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אוניברסיטת בן גוריון בנגב

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המחלקה למתמטיקה
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A COURSE ON DERIVED CATEGORIES

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Contents

0. Introduction

These are notes for an advanced course given at Ben Gurion University in the academic year 2015-16. In this course I am following various sources, mostly the earlier course notes [Ye5] and the mini-course [Ye6]. Some new material is also included. See the References for other sources. Some further material will be posted on the course web page [CWP].

I want to thank the participants of the course in Spring 2012 for correcting many of my mistakes, both in real time during the lectures, and afterwards when writing the notes [Ye5]. Thanks also to J. Lipman, P. Schapira, A. Neeman and C. Weibel for helpful discussions on the material in [Ye5].

0.1. **A motivating discussion: duality.** By way of introduction to the subject, let us consider *duality*. Take a field K. Given a K-module M (i.e. a vector space), let

$$D(M) := \mathrm{Hom}_K(M, K),$$

be the dual module. There is a canonical homomorphism

$$\eta_M : M \to D(D(M)),$$

$\eta_M(m)(\phi) := \phi(m)$ for $m \in M$ and $\phi \in D(M)$. If M is finitely generated then η_M is an isomorphism (actually this is "if and only if").

To formalize this situation, let $\mathsf{Mod}\, K$ denote the category of K-modules. Then

$$D : \mathsf{Mod}\, K \to \mathsf{Mod}\, K$$

is a contravariant functor, and

$$\eta : \mathrm{id} \to D \circ D$$

is a natural transformation. Here id is the identity functor of $\mathsf{Mod}\, K$.

Now let us replace K by any nonzero commutative ring A. Again we can define a contravariant functor

$$D : \mathsf{Mod}\, A \to \mathsf{Mod}\, A, \quad D(M) := \mathrm{Hom}_A(M, A),$$

and a natural transformation $\eta : \mathrm{id} \to D \circ D$. It is easy to see that $\eta_M : M \to D(D(M))$ is an isomorphism if M is a finitely generated free module. Of course we can't expect reflexivity (i.e. η_M being an isomorphism) if M is not finitely generated; but what about a finitely generated module that is not free?

In order to understand this better, let us concentrate on the ring $A = \mathbb{Z}$. Since $\mathbb{Z}$-modules are just abelian groups, the category $\mathsf{Mod}\, \mathbb{Z}$ is often denoted by Ab. Let Ab_{f} be the full subcategory of finitely generated abelian groups. Any finitely generated abelian group is of the form $M \cong T \oplus F$, with F free and T finite (the letter "T" stands for "torsion".). It is important to note that this is *not a canonical isomorphism.* There is a canonical short exact sequence

$$0 \to T \xrightarrow{\phi} M \xrightarrow{\psi} F \to 0, \tag{0.1.1}$$

but the decomposition $M \cong T \oplus F$ comes from *choosing a splitting* $\sigma : F \to M$ of this sequence.

Exercise 0.1.2. Prove that the exact sequence (0.1.5) is functorial (i.e. natural); namely there are functors $T, F : \mathsf{Ab}_{\mathrm{f}} \to \mathsf{Ab}_{\mathrm{f}}$, and natural transformations $\phi : T \to \mathrm{id}$ and $\psi : \mathrm{id} \to F$, such that for any $M \in \mathsf{Ab}_{\mathrm{f}}$, the group $T(M)$ is finite; the group $F(M)$ is free; and the sequence of homomorphisms

$$0 \to T(M) \xrightarrow{\phi_M} M \xrightarrow{\psi_M} F(M) \to 0 \tag{0.1.3}$$

is exact.

Next, prove that there does not exist a *functorial decomposition* of a finitely generated abelian group into a free part and a finite part. Namely, there is no natural transformation $\sigma : F \to \mathrm{id}$, such that for every M, the homomorphism $\sigma_M : F(M) \to M$ splits the sequence (0.1.3). (Hint: find a counterexample.)

We know that for the free abelian group F there is reflexivity, i.e. $\eta_F : F \to D(D(F))$ is an isomorphism. But for the finite abelian group T we have

$$D(T) = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Z}) = 0.$$

Thus, whenever $T \neq 0$, reflexivity fails: $\eta_M : M \to D(D(M))$ is not an isomorphism.

On the other hand, for an abelian group M we can define another sort of dual:

$$D'(M) := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}).$$

There is a natural transformation $\eta' : \mathrm{id} \to D' \circ D'$. For a finite abelian group T the homomorphism $\eta'_T : T \to D'(D'(T))$ is an isomorphism; this can be seen by decomposing T into cyclic groups, and for a finite cyclic group it is clear. So D' is a duality for finite abelian groups. (We may view the abelian group $\mathbb{Q}/\mathbb{Z}$ as the group of roots of 1 in $\mathbb{C}$, via the exponential function; and then D' becomes *Pontryagin Duality.*)

But for a finitely generated free abelian group F we get $D'(D'(F)) = \widehat{F}$, the profinite completion of F. So once more this is not a good duality for all finitely generated abelian groups.

We could try to be more clever and "patch" the two dualities D and D', into something that we will call $D \oplus D'$. This looks pleasing at first – but then we recall that the decomposition $M \cong T \oplus F$ of a finitely generated group is not functorial, so that $D \oplus D'$ can't be a functor.

Here is where the *derived category* enters. For any commutative ring A there is the derived category $\mathsf{D}(\mathsf{Mod}\, A)$. Here is a very quick explanation of it.

Recall that a *complex* of A-modules is a diagram

$$M^{\cdot} = \big(\cdots \to M^{-1} \xrightarrow{\mathrm{d}_M^{-1}} M^0 \xrightarrow{\mathrm{d}_M^{0}} M^1 \to \cdots\big) \tag{0.1.4}$$

in $\mathsf{Mod}\, A$. Here M^i are A-modules, the differentials (or coboundary operators) $\mathrm{d}_M^i : M^i \to M^{i+1}$ are A-linear homomorphisms, and $\mathrm{d}_M^{i+1} \circ \mathrm{d}_M^i = 0$.

Given a second complex

$$N^{\cdot} = \big(\cdots \to N^{-1} \xrightarrow{\mathrm{d}_N^{-1}} N^0 \xrightarrow{\mathrm{d}_N^{0}} N^1 \to \cdots\big),$$

a *homomorphism of complexes* $\phi^{\cdot} : M^{\cdot} \to N^{\cdot}$ is a collection of homomorphisms $\phi^i : M^i \to N^i$ satisfying

$$\phi^{i+1} \circ \mathrm{d}_M^i = \mathrm{d}_N^i \circ \phi^i.$$

The resulting category is denoted by $\mathsf{C}(\mathsf{Mod}\, A)$.

The i-th *cohomology* of $M^{\cdot}$ is

$$\mathrm{H}^i(M^{\cdot}) := \frac{\mathrm{Ker}(\mathrm{d}_M^i)}{\mathrm{Im}(\mathrm{d}_M^{i-1})} \in \mathsf{Mod}\, A.$$

A homomorphism $\phi^{\cdot} : M^{\cdot} \to N^{\cdot}$ induces homomorphisms

$$\mathrm{H}^i(\phi^{\cdot}) : \mathrm{H}^i(M^{\cdot}) \to \mathrm{H}^i(N^{\cdot}).$$

We call $\phi^{\cdot}$ a *quasi-isomorphism* if all the homomorphisms $\mathrm{H}^i(\phi^{\cdot})$ are isomorphisms.

The derived category $\mathsf{D}(\mathsf{Mod}\,A)$ has the same objects are $\mathsf{C}(\mathsf{Mod}\,A)$, namely the complexes. There is a functor

$$\mathsf{Q} : \mathsf{C}(\mathsf{Mod}\,A) \to \mathsf{D}(\mathsf{Mod}\,A)$$

that is the identity of objects. If $\psi^{\cdot}$ is quasi-isomorphism in $\mathsf{C}(\mathsf{Mod}\,A)$, the morphism $\mathsf{Q}(\psi^{\cdot})$ is invertible in $\mathsf{D}(\mathsf{Mod}\,A)$, i.e. it is an isomorphisms. The morphisms in $\mathsf{D}(\mathsf{Mod}\,A)$ are all of the form

$$\mathsf{Q}(\phi^{\cdot}) \circ \mathsf{Q}(\psi^{\cdot})^{-1},$$

where $\phi^{\cdot}$ is any morphism in $\mathsf{C}(\mathsf{Mod}\,A)$, and $\psi^{\cdot}$ is any quasi-morphism in $\mathsf{C}(\mathsf{Mod}\,A)$ (and of course these are composable morphisms, re. source and target).

A single A-module M can be viewed as a complex concentrated in degree 0:

$$M^{\cdot} = (\cdots \to 0 \xrightarrow{0} M \xrightarrow{0} 0 \to \cdots). \tag{0.1.5}$$

In other words, $M^0 = M$ and the rest are 0. This turns out to be fully faithful embedding

$$\mathsf{Mod}\,A \to \mathsf{D}(\mathsf{Mod}\,A). \tag{0.1.6}$$

Moreover, any complex $M^{\cdot}$ whose cohomology is concentrated in degree 0, (i.e. $\mathrm{H}^i(M^{\cdot}) = 0$ for all $i \neq 0$) is isomorphic in $\mathsf{D}(\mathsf{Mod}\,A)$ to the module $\mathrm{H}^0(M)$. In this way we have *enlarged* the category of A-modules.

Here is a very impotant kind of quasi-isomorphism. Suppose

$$\cdots \to P^{-2} \xrightarrow{\mathrm{d}_P^{-2}} P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \xrightarrow{\epsilon^0} M \to 0 \tag{0.1.7}$$

is a free resolution of a module M. Let $M^{\cdot}$ be the complex from (0.1.5), and let $P^{\cdot}$ be the complex

$$P^{\cdot} = (\cdots \to P^{-2} \xrightarrow{\mathrm{d}_P^{-2}} P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \to 0 \to \cdots).$$

Then, letting $\epsilon^i := 0$ for $i \neq 0$, we get a quasi-isomorphism

$$\epsilon^{\cdot} : P^{\cdot} \to M^{\cdot}$$

in $\mathsf{C}(\mathsf{Mod}\,A)$, and thus an isomorphism

$$\mathsf{Q}(\epsilon^{\cdot}) : P^{\cdot} \to M^{\cdot}$$

in $\mathsf{D}(\mathsf{Mod}\,A)$.

Let us now return to $A = \mathbb{Z}$. The functor $D = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ from $\mathsf{Mod}\,\mathbb{Z}$ to itself has a *right derived functor*

$$\mathrm{R}D = \mathrm{RHom}_{\mathbb{Z}}(-, \mathbb{Z}),$$

which is a contravariant *triangulated functor*

$$\mathrm{R}D : \mathsf{D}(\mathsf{Mod}\,\mathbb{Z}) \to \mathsf{D}(\mathsf{Mod}\,\mathbb{Z}).$$

And there is a natural transformation of triangulated functors

$$\eta : \mathrm{id} \to \mathrm{R}D \circ \mathrm{R}D.$$

Here is the way to calculate the functor $\mathrm{R}D$, at least for a finitely generated abelian group M. Let us choose a free resolution of M like in (0.1.7). To be easy on ourselves, we take it to be like this:

$$P^{\cdot} = (\cdots \to 0 \to P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \to 0 \to \cdots) = (\cdots \to 0 \to \mathbb{Z}^{r_1} \xrightarrow{\mathrm{d}} \mathbb{Z}^{r_0} \to 0 \cdots),$$

where $r_0, r_1 \in \mathbb{N}$ and d is a matrix of integers. Because $\mathsf{Q}(\epsilon^{\cdot}) : P^{\cdot} \to M^{\cdot}$ is an isomorphism in $\mathsf{D}(\mathsf{Mod}\,\mathbb{Z})$, it suffices to calculate $\mathrm{R}D(P^{\cdot})$.

Now by construction,

$$\mathrm{R}D(P^{\cdot}) = D(P^{\cdot}) = \mathrm{Hom}_{\mathbb{Z}}(P^{\cdot}, \mathbb{Z}),$$

where the complex $\mathrm{Hom}_{\mathbb{Z}}(P^{\cdot}, \mathbb{Z})$ is

$$\mathrm{Hom}_{\mathbb{Z}}(P^{\cdot}, \mathbb{Z}) = \big(\cdots \to 0 \to \mathbb{Z}^{r_0} \xrightarrow{\mathrm{d}^*} \mathbb{Z}^{r_1} \to 0 \cdots\big),$$

concentrated in degrees 0 and 1, with the transpose matrix d^* are the differential.

Because $\mathrm{R}D(P^{\cdot}) = D(P^{\cdot})$ is itself a bounded complex of finite free modules, its derived dual is

$$\mathrm{R}D(\mathrm{R}D(P^{\cdot})) = D(D(P^{\cdot})) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(P^{\cdot}, \mathbb{Z}), \mathbb{Z}).$$

The canonical morphism (in $\mathsf{C}(\mathsf{Mod}\,\mathbb{Z})$)

$$\eta_P : P^{\cdot} \to D(D(P^{\cdot}))$$

is an isomorphism; and therefore

$$\eta_M : M^{\cdot} \to \mathrm{R}D(\mathrm{R}D(M^{\cdot}))$$

is an isomorphism in $\mathsf{D}(\mathsf{Mod}\,\mathbb{Z})$.

We see that $\mathrm{R}D$ *is a duality that holds for all finitely generated* $\mathbb{Z}$*-modules* !

Here is the connection between the derived duality $\mathrm{R}D$ and the “classical” dualities D and D'. Take a finitely generated abelian group M, with short exact sequence (0.1.1). There are functorial isomorphisms

$$\mathrm{H}^0(\mathrm{R}D(M)) \cong \mathrm{Ext}^0_{\mathbb{Z}}(M, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \cong D(M)$$

and

$$\mathrm{H}^1(\mathrm{R}D(M)) \cong \mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z}) \cong D'(M).$$

The cohomologies $\mathrm{H}^i(\mathrm{R}D(M)) = 0$ for $i \neq 0, 1$.

Note that $D(M) \cong D(F)$ and $D'(M) \cong D'(T)$. We see that if M is neither free nor finite, then both $\mathrm{H}^0(\mathrm{R}D(M))$ and $\mathrm{H}^1(\mathrm{R}D(M))$ are both nonzero; so that the complex $D(M)$ is not isomorphic to an object of $\mathsf{Mod}\,\mathbb{Z}$, under the embedding (0.1.6).

This sort of duality holds for *many noetherian commutative rings* A. But the formula for the duality functor

$$\mathrm{R}D : \mathsf{D}(\mathsf{Mod}\,A) \to \mathsf{D}(\mathsf{Mod}\,A)$$

is somewhat different – it is

$$\mathrm{R}D(M) := \mathrm{RHom}_A(M, R),$$

where $R \in \mathsf{D}(\mathsf{Mod}\,A)$ is a *dualizing complex*. Such a dualizing complex is unique (up to shift and tensoring with an invertible module).

Interestingly, the structure of the dualizing complex R depends on the geometry of the ring A (i.e. of the scheme $\mathrm{Spec}\,A$). If A is a regular ring (like $\mathbb{Z}$) then $R = A$ is dualizing. If A is Cohen-Macaulay (and $\mathrm{Spec}\,A$ is connected) then R is a single A-module. But if A is a more complicated ring, then R must live in several degrees.

Example 0.1.8. Consider the affine algebraic variety $X \subseteq \mathbf{A}^3_{\mathbb{R}}$ which is the union of a plane and a line, with coordinate ring

$$A = \mathbb{R}[t_1, t_2, t_3]/(t_3 \cdot t_1,\ t_3 \cdot t_2).$$

See figure 1. The dualizing complex R must live in two adjacent degrees; namely there is some i s.t. $\mathrm{H}^i(R)$ and $\mathrm{H}^{i+1}(R)$ are nonzero.

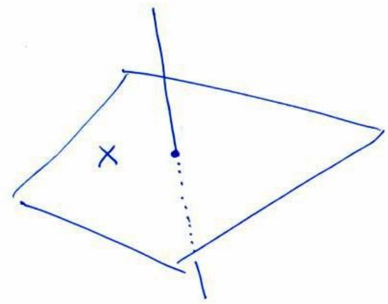

Figure 1. An algebraic variety that is connected but not equidimensional, and hence not Cohen-Macaulay.

One can also talk about dualizing complexes over *noncommutative rings*. (This is a favorite topic of mine!)

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