\varphi} \circ \overline{\psi} = 0$.

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There are homs $\bar{\sigma}$ and $\bar{\tau}$ s.t. $\boxed{\bar{\varphi} \circ \bar{\sigma} = 1\!\!1_{\bar{P}}}$ and $\boxed{\bar{\tau} \circ \bar{\varphi} = 1\!\!1_{\bar{N}}}$ and $\boxed{1\!\!1_{\mathrm{Ker}} = \bar{\sigma} \circ \bar{\psi} + \bar{\varphi} \circ \bar{\tau}.}$ Therefore (††) is split, and in particular it's exact.

Note that $\bar{\varphi}(\bar{a}_1, \dots) = \sum_i \bar{a}_i \bar{p}_i$.

Because $(\bar{p}_1, \dots, \bar{p}_r)$ is a basis, the hom. $\bar{\varphi}$ is bijective. Hence $\bar{N} = 0$.

Finally, by Nakayama, $\bar{N} = 0$ implies $N = 0$. So φ is bijective, and $(p_1, \dots, p_r)$ is a basis of P. $\square$

In the proof above we have also proved the next useful thing:

Prop. Let $F : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, B$ be an additive functor. If

$$0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$$

is a split exact seq. in $\underline{\mathrm{Mod}}\, A$, then

$$0 \to F(M') \xrightarrow{F(\varphi)} F(M) \xrightarrow{F(\psi)} F(M'') \to 0$$

is a split exact seq.

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