BEN GURION UNIVERSITY אוניברסיטת בן גוריון בנגב

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Derived Categories I

Course in 1st Semester 2015-16

Audience & Prerequisites. This is an advanced course, aimed at M.Sc. and Ph.D. students, post-docs and researchers. *Participants from outside the BGU community are welcome.* The lectures will be in English.

The prerequisite for this course is a solid knowledge of classical homological algebra. For some of the examples and applications, a familiarity with commutative algebra, ring theory, algebraic geometry and algebraic topology will be helpful.

Organization. The course will meet once a week for a 2 hour lecture.
Time: Wednesday 12:00-14:00
Location: building 58 room -101 (basement)
First Meeting: 28 October 2015
Course web page:
www.math.bgu.ac.il/~amyekut/teaching/2015-16/der-cats-I/course_page.html

If there is demand, we might add an exercise session. (Note that the BGU AG&NT Seminar is at Wednesday 15:00.) Potential participants are urged to get in touch regarding information about registration. The course is expected to continue in the 2nd semester. The course notes are likely to be published as a textbook.

On the subject. *Derived categories* were invented by Grothendieck and Verdier around 1960, not very long after the classical homological algebra (of derived functors between abelian categories) was established. This "new" homological algebra, of derived categories and derived functors between them, provides a significantly richer and more flexible machinery than the classical homological algebra.

Here are the main ideas of the theory. Consider an abelian category M (e.g. $\mathsf{M} = \mathsf{Mod}\, A$, the category of left modules over a ring A). The *objects* of the derived category $\mathsf{D}(\mathsf{M})$ are the *complexes* $M = (\cdots \to M^i \xrightarrow{\mathrm{d}} M^{i+1} \to \cdots)$, $M^i \in \mathsf{M}$. A homomorphism of complexes $\psi : M \to N$ is called a *quasi-isomorphism* if the induced homomorphisms $\mathrm{H}^i(\psi) : \mathrm{H}^i(M) \to \mathrm{H}^i(N)$ are all isomorphisms. The *morphisms* in $\mathsf{D}(\mathsf{M})$ are the fractions $\phi \circ \psi^{-1}$, where ϕ and ψ are homomorphisms of complexes, and the denominator ψ is a quasi-isomorphism. Now consider an additive functor $F : \mathsf{M} \to \mathsf{N}$ between abelian categories. Under suitable conditions (existence of resolutions), the functor F has *right and left derived functors* $\mathrm{R}F, \mathrm{L}F$:

$\mathsf{D}(\mathsf{M}) \to \mathsf{D}(\mathsf{N})$. The classical derived functors $\mathrm{R}^i F, \mathrm{L}^i F : \mathsf{M} \to \mathsf{N}$ can be recovered easily as $\mathrm{R}^i F = \mathrm{H}^i(\mathrm{R}F)$ and $\mathrm{L}^i F = \mathrm{H}^i(\mathrm{L}F)$. But the full derived functors $\mathrm{R}F$ and $\mathrm{L}F$ are much more potent; for instance, the spectral sequences for the composition of classical derived functors are mere shadows of the composed derived functors.

Some results from classical homological algebra have easier or better proofs using derived categories. But more importantly, there are operations and constructions (such as dualizing complexes or perverse sheaves) that cannot even be expressed in terms of classical homological algebra.

In recent years derived categories have come to play central roles in various areas of mathematics (and even in theoretical physics). The theory itself has developed in many ways (for instance with the advent of unbounded resolutions, around 1990). More recently, geometry is also becoming derived – researchers see a need for more flexible notions of spaces and operations between them, analogous to the passage from modules to complexes of modules.

In this course we intend to present the theory of derived categories from the point of view of research in algebraic geometry and ring theory. Thus, our emphasis will be on constructions and applications, rather than axiomatics.

Topics. Here is a tentative list of topics for the course (including the 2nd part). Some of the material is quite new and still not in textbooks.

(1) Review of abelian categories and additive functors.
(2) Differential graded (DG) rings, modules and categories.
(3) Triangulated categories and derived functors.
(4) The derived category of a pretriangulated additive DG category.
(5) Resolutions of DG modules (K-projective, K-injective and K-flat resolutions).
(6) Commutative algebra via derived categories. Dualizing complexes, local duality, MGM equivalence, rigid dualizing complexes.
(7) Geometric derived categories (of sheaves on spaces). Direct and inverse image functors, Grothendieck duality, Poicaré-Verdier duality, perverse sheaves.
(8) Derived categories associated to nocommutative rings. Dualizing complexes, tilting complexes, the derived Picard group, derived Morita theory.
(9) Survey of derived categories in modern algebraic geometry and mathematical physics. Survey of derived algebraic geometry.

Bibliography.

(1) Yekutieli, “A Course on Derived Categories”, `http://arxiv.org/abs/1206.6632v2`.
(2) Hartshorne, “Residues and Duality”, Springer.
(3) Kashiwara and Schapira, “Sheaves on Manifolds”, Springer.
(4) Kashiwara and Schapira, “Categories and Sheaves”, Springer.
(5) Lipman, in: “Foundations of Grothendieck duality for diagrams of schemes”, Springer.
(6) Weibel, “An introduction to homological algebra”, Cambridge.
(7) Gelfand and Manin, “Methods of Homological Algebra”, Springer.
(8) Neeman, “Triangulated Categories”, Princeton.
(9) de Jong (ed.), “Stacks Project”, `http://stacks.math.columbia.edu`.