BEN GURION UNIVERSITY OF THE NEGEV

אוניברסיטת בן גוריון בנגב

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המחלקה למתמטיקה
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A COURSE ON DERIVED CATEGORIES

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Contents

0. INTRODUCTION

These are notes for an advanced course given at Ben Gurion University in the academic year 2015-16. In this course I am following various sources, mostly the earlier course notes [Ye6] and the mini-course [Ye7]. Some new material is also included. See the References for other sources. Some further material will be posted on the course web page [CWP].

I want to thank the participants of the course in Spring 2012 for correcting many of my mistakes, both in real time during the lectures, and afterwards when writing the notes [Ye6]. Thanks also to J. Lipman, P. Schapira, A. Neeman and C. Weibel for helpful discussions on the material in [Ye6].

0.1. **A motivating discussion: duality.** By way of introduction to the subject, let us consider *duality.* Take a field K. Given a K-module M (i.e. a vector space), let

$$D(M) := \operatorname{Hom}_K(M, K),$$

be the dual module. There is a canonical homomorphism

$$\eta_M : M \to D(D(M)),$$

$\eta_M(m)(\phi) := \phi(m)$ for $m \in M$ and $\phi \in D(M)$. If M is finitely generated then η_M is an isomorphism (actually this is "if and only if").

To formalize this situation, let $\mathsf{Mod}\, K$ denote the category of K-modules. Then

$$D : \mathsf{Mod}\, K \to \mathsf{Mod}\, K$$

is a contravariant functor, and

$$\eta : \mathrm{id} \to D \circ D$$

is a natural transformation. Here id is the identity functor of $\mathsf{Mod}\, K$.

Now let us replace K by any nonzero commutative ring A. Again we can define a contravariant functor

$$D : \mathsf{Mod}\, A \to \mathsf{Mod}\, A, \quad D(M) := \operatorname{Hom}_A(M, A),$$

and a natural transformation $\eta : \mathrm{id} \to D \circ D$. It is easy to see that $\eta_M : M \to D(D(M))$ is an isomorphism if M is a finitely generated free module. Of course we can't expect reflexivity (i.e. η_M being an isomorphism) if M is not finitely generated; but what about a finitely generated module that is not free?

In order to understand this better, let us concentrate on the ring $A = \mathbb{Z}$. Since $\mathbb{Z}$-modules are just abelian groups, the category $\mathsf{Mod}\, \mathbb{Z}$ is often denoted by Ab. Let Ab_{f} be the full subcategory of finitely generated abelian groups. Any finitely generated abelian group is of the form $M \cong T \oplus F$, with F free and T finite (the letter "T" stands for "torsion".). It is important to note that this is *not a canonical isomorphism.* There is a canonical short exact sequence

$$0 \to T \xrightarrow{\phi} M \xrightarrow{\psi} F \to 0, \tag{0.1.1}$$

but the decomposition $M \cong T \oplus F$ comes from *choosing a splitting* $\sigma : F \to M$ of this sequence.

Exercise 0.1.2. Prove that the exact sequence (0.1.5) is functorial (i.e. natural); namely there are functors $T, F : \mathsf{Ab}_{\mathrm{f}} \to \mathsf{Ab}_{\mathrm{f}}$, and natural transformations $\phi : T \to \mathrm{id}$ and $\psi : \mathrm{id} \to F$, such that for any $M \in \mathsf{Ab}_{\mathrm{f}}$, the group $T(M)$ is finite; the group $F(M)$ is free; and the sequence of homomorphisms

$$0 \to T(M) \xrightarrow{\phi_M} M \xrightarrow{\psi_M} F(M) \to 0 \tag{0.1.3}$$

is exact.

Next, prove that there does not exist a *functorial decomposition* of a finitely generated abelian group into a free part and a finite part. Namely, there is no natural transformation $\sigma : F \to \mathrm{id}$, such that for every M, the homomorphism $\sigma_M : F(M) \to M$ splits the sequence (0.1.3). (Hint: find a counterexample.)

We know that for the free abelian group F there is reflexivity, i.e. $\eta_F : F \to D(D(F))$ is an isomorphism. But for the finite abelian group T we have

$$D(T) = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Z}) = 0.$$

Thus, whenever $T \neq 0$, reflexivity fails: $\eta_M : M \to D(D(M))$ is not an isomorphism.

On the other hand, for an abelian group M we can define another sort of dual:

$$D'(M) := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}).$$

There is a natural transformation $\eta' : \mathrm{id} \to D' \circ D'$. For a finite abelian group T the homomorphism $\eta'_T : T \to D'(D'(T))$ is an isomorphism; this can be seen by decomposing T into cyclic groups, and for a finite cyclic group it is clear. So D' is a duality for finite abelian groups. (We may view the abelian group $\mathbb{Q}/\mathbb{Z}$ as the group of roots of 1 in $\mathbb{C}$, via the exponential function; and then D' becomes *Pontryagin Duality.*)

But for a finitely generated free abelian group F we get $D'(D'(F)) = \widehat{F}$, the profinite completion of F. So once more this is not a good duality for all finitely generated abelian groups.

We could try to be more clever and "patch" the two dualities D and D', into something that we will call $D \oplus D'$. This looks pleasing at first – but then we recall that the decomposition $M \cong T \oplus F$ of a finitely generated group is not functorial, so that $D \oplus D'$ can't be a functor.

Here is where the *derived category* enters. For any commutative ring A there is the derived category $\mathbf{D}(\mathsf{Mod}\, A)$. Here is a very quick explanation of it.

Recall that a *complex* of A-modules is a diagram

$$M^{\cdot} = \big(\cdots \to M^{-1} \xrightarrow{\mathrm{d}_M^{-1}} M^0 \xrightarrow{\mathrm{d}_M^0} M^1 \to \cdots\big) \tag{0.1.4}$$

in $\mathsf{Mod}\, A$. Here M^i are A-modules, the differentials (or coboundary operators) $\mathrm{d}_M^i : M^i \to M^{i+1}$ are A-linear homomorphisms, and $\mathrm{d}_M^{i+1} \circ \mathrm{d}_M^i = 0$.

Given a second complex

$$N^{\cdot} = \big(\cdots \to N^{-1} \xrightarrow{\mathrm{d}_N^{-1}} N^0 \xrightarrow{\mathrm{d}_N^0} N^1 \to \cdots\big),$$

a *homomorphism of complexes* $\phi^{\cdot} : M^{\cdot} \to N^{\cdot}$ is a collection of homomorphisms $\phi^i : M^i \to N^i$ satisfying

$$\phi^{i+1} \circ \mathrm{d}_M^i = \mathrm{d}_N^i \circ \phi^i.$$

The resulting category is denoted by $\mathbf{C}(\mathsf{Mod}\, A)$.

The i-th *cohomology* of $M^{\cdot}$ is

$$\mathrm{H}^i(M^{\cdot}) := \frac{\mathrm{Ker}(\mathrm{d}_M^i)}{\mathrm{Im}(\mathrm{d}_M^{i-1})} \in \mathsf{Mod}\, A.$$

A homomorphism $\phi^{\cdot} : M^{\cdot} \to N^{\cdot}$ induces homomorphisms

$$\mathrm{H}^i(\phi^{\cdot}) : \mathrm{H}^i(M^{\cdot}) \to \mathrm{H}^i(N^{\cdot}).$$

We call $\phi^{\cdot}$ a *quasi-isomorphism* if all the homomorphisms $\mathrm{H}^i(\phi^{\cdot})$ are isomorphisms.

The derived category $\mathbf{D}(\mathsf{Mod}\,A)$ has the same objects are $\mathbf{C}(\mathsf{Mod}\,A)$, namely the complexes. There is a functor

$$\mathrm{Q} : \mathbf{C}(\mathsf{Mod}\,A) \to \mathbf{D}(\mathsf{Mod}\,A)$$

that is the identity of objects. If $\psi^{\cdot}$ is quasi-isomorphism in $\mathbf{C}(\mathsf{Mod}\,A)$, the morphism $\mathrm{Q}(\psi^{\cdot})$ is invertible in $\mathbf{D}(\mathsf{Mod}\,A)$, i.e. it is an isomorphisms. The morphisms in $\mathbf{D}(\mathsf{Mod}\,A)$ are all of the form

$$\mathrm{Q}(\phi^{\cdot}) \circ \mathrm{Q}(\psi^{\cdot})^{-1},$$

where $\phi^{\cdot}$ is any morphism in $\mathbf{C}(\mathsf{Mod}\,A)$, and $\psi^{\cdot}$ is any quasi-morphism in $\mathbf{C}(\mathsf{Mod}\,A)$ (and of course these are composable morphisms, re. source and target).

A single A-module M can be viewed as a complex concentrated in degree 0:

$$M^{\cdot} = (\cdots \to 0 \xrightarrow{0} M \xrightarrow{0} 0 \to \cdots). \tag{0.1.5}$$

In other words, $M^0 = M$ and the rest are 0. This turns out to be fully faithful embedding

$$\mathsf{Mod}\,A \to \mathbf{D}(\mathsf{Mod}\,A). \tag{0.1.6}$$

Moreover, any complex $M^{\cdot}$ whose cohomology is concentrated in degree 0, (i.e. $\mathrm{H}^i(M^{\cdot}) = 0$ for all $i \neq 0$) is isomorphic in $\mathbf{D}(\mathsf{Mod}\,A)$ to the module $\mathrm{H}^0(M)$. In this way we have *enlarged* the category of A-modules.

Here is a very impotant kind of quasi-isomorphism. Suppose

$$\cdots \to P^{-2} \xrightarrow{\mathrm{d}_P^{-2}} P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \xrightarrow{\epsilon^0} M \to 0 \tag{0.1.7}$$

is a free resolution of a module M. Let $M^{\cdot}$ be the complex from (0.1.5), and let $P^{\cdot}$ be the complex

$$P^{\cdot} = (\cdots \to P^{-2} \xrightarrow{\mathrm{d}_P^{-2}} P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \to 0 \to \cdots).$$

Then, letting $\epsilon^i := 0$ for $i \neq 0$, we get a quasi-isomorphism

$$\epsilon^{\cdot} : P^{\cdot} \to M^{\cdot}$$

in $\mathbf{C}(\mathsf{Mod}\,A)$, and thus an isomorphism

$$\mathrm{Q}(\epsilon^{\cdot}) : P^{\cdot} \to M^{\cdot}$$

in $\mathbf{D}(\mathsf{Mod}\,A)$.

Let us now return to $A = \mathbb{Z}$. The functor $D = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ from $\mathsf{Mod}\,\mathbb{Z}$ to itself has a *right derived functor*

$$\mathrm{R}D = \mathrm{RHom}_{\mathbb{Z}}(-, \mathbb{Z}),$$

which is a contravariant *triangulated functor*

$$\mathrm{R}D : \mathbf{D}(\mathsf{Mod}\,\mathbb{Z}) \to \mathbf{D}(\mathsf{Mod}\,\mathbb{Z}).$$

And there is a natural transformation of triangulated functors

$$\eta : \mathrm{id} \to \mathrm{R}D \circ \mathrm{R}D.$$

Here is the way to calculate the functor $\mathrm{R}D$, at least for a finitely generated abelian group M. Let us choose a free resolution of M like in (0.1.7). To be easy on ourselves, we take it to be like this:

$$P^{\cdot} = (\cdots \to 0 \to P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \to 0 \to \cdots) = (\cdots \to 0 \to \mathbb{Z}^{r_1} \xrightarrow{\mathrm{d}} \mathbb{Z}^{r_0} \to 0 \cdots),$$

where $r_0, r_1 \in \mathbb{N}$ and d is a matrix of integers. Because $\mathrm{Q}(\epsilon^{\cdot}) : P^{\cdot} \to M^{\cdot}$ is an isomorphism in $\mathbf{D}(\mathsf{Mod}\,\mathbb{Z})$, it suffices to calculate $\mathrm{R}D(P^{\cdot})$.

Now by construction,

$$\mathrm{R}D(P^{\cdot}) = D(P^{\cdot}) = \mathrm{Hom}_{\mathbb{Z}}(P^{\cdot}, \mathbb{Z}),$$

where the complex $\mathrm{Hom}_{\mathbb{Z}}(P^{\cdot}, \mathbb{Z})$ is

$$\mathrm{Hom}_{\mathbb{Z}}(P^{\cdot}, \mathbb{Z}) = \big(\cdots \to 0 \to \mathbb{Z}^{r_0} \xrightarrow{\mathrm{d}^*} \mathbb{Z}^{r_1} \to 0 \cdots\big),$$

concentrated in degrees 0 and 1, with the transpose matrix d^* are the differential.

Because $\mathrm{R}D(P^{\cdot}) = D(P^{\cdot})$ is itself a bounded complex of finite free modules, its derived dual is

$$\mathrm{R}D(\mathrm{R}D(P^{\cdot})) = D(D(P^{\cdot})) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(P^{\cdot}, \mathbb{Z}), \mathbb{Z}).$$

The canonical morphism (in $\mathbf{C}(\mathsf{Mod}\,\mathbb{Z})$)

$$\eta_P : P^{\cdot} \to D(D(P^{\cdot}))$$

is an isomorphism; and therefore

$$\eta_M : M^{\cdot} \to \mathrm{R}D(\mathrm{R}D(M^{\cdot}))$$

is an isomorphism in $\mathbf{D}(\mathsf{Mod}\,\mathbb{Z})$.

We see that $\mathrm{R}D$ *is a duality that holds for all finitely generated* $\mathbb{Z}$*-modules* !

Here is the connection between the derived duality $\mathrm{R}D$ and the "classical" dualities D and D'. Take a finitely generated abelian group M, with short exact sequence (0.1.1). There are functorial isomorphisms

$$\mathrm{H}^0(\mathrm{R}D(M)) \cong \mathrm{Ext}^0_{\mathbb{Z}}(M, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \cong D(M)$$

and

$$\mathrm{H}^1(\mathrm{R}D(M)) \cong \mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z}) \cong D'(M).$$

The cohomologies $\mathrm{H}^i(\mathrm{R}D(M)) = 0$ for $i \neq 0, 1$.

Note that $D(M) \cong D(F)$ and $D'(M) \cong D'(T)$. We see that if M is neither free nor finite, then both $\mathrm{H}^0(\mathrm{R}D(M))$ and $\mathrm{H}^1(\mathrm{R}D(M))$ are both nonzero; so that the complex $D(M)$ is not isomorphic to an object of $\mathsf{Mod}\,\mathbb{Z}$, under the embedding (0.1.6).

This sort of duality holds for *many noetherian commutative rings* A. But the formula for the duality functor

$$\mathrm{R}D : \mathbf{D}(\mathsf{Mod}\,A) \to \mathbf{D}(\mathsf{Mod}\,A)$$

is somewhat different – it is

$$\mathrm{R}D(M) := \mathrm{RHom}_A(M, R),$$

where $R \in \mathbf{D}(\mathsf{Mod}\,A)$ is a *dualizing complex.* Such a dualizing complex is unique (up to shift and tensoring with an invertible module).

Interestingly, the structure of the dualizing complex R depends on the geometry of the ring A (i.e. of the scheme $\mathrm{Spec}\,A$). If A is a regular ring (like $\mathbb{Z}$) then $R = A$ is dualizing. If A is Cohen-Macaulay (and $\mathrm{Spec}\,A$ is connected) then R is a single A-module. But if A is a more complicated ring, then R must live in several degrees.

Example 0.1.8. Consider the affine algebraic variety $X \subseteq \mathbf{A}^3_{\mathbb{R}}$ which is the union of a plane and a line, with coordinate ring

$$A = \mathbb{R}[t_1, t_2, t_3]/(t_3 \cdot t_1,\, t_3 \cdot t_2).$$

See figure 1. The dualizing complex R must live in two adjacent degrees; namely there is some i s.t. $\mathrm{H}^i(R)$ and $\mathrm{H}^{i+1}(R)$ are nonzero.

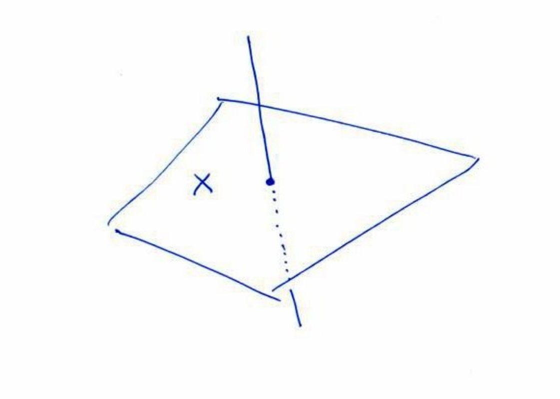

Figure 1. An algebraic variety that is connected but not equidimensional, and hence not Cohen-Macaulay.

One can also talk about dualizing complexes over *noncommutative rings*. (This is a favorite topic of mine!)

1. Basics Facts on Categories

1.1. **Set Theory.** In this course we will not try to be precise about issues of set theory. The blanket assumption is that we are given a *Grothendieck universe* U. This is a "large" infinite set. A *small set*, or a U-small set, is a set S that is an element of U. We want all the products $\prod_{i \in I} S_i$ and disjoint unions $\coprod_{i \in I} S_i$, with I and S_i small sets, to be small sets too. A U-*category* is a category C whose set of objects $\operatorname{Ob}(\mathsf{C})$ is a subset of U, and for every $C, D \in \operatorname{Ob}(\mathsf{C})$ the set of morphisms $\operatorname{Hom}_{\mathsf{C}}(C, D)$ is small. See [SGA 4] or [KS2, Section 1.1]. Another approach, involving "sets" vs "classes", can be found in [Ne].

We denote by Set the category of all small sets. So $\operatorname{Ob}(\mathsf{Set}) = \mathsf{U}$, and Set is a U-category. A group (or a ring, etc.) is called small if its underlying set is small. We denote by Grp, Ab, Ring and $\mathsf{Ring}_{\mathsf{c}}$ the categories of small groups, small abelian groups, small rings and small commutative rings respectively. For a small ring A we denote by $\mathsf{Mod}\, A$ the category of all small left A-modules.

By default we work with U-categories, and from now on U will remain implicit. The one exception is when we deal with localization of categories, where we shall briefly encounter a set theoretical issue; but for most interesting cases this issue has an easy solution.

1.2. **Notation.** Let C be a category. We often write $C \in \mathsf{C}$ as an abbreviation for $C \in \operatorname{Ob}(\mathsf{C})$. For an object C, its identity automorphism is denoted by id_C. The identity functor of C is denoted by id_{C}.

The opposite category of C is C^{op}. It has the same objects as C, but the morphism sets are

$$\operatorname{Hom}_{\mathsf{C}^{\mathrm{op}}}(C_0, C_1) := \operatorname{Hom}_{\mathsf{C}}(C_1, C_0),$$

and composition is reversed. The identity functor of C can be viewed as a contravariant functor $\mathrm{id}_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}^{\mathrm{op}}$. A contravariant functor $F : \mathsf{C} \to \mathsf{D}$ is the same as a covariant functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$. Since we prefer dealing only with covariant functors, we make the following convention:

Convention 1.2.1. By default all functors will be covariant, unless explicitly mentioned otherwise.

We will try to keep the following font and letter conventions:

- $f : C \to D$ is a morphism between objects in a category.
- $F : \mathsf{C} \to \mathsf{D}$ is a functor between categories.
- $\eta : F \to G$ is morphism of functors (i.e. a natural transformation) between functors $F, G : \mathsf{C} \to \mathsf{D}$.
- $f, \phi, \alpha : M \to N$ are morphisms between objects in an abelian category M.
- $F : \mathsf{M} \to \mathsf{N}$ is an additive functor between abelian categories.
- The derived category of an abelian category M is $\mathbf{D}(\mathsf{M})$.
- If M is a module category, and $M \in \operatorname{Ob}(\mathsf{M})$, then elements of M will be denoted by $m, n, m_i, \ldots$.

1.3. **Zero objects.** Let C be a category. Recall that a morphism $f : C \to D$ in C is called an *isomorphism* if there is a morphism $g : D \to C$ such that $f \circ g = \mathrm{id}_D$ and $g \circ f = \mathrm{id}_C$. The morphism g is called the *inverse* of f, it is unique (if it exists), and it is denoted by f^{-1}.

The morphism $f : C \to D$ in C is called an *epimorphism* if it has the right cancellation property: for any $g, g' : D \to E$, $g \circ f = g' \circ f$ implies $g = g'$. The morphism $f : C \to D$

is called a *monomorphism* if it has the left cancellation property: for any $g, g' : E \to C$, $f \circ g = f \circ g'$ implies $g = g'$.

Example 1.3.1. In Set the monomorphisms are the injections, and the epimorphisms are the surjections. A morphism $f : C \to D$ in Set that is both a monomorphism and an epimorphism is an isomorphism. The same holds in the category $\mathsf{Mod}\, A$ of left modules over a ring A.

This example could be misleading, because the property of being an epimorphism is often not preserved by forgetful functors, as the next exercise shows.

Exercise 1.3.2. Consider the category of rings Ring. (All rings have units, and ring homomorphisms are unital.) Show that the forgetful functor $\mathsf{Ring} \to \mathsf{Set}$ respects monomorphisms, but it does not respect epimorphisms. (Hint: show that the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in Ring.) [1]

By a *subobject* of an object $C \in \mathsf{C}$ we mean a monomorphism $f : C' \to C$ in C. We sometimes write $C' \hookrightarrow C$ or $C' \subseteq C$ in this situation, but this is only notational (and does not mean inclusion of sets). Likewise, by a *quotient* of C we mean an epimorphism $g : C \to \bar{C}$ in C, and then sometimes write $C' \twoheadrightarrow C$.

An *initial object* in a category C is an object $C_0 \in \mathsf{C}$, such that for every object $C \in \mathsf{C}$ there is exactly one morphism $C_0 \to C$. Thus the set $\operatorname{Hom}_{\mathsf{C}}(C_0, C)$ is a singleton. A *terminal object* in C is an object $C_\infty \in \mathsf{C}$, such that for every object $C \in \mathsf{C}$ there is exactly one morphism $C \to C_\infty$.

Definition 1.3.3. A *zero object* in a category C is an object which is both initial and terminal.

Initial, terminal and zero objects are unique up to unique isomorphisms (but they need not exist).

Example 1.3.4. In Set, $\varnothing$ is an initial object, and any singleton is a terminal object. There is no zero object.

Example 1.3.5. In $\mathsf{Mod}\, A$, any trivial module (with only the zero element) is a zero object, and we denote this module by 0. This is allowed, since any other zero module is uniquely isomorphic to it.

1.4. **Products and Coproducts.** Let C be a category. By a *collection of objects of* C indexed by a (small) set I, we mean a function $I \to \operatorname{Ob}(\mathsf{C})$, $i \mapsto C_i$. We usually denote this collection like this: $\{C_i\}_{i \in I}$.

Given a a collection $\{C_i\}_{i \in I}$ of objects of C, its *product* is a pair $(C, \{p_i\}_{i \in I})$ consisting of an object $C \in \mathsf{C}$, and morphisms $p_i : C \to C_i$ for all $i \in I$, called *projections.* The pair $(C, \{p_i\}_{i \in I})$ must have this universal property: given any object D and morphisms $f_i : D \to C_i$, there is a unique morphism $f : D \to C$ s.t. $f_i = p_i \circ f$. Of course if a product $(C, \{p_i\}_{i \in I})$ exists, then it is unique up to a unique isomorphism; and we usualy write $\prod_{i \in I} C_i := C$, leaving the projection morphisms implicit.

Example 1.4.1. In Set and $\mathsf{Mod}\, A$ all products exist, and they are the usual cartesian products.

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[1] In an early version of [Ye6] we claimed that this happens for the category Grp; but this is false. We thank Vincent Beck for this correction.

For a collection $\{C_i\}_{i\in I}$ of objects of C, their *coproduct* is a pair $(C, \{e_i\}_{i\in I})$, consisting of an object C and morphisms $e_i : C_i \to C$, called *embeddings*. The pair $(C, \{e_i\}_{i\in I})$ must have this universal property: given any object D and morphisms $f_i : C_i \to D$, there is a unique morphism $f : C \to D$ s.t. $f_i = f \circ e_i$. If a product $(C, \{e_i\}_{i\in I})$ exists, then it is unique up to a unique isomorphism; and we write $\coprod_{i\in I} C_i := C$, leaving the embeddings implicit.

Example 1.4.2. In Set the coproduct is the disjoint union. In $\mathsf{Mod}\, A$ the coproduct is the direct sum.

1.5. **Equivalence.** Recall that a functor $F : \mathsf{C} \to \mathsf{D}$ is an *equivalence* if there exists a functor $G : \mathsf{D} \to \mathsf{C}$, and isomorphisms of functors (i.e. natural isomorphisms) $G \circ F \xrightarrow{\simeq} \mathrm{id}_{\mathsf{C}}$ and $F \circ G \xrightarrow{\simeq} \mathrm{id}_{\mathsf{D}}$. Such a functor G is called a *quasi-inverse* of F, and it is unique up to isomorphism (if it exists), and it is denoted by F^{-1}.

The functor $F : \mathsf{C} \to \mathsf{D}$ is *full* (resp. *faithful*) if every $C_0, C_1 \in \mathsf{C}$ the function

$$F : \mathrm{Hom}_{\mathsf{C}}(C_0, C_1) \to \mathrm{Hom}_{\mathsf{D}}(F(C_0), F(C_1))$$

is surjective (resp. injective).

We know that $F : \mathsf{C} \to \mathsf{D}$ is an equivalence iff these two conditions hold:

(i) F is essentially surjective on objects. This means that for every $D \in \mathsf{D}$ there is some $C \in \mathsf{C}$ and an isomorphism $F(C) \xrightarrow{\simeq} D$.
(ii) F is fully faithful (i.e. full and faithful).

Exercise 1.5.1. If you are not sure about the last claim (characterization of equivalences), then prove it. (Hint: use the axiom of choice to construct a quasi-inverse of F.)

Example 1.5.2. Let C and D be categories. A functor $F : \mathsf{C} \to \mathsf{D}$ is an *isomorphism* if it is bijective of sets of objects and on sets of morphisms. It is clear that an isomorphic of categories is an equivalence. In practice, it is very rare to find an isomorphism of categories.

1.6. **Bifunctors.** Let C and D be categories. Their product is the category $\mathsf{C} \times \mathsf{D}$ defined as follows: the set of objects is

$$\mathrm{Ob}(\mathsf{C} \times \mathsf{D}) := \mathrm{Ob}(\mathsf{C}) \times \mathrm{Ob}(\mathsf{D}).$$

The sets of morphisms are

$$\mathrm{Hom}_{\mathsf{C}\times\mathsf{D}}\left((C_0, D_0), (C_1, D_1)\right) := \mathrm{Hom}_{\mathsf{C}}(C_0, C_1) \times \mathrm{Hom}_{\mathsf{D}}(D_0, D_1).$$

The composition is

$$(f_1, g_1) \circ (f_0, g_0) := (f_1 \circ f_0, g_1 \circ g_0),$$

and the identity morphisms are $(\mathrm{id}_C, \mathrm{id}_D)$.

A *bifunctor*

$$F : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$$

is by definition a functor from the product category $\mathsf{C} \times \mathsf{D}$ to E. We say “bifunctor” because it is a functor of two arguments: $F(C, D) \in \mathsf{E}$. This will be especially useful when considering additive categories, because then we can talk about “bi-additive bifunctors”.

1.7. **Representable Funtors.** Let C be a category and $C \in \mathsf{C}$ an object. We get a functor

$$Y_C : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}, \quad Y_C := \mathrm{Hom}_{\mathsf{C}}(-, C),$$

called the *Yoneda functor*. This functor sends a morphism $\phi : C_0 \to C_1$ in C to

$$Y(\phi) := \mathrm{Hom}(\phi, \mathrm{id}_C) : Y_{C_1} \to Y_{C_0}.$$

Here is the first formulation of the *Yoneda Lemma.*

Proposition 1.7.1 (Yoneda Lemma 1)**.** *Let* C *be a category, let* $C_0, C_1 \in \mathsf{C}$ *be objects, and let* $\eta : Y_{C_1} \to Y_{C_0}$ *be a morphism of functors* $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$*.*

(1) *There exists a unique morphism* $\phi : C_0 \to C_1$ *in* C *such that* $Y(\phi) = \eta$*.*
(2) *If* $\eta : Y_{C_1} \to Y_{C_0}$ *is an isomorphism of functors, then* $\phi : C_0 \to C_1$ *is an isomorphism in* C*.*

See [KS2] for a proof.

A functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is called *representable* if $F \cong Y_C$ for some object $C \in \mathsf{C}$. The object C is said to represent the functor F. In this case there is a universal element $f \in F(C)$, corresponding to $\mathrm{id}_C \in Y_C(C)$. By the Yoneda Lemma, The pair (C, f) is unique up to a unique isomorphism (if it exists).

Here is a fancier way to state this result. Consider the category $\mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$, whose objects are the functors $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$, and whose morphisms are the morphisms of functors (the natural transformations). There is a set-theoretic difficulty here: the sets of objects and morphisms of $\mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ are too big (unless C is a small category); so this is not a U-category, and we must enlarge the universe.

Proposition 1.7.2 (Yoneda Lemma 2)**.** *The Yoneda functor*

$$Y : \mathsf{C} \to \mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$$

is fully faithful.

In other words, the Yoneda Lemma says that the functor Y is an equivalence from C to the category of representable functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$.

Dually, any $C \in \mathsf{C}$ gives rise to a functor

$$Y'_C : \mathsf{C} \to \mathsf{Set}, \quad Y'_C := \mathrm{Hom}_{\mathsf{C}}(C, -).$$

The identity automorphism id_C is a special element of the set $Y'_C(C)$.

A functor $F : \mathsf{C} \to \mathsf{Set}$ is called *corepresentable* if $F \cong Y'_C$ for some object $C \in \mathsf{C}$. The object C is said to corepresent the functor F. In this case there is a universal element $f \in F(C)$, corresponding to $\mathrm{id}_C \in Y'_C(C)$. The pair (C, f) is unique up to a unique isomorphism (if it exists). The dual Yoneda Lemma says that the functor $C \mapsto F'_C$ is an equivalence from C^{op} to the category of representable functors $\mathsf{C} \to \mathsf{Set}$.

2. ABELIAN CATEGORIES

The concept of *abelian category* is an extremely useful abstraction of module categories, introduced by Grothendieck in 1957. Before definint it (in Definition 2.3.8), we need some preparation.

2.1. Linear categories.

Definition 2.1.1. Let $\mathbb{K}$ be a commutative ring. A $\mathbb{K}$-*linear category* is a category M, endowed with a $\mathbb{K}$-module structure on each of the morphism sets $\mathrm{Hom}_{\mathsf{M}}(M_0, M_1)$. The condition is this:

- For all $M_0, M_1, M_2 \in \mathsf{M}$ the composition function
$$\begin{aligned}&\mathrm{Hom}_{\mathsf{M}}(M_1, M_2) \times \mathrm{Hom}_{\mathsf{M}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{M}}(M_0, M_2)\\ &(\phi_2, \phi_1) \mapsto \phi_2 \circ \phi_1\end{aligned}$$
is $\mathbb{K}$-bilinear.

If $\mathbb{K} = \mathbb{Z}$, we say that M is a *linear category*.

Let $\mathbb{K}$ be a commutative ring. By *central* $\mathbb{K}$-*ring* we mean a ring A, with a ring homomorphism $\mathbb{K} \to A$, such that the image of $\mathbb{K}$ is inside the center of A. (Many texts would call such A a "unital associative $\mathbb{K}$-algebra".)

Example 2.1.2. Let $\mathbb{K}$ be any nonzero commutative ring, and let n be a positive integer. Then the ring of matrices $A := \mathrm{Mat}_n(\mathbb{K})$ is a central $\mathbb{K}$-ring.

Proposition 2.1.3. *Let* M *be a* $\mathbb{K}$-*linear category.*

1. *For any object* $M \in \mathsf{M}$*, the set*
$$\mathrm{End}_{\mathsf{M}}(M) := \mathrm{Hom}_{\mathsf{M}}(M, M),$$
with its given addition operation, and with the operation of composition, is a central $\mathbb{K}$-*ring.*
2. *For any two objects* $M_0, M_1 \in \mathsf{M}$*, the set* $\mathrm{Hom}_{\mathsf{M}}(M_0, M_1)$*, with its given addition operation, and with the operations of composition, is a left module over the ring* $\mathrm{End}_{\mathsf{M}}(M_1)$*, and a right module over the ring* $\mathrm{End}_{\mathsf{M}}(M_0)$*. Furthermore, these left and right actions commute with each other.*

Proof. Exercise. □

This result can be reversed:

Example 2.1.4. Let A be a central $\mathbb{K}$-ring. Define a category M like this: there is a single object M, and its set of morphisms is
$$\mathrm{Hom}_{\mathsf{M}}(M, M) := A.$$
Composition in M is the multiplication of A. Then M is a $\mathbb{K}$-linear category.

Because of the above, in a linear category M, we often denote the identity automorphism of an object M by $1_M := \mathrm{id}_M \in \mathrm{End}_{\mathsf{M}}(M)$.

For a central $\mathbb{K}$-ring A, the opposite ring A^{op} has the same $\mathbb{K}$-module structure as A, but the multiplication is reversed.

Exercise 2.1.5. Let A be a nonzero ring. Let $P, Q \in \mathsf{Mod}\, A$ be distinct free A-modules of rank 1.

(1) Prove that there is a ring isomorphism $\mathrm{End}_{\mathsf{Mod}\, A}(P) \cong A^{\mathrm{op}}$. Is this ring isomorphism canonical?

(2) Let M be the full subcategory of $\mathsf{Mod}\, A$ on the set of objects $\{P, Q\}$. Compare the linear category M to the ring of matrices $\mathrm{Mat}_2(A^{\mathrm{op}})$.

2.2. **Additive categories.**

Definition 2.2.1. An *additive category* is a linear category M satisfying these conditions:

(i) M has a zero object 0.

(ii) M has finite coproducts.

Observe that $\mathrm{Hom}_{\mathsf{M}}(M, N) \neq \varnothing$ for any $M, N \in \mathsf{M}$, since this is an abelian group. Also

$$\mathrm{Hom}_{\mathsf{M}}(M, 0) = \mathrm{Hom}_{\mathsf{M}}(0, M) = 0,$$

the zero abelian group. We denote the unique arrows $0 \to M$ and $M \to 0$ also by 0. So the numeral 0 has a lot of meanings; but they are (hopefully) clear from the contexts. The coproduct in a linear category M is usually denoted by $\bigoplus$; cf. Example 1.4.2.

Example 2.2.2. Let A be a $\mathbb{K}$-central ring. The category $\mathsf{Mod}\, A$ is a $\mathbb{K}$-linear additive category. The full subcategory $\mathsf{F} \subseteq \mathsf{Mod}\, A$ on the free modules is also additive.

Proposition 2.2.3. *Let M be a linear category. Let $\{M_i\}_{i \in I}$ be a finite collection of objects of M, and assume the coproduct $M = \bigoplus_{i \in I} M_i$ exists, with embeddings $e_i : M_i \to M$.*

(1) *For any i let $p_i : M \to M_i$ be the unique morphism s.t. $p_i \circ e_i = 1_{M_i}$, and $p_i \circ e_j = 0$ for $j \neq i$. Then $(M, \{p_i\}_{i \in I})$ is a product of the collection $\{M_i\}_{i \in I}$.*

(2) $\sum_{i \in I} e_i \circ p_i = 1_M$.

Proof. Exercise. □

Part (1) directly implies:

Corollary 2.2.4. *An additive category has finite products.*

Definition 2.2.5. Let M be an additive category, and let N be a full subcategory of M. We say that N is a *full additive subcategory* of M if N is closed under finite direct sums.

Exercise 2.2.6. In the situation of Definition 2.2.5, the category N is itself additive.

Example 2.2.7. Consider the linear category M from Example 2.1.4, built from a ring A. It does not have a zero object (unless the ring A is the zero ring), so it is not additive.

A more puzzling question is this: Does M have finite direct sums? This turns out to be equivalent to whether or not $A \cong A \oplus A$ as right A-modules. To see why, choose a fully faithful additive functor $F : \mathsf{M} \to \mathsf{Mod}\, A^{\mathrm{op}}$, that sends the unique object $M \in \mathsf{M}$ to a rank 1 free right A-module P. (We identify right A-modules with left A^{op}-modules.) Compare to Exercise 2.1.5.

Let $I := \{1, 2\}$, and let $\{M_i\}_{i \in I}$ be the only possible collection in M indexed by I (i.e. $M_i = M$). If there is a coproduct in M, then it must be $M_1 \oplus M_2 \cong M$. According to Proposition 2.4.2, we get

$$P \oplus P \cong F(M_1) \oplus F(M_2) \cong F(M) \cong P$$

in $\mathsf{Mod}\, A^{\mathrm{op}}$.

We know that when A is nonzero and commutative, or nonzero and noetherian, then $A \not\cong A \oplus A$ in $\mathsf{Mod}\, A^{\mathrm{op}}$. On the other hand, if we take a field $\mathbb{K}$, and a countable rank $\mathbb{K}$-module N, then $A := \mathrm{End}_{\mathbb{K}}(N)$ will satisfy $A \cong A \oplus A$.

Proposition 2.2.8. *Let* M *be a linear category, and* $N \in \mathsf{M}$*. The following conditions are equivalent*:

(i) *The ring* $\mathrm{End}_{\mathsf{M}}(N)$ *is trivial.*
(ii) N *is a zero object of* M.

Proof. (ii) ⇒ (i): Since the set $\mathrm{End}_{\mathsf{M}}(N)$ is a singleton, it must be the trivial ring $(1 = 0)$.

(i) ⇒ (ii): If the ring $\mathrm{End}_{\mathsf{M}}(N)$ is trivial, then all left and right modules over it must be trivial. Now use Proposition 2.1.3(2). □

2.3. **Abelian categories.**

Definition 2.3.1. Let M be an additive category, and let $f : M \to N$ be a morphism in M. A *kernel* of f is a pair (K, k), consisting of an object $K \in \mathsf{M}$ and a morphism $k : K \to M$, with these properties:

(i) $f \circ k = 0$.
(ii) If $k' : K' \to M$ is a morphism in M such that $f \circ k' = 0$, then there is a unique morphism $g : K' \to K$ such that $k' = k \circ g$.

In other words, the object K represents the functor $\mathsf{M}^{\mathrm{op}} \to \mathsf{Ab}$,

$$K' \mapsto \{k' \in \mathrm{Hom}_{\mathsf{M}}(K', M) \mid f \circ k' = 0\}.$$

The kernel of f is of course unique up to a unique isomorphism (if it exists), and we denote if by $\mathrm{Ker}(f)$. Sometimes $\mathrm{Ker}(f)$ refers only to the object K, and other times it refers only to the morphism k; as usual, this should be clear from the context.

Definition 2.3.2. Let M be an additive category, and let $f : M \to N$ be a morphism in M. A *cokernel* of f is a pair (C, c), consisting of an object $C \in \mathsf{M}$ and a morphism $c : N \to C$, with these properties:

(i) $c \circ f = 0$.
(ii) If $c' : N \to C'$ is a morphism in M such that $c' \circ f = 0$, then there is a unique morphism $g : C \to C'$ such that $c' = g \circ c$.

In other words, the object C corepresents the functor $\mathsf{M} \to \mathsf{Ab}$,

$$C' \mapsto \{c' \in \mathrm{Hom}_{\mathsf{M}}(M, C') \mid c' \circ f = 0\}.$$

The cokernel of f is of course unique up to a unique isomorphism (if it exists), and we denote if by $\mathrm{Coker}(f)$. Sometimes $\mathrm{Coker}(f)$ refers only to the object C, and other times it refers only to the morphism c; as usual, this should be clear from the context.

Example 2.3.3. In $\mathsf{Mod}\, A$ all kernels and cokernels exist. Given $f : M \to N$, the kernel is $k : K \to M$, where

$$K := \{m \in M \mid f(m) = 0\},$$

and the k is the inclusion. The cokernel is $c : N \to C$, where $C := N/f(M)$, and c is the canonical projection.

Proposition 2.3.4. *Let* M *be an additive category, and let* $f : M \to N$ *be a morphism in* M.

(1) *If* $k : K \to M$ *is a kernel of* f*, then* k *is a monomorphism.*
(2) *If* $c : N \to C$ *is a cokernel of* f*, then* c *is an epimorphism.*

Proof. Exercise. □

Definition 2.3.5. Assume the additive category M has kernels and cokernels. Let $f : M \to N$ be a morphism in M.

(1) Define the *image* of f to be
$$\operatorname{Im}(f) := \operatorname{Ker}(\operatorname{Coker}(f)).$$

(2) Define the *coimage* of f to be
$$\operatorname{Coim}(f) := \operatorname{Coker}(\operatorname{Ker}(f)).$$

The image is familiar, but the coimage is not. The next diagram should help. We start with a morphism $f : M \to N$ in M. The kernel and cokernel of f fit into this diagram:
$$K \xrightarrow{k} M \xrightarrow{f} N \xrightarrow{c} C.$$
Inserting $\alpha := \operatorname{Coker}(k) = \operatorname{Coim}(f)$ and $\beta := \operatorname{Ker}(c) = \operatorname{Im}(f)$ we get the following commutative diagram (solid arrows):

(2.3.6)
$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[dr, "0"'] & M \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, dashed, "\gamma"] & N \arrow[r, "c"] & C \\
& M' \arrow[r, dashed, "f'"'] & N' \arrow[u, "\beta"'] \arrow[ur, "0"'] &
\end{tikzcd}$$

Since $c \circ f = 0$ there is a unique morphism γ making the diagram commutative. Now $\beta \circ \gamma \circ k = f \circ k = 0$; and β is a monomorphism; so $\gamma \circ k = 0$. Hence there is a unique morphism $f' : M' \to N'$ making the diagram commutative. We conclude that $f : M \to N$ induces a morphism

(2.3.7)
$$f' : \operatorname{Coim}(f) \to \operatorname{Im}(f).$$

Definition 2.3.8. An *abelian category* is an additive category M with these extra properties:

(i) All morphisms in M admit kernels and cokernels.
(ii) For any morphism $f : M \to N$ in M, the induced morphism f' in equation (2.3.7) is an isomorphism.

Here is a less precise but (maybe) easier to remember way to state property (ii). Because $M' = \operatorname{Coker}(\operatorname{Ker}(f))$ and $N' = \operatorname{Ker}(\operatorname{Coker}(f))$, we see that

(2.3.9)
$$\operatorname{Coker}(\operatorname{Ker}(f)) = \operatorname{Ker}(\operatorname{Coker}(f)).$$

From now on we forget all about the coimage.

Exercise 2.3.10. For any ring A, the category $\mathsf{Mod}\, A$ is abelian.

This includes the category $\mathsf{Ab} = \mathsf{Mod}\, \mathbb{Z}$, from which the name derives.

Definition 2.3.11. Let M be an abelian category, and let N be a full subcategory of M. We say that N is a *full abelian subcategory* of M if N is closed under finite direct sums, kernels and cokernels.

Exercise 2.3.12. In the situation of Definition 2.3.11, the category N is itself abelian.

Example 2.3.13. Let M_1 be the category of finitely generated abelian groups, and let M_0 be the category of finite abelian groups. Then M_1 is a full abelian subcategory of Ab, and M_0 is a full abelian subcategory of M_1.

Exercise 2.3.14. Let N be the full subcategory of Ab whose objects are the finitely generated free abelian groups. It is an additive subcategory of Ab (since it is closed under direct sums).

(1) Show that N is closed under kernels in Ab.
(2) Show that N is not closed under cokernels in Ab, so it is not a full abelian subcategory of Ab.
(3) Show that N has cokernels (not the same as those of Ab). Still, it fails to be an abelian category.

Exercise 2.3.15. Of course Grp is not an additive category. Still it has a zero object (the trivial group). Show that Grp has kernels and cokernels, but condition (ii) of Definition 2.3.8 fails.

Exercise 2.3.16. Let Hilb be the category of Hilbet spaces over $\mathbb{C}$. The morphisms are the $\mathbb{C}$-linear homomorphisms $f : M \to N$ that respect the inner products. Show that Hilb is a $\mathbb{C}$-linear additive category with kernels and cokernels, but it is not an abelian category.

Exercise 2.3.17. Let A be a ring. Show that A is *left noetherian* iff the category $\mathsf{Mod}_{\mathrm{f}}\, A$ of finitely generated left modules is a full abelian subcategory of $\mathsf{Mod}\, A$.

Example 2.3.18. Let $(X, \mathcal{A})$ be a ringed space; namely X is a topological space and $\mathcal{A}$ is a sheaf of rings on X. We denote by $\mathsf{PMod}\ \mathcal{A}$ the category of presheaves of left $\mathcal{A}$-modules on X. This is an abelian category. Given a morphism $f : \mathcal{M} \to \mathcal{N}$ in $\mathsf{PMod}\ \mathcal{A}$, its kernel is the presheaf $\mathcal{K}$ defined by

$$\Gamma(U, \mathcal{K}) := \mathrm{Ker}\,(f : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N}))$$

on every open set $U \subseteq X$. The cokernel is the presheaf $\mathcal{C}$ defined by

$$\Gamma(U, \mathcal{C}) := \mathrm{Coker}\,(f : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})).$$

Now let $\mathsf{Mod}\, \mathcal{A}$ be the full subcategory of $\mathsf{PMod}\ \mathcal{A}$ consisting of sheaves. It is a full additive subcategory of $\mathsf{PMod}\ \mathcal{A}$, closed under kernels. We know that $\mathsf{Mod}\, \mathcal{A}$ is not closed under cokernels inside $\mathsf{PMod}\ \mathcal{A}$, and hence it is not a full abelian subcategory.

However $\mathsf{Mod}\, \mathcal{A}$ is itself an abelian category, but with different cokernels. Indeed, for a morphism $f : \mathcal{M} \to \mathcal{N}$ in $\mathsf{Mod}\, \mathcal{A}$, its cokernel $\mathrm{Coker}_{\mathsf{Mod}\, \mathcal{A}}(f)$ is the sheafification of the presheaf $\mathrm{Coker}_{\mathsf{PMod}\ \mathcal{A}}(f)$.

For educational purposes we state without proof:

Theorem 2.3.19 (Freyd & Mitchell)**.** *Let M be a small abelian category. Then M is equivalent to a full abelian subcategory of $\mathsf{Mod}\, A$, for a suitable ring A.*

This means that most of the time we can pretend that $\mathsf{M} \subseteq \mathsf{Mod}\, A$; this could be a helpful heuristic.

Proposition 2.3.20. (1) *Let M be an additive category. Then the opposite category M^{op} is also additive.*
(2) *Let M be an abelian category. Then the opposite category M^{op} is also abelian.*

Proof. (1) First note that

$$\mathrm{Hom}_{\mathsf{M}^{\mathrm{op}}}(M, N) = \mathrm{Hom}_{\mathsf{M}}(N, M),$$

so this is an abelian group. The bilinearity of the composition in M^{op} is clear, and the zero objects are the same. Existence of finite coproducts in M^{op} is because of existence of finite products in M; see Proposition 2.2.3(1).

(2) M^{op} has kernels and cokernels, since $\mathrm{Ker}_{\mathsf{M}^{\mathrm{op}}}(f) = \mathrm{Coker}_{\mathsf{M}}(f)$ and vice versa. Also the symmetric condition (ii) of Definition 2.3.8 holds. ☐

Proposition 2.3.21. *Let $f : M \to N$ be a morphism in an abelian category M.*

1. *f is a monomorphism iff $\mathrm{Ker}(f) = 0$.*
2. *f is an epimorphism iff $\mathrm{Coker}(f) = 0$.*
3. *f is an isomorphism iff it is both a monomorphism and an epimorphism.*

Proof. Exercise. ☐

2.4. **Additive Functors.**

Definition 2.4.1. Let M and N be $\mathbb{K}$-linear categories. A functor $F : \mathsf{M} \to \mathsf{N}$ is called a $\mathbb{K}$-*linear functor* if for every $M_0, M_1 \in \mathsf{M}$ the function

$$F : \mathrm{Hom}_{\mathsf{M}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{N}}(F(M_0), F(M_1))$$

is a $\mathbb{K}$-linear homomorphism.

A $\mathbb{Z}$-linear functor is also called an *additive functor*.

Additive functors commute with finite direct sums. More precisely:

Proposition 2.4.2. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between linear categories, let $\{M_i\}_{i \in I}$ be a finite collection of objects of M, and assume that the direct sum $(M, \{e_i\}_{i \in I})$ of the collection $\{M_i\}_{i \in I}$ exists in M. Then $(F(M), \{F(e_i)\}_{i \in I})$ is a direct sum of the collection $\{F(M_i)\}_{i \in I}$ in N.*

Proof. Exercise. (Hint: use Proposition 2.2.3.) ☐

Note that the proposition above also talks about finite products, because of Proposition 2.2.3.

Example 2.4.3. Let $A \to B$ be a ring homomorphism. The corresponding forgetful functor

$$F : \mathsf{Mod}\, B \to \mathsf{Mod}\, A$$

(also called restriction of scalars) is additive. The functor

$$G : \mathsf{Mod}\, A \to \mathsf{Mod}\, B$$

defined by $G(M) := B \otimes_A M$, called extension of scalars, is also additive.

Proposition 2.4.4. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between linear categories. Then*:

1. *For any $M \in \mathsf{M}$ the function*

$$F : \mathrm{End}_{\mathsf{M}}(M) \to \mathrm{End}_{\mathsf{N}}(F(M))$$

   is a ring homomorphism.
2. *For any $M_0, M_1 \in \mathsf{M}$ the function*

$$F : \mathrm{Hom}_{\mathsf{M}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{N}}(F(M_0), F(M_1))$$

   is a homomorphism of left $\mathrm{End}_{\mathsf{M}}(M_1)$-modules, and of right $\mathrm{End}_{\mathsf{M}}(M_0)$-modules.
3. *If M is a zero object of M, then $F(M)$ is a zero object of N.*

Proof. (1) By Definition 2.4.1 the function F respect addition. By definition of a functor, it respects multiplication and units.

(2) Immediate from the definitions, like (1).

(3) Combine part (1) with Proposition 2.2.8. □

Definition 2.4.5. Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between abelian categories.

1. F is called *left exact* if it commutes with kernels. Namely for any morphism $\phi : M_0 \to M_1$ in M, with kernel $k : K \to M_0$, the morphism $F(k) : F(K) \to F(M_0)$ is a kernel of $F(\phi) : F(M_0) \to F(M_1)$.
2. F is called *right exact* if it commutes with cokernels. Namely for any morphism $\phi : M_0 \to M_1$ in M, with cokernel $c : M_1 \to C$, the morphism $F(c) : F(M_1) \to F(C)$ is a cokernel of $F(\phi) : F(M_0) \to F(M_1)$.
3. F is called *exact* if it both left exact and right exact.

This is illustrated in the following diagrams. Suppose $\phi : M_0 \to M_1$ is a morphism in M, with kernel K and cokernel C. Applying F to the diagram

$$K \xrightarrow{k} M_0 \xrightarrow{\phi} M_1 \xrightarrow{c} C$$

we get the solid arrows in

$$\begin{tikzcd}
F(K) \arrow[r, "F(k)"] \arrow[dr, dashed, "\psi"'] & F(M_0) \arrow[r, "F(\phi)"] & F(M_1) \arrow[r, "F(c)"] \arrow[d, dashed] & F(C) \\
& \operatorname{Ker}_{\mathsf{N}}(F(\phi)) \arrow[u, dashed] & \operatorname{Coker}_{\mathsf{N}}(F(\phi)) \arrow[ur, dashed, "\chi"'] &
\end{tikzcd}$$

Bacause N is abelian, we get the vertical dashed arrows: the kernel and cokernel of $F(\phi)$. The slanted dashed arrows exist and are unique because $F(\phi) \circ F(k) = 0$ and $F(c) \circ F(\phi) = 0$. Left exactness requires ψ to be an isomorphism, and right exactness requires χ to be an isomorphism.

Definition 2.4.6. Let M be an abelian category. An *exact sequence* in M is a diagram

$$\cdots M_0 \xrightarrow{\phi_0} M_1 \xrightarrow{\phi_1} M_2 \cdots$$

(finite or infinite on either side) s.t. $\operatorname{Ker}(\phi_i) = \operatorname{Im}(\phi_{i-1})$ for all i (for which ϕ_i and ϕ_{i-1} are defined).

As usual, a *short exact sequence* is one of the form

$$0 \to M_0 \to M_1 \to M_2 \to 0. \tag{2.4.7}$$

Proposition 2.4.8. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between abelian categories.*

1. *The functor F is left exact iff for every short exact sequence* (2.4.7) *in M, the sequence*
$$0 \to F(M_0) \to F(M_1) \to F(M_2)$$
is exact in N.
2. *The functor F is right exact iff for every short exact sequence* (2.4.7) *in M, the sequence*
$$F(M_0) \to F(M_1) \to F(M_2) \to 0$$
is exact in N.

Proof. Exercise. (Hint: $M_0 \cong \operatorname{Ker}(M_1 \to M_2)$ etc.) $\square$

Example 2.4.9. Let A be a commutative ring, and let M be a fixed A-module. Define functors $F, G : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$ and $H : (\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathsf{Mod}\, A$ like this: $F(N) := M \otimes_A N$, $G(N) := \operatorname{Hom}_A(M, N)$ and $H(N) := \operatorname{Hom}_A(N, M)$. Then F is right exact, and G and H are left exact.

Proposition 2.4.10. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between abelian categories. If F is an equivalence then it is exact.*

Proof. We will prove that F respects kernels; the proof for cokernels is similar. Take a morphism $\phi : M_0 \to M_1$ in M, with kernel K. We have this diagram (solid arrows):

$$\begin{tikzcd}
M \arrow[d, dashed, "\psi"'] \arrow[dr, dashed, "\theta"] & & \\
K \arrow[r, "k"] & M_0 \arrow[r, "\phi"] & M_1
\end{tikzcd}$$

Applying F we obtain this diagram (solid arrows):

$$\begin{tikzcd}
N = F(M) \arrow[d, dashed, "F(\psi)"'] \arrow[dr, dashed, "\bar{\theta}"] & & \\
F(K) \arrow[r, "F(k)"] & F(M_0) \arrow[r, "F(\phi)"] & F(M_1)
\end{tikzcd}$$

in N. Suppose $\bar{\theta} : N \to F(M_0)$ is a morphism in N s.t. $F(\phi) \circ \bar{\theta} = 0$. Since F is essentially surjective on objects, there is some $M \in \mathsf{M}$ with an isomorphism $\alpha : F(M) \xrightarrow{\simeq} N$. After replacing N with $F(M)$ and $\bar{\theta}$ with $\bar{\theta} \circ \alpha$, we can assume that $N = F(M)$.

Now since F is fully faithful, there is a unique $\theta : M \to M_0$ s.t. $F(\theta) = \bar{\theta}$; and $\phi \circ \theta = 0$. So there is a unique $\psi : M \to K$ s.t. $\theta = k \circ \psi$. It follows that $F(\psi) : F(M) \to F(M_0)$ is the unique morphism s.t. $\bar{\theta} = F(k) \circ F\psi)$. $\square$

Here is a result that could afford another proof of the previous proposition.

Proposition 2.4.11. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between linear categories. Assume F is an equivalence, with quasi-inverse G. Then $G : \mathsf{N} \to \mathsf{M}$ is an additive functor.*

Proof. Exercise. $\square$

3. Projective and Injective Objects

Here M is an abelian category.

3.1. **Projectives.** A *splitting* of an epimorphism $\psi : M \to M''$ in M is a morphism $\alpha : M'' \to M$ s.t. $\psi \circ \alpha = 1_{M''}$. A splitting of a monomorphism $\phi : M' \to M$ is a morphism $\beta : M \to M'$ s.t. $\beta \circ \phi = 1_{M'}$. A splitting of a short exact sequence

$$0 \to M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \to 0 \tag{3.1.1}$$

is a splitting of the epimorphism ψ, or equivalently a splitting of the monomorphism ϕ. The short exact sequence is said to be *split* if it has some splitting.

Exercise 3.1.2. Show how to get from a splitting of ϕ to a splitting of ψ, and vice versa. Show how any of those gives rise to an isomorphism $M \cong M' \oplus M''$.

Definition 3.1.3. An object $P \in \mathsf{M}$ is called a *projective object* if any diagram (solid arrows)

$$\begin{tikzcd} & P \arrow[ld, dashed, "\tilde{\gamma}"'] \arrow[d, "\gamma"] \\ M \arrow[r, "\psi"'] & N \end{tikzcd}$$

in M, in which ψ is an epimorphism, can be completed (dashed arrow) to a commutative diagram.

Proposition 3.1.4. *The following conditions are equivalent for $P \in \mathsf{M}$:*

(i) *P is projective.*

(ii) *The additive functor*
$$\mathrm{Hom}_{\mathsf{M}}(P, -) : \mathsf{M} \to \mathsf{Ab}$$
is exact.

Proof. Exercise. □

Definition 3.1.5. We say M *has enough projectives* if every $M \in \mathsf{M}$ admits an epimorphism $P \to M$ from a projective object P.

Example 3.1.6. Let A be a ring. An A-module P is projective iff it is a direct summand of a free module; i.e. $P \oplus P' \cong Q$ for some module P' and free module Q. The category $\mathsf{Mod}\, A$ has enough projectives.

Example 3.1.7. Let M be the category of finite abelian groups. The only projective object in M is 0. So M does not have enough projectives.

Example 3.1.8. Consider the scheme $X := \mathbf{P}^1_{\mathbb{K}}$, the projective line over a field $\mathbb{K}$. (If the reader prefers, he/she can assume $\mathbb{K}$ is algebraically closed, so X is a classical algebraic variety.) The structure sheaf (sheaf of functions) is $\mathcal{O}_X$. The category $\mathsf{Coh}\ \mathcal{O}_X$ of coherent $\mathcal{O}_X$-modules is abelian (it is a full abelian subcategory of $\mathsf{Mod}\, \mathcal{O}_X$, cf. Example 2.3.18). One can show that the only projective object of $\mathsf{Coh}\ \mathcal{O}_X$ is 0, but this is quite involved.

Let us only indicate why $\mathcal{O}_X$ is not projective. Denote by t_0, t_1 the homogenous coordinates of X. These belong to $\Gamma(X, \mathcal{O}_X(1))$, so each determines a homomorphism of sheaves $t_j : \mathcal{O}_X(i) \to \mathcal{O}_X(i+1)$. We get a sequence

$$0 \to \mathcal{O}_X(-2) \xrightarrow{\left[t_0\ t_1 \right]} \mathcal{O}_X(-1)^2 \xrightarrow{\begin{bmatrix} -t_1 \\ t_0 \end{bmatrix}} \mathcal{O}_X \to 0$$

in $\mathsf{Coh}\, \mathcal{O}_X$, which is known to be exact. Because $\Gamma(X, \mathcal{O}_X) = \mathbb{K}$, and $\Gamma(X, \mathcal{O}_X(-1)) = 0$, this sequence is not split.

3.2. **Injectives.**

Definition 3.2.1. An object $I \in \mathsf{M}$ is called an *injective object* if any diagram (solid arrows)

$$\begin{tikzcd} I & \\ M \arrow[u, "\gamma"] \arrow[r, "\psi"'] & N \arrow[ul, dashed, "\tilde{\gamma}"'] \end{tikzcd}$$

in M, in which ψ is a monomorphism, can be completed (dashed arrow) to a commutative diagram.

Proposition 3.2.2. *The following conditions are equivalent for $I \in \mathsf{M}$:*

(i) *I is injective.*

(ii) *The additive functor*

$$\operatorname{Hom}_{\mathsf{M}}(-, I) : \mathsf{M}^{\mathrm{op}} \to \mathsf{Ab}$$

is exact.

Proof. Exercise. □

Example 3.2.3. Let A be a ring. Unlike projectives, the structure of injective objects in $\mathsf{Mod}\, A$ is very complicated, and not much is known (except that they exist). However if A is a commutative noetherian ring then we know this: every injective module I is a direct sum of indecomposable injective modules. And these indecomposables are parametrized by $\operatorname{Spec} A$, the set of prime ideals of A. These facts are due to Matlis; see [RD, pages 120-122] for details.

Definition 3.2.4. We say M *has enough injectives* if every $M \in \mathsf{M}$ admits a monomorphism $M \to I$ to an injective object I.

Here are a few results about injective objects. Recall that modules over a ring are always left modules by default.

Proposition 3.2.5. *Let $f : A \to B$ be a ring homomorphism, and let I be an injective A-module. Then $J := \operatorname{Hom}_A(B, I)$ is an injective B-module.*

Proof. Note that B is a left A-module via f, and a right B-module. This makes J into a left B-module. In a formula: for $\phi \in J$ and $b, b' \in B$ we have $(b \cdot \phi)(b') = \phi(b' \cdot b)$.

Now given any $N \in \mathsf{Mod}\, B$ there is an isomorphism

$$\operatorname{Hom}_B(N, J) = \operatorname{Hom}_B(N, \operatorname{Hom}_A(B, I)) \cong \operatorname{Hom}_A(N, I). \tag{3.2.6}$$

This is a natural isomorphism (of functors in N). So the functor $\operatorname{Hom}_B(-, J)$ is exact, and hence J is injective. □

Theorem 3.2.7 (Baer Criterion). *Let A be a ring and I an A-module. Assume that every A-module homomorphism $\mathfrak{a} \to I$ from a left ideal $\mathfrak{a} \subseteq A$ extends to a homomorphism $A \to I$. Then the module I is injective.*

Proof. Consider an A-module M, a submodule $N \subseteq M$, and a homomorphism $\gamma : N \to I$. We have to prove that γ extends to a homomorphism $M \to I$. Look at the pairs (N', γ') consisting of a submodule $N' \subseteq M$ that contains N, and a homomorphism $\gamma' : N' \to I$ that extends γ. The set of all such pairs is ordered by inclusion, and it satisfies the conditions of Zorn's Lemma. Therefore there exists a maximal pair (N', γ'). We claim that $N' = M$.

Otherwise, there is an element $m \in M$ that does not belong to N'. Define $N'' := N' + A \cdot m \subseteq M$. Let

$$\mathfrak{a} := \{a \in A \mid a \cdot m \in N'\},$$

which is a left ideal of A. There is a short exact sequence

$$0 \to \mathfrak{a} \to N' \oplus A \to N'' \to 0$$

of A-modules. Let $\phi : \mathfrak{a} \to I$ be the homomorphism induced by $\gamma' : N' \to I$. By assumption, it extends to a homomorphism $\tilde{\phi} : A \to I$. We get a homomorphism

$$(\gamma', \tilde{\phi}) : N' \oplus A \to I$$

that agrees on $\mathfrak{a}$; and thus there is an induced homomorphism $\gamma'' : N'' \to I$. This contradicts the maximality of (N', γ'). ☐

Lemma 3.2.8. *The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is injective.*

Proof. By the Baer criterion, it is enough to consider a homomorphism $\gamma : \mathfrak{a} \to \mathbb{Q}/\mathbb{Z}$ for an ideal $\mathfrak{a} = n \cdot \mathbb{Z} \subseteq \mathbb{Z}$. We may assume that $n \neq 0$. Say $\gamma(n) = r + \mathbb{Z}$ with $r \in \mathbb{Q}$. Then we can extend γ to $\tilde{\gamma} : \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ with $\tilde{\gamma}(1) := r/n + \mathbb{Z}$. ☐

Lemma 3.2.9. *Let $\{I_x\}_{x \in X}$ be a collection of injective objects of M. If the product $\prod_{x \in X} I_x$ exists in M, then it is an injective object.*

Proof. Exercise. ☐

Theorem 3.2.10. *Let A be any ring. The category $\mathsf{Mod}\, A$ has enough injectives.*

Proof. Step 1. Here $A = \mathbb{Z}$. Take any nonzero $\mathbb{Z}$-module M and any nonzero $m \in M$. Consider the cyclic submodule $M' := \mathbb{Z} \cdot m \subseteq M$. There is a homomorphism $\gamma' : M' \to \mathbb{Q}/\mathbb{Z}$ s.t. $\gamma'(m) \neq 0$. Indeed, if $M' \cong \mathbb{Z}$, then we take any $r \in \mathbb{Q} - \mathbb{Z}$; and if $M' \cong \mathbb{Z}/(n)$ for some $n > 0$, then we take $r := 1/n$. In either case, we define $\gamma'(m) := r + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$. Since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, γ' extends to a homomorphism $\gamma : M \to \mathbb{Q}/\mathbb{Z}$. By construction we have $\gamma(m) \neq 0$.

Step 2. Now A is any ring, M is any nonzero A-module, and $m \in M$ a nonzero element. Define the A-module $I := \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$, which, according to Lemma 3.2.8 and Proposition 3.2.5, is an injective A-module. Let $\gamma : M \to \mathbb{Q}/\mathbb{Z}$ be a $\mathbb{Z}$-linear homomorphism such that $\gamma(m) \neq 0$. Such γ exists by step 1. Let $\tau : I \to \mathbb{Q}/\mathbb{Z}$ be the $\mathbb{Z}$-linear homomorphism that sends an element $\chi \in I$ to $\chi(1) \in \mathbb{Q}/\mathbb{Z}$. The adjunction formula (3.2.6) gives an A-module homomorphism $\psi : M \to I$ s.t. $\tau \circ \psi = \gamma$. We note that $(\tau \circ \psi)(m) = \gamma(m) \neq 0$, and hence $\psi(m) \neq 0$.

Step 3. Here A and M are arbitrary. Let I be as in step 2. For any nonzero $m \in M$ there is an A-linear homomorphism $\psi_m : M \to I$ such that $\psi_m(m) \neq 0$. For $m = 0$ let $\psi_0 : M \to I$ be an arbitrary homomorphism (e.g. $\psi_0 = 0$). Define the A-module $J := \prod_{m \in M} I$. There is a homomorphism $\psi := \prod_{m \in M} \psi_m : M \to J$, and it is easy to check that ψ is a monomorphism. By Lemma 3.2.9, J is an injective A-module. ☐

Exercise 3.2.11. At the price of getting a bigger injective module, we can make the construction of injective resolutions functorial. Let $I := \operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ as above. Given an A-module M, consider the set

$$X(M) := \operatorname{Hom}_A(M, I) \cong \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}).$$

Let $J(M) := \prod_{\psi \in X(M)} I$. There is a "tautological" homomorphism $\phi_M : M \to J(M)$. Show that ϕ_M is a monomorphism, $J : M \mapsto J(M)$ is a functor, and $\phi : \mathrm{id} \to J$ is a natural transformation.

Is the functor $J : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$ additive?

Example 3.2.12. Let N be the category of torsion abelian groups, and M the category of finite abelian groups. Then $\mathsf{N} \subseteq \mathsf{Ab}$ and $\mathsf{M} \subseteq \mathsf{N}$ are full abelian subcategories. M has no projectives nor injectives except 0. The only projective in N is 0. But N has enough injectives: this is because $\mathbb{Q}/\mathbb{Z} \in \mathsf{N}$, N is closed under infinite direct sums in Ab, and the next proposition.

Proposition 3.2.13. *If A is a left noetherian ring, then any direct sum of injective A-modules is an injective module.*

Proof. Exercise. (Hint: use the Baer criterion.) □

Remark 3.2.14. Actually, the converse of Proposition 3.2.13 is also true: if every direct sum of injective A-modules is injective, then A is left noetherian. But experience tells us that this fact is not very important...

Exercise 3.2.15. Here we study injectives in the category $\mathsf{Ab} = \mathsf{Mod}\, \mathbb{Z}$. By Lemma 3.2.8, the module $I := \mathbb{Q}/\mathbb{Z}$ is injective. For a (positive) prime number p, we denote by $\widehat{\mathbb{Z}}_p$ the ring of p-adic integers, and by $\widehat{\mathbb{Q}}_p$ its field of fractions (namely the p-adic completions of $\mathbb{Z}$ and $\mathbb{Q}$ respectively). Define the abelian group $I_p := \widehat{\mathbb{Q}}_p / \widehat{\mathbb{Z}}_p$.

1. Show that I_p is an injective object of Ab.
2. Show that I_p is indecomposable (i.e. it is not the direct sum of two nonzero objects).
3. Show that $I \cong \bigoplus_p I_p$.
4. The theory tells us that there is another indecomposable injective object in Ab, besides the I_p. Try to identify it.

Remark 3.2.16. Let $\mathbb{K}$ be a field and $A := \mathbb{K}[t]$, the polymomial ring in one variable. As we very well know, the categories $\mathsf{Mod}\, A$ and $\mathsf{Mod}\, \mathbb{Z}$ share many properties. Let $A^* := \operatorname{Hom}_{\mathbb{K}}(A, \mathbb{K})$, which is of course an injective A-module (because $\mathbb{K}$ is an injective $\mathbb{K}$-module). The structure of A^*, as a direct sum of indecomposable injectives, was used to cook up a counterexample in [Ye5, Section 6].

The abelian category $\mathsf{Mod}\, \mathcal{A}$ associated to a ringed space $(X, \mathcal{A})$ was introduced in Example 2.3.18.

Proposition 3.2.17. *Let $(X, \mathcal{A})$ be a ringed space. The category $\mathsf{Mod}\, \mathcal{A}$ has enough injectives.*

Proof. Let $\mathcal{M}$ be an $\mathcal{A}$-module. Take a point $x \in X$. The stalk $\mathcal{M}_x$ is a module over the ring $\mathcal{A}_x$, and by Theorem 3.2.10 we can find an embedding $\phi_x : \mathcal{M}_x \to I_x$ into an injective $\mathcal{A}_x$-module. Let $g_x : \{x\} \to X$ be the inclusion, which we may view as a map of ringed spaces from $(\{x\}, \mathcal{A}_x)$ to $(X, \mathcal{A})$. Define $\mathcal{I}_x := g_{x*}(I_x)$, which is an $\mathcal{A}$-module (in fact it is a constant sheaf supported on the closed set $\overline{\{x\}} \subseteq X$). The adjunction formula gives rise to a sheaf homomorphism $\psi_x : \mathcal{M} \to \mathcal{I}_x$. Since the functor $g_x^* : \mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}_x$ is exact, the adjunction formula shows that $\mathcal{I}_x$ is an injective object of $\mathsf{Mod}\,\mathcal{A}$.

Finally let $\mathcal{J} := \prod_{x \in X} \mathcal{I}_x$. This is an injective $\mathcal{A}$-module. There is a homomorphism $\psi := \prod_{x \in X} \psi_x : \mathcal{M} \to \mathcal{J}$ in $\mathsf{Mod}\,\mathcal{A}$. This is a monomorphism, since for every point, letting $\mathcal{J}_x$ be the stalk of the sheaf $\mathcal{J}$, the composition $\mathcal{M}_x \xrightarrow{\psi_x} \mathcal{J}_x \xrightarrow{p_x} \mathcal{I}_x$ is the embedding $\phi_x : \mathcal{M}_x \to I_x$. □

4. DG RINGS AND MODULES

In this section we fix a nonzero commutative base ring $\mathbb{K}$. (It seems more relaxing to have a base ring $\mathbb{K}$ around, rather than working with the universal base $\mathbb{K} = \mathbb{Z}$.) Throughout, "DG" stands for "differential graded".

I do not know any good references for this very basic material. There is some in [ML], in [SP, Chapter 09JD], and also in the not-yet-published [AFH]. Some of the material is taken directly from my recent papers [Ye8] and [Ye9].

4.1. **DG $\mathbb{K}$-modules.**

Definition 4.1.1. A *DG $\mathbb{K}$-module* is a graded $\mathbb{K}$-module $M = \bigoplus_{i \in \mathbb{Z}} M^i$, together with a $\mathbb{K}$-linear operator $\mathrm{d} : M \to M$ of degree 1, called the differential, satisfying $\mathrm{d} \circ \mathrm{d} = 0$.

Definition 4.1.2. Let M and N be DG $\mathbb{K}$-modules. A *strict homomorphism of DG $\mathbb{K}$-modules* is a $\mathbb{K}$-linear homomorphism $\phi : M \to N$ that commutes with the differentials and respects the gradings. The resulting category is denoted by $\mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K}$.

Remark 4.1.3. The name "strict morphism of DG modules", and the corresponding notation $\mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K}$, are new. We introduced them to distinguish $\mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K}$ from the DG category $\mathsf{DGMod}\, \mathbb{K}$ that contains it; cf. Definitions 4.2.1 and 4.2.4.

Suppose M and N are DG $\mathbb{K}$-modules. For any integer i let

$$(M \otimes_{\mathbb{K}} N)^i := \bigoplus_{j \in \mathbb{Z}} (M^j \otimes_{\mathbb{K}} N^{i-j}).$$

Then

$$M \otimes_{\mathbb{K}} N = \bigoplus_{i \in \mathbb{Z}} (M \otimes_{\mathbb{K}} N)^i, \tag{4.1.4}$$

so it is a graded $\mathbb{K}$-module. We put on it the differential

$$\mathrm{d}(m \otimes n) := \mathrm{d}_M(m) \otimes n + (-1)^i \cdot m \otimes \mathrm{d}_N(n) \tag{4.1.5}$$

for $m \in M^i$ and $n \in N^j$. In this way $M \otimes_{\mathbb{K}} N$ becomes a DG $\mathbb{K}$-module.

For DG $\mathbb{K}$-modules M, N, we let $\mathrm{Hom}_{\mathbb{K}}(M,N)^i$ be the $\mathbb{K}$-module of degree i homomorphisms $\phi : M \to N$; namely

$$\mathrm{Hom}_{\mathbb{K}}(M,N)^i = \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{K}}(M^j, N^{j+i}).$$

We then define

$$\mathrm{Hom}_{\mathbb{K}}(M,N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{K}}(M,N)^i. \tag{4.1.6}$$

This is also a DG $\mathbb{K}$-module. The differential is

$$\mathrm{d}(\phi) := \mathrm{d}_N \circ \phi - (-1)^i \cdot \phi \circ \mathrm{d}_M \tag{4.1.7}$$

for $\phi \in \mathrm{Hom}_{\mathbb{K}}(M,N)^i$. When we need to emphasize where d acts, we sometimes denote it by $\mathrm{d}_{\mathrm{hom}}$ or $\mathrm{d}_{\mathrm{Hom}_{\mathbb{K}}(M,N)}$.

Let M be a DG $\mathbb{K}$-module. The set of degree i cocycles is

$$\mathrm{Z}^i(M) := \ker(\mathrm{d}|_{M^i}) \subseteq M^i,$$

and the set of degree i coboundaries is

$$\mathrm{B}^i(M) := \mathrm{im}(\mathrm{d}|_{M^{i-1}}) \subseteq M^i.$$

The i-th cohomology is

$$\mathrm{H}^i(M) := \mathrm{Z}^i(M) / \mathrm{B}^i(M).$$

These are all $\mathbb{K}$-modules, and in fact they are functors

$$\mathrm{Z}^i, \mathrm{B}^i, \mathrm{H}^i : \mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K} \to \mathsf{Mod}\, \mathbb{K}.$$

Let us record the following result, whose easy proof we leave out.

Proposition 4.1.8. *For DG $\mathbb{K}$-modules M and N, there is equality*

$$\mathrm{Hom}_{\mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K}}(M, N) = \mathrm{Z}^0(\mathrm{Hom}_{\mathbb{K}}(M, N))$$

of these submodules of $\mathrm{Hom}_{\mathbb{K}}(M, N)$.

4.2. **DG Categories.** Suppose C is a $\mathbb{K}$-linear category (Definition 2.1.1). Since the composition of morphisms is $\mathbb{K}$-bilinear, for any triple of objects $M_0, M_1, M_2 \in \mathsf{C}$, composition can be expressed as a $\mathbb{K}$-linear homomorphism

$$\begin{aligned} &\mathrm{Hom}_{\mathsf{C}}(M_1, M_2) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathsf{C}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{C}}(M_0, M_2) \\ &\phi_2 \otimes \phi_1 \mapsto \phi_2 \circ \phi_1. \end{aligned}$$

We refer to it as the composition homomorphism. It will be used in the following definition.

Definition 4.2.1. A $\mathbb{K}$-*linear DG category* is a $\mathbb{K}$-linear category C, endowed with a DG $\mathbb{K}$-module structure on each of the morphism $\mathbb{K}$-modules $\mathrm{Hom}_{\mathsf{C}}(M_0, M_1)$. The conditions are these:

(a) For any object M, the identity morphism 1_M is a degree 0 cocycle in $\mathrm{Hom}_{\mathsf{C}}(M, M)$.

(b) For any triple of objects $M_0, M_1, M_2 \in \mathsf{C}$, the composition homomorphism

$$\mathrm{Hom}_{\mathsf{C}}(M_1, M_2) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathsf{C}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{C}}(M_0, M_2)$$

is a strict homomorphism of DG $\mathbb{K}$-modules.

If $\mathbb{K} = \mathbb{Z}$, we say that C is a *DG category.*

In the condition (b) of the definition we use formula (4.1.4) for the DG module structure on a tensor product of DG $\mathbb{K}$-modules. It is possible that condition (a) is redundant (cf. Proposition 4.4.2 below). A morphism $\phi : M \to N$ in C is called a degree i morphism if $\phi \in \mathrm{Hom}_{\mathsf{C}}(M, N)^i$.

Definition 4.2.2. Let C be a $\mathbb{K}$-linear DG category. A morphism $\phi : M \to N$ in C is called a strict morphism if it is a degree 0 cocycle in $\mathrm{Hom}_{\mathsf{C}}(M, N)$.

Lemma 4.2.3. *Let C be a $\mathbb{K}$-linear DG category, and for $i = 1, 2, 3$ let $\phi_i : M_{i-1} \to M_i$ be morphisms in C of degree k_i.*

(1) *The morphism $\phi_2 \circ \phi_1$ has degree $k_1 + k_2$, and*

$$\mathrm{d}(\phi_2 \circ \phi_1) = \mathrm{d}(\phi_2) \circ \phi_1 + (-1)^{k_2} \cdot \phi_2 \circ \mathrm{d}(\phi_1).$$

(2) *If ϕ_1 and ϕ_2 are cocycles, then so is $\phi_2 \circ \phi_1$.*

(3) *If ϕ_2 is a coboundary, and ϕ_1 and ϕ_3 are cocycles, then $\phi_3 \circ \phi_2 \circ \phi_1$ is a coboundary.*

Proof. (1) This is just a rephrasing of item (b) in Definition 4.2.1.

(2) This is immediate from (1).

(3) Say $\phi_2 = \mathrm{d}(\psi_2)$ for some degree $k_2 - 1$ morphism $\psi_2 : M_1 \to M_2$. Then

$$\phi_3 \circ \phi_2 \circ \phi_1 = \mathrm{d}(\phi_3 \circ \psi_2 \circ \phi_1).$$

□

The lemma makes the next definition possible.

Definition 4.2.4. Let C be a $\mathbb{K}$-linear DG category.

(1) The *strict category* of C is the category $\mathsf{C}_{\mathrm{str}} = \mathrm{Z}^0(\mathsf{C})$, with the same objects as C, but with strict morphisms only. Thus

$$\mathrm{Hom}_{\mathsf{C}_{\mathrm{str}}}(M, N) = \mathrm{Z}^0\big(\mathrm{Hom}_{\mathsf{C}}(M, N)\big).$$

(2) The *homotopy category* of C is the category $\mathrm{H}^0(\mathsf{C})$, with the same objects as C, and with morphism sets

$$\mathrm{Hom}_{\mathrm{H}^0(\mathsf{C})}(M, N) := \mathrm{H}^0\big(\mathrm{Hom}_{\mathsf{C}}(M, N)\big).$$

The categories $\mathsf{C}_{\mathrm{str}}$ and $\mathrm{H}^0(\mathsf{C})$ are $\mathbb{K}$-linear. There are $\mathbb{K}$-linear functors $\mathsf{C}_{\mathrm{str}} \to \mathsf{C}$ and $\mathsf{C}_{\mathrm{str}} \to \mathrm{H}^0(\mathsf{C})$. Both functors are the identity on objects, but the first is faithful (injective on morphisms), and the second is full (surjective on morphisms).

Remark 4.2.5. There is a vast theory on DG categories. See [Ke], [BK]. In this course we shall be exclusively concerned with the categories $\mathbf{C}(\mathsf{M}, A)$, to be introduced in Subsection 4.6, that have a lot more structure than other DG categories; most importantly, they are additive, and their objects (the DG A-modules in M) have a DG structure too.

Here is a useful result.

Proposition 4.2.6. *Let $\phi : M \to N$ be a degree i isomorphism in the $\mathbb{K}$-linear DG category C. Assume ϕ is a cocycle, namely $\mathrm{d}(\phi) = 0$. Then its inverse $\phi^{-1} : N \to M$ is also a cocycle.*

Proof. According the Leibniz rule (Lemma 4.2.3(1)), and the fact that 1_M is a cocycle, we have

$$0 = \mathrm{d}(1_M) = \mathrm{d}(\phi^{-1} \circ \phi) = \mathrm{d}(\phi^{-1}) \circ \phi - (-1)^{-i} \cdot \phi^{-1} \circ \mathrm{d}(\phi) = \mathrm{d}(\phi^{-1}) \circ \phi.$$

Because ϕ is an isomorphism, we conclude that $\mathrm{d}(\phi^{-1}) = 0$. ☐

4.3. **Complexes.** Before delving into the deeper waters of DG modules, let's recall facts about complexes from the classical homological theory.

Let M be a $\mathbb{K}$-linear category. A *complex* of objects of M (or a complex in M) is a diagram

$$(\cdots \to M^{-1} \xrightarrow{\mathrm{d}_M^{-1}} M^0 \xrightarrow{\mathrm{d}_M^{0}} M^1 \xrightarrow{\mathrm{d}_M^{1}} M^2 \to \cdots) \tag{4.3.1}$$

of objects and morphisms in M, such that $\mathrm{d}_M^{i+1} \circ \mathrm{d}_M^i = 0$. Alternative notations for this complex are

$$\big(\{M^i\}_{i \in \mathbb{Z}}, \{\mathrm{d}_M^i\}_{i \in \mathbb{Z}}\big),$$

or just (M, d_M). The collection of morphisms d_M is called the *differential* of M, or the *coboundary operator*. We sometimes write d instead of d_M or d_M^i; and often we write M instead of (M, d_M), leaving the differential implicit. [2]

Let N be another such complex. A *strict morphism of complexes* $\phi : M \to N$ is a collection $\phi = \{\phi^i\}_{i \in \mathbb{Z}}$ of morphisms $\phi^i : M^i \to N^i$ in M, such that

$$\mathrm{d}_N^i \circ \phi^i = \phi^{i+1} \circ \mathrm{d}_M^i.$$

---

[2] Thanks to Stephan Snigerov for correcting typos here.

Note that the strict morphism $\phi : M \to N$ can be viewed as a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & M^i \arrow[r, "\mathrm{d}_M^i"] \arrow[d, "\phi^i"'] & M^{i+1} \arrow[r] \arrow[d, "\phi^{i+1}"'] & \cdots \\
\cdots \arrow[r] & N^i \arrow[r, "\mathrm{d}_N^i"] & N^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Of course the identity morphism 1_M is a strict morphism.

Remark 4.3.2. In most textbooks, what we call "strict morphism of complexes" is simply called a "morphism of complexes". See Remark 4.1.3 for an explanation.

Let us denote by $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$ the category of complexes in M, with strict morphisms. This is a $\mathbb{K}$-linear category. If M happens to be additive, then so is $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$. Indeed, direct sums of complexes are degree-wise, i.e. $(M \oplus N)^i = M^i \oplus N^i$. If M is abelian, then so is $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, again with kernels and cokernels made degree-wise. If N is a full subcategory of M, then $\mathbf{C}_{\mathrm{str}}(\mathsf{N})$ is a full subcategory of $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$.

Any single object $M \in \mathsf{M}$ can be viewed as a complex

$$M' := (\cdots \to 0 \to M \to 0 \to \cdots),$$

where M is in degree 0; the differential of this complex is of course zero. The assignment $M \mapsto M'$ is a fully faithful $\mathbb{K}$-linear functor $\mathsf{M} \to \mathbf{C}_{\mathrm{str}}(\mathsf{M})$.

Let M, N be complexes in M. For any integer i we define

$$\mathrm{Hom}_{\mathsf{M}}(M, N)^i := \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{M}}(M^j, N^{j+i}) \in \mathsf{Mod}\,\mathbb{K}.$$

The graded $\mathbb{K}$-module

$$\mathrm{Hom}_{\mathsf{M}}(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{M}}(M, N)^i \tag{4.3.3}$$

is a DG $\mathbb{K}$-module with differential

$$\mathrm{d} : \mathrm{Hom}_{\mathsf{M}}(M, N)^i \to \mathrm{Hom}_{\mathsf{M}}(M, N)^{i+1}$$

given by

$$\mathrm{d}(\phi) := \mathrm{d}_N \circ \phi - (-1)^i \cdot \phi \circ \mathrm{d}_M. \tag{4.3.4}$$

It is easy to check that $\mathrm{d} \circ \mathrm{d} = 0$. We sometimes denote this differential by $\mathrm{d}_{\mathrm{hom}}$ or $\mathrm{d}_{\mathrm{Hom}_{\mathsf{M}}(M,N)}$.

Thus, an element $\phi \in \mathrm{Hom}_{\mathsf{M}}(M, N)^i$ is a collection $\phi = \{\phi^j\}_{j \in \mathbb{Z}}$ of morphisms $\phi^j : M^j \to N^{j+i}$. In a diagram, for $i = 2$, it looks like this:

$$\begin{tikzcd}
\cdots \arrow[r] & M^j \arrow[r, "\mathrm{d}"] \arrow[drr, "\phi^j"] & M^{j+1} \arrow[r, "\mathrm{d}"] \arrow[drr, "\phi^{j+1}"] & M^{j+2} \arrow[r, "\mathrm{d}"] & M^{j+3} \arrow[r] & \cdots \\
\cdots \arrow[r] & N^j \arrow[r, "\mathrm{d}"] & N^{j+1} \arrow[r, "\mathrm{d}"] & N^{j+2} \arrow[r, "\mathrm{d}"] & N^{j+3} \arrow[r] & {}
\end{tikzcd}$$

Since ϕ does not have to commute with the differentials, this is usually not a commutative diagram!

For a triple of complexes M_0, M_1, M_2, there are $\mathbb{K}$-linear homomorphisms

$$\mathrm{Hom}_{\mathsf{M}}(M_1, M_2)^{i_2} \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathsf{M}}(M_0, M_1)^{i_1} \to \mathrm{Hom}_{\mathsf{M}}(M_0, M_2)^{i_1 + i_2},$$

$\phi_2 \otimes \phi_1 \mapsto \phi_2 \circ \phi_1$.

Lemma 4.3.5. *The composition homomorphism*

$$\mathrm{Hom}_{\mathsf{M}}(M_1, M_2) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathsf{M}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{M}}(M_0, M_2)$$

is a strict homomorphism of DG $\mathbb{K}$*-modules.*

Proof. This is a good exercise. □

Definition 4.3.6. Let $\mathbf{C}(\mathsf{M})$ be the $\mathbb{K}$-linear DG category whose objects are the complexes in M, and the morphism sets are $\mathrm{Hom}_{\mathsf{M}}(M, N)$ from formula (4.3.3).

We will later see that $\mathbf{C}(\mathsf{M})$ is a DG category. For now, we only observe that $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$ is a subcategories of $\mathbf{C}(\mathsf{M})$. The relation between them is explained in Proposition 4.3.8 below.

Let N be an abelian category. For a complex $N \in \mathbf{C}(\mathsf{N})$ we write

$$\mathrm{Z}^i(N) := \mathrm{Ker}(\mathrm{d}_N^i) \subseteq N^i$$

and

$$\mathrm{B}^i(N) := \mathrm{Im}(\mathrm{d}_N^{i-1}) \subseteq N^i;$$

these are the objects of *i-cocycles* and *i-coboundaries* of N, respectively. Since $\mathrm{d} \circ \mathrm{d} = 0$ we have $\mathrm{B}^i(N) \subseteq \mathrm{Z}^i(N)$, and we let

$$\mathrm{H}^i(N) := \mathrm{Z}^i(N) / \mathrm{B}^i(N). \tag{4.3.7}$$

This is the *i-th cohomology* of the complex N.

Proposition 4.3.8. *Let* M *be a* $\mathbb{K}$*-linear category, and let* $M, N \in \mathbf{C}(\mathsf{M})$*. Then there is equality of* $\mathbb{K}$*-modules*

$$\mathrm{Hom}_{\mathbf{C}_{\mathrm{str}}(\mathsf{M})}(M, N) = \mathrm{Z}^0(\mathrm{Hom}_{\mathsf{M}}(M, N)).$$

In other words, a strict morphism of complexes $\phi : M \to N$ *is the same as a* 0*-cocycle in the DG* $\mathbb{K}$*-module* $\mathrm{Hom}_{\mathsf{M}}(M, N)$*.*

Proof. Exercise. □

Remark 4.3.9. A possible ambiguity could arise in the meaning of $\mathrm{Hom}_{\mathsf{M}}(M, N)$ if $M, N \in \mathsf{M}$: does it mean the $\mathbb{K}$-module of morphisms in the category M ? Or, if we view M and N as complexes by the canonical embedding $\mathsf{M} \subseteq \mathbf{C}(\mathsf{M})$, does $\mathrm{Hom}_{\mathsf{M}}(M, N)$ mean the complex of $\mathbb{K}$-modules defined for complexes? It turns out that there is no actual difficulty: since the complex of $\mathbb{K}$-modules $\mathrm{Hom}_{\mathsf{M}}(M, N)$ is concentrated in degree 0, we may view it as a single $\mathbb{K}$-module, and this is precisely the $\mathbb{K}$-module of morphisms in the category M.

4.4. **DG Rings.**

Definition 4.4.1. A *DG ring* is a graded ring $A = \bigoplus_{i \in \mathbb{Z}} A^i$, together with an operator $\mathrm{d} : A \to A$ of degree 1 called the differential, satisfying the equation $\mathrm{d} \circ \mathrm{d} = 0$, and the graded Leibniz rule

$$\mathrm{d}(a \cdot b) = \mathrm{d}(a) \cdot b + (-1)^i \cdot a \cdot \mathrm{d}(b)$$

for all $a \in A^i$ and $b \in A^j$.

When we have to emphasize that d is the differential of A, we write d_A. Forgetting the multiplication, the DG ring A becomes a DG $\mathbb{Z}$-module.

Proposition 4.4.2. *Let A be a DG ring. The unit element 1_A of A is a 0-cocycle, namely $1_A \in \mathrm{Z}^0(A)$.*

Proof. Exercise. [3] □

Definition 4.4.3. Let A and B be DG rings. A *homomorphism of DG rings $f : A \to B$* is a ring homomorphism that commutes with the differentials and respects the gradings. The resulting category is denoted by DGRing.

As always for ring homomorphisms, f must preserve units, i.e. $f(1_A) = 1_B$.

Rings are viewed as DG rings concentrated in degree 0. Thus the category of rings Ring is a full subcategory of DGRing.

Definition 4.4.4. We say that A is a *central DG $\mathbb{K}$-ring* if there is a given DG ring homomorphism $\mathbb{K} \to A$, whose image is central in A.

We denote by $\mathsf{DGRing} /_{\mathrm{ce}} \mathbb{K}$ the category of central DG $\mathbb{K}$-rings, in which the morphisms $f : A \to B$ are the homomorphisms in DGRing that respect the given structural homomorphisms from $\mathbb{K}$.

Of couse when $\mathbb{K} = \mathbb{Z}$ we have $\mathsf{DGRing} /_{\mathrm{ce}} \mathbb{K} = \mathsf{DGRing}$.

Here are few exmaples of DG rings. First a silly example.

Example 4.4.5. Let A be a central graded $\mathbb{K}$-ring. Then A is a central DG $\mathbb{K}$-ring, with trivial differential.

Example 4.4.6. Let X be a differentiable (i.e. of type C^∞) real manifold. The de Rham complex A of X is a central DG $\mathbb{R}$-ring, with the wedge product and the exterior differential.

The next example is the algebraic analogue of the previous one.

Example 4.4.7. Let C be a commutative $\mathbb{K}$-ring. Then the algebraic de Rham complex $A := \Omega_{C/\mathbb{K}} = \bigoplus_{p \geq 0} \Omega^p_{C/\mathbb{K}}$ is a central DG $\mathbb{K}$-ring.

Example 4.4.8. Let M be a DG $\mathbb{K}$-module. Consider the DG $\mathbb{K}$-module

$$\mathrm{End}_{\mathbb{K}}(M) := \mathrm{Hom}_{\mathbb{K}}(M, M)$$

from (4.1.6). Composition of endomorphisms is an associative multiplication on $\mathrm{End}_{\mathbb{K}}(M)$ that respects the grading, and the graded Leibniz rule holds. We see that $\mathrm{End}_{\mathbb{K}}(M)$ is a central DG $\mathbb{K}$-ring.

Example 4.4.9. Let M be a $\mathbb{K}$-linear category. For a complex $M \in \mathbf{C}(\mathsf{M})$, the DG $\mathbb{K}$-module

$$\mathrm{End}_{\mathsf{M}}(M) := \mathrm{Hom}_{\mathsf{M}}(M, M)$$

from (4.3.3) is a central DG $\mathbb{K}$-ring, where multiplication is composition.

Example 4.4.10. Let C be a commutative ring and let $c \in C$ be an element. The *Koszul complex* of c is the DG C-module $\mathrm{K}(C; c)$ defined as follows. In degree 0 we let $\mathrm{K}^0(C; c) := C$. In degree -1, $\mathrm{K}^{-1}(C; c)$ is a free C-module of rank 1, with basis element x. All other homogenous components are trivial. The differential d is determined by what it does to the basis element $x \in \mathrm{K}^{-1}(C; c)$, and we let $\mathrm{d}(x) := c \in \mathrm{K}^0(C; c)$.

To make $\mathrm{K}(C; c)$ into a DG ring, we treat x as an *odd variable* (in the suitable sense – see *strongly commutative DG rings* in [Ye9, Definitions 3.2 and 3.10]). This dictates

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[3] Thanks to Rishy Vyas and Shai Shechter for explaining to me why 1_A must have degree 0.

$x^2 = 0$. (Of course this is also dictated by the fact that $\mathrm{K}^{-2}(C; c) = 0$.) It is easy to verify that $\mathrm{K}(C; c)$ is a central DG C-ring.

Example 4.4.11. Let A and B be central DG $\mathbb{K}$-rings. The DG $\mathbb{K}$-module $A \otimes_{\mathbb{K}} B$ from 4.1.4 has a graded ring structure, with formula

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := (-1)^{k_2 \cdot l_1} \cdot (a_1 \cdot a_2) \otimes (b_1 \cdot b_2)$$

for $a_i \in A^{k_i}$ and $b_i \in B^{l_i}$. It is easy to verify that $A \otimes_{\mathbb{K}} B$ is a central DG $\mathbb{K}$-ring.

Example 4.4.12. Let C be a commutative ring and let $\boldsymbol{c} = (c_1, \ldots, c_n)$ be a sequence of elements in C. By combining Examples 4.4.10 and 4.4.11 we obtain the Koszul complex

$$\mathrm{K}(C; \boldsymbol{c}) := \mathrm{K}(C; c_1) \otimes_C \cdots \otimes_C \mathrm{K}(C; c_n).$$

This is a central DG C-ring.

Definition 4.4.13. Let A be a central DG $\mathbb{K}$-ring. The *opposite DG ring* A^{op} is the same DG $\mathbb{K}$-module as A, but the multiplication $\cdot^{\mathrm{op}}$ is reversed and twisted by signs:

$$a \cdot^{\mathrm{op}} b := (-1)^{ij} \cdot b \cdot a$$

for $a \in A^i$ and $b \in A^j$.

4.5. **DG A-modules.**

Definition 4.5.1. Let A be a central DG $\mathbb{K}$-ring. A *left DG A-module* is a graded left A-module $M = \bigoplus_{i \in \mathbb{Z}} M^i$, with an operator $\mathrm{d} : M \to M$ of degree 1 called the differential, satisfying $\mathrm{d} \circ \mathrm{d} = 0$ and

$$\mathrm{d}(a \cdot m) = \mathrm{d}(a) \cdot m + (-1)^i \cdot a \cdot \mathrm{d}(m)$$

for $a \in A^i$ and $m \in M^j$.

As usual, when we have to emphasize that d is the differential of M, we write d_M. When we forget the action of A, M remains a DG $\mathbb{K}$-module.

Right DG A-modules are defined likewise, but we won't deal with them much. This is because right DG A-modules are left DG modules over the opposite DG ring A^{op}. More precisely, if M is a right DG A-module, then the formula

$$a \cdot m := (-1)^{ij} \cdot m \cdot a, \tag{4.5.2}$$

for $a \in A^i$ and $m \in M^j$, makes M in to a left DG A^{op}-module.

So we make this convention for the rest of the course (anlogous to Convention 1.2.1):

Convention 4.5.3. By default, DG modules are *left DG modules.* In particular, a module over a ring is by default a left module.

Proposition 4.5.4. *Let A be a central DG $\mathbb{K}$-ring, and let M be a DG $\mathbb{K}$-module.*

1. *Suppose $f : A \to \mathrm{End}_{\mathbb{K}}(M)$ is a DG $\mathbb{K}$-ring homomorphism. Then the formula $a \cdot m := f(a)(m)$, for $a \in A^i$ and $m \in M^j$, makes M into a DG A-module.*
2. *Conversely, any DG A-module structure on M, that's compatible with the DG $\mathbb{K}$-module structure, arises in this way from a DG $\mathbb{K}$-ring homomorphism $f : A \to \mathrm{End}_{\mathbb{K}}(M)$.*

Proof. Exercise. □

Exercise 4.5.5. Let A be a DG ring. Show that the cocycles $\mathrm{Z}(A) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Z}^i(A)$ are a graded subring of A, and the coboundaries $\mathrm{B}(A) := \bigoplus_{i \in \mathbb{Z}} \mathrm{B}^i(A)$ are a two-sided ideal of $\mathrm{Z}(A)$. Thus the cohomology $\mathrm{H}(A) := \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^i(A)$ is a graded ring.

Next show that given a DG A-module M, its cohomology $\mathrm{H}(M)$ is a graded $\mathrm{H}(A)$-module.

Lemma 4.5.6. *Let A be a central graded $\mathbb{K}$-ring, let M be a right graded A-module, and let N be a left graded A-module. Then*

$$M \otimes_A N = \bigoplus_{i \in \mathbb{Z}} (M \otimes_A N)^i,$$

where $(M \otimes_A N)^i$ is the $\mathbb{K}$-linear span of the tensors $m \otimes n$ with $m \in M^j$, $n \in N^k$ and $j + k = i$.

Proof. There is a canonical surjection of $\mathbb{K}$-modules

$$M \otimes_{\mathbb{K}} N \to M \otimes_A N.$$

Its kernel is the $\mathbb{K}$-submodule $L \subseteq M \otimes_{\mathbb{K}} N$ generated by the elements

$$(m \cdot a) \otimes n - m \otimes (a \cdot n),$$

for $m \in M^j$, $n \in N^k$ and $a \in A^l$. Since L is a graded submodule of $M \otimes_{\mathbb{K}} N$, so is the quotient. Finally we see that the i-th homogeneous component of $M \otimes_A N$ is precisely $(M \otimes_A N)^i$. □

Definition 4.5.7. Let A be a central DG $\mathbb{K}$-ring, let $M \in \mathsf{DGMod}\, A^{\mathrm{op}}$, and let $N \in \mathsf{DGMod}\, A$. By Lemma 4.5.6, $M \otimes_A N$ is a graded $\mathbb{K}$-module. We make it into a DG $\mathbb{K}$-module with the differential from formula (4.1.5).

Definition 4.5.8. Let A be a central DG $\mathbb{K}$-ring, and let $M, N \in \mathsf{DGMod}\, A$. For any $i \in \mathbb{Z}$, define $\mathrm{Hom}_A(M, N)^i$ to be the subset of $\mathrm{Hom}_{\mathbb{K}}(M, N)^i$ consisting of the homomorphisms $\phi : M \to N$ such that

$$\phi(a \cdot m) = (-1)^{ik} \cdot a \cdot \phi(m) \tag{4.5.9}$$

for all $a \in A^k$. Next let

$$\mathrm{Hom}_A(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_A(M, N)^i.$$

This is a graded $\mathbb{K}$-module, and we endow it with the differential (4.1.7).

Remark 4.5.10. The reader might wonder why a sign occurs in formula (4.5.9). The reason is that we view $a \in A^k$ and $\phi \in \mathrm{Hom}_{\mathbb{K}}(M, N)^i$ as graded operators (on $M \oplus N$) that *commute in the graded sense*: $\phi \circ a = (-1)^{ik} \cdot a \circ \phi$. Note that $\mathrm{Hom}_A(M, N)$ is a subobject, in $\mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K}$, of $\mathrm{Hom}_{\mathbb{K}}(M, N)$.

Definition 4.5.11. Let $\mathsf{DGMod}\, A$ be the $\mathbb{K}$-linear DG category whose objects are the DG A-modules, and the morphism modules are $\mathrm{Hom}_A(M, N)$ from Definition 4.5.8. Its strict category is denoted by $\mathsf{DGMod}_{\mathrm{str}}\, A$.

The analogue of Proposition 4.3.8 holds here too: a homomorphism $\phi : M \to N$ in $\mathsf{DGMod}\, A$ is strict iff it respects the gradings and commutes with the differentials.

For a ring A there is no essential distinction between complexes and DG modules:

Proposition 4.5.12. *Let A be a central $\mathbb{K}$-ring. Given a complex $M \in \mathbf{C}(\mathsf{Mod}\, A)$, with notation as in* (4.3.1)*, define the DG A-module*

$$F(M) := \bigoplus_{i \in \mathbb{Z}} M^i,$$

with differential $\mathrm{d} := \sum_{i \in \mathbb{Z}} \mathrm{d}_M^i$. *Then the functor*

$$F : \mathbf{C}(\mathsf{Mod}\, A) \to \mathsf{DGMod}\, A$$

is a $\mathbb{K}$-linear equivalence.

The proof is an exercise. The only hard part in it is to choose good notation. In fact, the functor F is an equivalence of DG categories; but we shall not try to make this notion precise (cf. Definition 5.2.3).

4.6. **DG A-modules in M.** We now combine the material from Subsections 4.3 and 4.5.

Definition 4.6.1. Let M be a $\mathbb{K}$-linear category, and let A be a central DG $\mathbb{K}$-ring. A *DG A-module in* M is an object $M \in \mathbf{C}(\mathsf{M})$, together with DG $\mathbb{K}$-ring homomorphism $f : A \to \mathrm{End}_{\mathsf{M}}(M)$. The set of DG A-modules in M is denoted by $\mathbf{C}(\mathsf{M}, A)$.

The DG ring structure of $\mathrm{End}_{\mathsf{M}}(M)$ is explained in Example 4.4.9. What the definition says is that any element $a \in A^i$ gives rise to a degree i endomorphism $f(a)$ of the complex M. In turn, this means that for every j, $f(a) : M^j \to M^{j+i}$ is a morphism in M. The operation f satisfies $f(1_A) = 1_M$, $f(a_1 \cdot a_2) = f(a_1) \circ f(a_2)$, and $f(\mathrm{d}(a)) = \mathrm{d}(f(a))$.

Example 4.6.2. If $A = \mathbb{K}$, then $\mathbf{C}(\mathsf{M}, A) = \mathbf{C}(\mathsf{M})$; and if $\mathsf{M} = \mathsf{Mod}\, \mathbb{K}$, then $\mathbf{C}(\mathsf{M}, A) = \mathsf{DGMod}\, A$. Because of this, we sometimes write $\mathbf{C}(A) = \mathsf{DGMod}\, A$.

It is sometimes convenient to have notation for partial structures related to $\mathbf{C}(\mathsf{M}, A)$. Let M be a $\mathbb{K}$-linear category. A *graded object of* M, or a *graded module in* M, is a collection $M = \{M^i\}_{i \in \mathbb{Z}}$ of objects of M. (It is like a complex, but without a differential.) The set of graded modules in M is denoted by $\mathbf{G}(\mathsf{M})$.

Now let A be a central graded $\mathbb{K}$-ring. A *graded A-module in* M is a graded module M in M, together with a graded $\mathbb{K}$-ring homomorphism $f : A \to \mathrm{End}_{\mathsf{M}}(M)$. The set of graded modules in M is denoted by $\mathbf{G}(\mathsf{M}, A)$. It is a category, and there are there is a faithful functor $\mathbf{C}(\mathsf{M}, A) \to \mathbf{G}(\mathsf{M}, A)$ that forgets the differentials.

The next definition is a variant of Definition 4.5.8.

Definition 4.6.3. Let M be a $\mathbb{K}$-linear category, and let A be a central DG $\mathbb{K}$-ring. For $M, N \in \mathbf{C}(\mathsf{M}, A)$ and $i \in \mathbb{Z}$ we define $\mathrm{Hom}_{\mathsf{M},A}(M, N)^i$ to be the subset of $\mathrm{Hom}_{\mathsf{M}}(M, N)^i$ consisting of the degree i morphism $\phi : M \to N$ such that

$$\phi \circ f_M(a) = (-1)^{ik} \cdot f_N(a) \circ \phi$$

for all $a \in A^k$.

Next let

$$\mathrm{Hom}_{\mathsf{M},A}(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{M},A}(M, N)^i.$$

This is a graded $\mathbb{K}$-module, and we endow it with the differential

$$\mathrm{d} : \mathrm{Hom}_{\mathsf{M},A}(M, N)^i \to \mathrm{Hom}_{\mathsf{M},A}(M, N)^{i+1},$$

$$\mathrm{d}(\phi) := \mathrm{d}_N \circ \phi - (-1)^i \cdot \phi \circ \mathrm{d}_M,$$

making it into a DG $\mathbb{K}$-module. When we have to be specific, we denote the differential d by $\mathrm{d}_{\mathrm{hom}}$, $\mathrm{d}_{\mathsf{M},A}$, or $\mathrm{d}_{\mathrm{Hom}_{\mathsf{M},A}(M,N)}$.

As we have seen before (in Lemma 4.3.5, and Conversely in Lemma 4.2.3), given $\phi_k \in \mathrm{Hom}_{\mathsf{M},A}(M_{k-1}, M_k)^{i_k}$ for $k = 1, 2$, we have

$$\phi_2 \circ \phi_1 \in \mathrm{Hom}_{\mathsf{M},A}(M_0, M_2)^{i_1+i_2},$$

and

$$\mathrm{d}(\phi_2 \circ \phi_1) = \mathrm{d}(\phi_2) \circ \phi_1 + (-1)^{i_2} \cdot \phi_2 \circ \mathrm{d}(\phi_1).$$

Also the identity 1_M belongs to $\mathrm{Hom}_{\mathsf{M},A}(M, M)^0$, and $\mathrm{d}(1_M) = 0$. Therefore the next definition is legitimate.

Definition 4.6.4. Let M be a $\mathbb{K}$-linear category, and let A be a central DG $\mathbb{K}$-ring. We put a $\mathbb{K}$-linear DG category structure on the set of objects $\mathbf{C}(\mathsf{M}, A)$, with morphisms DG modules

$$\mathrm{Hom}_{\mathbf{C}(\mathsf{M},A)}(M_0, M_1) := \mathrm{Hom}_{\mathsf{M},A}(M_0, M_1).$$

Definition 4.6.5. In the situation of Definition 4.6.4:

(1) The strict category of $\mathbf{C}(\mathsf{M}, A)$ (see Definition 4.2.4(1)) is denoted by $\mathbf{C}_{\mathrm{str}}(\mathsf{M}, A)$.
(2) The *homotopy category* of $\mathbf{C}(\mathsf{M}, A)$ (see Definition 4.2.4(2)) is denoted by $\mathbf{K}(\mathsf{M}, A)$.

The next proposition is merely an interpretation of the definitions; but it is worthy of mentioning.

Proposition 4.6.6. *Let $\phi : M \to N$ be a degree 0 morphism in $\mathbf{C}(\mathsf{M}, A)$. The next two conditions are equivalent*:

(i) *ϕ is strict.*
(ii) *$\phi \circ \mathrm{d}_M = \mathrm{d}_N \circ \phi$.*

In the last definition in this subsection, we assume M is abelian. Then complexes have cohomology objects.

Definition 4.6.7. Let M be a $\mathbb{K}$-linear abelian category, and let A be a central DG $\mathbb{K}$-ring. For any integer i let

$$\mathrm{H}^i : \mathbf{C}_{\mathrm{str}}(\mathsf{M}, A) \to \mathsf{M}$$

be the $\mathbb{K}$-linear functor, that sends a complex M to its i-th cohomology $\mathrm{H}^i(M) \in \mathsf{M}$ as in (4.3.7), and that sends a strict morphism $\phi : M_0 \to M_1$ to the morphism

$$\mathrm{H}^i(\phi) : \mathrm{H}^i(M_0) \to \mathrm{H}^i(M_1).$$

4.7. **Translations.** This concept goes back to the beginnings of derived categories. The treatment in this subsection (with the operator t) is taken from [Ye9, Section 1]. We fix a $\mathbb{K}$-linear category M and a central DG $\mathbb{K}$-ring A.

Definition 4.7.1. Let $M = \{M^i\}_{i\in\mathbb{Z}}$ be a DG A-module in M, i.e. an object of $\mathbf{C}(\mathsf{M}, A)$, with differential $\mathrm{d}_M = \{\mathrm{d}^i_M\}_{i\in\mathbb{Z}}$, as in Subsection 4.3. The *translation* (or twist, or shift, or suspension) of M is the object

$$\mathrm{T}(M) = \{\mathrm{T}(M)^i\}_{i\in\mathbb{Z}} \in \mathbf{C}(\mathsf{M}, A)$$

defined as follows. First we say what $\mathrm{T}(M)$ is as a graded object of M. The graded component of degree i of $\mathrm{T}(M)$ is $\mathrm{T}(M)^i := M^{i+1}$. The identity morphisms $\mathrm{t} : M^{i+1} \to \mathrm{T}(M)^i$ combine to give a degree -1 morphism $\mathrm{t} : M \to \mathrm{T}(M)$ of graded objects of M.

Next we make $\mathrm{T}(M)$ into a complex, namely an object of $\mathbf{C}(\mathsf{M})$. The differential $\mathrm{d}_{\mathrm{T}(M)} = \{\mathrm{d}^i_{\mathrm{T}(M)}\}_{i\in\mathbb{Z}}$ is defined by the formula

$$\mathrm{d}^i_{\mathrm{T}(M)} := -\,\mathrm{t} \circ \mathrm{d}^{i+1}_M \circ \mathrm{t}^{-1}\,.$$

Finally we put a left action of A on the complex $\mathrm{T}(M)$. Let $f_M : A \to \mathrm{End}_{\mathsf{M}}(M)$ be the DG ring homomorphism that determines the action of A on M. Then

$$f_{\mathrm{T}(M)} : A \to \mathrm{End}_{\mathsf{M}}(\mathrm{T}(M))$$

is defined by

$$f_{\mathrm{T}(M)}(a) := (-1)^j \cdot \mathrm{t} \circ f_M(a) \circ \mathrm{t}^{-1}$$

for $a \in A^j$.

Thus, the differential $\mathrm{d}_{\mathrm{T}(M)}$ makes this diagram in M commutative for every i :

$$\begin{tikzcd}
\mathrm{T}(M)^i \arrow[r, "\mathrm{d}^i_{\mathrm{T}(M)}"] & \mathrm{T}(M)^{i+1} \\
M^{i+1} \arrow[u, "\mathrm{t}"] \arrow[r, "-\mathrm{d}^{i+1}_M"] & M^{i+2} \arrow[u, "\mathrm{t}"']
\end{tikzcd}$$

And the left A-module structure makes this diagram in M commutative for every i and every $a \in A^j$:

$$\begin{tikzcd}
\mathrm{T}(M)^i \arrow[r, "f_{\mathrm{T}(M)}(a)"] & \mathrm{T}(M)^{i+j} \\
M^{i+1} \arrow[u, "\mathrm{t}"] \arrow[r, "(-1)^j \cdot f_M(a)"] & M^{i+j+1} \arrow[u, "\mathrm{t}"']
\end{tikzcd}$$

When we need to emphasize the category $\mathbf{C}(\mathsf{M}, A)$ or the object M, we will write

$$\mathrm{t}_M : M \to \mathrm{T}_{\mathsf{M},A}(M). \tag{4.7.2}$$

This is an isomorphism in $\mathbf{C}(\mathsf{M}, A)$. Warning: t_M is not a morphism in $\mathbf{C}_{\mathrm{str}}(\mathsf{M}, A)$, because it has degree -1. Indeed,

$$\mathrm{t}_M \in \mathrm{Hom}_{\mathsf{M},A}\big(M, \mathrm{T}(M)\big)^{-1}$$

and its inverse

$$\mathrm{t}_M^{-1} \in \mathrm{Hom}_{\mathsf{M},A}\big(\mathrm{T}(M), M\big)^{1}.$$

Proposition 4.7.3. *The morphisms* t_M *and* t_M^{-1} *are cocycles, in the DG* $\mathbb{K}$*-modules* $\mathrm{Hom}_{\mathsf{M},A}\big(M, \mathrm{T}(M)\big)$ *and* $\mathrm{Hom}_{\mathsf{M},A}\big(\mathrm{T}(M), M\big)$ *respectively.*

Proof. We leave the first assertion as an exercise. Note that the sign appearing in the definition of $\mathrm{d}_{\mathrm{T}(M)}$ is essential here. [4]

As for t_M^{-1} : since t_M is an isomorphism, we can use Proposition 4.2.6. □

Definition 4.7.4. Given a morphism

$$\phi = \{\phi^j\}_{j \in \mathbb{Z}} \in \mathrm{Hom}_{\mathsf{M},A}(M, N)^i,$$

we define the morphism

$$\mathrm{T}(\phi) = \{\mathrm{T}(\phi)^j\}_{j \in \mathbb{Z}} \in \mathrm{Hom}_{\mathsf{M},A}\big(\mathrm{T}(M), \mathrm{T}(N)\big)^i$$

as follows:

$$\mathrm{T}(\phi)^j := (-1)^i \cdot \mathrm{t}_N \circ \phi^{j+1} \circ \mathrm{t}_M^{-1} .$$

---

[4] Thanks to Rishi Vyas for pointing this out.

To clarify this definition, let us note that the corresponding commutative diagram in M, for each i, j, is:

$$\begin{tikzcd} \mathrm{T}(M)^{j} \arrow[r, "\mathrm{T}(\phi)^{j}"] & \mathrm{T}(N)^{j+i} \\ M^{j+1} \arrow[u, "\mathrm{t}_M"] \arrow[r, "(-1)^{i} \cdot \phi^{j+1}"] & N^{j+i} \arrow[u, "\mathrm{t}_N"'] \end{tikzcd} \tag{4.7.5}$$

Proposition 4.7.6.

(1) T *respects composition of morphisms in* $\mathbf{C}(\mathsf{M}, A)$. *Namely, given morphisms* $\phi_1 : M_0 \to M_1$ *and* $\phi_2 : M_1 \to M_2$*, of degrees* i_1 *and* i_2 *respectively, we have*

$$\mathrm{T}(\phi_2 \circ \phi_1) = \mathrm{T}(\phi_2) \circ \mathrm{T}(\phi_1) \in \mathrm{Hom}_{\mathsf{M},A}\big(\mathrm{T}(M_0), \mathrm{T}(M_2)\big)^{i_1+i_2}.$$

(2) T *commutes with* d *on morphisms; namely for any morphism* ϕ *in* $\mathbf{C}(\mathsf{M}, A)$*, we have* $\mathrm{T}(\mathrm{d}(\phi)) = \mathrm{d}(\mathrm{T}(\phi))$.

(3) T *is a* $\mathbb{K}$*-linear automorphism of the category* $\mathbf{C}(\mathsf{M}, A)$.

(4) T *restricts to a* $\mathbb{K}$*-linear automorphism of the category* $\mathbf{C}_{\mathrm{str}}(\mathsf{M}, A)$.

(5) T *induces a* $\mathbb{K}$*-linear automorphism of the category* $\mathbf{K}(\mathsf{M}, A)$.

Proof. (1) Here is the calculation:

$$\begin{aligned} \mathrm{T}(\phi_2 \circ \phi_1) &= (-1)^{i_1+i_2} \cdot \mathrm{t}_{M_2} \circ (\phi_2 \circ \phi_1) \circ \mathrm{t}_{M_0}^{-1} \\ &= (-1)^{i_1+i_2} \cdot \mathrm{t}_{M_2} \circ \phi_2 \circ (\mathrm{t}_{M_1}^{-1} \circ \mathrm{t}_{M_1}) \circ \phi_1 \circ \mathrm{t}_{M_0}^{-1} \\ &= \big((-1)^{i_2} \cdot \mathrm{t}_{M_2} \circ \phi_2 \circ \mathrm{t}_{M_1}^{-1}\big) \circ \big((-1)^{i_1} \cdot \mathrm{t}_{M_1} \circ \phi_1 \circ \mathrm{t}_{M_0}^{-1}\big) \\ &= \mathrm{T}(\phi_2) \circ \mathrm{T}(\phi_1). \end{aligned}$$

(2) A similar calcuation, this time using the fact that $\mathrm{d} \circ \mathrm{t} = -\,\mathrm{t} \circ \mathrm{d}$ and $\mathrm{d} \circ \mathrm{t}^{-1} = -\,\mathrm{t}^{-1} \circ \mathrm{d}$, which is a consequence of Proposition 4.7.3.

(3) That T is a functor is just item (1). And clearly

$$\mathrm{T}(\lambda \cdot \phi_1 + \phi_1') = \lambda \cdot \mathrm{T}(\phi_1) + \mathrm{T}(\phi_1')$$

for $\lambda \in \mathbb{K}$ and $\phi_1, \phi_1' \in \mathrm{Hom}_{\mathsf{M},A}(M_0, M_1)^{i_1}$.

(4) By item (2), T sends 0-cocycles to 0-cocycles.

(5) Likewise, T sends 0-coboundaries to 0-coboundaries. Compare to Exercise 4.5.5. □

Proposition 4.7.7. *As* M *varies, the isomorphisms* $\mathrm{t}_M : M \to \mathrm{T}(M)$ *in* $\mathbf{C}(\mathsf{M}, A)$ *become an isomorphism*

$$\mathrm{t} : \mathrm{id} \to \mathrm{T}$$

of functors

$$\mathbf{C}_{\mathrm{str}}(\mathsf{M}, A) \to \mathbf{C}(\mathsf{M}, A).$$

Proof. We have to prove that for every morphism $\phi : M \to N$ in $\mathbf{C}_{\mathrm{str}}(\mathsf{M}, A)$, the diagram

$$\begin{tikzcd} M \arrow[r, "\mathrm{t}_M"] \arrow[d, "\phi"'] & \mathrm{T}(M) \arrow[d, "\mathrm{T}(\phi)"] \\ N \arrow[r, "\mathrm{t}_N"] & \mathrm{T}(N) \end{tikzcd}$$

in $\mathbf{C}(\mathsf{M}, A)$ is commutative. The can be seen immeduately from Definition 4.7.4, since $i = 0$ here. □

For any $k \in \mathbb{Z}$, the k-th power of the translation T is denoted by T^k. There is a corresponding element

(4.7.8) $$\mathrm{t}^k \in \mathrm{Hom}_{\mathsf{M},A}(M, \mathrm{T}^k(M))^{-k}.$$

Note that $\mathrm{T}^l(\mathrm{T}^k(M)) = \mathrm{T}^{k+l}(M)$. As usual in the literature (going back to [RD]), we will later write $M[k] := \mathrm{T}^k(M)$.

4.8. **Cones.** Again we fix a $\mathbb{K}$-linear category M and a central DG $\mathbb{K}$-ring A. Here is the *cone* construction in $\mathbf{C}(\mathsf{M}, A)$, as it looks using the operator t.

Definition 4.8.1. Let $\phi : M \to N$ be a strict morphism in $\mathbf{C}(\mathsf{M}, A)$. The cone of ϕ is the object $\mathrm{Cone}(\phi) \in \mathbf{C}(\mathsf{M}, A)$ defined as follows. As a graded A-module in M we let

$$\mathrm{Cone}(\phi) := N \oplus \mathrm{T}(M).$$

The differential $\mathrm{d}_{\mathrm{cone}}$ is this: if we express the graded module $\mathrm{Cone}(\phi)$ as a column $\left[\begin{smallmatrix} N \\ \mathrm{T}(M) \end{smallmatrix}\right]$, then $\mathrm{d}_{\mathrm{cone}}$ is left multiplication by the matrix

$$\begin{bmatrix} \mathrm{d}_N & \phi \circ \mathrm{t}_M^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}(M)} \end{bmatrix}$$

of degree 1 morphisms of graded objects of M.

In other words,

$$\mathrm{d}^i_{\mathrm{cone}} : \mathrm{Cone}(\phi)^i \to \mathrm{Cone}(\phi)^{i+1}$$

is

$$\mathrm{d}^i_{\mathrm{cone}} = \mathrm{d}^i_N + \mathrm{d}^i_{\mathrm{T}(M)} + \phi^{i+1} \circ \mathrm{t}_M^{-1},$$

where $\phi^{i+1} \circ \mathrm{t}_M^{-1}$ is the composed morphism

$$\mathrm{T}(M)^i \xrightarrow{\mathrm{t}_M^{-1}} M^{i+1} \xrightarrow{\phi^{i+i}} N^{i+1}.$$

Let us denote by

(4.8.2) $$e_\phi : N \to N \oplus \mathrm{T}(M)$$

the embedding, and by

(4.8.3) $$p_\phi : N \oplus \mathrm{T}(M) \to \mathrm{T}(M)$$

the projection.

Definition 4.8.4. Let $\phi : M \to N$ be a strict morphism in $\mathbf{C}(\mathsf{M}, A)$. The diagram

$$M \xrightarrow{\phi} N \xrightarrow{e_\phi} \mathrm{Cone}(\phi) \xrightarrow{p_\phi} \mathrm{T}(M)$$

in $\mathbf{C}_{\mathrm{str}}(\mathsf{M}, A)$ is called the *standard triangle* associated to ϕ.

The cone construction is functorial, in the following sense.

Proposition 4.8.5. *Let*

$$\begin{tikzcd} M_0 \arrow[r, "\phi_0"] \arrow[d, "\psi"'] & N_0 \arrow[d, "\chi"] \\ M_1 \arrow[r, "\phi_1"] & N_1 \end{tikzcd}$$

be a commutative diagram in $\mathbf{C}_{\mathrm{str}}(\mathsf{M}, A)$. *Then*

(4.8.6) $$(\chi, \mathrm{T}(\psi)) : \mathrm{Cone}(\phi_0) \to \mathrm{Cone}(\phi_1)$$

is a morphism in $\mathsf{C}_{\mathrm{str}}(\mathsf{M}, A)$*, and the diagram*

$$\begin{tikzcd}
M_0 \arrow[r, "\phi_0"] \arrow[d, "\psi"'] & N_0 \arrow[r, "e_{\phi_0}"] \arrow[d, "\chi"'] & \mathrm{Cone}(\phi_0) \arrow[r, "p_{\phi_0}"] \arrow[d, "{(\chi, \mathrm{T}(\psi))}"'] & \mathrm{T}(M_0) \arrow[d, "\mathrm{T}(\psi)"'] \\
M_1 \arrow[r, "\phi_1"] & N_1 \arrow[r, "e_{\phi_1}"] & \mathrm{Cone}(\phi_1) \arrow[r, "p_{\phi_1}"] & \mathrm{T}(M_1)
\end{tikzcd}$$

in $\mathsf{C}_{\mathrm{str}}(\mathsf{M}, A)$ *is commutative.*

Proof. This is a simple consequence of the definitions. ☐

5. DG Functors

★ public 14 | 24 Dec 2015 :
big change, see Thm 5.3.2. ★

In this section we fix a commutative base ring $\mathbb{K}$; a pair of $\mathbb{K}$-linear categories M and N; and a pair of central DG $\mathbb{K}$-rings A and B. We will deal with the DG $\mathbb{K}$-linear categories $\mathbf{C}(\mathsf{M}, A)$ and $\mathbf{C}(\mathsf{N}, B)$ that were introduced in subsection 4.6.

5.1. **Graded Functors.**

Definition 5.1.1. A functor

$$F : \mathbf{C}(\mathsf{M}, A) \to \mathbf{C}(\mathsf{N}, B)$$

is called a $\mathbb{K}$-*linear graded functor* if it satisfies this condition:

- For any pair of objects $M_0, M_1 \in \mathbf{C}(\mathsf{M}, A)$, the function
$$F : \mathrm{Hom}_{\mathsf{M},A}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{N},A}(F(M_0), F(M_1))$$
is a degree 0 homomorphism of graded $\mathbb{K}$-modules.

Recall that "morphism of functors" is synonymous with "natural transformation".

Definition 5.1.2. Let

$$F, G : \mathbf{C}(\mathsf{M}, A) \to \mathbf{C}(\mathsf{N}, B)$$

be $\mathbb{K}$-linear graded functors, and let $i \in \mathbb{Z}$. A *degree i morphism of graded functors* $\eta : F \to G$ is a collection $\eta = \{\eta_M\}$ of morphisms

$$\eta_M \in \mathrm{Hom}_{\mathsf{N},B}(F(M), G(M))^i,$$

indexed by objects $M \in \mathbf{C}(\mathsf{M}, A)$, such that for any morphism $\phi \in \mathrm{Hom}_{\mathsf{M},A}(M_0, M_1)^j$, there is equality

$$G(\phi) \circ \eta_{M_0} = (-1)^{ij} \cdot \eta_{M_1} \circ F(\phi)$$

inside

$$\mathrm{Hom}_{\mathsf{N},B}(F(M_0), G(M_1))^{i+j}.$$

5.2. **DG Functors.** Recall the meaning of a strict homomorphism of DG $\mathbb{K}$-modules: it has degree 0 and commutes with the differentials. (See Proposition 4.6.6 where this is emphasized.)

The differential of the DG $\mathbb{K}$-module $\mathrm{Hom}_{\mathsf{M},A}(M_0, M_1)$, for $M_0, M_1 \in \mathbf{C}(\mathsf{M}, A)$, will be denoted by $\mathrm{d}_{\mathsf{M},A}$. Likewise for $\mathbf{C}(\mathsf{N}, B)$. See Definition 4.6.3.

Definition 5.2.1. A functor

$$F : \mathbf{C}(\mathsf{M}, A) \to \mathbf{C}(\mathsf{N}, B)$$

is called a $\mathbb{K}$-*linear DG functor* if it satisfies this condition:

- For any pair of objects $M_0, M_1 \in \mathbf{C}(\mathsf{M}, A)$, the function
$$F : \mathrm{Hom}_{\mathsf{M},A}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{N},A}(F(M_0), F(M_1))$$
is a strict homomorphism of DG $\mathbb{K}$-modules.

In other words, F is a DG functor if it is a graded functor, and

$$\mathrm{d}_{\mathsf{N},B} \circ F = F \circ \mathrm{d}_{\mathsf{M},A}. \tag{5.2.2}$$

Examples of DG functors will be given in Subsection 5.6.

Definition 5.2.3. Let

$$F, G : \mathsf{C}(\mathsf{M}, A) \to \mathsf{C}(\mathsf{N}, B)$$

be $\mathbb{K}$-linear DG functors.

1. A *degree i morphism of DG functors* $\eta : F \to G$ is just a degree i morphism of graded functors, as in Definition 5.1.1).
2. A *strict morphism of DG functors* is a degree 0 morphism of graded functors $\eta : F \to G$, such that each $\eta_M : F(M) \to G(M)$ is a strict morphism in $\mathsf{C}(\mathsf{N}, B)$.

Proposition 5.2.4. *Let*

$$F : \mathsf{C}(\mathsf{M}, A) \to \mathsf{C}(\mathsf{N}, B)$$

be a $\mathbb{K}$-linear DG functor. Then F induces $\mathbb{K}$-linear functors

$$\mathrm{Z}^0(F) : \mathsf{C}_{\mathrm{str}}(\mathsf{M}, A) \to \mathsf{C}_{\mathrm{str}}(\mathsf{N}, B)$$

and

$$\mathrm{H}^0(F) : \mathsf{K}(\mathsf{M}, A) \to \mathsf{K}(\mathsf{N}, B).$$

Proof. Because F is a DG functor, it sends 0-cocycles in $\mathrm{Hom}_{\mathsf{M},A}(M_0, M_1)$ to 0-cocycles in $\mathrm{Hom}_{\mathsf{N},B}(F(M_0), F(M_1))$. The same for 0-coboundaries. □

By abuse of notation, when there is no danger for confusion, we will often write F instead of $\mathrm{Z}^0(F)$.

Remark 5.2.5. Let F be a DG functor as in Definition 5.2.1. The differential d_M of an object $M \in \mathsf{C}(\mathsf{M}, A)$ lives in $\mathrm{Hom}_{\mathsf{M},A}(M, M)^1$. Thus, in $\mathrm{Hom}_{\mathsf{N},B}(F(M), F(M))^1$ we have the operators $F(\mathrm{d}_M)$ and $\mathrm{d}_{F(M)}$. A naive claim is that $F(\mathrm{d}_M) = \mathrm{d}_{F(M)}$. This is true sometimes, like in Example 5.6.1 below. However, in many situations, such as the one in Example 5.6.2, this is false! The discrepancy is what we will refer to below as the *gauge.* [5]

5.3. The Gauge of a Graded Functor.

Definition 5.3.1. Let

$$F : \mathsf{C}(\mathsf{M}, A) \to \mathsf{C}(\mathsf{N}, B)$$

be a $\mathbb{K}$-linear graded functor (Definition 5.1.1). For any object $M \in \mathsf{C}(\mathsf{M}, A)$ let

$$\gamma_{F,M} := \mathrm{d}_{F(M)} - F(\mathrm{d}_M) \in \mathrm{Hom}_{\mathsf{N},B}\big(F(M), F(M)\big)^1.$$

The collection of morphisms

$$\gamma_F := \{\gamma_{F,M}\}_{M \in \mathsf{C}(\mathsf{M},A)}$$

is called the *gauge of F*.

Theorem 5.3.2. [6] *The following two conditions are equivalent for a $\mathbb{K}$-linear graded functor*

$$F : \mathsf{C}(\mathsf{M}, A) \to \mathsf{C}(\mathsf{N}, B).$$

- (i) *F is a DG functor (Definition 5.2.1).*
- (ii) *The gauge γ_F is a degree 1 morphism of graded functors $\gamma_F : F \to F$.*

---

[5] Thanks to Asaf Yekutieli for a helpful discussion of this difficulty.

[6] This is due to Rishi Vyas

Proof. Recall that F is a DG functor (condition (i)) iff

$$(F \circ \mathrm{d}_{\mathsf{M},A})(\phi) = (\mathrm{d}_{\mathsf{N},B} \circ F)(\phi) \tag{5.3.3}$$

for every $\phi \in \mathrm{Hom}_{\mathsf{M},A}(M_0, M_1)^i$. And γ_F is a degree 1 morphism of graded functors (condition (ii)) iff

$$\gamma_{F,M_1} \circ F(\phi) = (-1)^i \cdot F(\phi) \circ \gamma_{F,M_0} \tag{5.3.4}$$

for every such ϕ.

Here is the calculation. Because F is a graded functor, we get

$$\begin{aligned} F\big(\mathrm{d}_{\mathsf{M},A}(\phi)\big) &= F\big(\mathrm{d}_{M_1} \circ \phi - (-1)^i \cdot \phi \circ \mathrm{d}_{M_0}\big) \\ &= F(\mathrm{d}_{M_1}) \circ F(\phi) - (-1)^i \cdot F(\phi) \circ F(\mathrm{d}_{M_0}) \end{aligned} \tag{5.3.5}$$

and

$$\mathrm{d}_{\mathsf{N},B}\big(F(\phi)\big) = \mathrm{d}_{F(M_1)} \circ F(\phi) - (-1)^i \cdot F(\phi) \circ \mathrm{d}_{F(M_0)}. \tag{5.3.6}$$

Using equations (5.3.5) and (5.3.6), and the definition of γ_F, we obtain

$$\begin{aligned} (F \circ \mathrm{d}_{\mathsf{M},A} - \mathrm{d}_{\mathsf{N},B} \circ F)(\phi) &= F\big(\mathrm{d}_{\mathsf{M},A}(\phi)\big) - \mathrm{d}_{\mathsf{N},B}\big(F(\phi)\big) \\ &= \big(F(\mathrm{d}_{M_1}) - \mathrm{d}_{F(M_1)}\big) \circ F(\phi) - (-1)^i \cdot F(\phi) \circ \big(F(\mathrm{d}_{M_0}) - \mathrm{d}_{F(M_0)}\big) \\ &= \gamma_{F,M_1} \circ F(\phi) - (-1)^i \cdot F(\phi) \circ \gamma_{F,M_0}. \end{aligned} \tag{5.3.7}$$

Finally, the vanishing of the first expression in (5.3.7) is the same as equality in (5.3.3); whereas the vanishing of the last expression in (5.3.7) is the same as equality in (5.3.4). □

5.4. **DG Functors and Translations.** Recall that for an object $M \in \mathbf{C}(\mathsf{M}, A)$, we have the little t operator

$$\mathrm{t}_M \in \mathrm{Hom}_{\mathsf{M},A}\big(M, \mathrm{T}_{\mathsf{M},A}(M)\big)^{-1}.$$

This is an isomorphism in $\mathbf{C}(\mathsf{M}, A)$.

Definition 5.4.1. Let

$$F : \mathbf{C}(\mathsf{M}, A) \to \mathbf{C}(\mathsf{N}, B)$$

be a $\mathbb{K}$-linear DG functor. For an object $M \in \mathbf{C}(\mathsf{M}, A)$, let

$$\zeta_M : F(\mathrm{T}_{\mathsf{M},A}(M)) \to \mathrm{T}_{\mathsf{N},B}(F(M))$$

be the isomorphism

$$\zeta_M := \mathrm{t}_{F(M)} \circ F(\mathrm{t}_M)^{-1}$$

in $\mathbf{C}(\mathsf{N}, B)$, called the *translation isomorphism.*

The isomorphism ζ_M sits in the following commutative diagram

$$\begin{tikzcd} F(\mathrm{T}_{\mathsf{M},A}(M)) \arrow[r, "\zeta_M"] & \mathrm{T}_{\mathsf{N},B}(F(M)) \\ F(M) \arrow[u, "F(\mathrm{t}_M)"] \arrow[ur, "\mathrm{t}_{F(M)}"'] & \end{tikzcd}$$

of isomorphisms in the category $\mathbf{C}(\mathsf{N}, B)$.

Proposition 5.4.2. *ζ_M is an isomorphism in $\mathbf{C}_{\mathrm{str}}(\mathsf{N}, B)$.*

Proof. We know that ζ_M is an isomorphism in $\mathbf{C}(\mathsf{N}, B)$. It sufficed to prove that both ζ_M and its inverse ζ_M^{-1} are strict morphisms. Now by Proposition 4.7.3, t_M and t_M^{-1} are cocycles. Therefore, $F(\mathrm{t}_M)$ and $F(\mathrm{t}_M)^{-1} = F(\mathrm{t}_M^{-1})$ are cocycles. For the same reason, $\mathrm{t}_{F(M)}$ and $\mathrm{t}_{F(M)}^{-1}$ are cocycles. But $\zeta_M = \mathrm{t}_{F(M)} \circ F(\mathrm{t}_M)^{-1}$, and $\zeta_M^{-1} = F(\mathrm{t}_M) \circ \mathrm{t}_{F(M)}^{-1}$. □

Theorem 5.4.3. *Let*

$$F : \mathbf{C}(\mathsf{M}, A) \to \mathbf{C}(\mathsf{N}, B)$$

be a $\mathbb{K}$-linear DG functor. Then the collection $\zeta := \{\zeta_M\}_{M \in \mathbf{C}(\mathsf{M},A)}$ is an isomorphism

$$\zeta : F \circ \mathrm{T}_{\mathsf{M},A} \xrightarrow{\simeq} \mathrm{T}_{\mathsf{N},B} \circ F$$

of functors

$$\mathbf{C}_{\mathrm{str}}(\mathsf{M}, A) \to \mathbf{C}_{\mathrm{str}}(\mathsf{N}, B).$$

The slogan summarizing this theorem is “A DG functor commutes with translations”.

Proof. In view of Proposition 5.4.2, all we need to prove is that ζ_M is a morphism of functors (i.e. it is a natural transformation).

Let $\phi : M_0 \to M_1$ be a morphism in $\mathbf{C}_{\mathrm{str}}(\mathsf{M}, A)$. We must prove that the diagram

$$\begin{tikzcd}
(F \circ \mathrm{T}_{\mathsf{M},A})(M_0) \arrow[r, "\zeta_{M_0}"] \arrow[d, "(F \circ \mathrm{T}_{\mathsf{M},A})(\phi)"'] & (\mathrm{T}_{\mathsf{N},B} \circ F)(M_0) \arrow[d, "(\mathrm{T}_{\mathsf{N},B} \circ F)(\phi)"] \\
(F \circ \mathrm{T}_{\mathsf{M},A})(M_1) \arrow[r, "\zeta_{M_1}"] & (\mathrm{T}_{\mathsf{N},B} \circ F)(M_1)
\end{tikzcd}$$

in $\mathbf{C}_{\mathrm{str}}(\mathsf{N}, B)$ is commutative. This will be true if the next diagram

$$\begin{tikzcd}
(F \circ \mathrm{T}_{\mathsf{M},A})(M_0) \arrow[d, "(F \circ \mathrm{T}_{\mathsf{M},A})(\phi)"'] & F(M_0) \arrow[l, "F(\mathrm{t}_{M_0})"'] \arrow[r, "\mathrm{t}_{F(M_0)}"] \arrow[d, "F(\phi)"'] & (\mathrm{T}_{\mathsf{N},B} \circ F)(M_0) \arrow[d, "(\mathrm{T}_{\mathsf{N},B} \circ F)(\phi)"] \\
(F \circ \mathrm{T}_{\mathsf{M},A})(M_1) & F(M_1) \arrow[l, "F(\mathrm{t}_{M_1})"'] \arrow[r, "\mathrm{t}_{F(M_1)}"] & (\mathrm{T}_{\mathsf{N},B} \circ F)(M_1)
\end{tikzcd}$$

in $\mathbf{C}(\mathsf{N}, B)$, whose horizontal arrows are isomorphisms, is commutative. For this to be true, it is enough to prove that both squares in this diagram are commutative. This is true by Proposition 4.7.7. □

5.5. **DG Functors and Cones.** Recall that M and N are $\mathbb{K}$-linear categories, and A and B are central DG $\mathbb{K}$-rings.

Definition 5.5.1. The subcategory $\mathbf{C}^0(\mathsf{M}, A)$ of $\mathbf{C}(\mathsf{M}, A)$ is defined to be the subcategory on all objects, but with degree 0 morphisms only.

There are inclusions (faithful functors, identities on objects)

$$\mathbf{C}_{\mathrm{str}}(\mathsf{M}, A) \xrightarrow{\subseteq} \mathbf{C}^0(\mathsf{M}, A) \xrightarrow{\subseteq} \mathbf{C}(\mathsf{M}, A).$$

Let

$$F : \mathbf{C}(\mathsf{M}, A) \to \mathbf{C}(\mathsf{N}, B)$$

be a $\mathbb{K}$-linear DG functor. Given a morphism $\phi : M_0 \to M_1$ in $\mathbf{C}_{\mathrm{str}}(\mathsf{M}, A)$, we have a morphism

$$F(\phi) : F(M_0) \to F(M_1)$$

in $\mathsf{C}_{\mathrm{str}}(\mathsf{N}, B)$, and objects $F(\mathrm{Cone}_{\mathsf{M},A}(\phi))$ and $\mathrm{Cone}_{\mathsf{N},B}(F(\phi))$ in $\mathsf{C}_{\mathrm{str}}(\mathsf{N}, B)$. By definition

$$\mathrm{Cone}_{\mathsf{M},A}(\phi) = M_1 \oplus \mathrm{T}_{\mathsf{M},A}(M_0) \tag{5.5.2}$$

in $\mathsf{C}^0(\mathsf{M}, A)$. Since F is an additive functor, it commutes with finite direct sums, and therefore there is a canonical isomorphism

$$F(\mathrm{Cone}_{\mathsf{M},A}(\phi)) \cong F(M_1) \oplus F(\mathrm{T}_{\mathsf{M},A}(M_0)) \tag{5.5.3}$$

in $\mathsf{C}^0(\mathsf{N}, B)$. And by definition,

$$\mathrm{Cone}_{\mathsf{N},B}(F(\phi)) = F(M_1) \oplus \mathrm{T}_{\mathsf{N},B}(F(M_0)) \tag{5.5.4}$$

in $\mathsf{C}^0(\mathsf{N}, B)$. Warning: the isomorphisms (5.5.2), (5.5.3) and (5.5.4) are usually not strict! Namely (see Proposition 4.6.6) they might not commute with the differentials. The differentials on the right sides are diagonal matrices, but not so on the left sides (see Definition 4.8.1).

Lemma 5.5.5. *Let*

$$F, G : \mathsf{C}(\mathsf{M}, A) \to \mathsf{C}(\mathsf{N}, B)$$

be $\mathbb{K}$-linear graded functors, and let $\eta : F \to G$ be a degree j morphism of graded functors. Suppose $M \cong M_0 \oplus M_1$ in $\mathsf{C}^0(\mathsf{M}, A)$, with embeddings $e_i : M_i \to M$ and projections $p_i : M \to M_i$. Then

$$\eta_M = \big(G(e_0), G(e_1)\big) \circ (\eta_{M_0}, \eta_{M_1}) \circ \big(F(p_0), F(p_1)\big),$$

as degree j morphisms $F(M) \to G(M)$ in $\mathsf{C}(\mathsf{N}, B)$.

The lemma says that the diagram

$$\begin{tikzcd}
F(M) \arrow[r, "\big(F(p_0){,}F(p_1)\big)"] \arrow[d, "\eta_M"'] & F(M_0) \oplus F(M_1) \arrow[d, "(\eta_{M_0}{,}\eta_{M_1})"] \\
G(M) & G(M_0) \oplus G(M_1) \arrow[l, "\big(G(e_0){,}G(e_1)\big)"']
\end{tikzcd}$$

in $\mathsf{C}(\mathsf{N}, B)$ is commutative.

Proof. It suffice to prove that the diagram below is commutative for $i = 0, 1$:

$$\begin{tikzcd}
F(M_i) \arrow[r, "F(e_i)"] \arrow[d, "\eta_{M_i}"'] \arrow[rr, bend left, "\mathrm{id}"] & F(M) \arrow[r, "F(p_i)"] \arrow[d, "\eta_M"'] & F(M_i) \arrow[d, "\eta_{M_i}"'] \\
G(M_i) \arrow[r, "G(e_i)"] \arrow[rr, bend right, "\mathrm{id}"'] & G(M) \arrow[r, "G(p_i)"] & G(M_i)
\end{tikzcd}$$

This is true because η is a morphism of functors (a natural transformation). ☐

★ public 14: big change – using Thm 5.3.2 instead of the "pretriangulated" condition, that is no longer needed! ★

Theorem 5.5.6. *Let*

$$F : \mathbf{C}(\mathsf{M}, A) \to \mathbf{C}(\mathsf{N}, B)$$

be a $\mathbb{K}$-linear DG functor, and let $\phi : M_0 \to M_1$ be a morphism in $\mathbf{C}_{\mathrm{str}}(\mathsf{M}, A)$. Define the the isomorphism

$$\theta : F(\mathrm{Cone}_{\mathsf{M},A}(\phi)) \to \mathrm{Cone}_{\mathsf{N},B}(F(\phi))$$

in $\mathbf{C}^0(\mathsf{N}, B)$ to be

$$\theta := (\mathrm{id}_{F(M_1)}, \zeta_{M_0}).$$

Then:

1. *The isomorphism θ is strict; namely it commutes with the differentials.*
2. *The diagram*

$$\begin{tikzcd} F(M_0) \arrow[r, "F(\phi)"] \arrow[d, "="'] & F(M_1) \arrow[r, "F(e_\phi)"] \arrow[d, "="'] & F(\mathrm{Cone}_{\mathsf{M},A}(\phi)) \arrow[r, "F(p_\phi)"] \arrow[d, "\theta"'] & F(\mathrm{T}_{\mathsf{M},A}(M_0)) \arrow[d, "\zeta_{M_0}"'] \\ F(M_0) \arrow[r, "F(\phi)"] & F(M_1) \arrow[r, "e_{F(\phi)}"] & \mathrm{Cone}_{\mathsf{N},B}(F(\phi)) \arrow[r, "p_{F(\phi)}"] & \mathrm{T}_{\mathsf{N},B}(F(M_0)) \end{tikzcd}$$

in $\mathbf{C}_{\mathrm{str}}(\mathsf{N}, B)$ is commutative.

When defining θ above, we are using the decompositions (5.5.3) and (5.5.4) in the category $\mathbf{C}^0(\mathsf{N}, B)$, and the isomorphism ζ_{M_0} from Definition 5.4.1.

The slogan summarizing this theorem is “A DG functor sends standard triangles to standard triangles”.

Proof. We have to prove that $\mathrm{d}_{\mathsf{N},B}(\theta) = 0$. Let's write $P := \mathrm{Cone}_{\mathsf{M},A}(\phi)$ and $Q := \mathrm{Cone}_{\mathsf{N},B}(F(\phi))$. Recall that

$$\mathrm{d}_{\mathsf{N},B}(\theta) = \mathrm{d}_Q \circ \theta - \theta \circ \mathrm{d}_{F(P)}.$$

We have to prove that this is the zero element in $\mathrm{Hom}_{\mathsf{N},B}(F(P), Q)^1$.

Writing the cones as column modules:

$$P = \begin{bmatrix} M_1 \\ \mathrm{T}_{\mathsf{M},A}(M_0) \end{bmatrix} , \ Q = \begin{bmatrix} F(M_1) \\ \mathrm{T}_{\mathsf{N},B}(F(M_0)) \end{bmatrix} ,$$

the matrices representing the morphisms in question are

$$\theta = \begin{bmatrix} \mathrm{id}_{F(M_1)} & 0 \\ 0 & \zeta_{M_0} \end{bmatrix} , \ \mathrm{d}_P = \begin{bmatrix} \mathrm{d}_{M_1} & \phi \circ \mathrm{t}_{M_0}^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}_{\mathsf{M},A}(M_0)} \end{bmatrix} , \ \mathrm{d}_Q = \begin{bmatrix} \mathrm{d}_{F(M_1)} & F(\phi) \circ \mathrm{t}_{F(M_0)}^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}_{\mathsf{N},B}(F(M_0))} \end{bmatrix} .$$

Let us write $\gamma := \gamma_F$ for simplicity. According to Theorem 5.3.2, the gauge $\gamma : F \to F$ is a degree 1 morphism of functors $\mathbf{C}(\mathsf{M}, A) \to \mathbf{C}(\mathsf{N}, B)$. Because the decomposition (5.5.2) is in the category $\mathbf{C}^0(\mathsf{M}, A)$, Lemma 5.5.5 tells us that γ_P decomposes too, i.e.

$$\gamma_P = \begin{bmatrix} \gamma_{M_1} & 0 \\ 0 & \gamma_{\mathrm{T}_{\mathsf{M},A}(M_0)} \end{bmatrix} .$$

By definition of γ_P we have

$$\mathrm{d}_{F(P)} = F(\mathrm{d}_P) + \gamma_P \in \mathrm{Hom}_{\mathsf{N},B}(F(P), F(P))^1 .$$

It follows that

$$\begin{aligned}
\mathrm{d}_{F(P)} &= F(\mathrm{d}_P) + \gamma_P \\
&= \begin{bmatrix} F(\mathrm{d}_{M_1}) & F(\phi \circ \mathrm{t}_{M_0}^{-1}) \\ 0 & F(\mathrm{d}_{\mathrm{T}_{\mathsf{M},A}(M_0)}) \end{bmatrix} + \begin{bmatrix} \gamma_{M_1} & 0 \\ 0 & \gamma_{\mathrm{T}_{\mathsf{M},A}(M_0)} \end{bmatrix} \\
&= \begin{bmatrix} F(\mathrm{d}_{M_1}) + \gamma_{M_1} & F(\phi \circ \mathrm{t}_{M_0}^{-1}) \\ 0 & F(\mathrm{d}_{\mathrm{T}_{\mathsf{M},A}(M_0)}) + \gamma_{\mathrm{T}_{\mathsf{M},A}(M_0)} \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{d}_{F(M_1)} & F(\phi \circ \mathrm{t}_{M_0}^{-1}) \\ 0 & \mathrm{d}_{F(\mathrm{T}_{\mathsf{M},A}(M_0))} \end{bmatrix} .
\end{aligned}$$

Finally we will check that $\theta \circ \mathrm{d}_{F(P)}$ and $\mathrm{d}_Q \circ \theta$ are equal as matrices of morphisms. We do that in each matrix position separately. The two left positions in the matrices $\theta \circ \mathrm{d}_{F(P)}$ and $\mathrm{d}_Q \circ \theta$ agree trivially. The bottom right positions in these matrices are $\zeta_{M_0} \circ \mathrm{d}_{F(\mathrm{T}_{\mathsf{M},A}(M_0))}$ and $\mathrm{d}_{\mathrm{T}_{\mathsf{N},B}(F(M_0))} \circ \zeta_{M_0}$ respectively; they are equal by Proposition 5.4.2. And in the top right positions we have $F(\phi \circ \mathrm{t}_{M_0}^{-1})$ and $F(\phi) \circ \mathrm{t}_{F(M_0)}^{-1} \circ \zeta_{M_0}$ respectively. Now $F(\phi \circ \mathrm{t}_{M_0}^{-1}) = F(\phi) \circ F(\mathrm{t}_{M_0}^{-1})$; so it suffices to prove that $F(\mathrm{t}_{M_0}^{-1}) = \mathrm{t}_{F(M_0)}^{-1} \circ \zeta_{M_0}$. This is immediate from the definition of ζ_{M_0}.

(2) By definition of θ, the diagram is commutative in $\mathbf{C}^0(\mathsf{N}, B)$. But by part (1) we know that all morphisms in it lie in $\mathbf{C}_{\mathrm{str}}(\mathsf{N}, B)$. □

5.6. **Examples.**

Example 5.6.1. Here $A = B = \mathbb{K}$, so $\mathbf{C}(\mathsf{M}, A) = \mathbf{C}(\mathsf{M})$ and $\mathbf{C}(\mathsf{N}, B) = \mathbf{C}(\mathsf{N})$. Let $F : \mathsf{M} \to \mathsf{N}$ be a $\mathbb{K}$-linear functor. It extends to a functor

$$\mathbf{C}(F) : \mathbf{C}(\mathsf{M}) \to \mathbf{C}(\mathsf{N})$$

as follows: on objects, a complex

$$M = \big(\{M^i\}_{i\in\mathbb{Z}}, \{\mathrm{d}_M^i\}_{i\in\mathbb{Z}}\big) \in \mathbf{C}(\mathsf{M})$$

goes to the complex

$$\mathbf{C}(F)(M) := \big(\{F(M^i)\}, \{F(\mathrm{d}_M^i)\}\big) \in \mathbf{C}(\mathsf{N}).$$

A morphism $\phi = \{\phi^j\}$ in $\mathbf{C}(\mathsf{M})$ goes to the morphism $\mathbf{C}(\phi) := \{F(\phi^j)\}$ in $\mathbf{C}(\mathsf{N})$. A slightly tedious calculation shows that $\mathbf{C}(F)$ is a $\mathbb{K}$-linear DG functor.

Given a complex $M \in \mathbf{C}(\mathsf{M})$, let $N := \mathbf{C}(F)(M) \in \mathbf{C}(\mathsf{N})$. Then the translations are

$$\mathrm{T}_{\mathsf{N}}(N) = \mathbf{C}(F)(\mathrm{T}_{\mathsf{M}}(M));$$

and $\mathbf{C}(F)(\mathrm{t}_M) = \mathrm{t}_N$. So the translation isomorphism

$$\zeta : \mathbf{C}(F) \circ \mathrm{T}_{\mathsf{M}} \xrightarrow{\simeq} \mathrm{T}_{\mathsf{N}} \circ \mathbf{C}(F)$$

of functors $\mathbf{C}_{\mathrm{str}}(\mathsf{M}) \to \mathbf{C}_{\mathrm{str}}(\mathsf{N})$ is equality.

Let $\phi : M_0 \to M_1$ be a morphism in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, whose image under $\mathbf{C}(F)$ is the morphism $\psi : N_0 \to N_1$ in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$. Then

$$\mathrm{Cone}(\psi) = N_1 \oplus \mathrm{T}_{\mathsf{N}}(N_0) = \mathbf{C}(F)(\mathrm{Cone}(\phi))$$

as graded objects of N, with differential

$$\mathrm{d}_{\mathrm{Cone}(\psi)} = \begin{bmatrix} \mathrm{d}_{N_1} & \psi \circ \mathrm{t}_{N_0}^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}(N_0)} \end{bmatrix} = \mathbf{C}(F)\Big(\begin{bmatrix} \mathrm{d}_{M_1} & \phi \circ \mathrm{t}_{M_0}^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}(M_0)} \end{bmatrix}\Big) = \mathbf{C}(F)(\mathrm{d}_{\mathrm{Cone}(\phi)}).$$

We see that the cone isomorphism θ is equality, and the gauge $\gamma_{\mathsf{C}(F)}$ is zero.

The next example is much more complicated, and we work out the full details (only once – later on, such details will be left to the reader).

Example 5.6.2. Let A and B be central DG $\mathbb{K}$-rings, and fix some

$$N \in \mathsf{DGMod}(B \otimes_{\mathbb{K}} A^{\mathrm{op}}).$$

In other words, N is a DG B-A-bimodule. For any $M \in \mathsf{DGMod}\, A$ we have a DG $\mathbb{K}$-module

$$F(M) := N \otimes_A M,$$

as in Definition 4.5.7. The differential of $F(M)$ is

$$\mathrm{d}_{F(M)} = \mathrm{d}_N \otimes \mathrm{id}_M + \mathrm{id}_N \otimes \mathrm{d}_M. \tag{5.6.3}$$

But $F(M)$ has a structure of a DG B-module: for any $b \in B$, $n \in N$ and $m \in M$, the action is

$$b \cdot (n \otimes m) := (b \cdot n) \otimes m.$$

Clearly

$$F : \mathsf{C}(A) = \mathsf{DGMod}\, A \to \mathsf{C}(B) = \mathsf{DGMod}\, B$$

is a $\mathbb{K}$-linear functor. We will show that it is actually a pretriangulated DG functor.

Let $M_0, M_1 \in \mathsf{C}(A)$, and consider the $\mathbb{K}$-linear homomorphism

$$F : \mathrm{Hom}_A(M_0, M_1) \to \mathrm{Hom}_B(N \otimes_A M_0, N \otimes_A M_1). \tag{5.6.4}$$

We must show that it is a homomorphism of DG $\mathbb{K}$-modules. Here is the calculation.

Take any $\phi \in \mathrm{Hom}_A(M_0, M_1)^i$. Then

$$F(\phi) \in \mathrm{Hom}_B(N \otimes_A M_0, N \otimes_A M_1)$$

is the homomorphism that on a homogenous tensor $n \otimes m \in (N \otimes_A M_0)^{k+j}$, with $n \in N^k$ and $m \in M_0^j$, has the value

$$F(\phi)(n \otimes m) = (-1)^{ik} \cdot n \otimes \phi(m) \in (N \otimes_A M_1)^{k+j+i}.$$

In other words,

$$F(\phi) = \mathrm{id}_N \otimes \phi. \tag{5.6.5}$$

We see that the homomorphism $F(\phi)$ has degree i. So the $\mathbb{K}$-linear homomorphism F from (5.6.4) respects gradings.

Next we must prove that F commutes with the differentials, namely that $F \circ \mathrm{d}_{\mathrm{hom}}$ equals $\mathrm{d}_{\mathrm{hom}} \circ F$. Take $\phi \in \mathrm{Hom}_A(M_0, M_1)^i$. Then, by (5.6.5), we have

$$\begin{aligned} F(\mathrm{d}_{\mathrm{hom}}(\phi)) &= \mathrm{id}_N \otimes \big(\mathrm{d}_{M_1} \circ \phi - (-1)^i \cdot \phi \circ \mathrm{d}_{M_0}\big) \\ &= \mathrm{id}_N \otimes (\mathrm{d}_{M_1} \circ \phi) - (-1)^i \cdot \mathrm{id}_N \otimes (\phi \circ \mathrm{d}_{M_0}). \end{aligned} \tag{5.6.6}$$

On the other hand,

$$\begin{aligned} \mathrm{d}_{\mathrm{hom}}(F(\phi)) &= \mathrm{d}_{F(M_1)} \circ F(\phi) - (-1)^i \cdot F(\phi) \circ \mathrm{d}_{F(M_0)} \\ &= (\mathrm{d}_N \otimes \mathrm{id}_{M_1} + \mathrm{id}_N \otimes \mathrm{d}_{M_1}) \circ (\mathrm{id}_N \otimes \phi) \\ &\qquad - (-1)^i \cdot (\mathrm{id}_N \otimes \phi) \circ (\mathrm{d}_N \otimes \mathrm{id}_{M_0} + \mathrm{id}_N \otimes \mathrm{d}_{M_0}) \\ &= \mathrm{d}_N \otimes \phi + \mathrm{id}_N \otimes (\mathrm{d}_{M_1} \circ \phi) \\ &\qquad - (-1)^i \cdot (-1)^i \cdot \mathrm{d}_N \otimes \phi - (-1)^i \cdot \mathrm{id}_N \otimes (\phi \circ \mathrm{d}_{M_0}) \\ &= \mathrm{id}_N \otimes (\mathrm{d}_{M_1} \circ \phi) - (-1)^i \cdot \mathrm{id}_N \otimes (\phi \circ \mathrm{d}_{M_0}). \end{aligned} \tag{5.6.7}$$

Therefore (5.6.6) and (5.6.7) are equal. So far we have established that F is a DG functor.

Let us figure out what is its translation isomorphism ζ. Take $M \in \mathbf{C}(A)$. Then

$$\mathrm{T}_B(F(M)) = \mathrm{T}_B(N \otimes_A M) = N \otimes_A \mathrm{T}_A(M) = F(\mathrm{T}_A(M))$$

as DG B-modules. The little t operators

$$\mathrm{t}_{F(M)}, F(\mathrm{t}_M) : F(M) \to \mathrm{T}_B(F(M)) = F(\mathrm{T}_A(M))$$

are

$$\mathrm{t}_{F(M)}(n \otimes m) = (-1)^k \cdot n \otimes \mathrm{t}_M(m) = F(\mathrm{t}_M)(n \otimes m)$$

for $n \in N^k$ and $m \in M^j$. We see that $\mathrm{t}_{F(M)} = F(\mathrm{t}_M)$. Therefore

$$\zeta_M : F(\mathrm{T}_A(M)) \xrightarrow{\simeq} \mathrm{T}_B(F(M))$$

is the identity automorphism.

Finally let us calculate γ_F, the gauge of F. From (5.6.5) and (5.6.3) we get

$$\gamma_{F,M} = \mathrm{d}_N \otimes \mathrm{id}_M,$$

which is often nonzero!

Example 5.6.8. Let A and B be central DG $\mathbb{K}$-rings, and fix some

$$N \in \mathsf{DGMod}(A \otimes_{\mathbb{K}} B^{\mathrm{op}}).$$

For any $M \in \mathsf{DGMod}\, A$ we define

$$F(M) := \mathrm{Hom}_A(N, M).$$

This is a DG B-module: for any $b \in B^i$ and $\phi \in \mathrm{Hom}_A(N, M)^j$, the homomorphism $b \cdot \phi \in \mathrm{Hom}_A(N, M)^{i+j}$ is

$$(b \cdot \phi)(n) := (-1)^{i \cdot (j+k)} \cdot \phi(n \cdot b) \in M$$

on $n \in N^k$. As in the previous example,

$$F : \mathbf{C}(A) = \mathsf{DGMod}\, A \to \mathbf{C}(B) = \mathsf{DGMod}\, B$$

is a $\mathbb{K}$-linear DG functor. The value of the gauge γ_F at $M \in \mathbf{C}(A)$ is $\gamma_{F,M} = \mathrm{Hom}(\mathrm{d}_N, \mathrm{id}_M)$. Namely for

$$\psi \in F(M)^j = \mathrm{Hom}_A(N, M)^j$$

we have

$$\gamma_{F,M}(\psi) = (-1)^j \cdot \psi \circ \mathrm{d}_N.$$

6. Triangulated Categories and Functors

In this section we do not have an explicit base ring $\mathbb{K}$. Everything we say can be easily upgraded to a "$\mathbb{K}$-linear" variant.

6.1. **Additive Categories with Translations.**

Definition 6.1.1. Let K be an additive category. A *translation* on K is an additive automorphism T of K. The pair (K, T) is called an *additive category with translation*, or a T-*additive category.*

Remark 6.1.2. Some texts give a more relaxed definition: T is only required to be an additive autoequivalence of K.

Often we will write $M[k] := \mathrm{T}^k(M)$, the k-th translation of an object M.

Definition 6.1.3. Suppose $(\mathsf{K}, \mathrm{T}_{\mathsf{K}})$ and $(\mathsf{L}, \mathrm{T}_{\mathsf{L}})$ are T-additive categories. A T-*additive functor* between them is a pair (F, ξ), consisting of an additive functor $F : \mathsf{K} \to \mathsf{L}$, together with an isomorphism

$$\xi : F \circ \mathrm{T}_{\mathsf{K}} \xrightarrow{\simeq} \mathrm{T}_{\mathsf{L}} \circ F$$

of functors $\mathsf{K} \to \mathsf{L}$.

Definition 6.1.4. Suppose $(\mathsf{K}, \mathrm{T}_{\mathsf{K}})$ and $(\mathsf{L}, \mathrm{T}_{\mathsf{L}})$ are T-additive categories, and

$$(F, \xi), (G, \nu) : (\mathsf{K}, \mathrm{T}_{\mathsf{K}}) \to (\mathsf{L}, \mathrm{T}_{\mathsf{L}})$$

are T-additive functors. A *morphism of* T-*additive functors*

$$\eta : (F, \xi) \to (G, \nu)$$

is a morphism of functors $\eta : F \to G$, such that this diagram of morphisms of functors is commutative:

$$\begin{tikzcd}
F \circ \mathrm{T}_{\mathsf{K}} \arrow[r, "\xi"] \arrow[d, "\eta \circ \mathrm{id}"'] & \mathrm{T}_{\mathsf{L}} \circ F \arrow[d, "\mathrm{id} \circ \eta"] \\
G \circ \mathrm{T}_{\mathsf{K}} \arrow[r, "\nu"] & \mathrm{T}_{\mathsf{L}} \circ G\ .
\end{tikzcd}$$

Definition 6.1.5. Let (K, T) be an additive category with translation. A *triangle* in (K, T) is a diagram

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

in K.

Definition 6.1.6. Let (K, T) be an additive category with translation. Suppose

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

and

$$L' \xrightarrow{\alpha'} M' \xrightarrow{\beta'} N' \xrightarrow{\gamma'} \mathrm{T}(L')$$

are triangles in (K, T). A *morphism of triangles* between them is a commutative diagram

$$\begin{tikzcd}
L \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[r, "\beta"] \arrow[d, "\psi"'] & N \arrow[r, "\gamma"] \arrow[d, "\chi"'] & \mathrm{T}(L) \arrow[d, "\mathrm{T}(\phi)"'] \\
L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L')
\end{tikzcd}$$

in K.

The morphism of triangles (ϕ, ψ, χ) is called an isomorphism if ϕ, ψ and χ are all isomorphisms.

Remark 6.1.7. Why "triangle"? This is because sometimes a triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

is written as a diagram

$$\begin{tikzcd}
 & N \arrow[dl, "\gamma"'] & \\
L \arrow[rr, "\alpha"] & & M \arrow[ul, "\beta"']
\end{tikzcd}$$

But here γ is a morphism "of degree 1".

6.2. **Triangulated Categories.**

Definition 6.2.1. A *triangulated category* is additive category with translation (K, T), equipped with a set of triangles called *distinguished triangles*. The following axioms have to be satisfied:

(TR1) (a) Any triangle that is isomorphic to a distinguished triangle is also a distinguished triangle.

(b) For every morphism $\alpha : L \to M$ in K there is a distinguished triangle

$$L \xrightarrow{\alpha} M \to N \to \mathrm{T}(L).$$

(c) For every object M the triangle

$$M \xrightarrow{1_M} M \to 0 \to \mathrm{T}(M)$$

is distinguished.

(TR2) A triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

is distinguished iff the triangle

$$M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L) \xrightarrow{-\mathrm{T}(\alpha)} T(M)$$

is distinguished.

★ $\boxed{\text{to here 23 Dec 2015}}$ ★

(TR3) Suppose

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

and

$$L' \xrightarrow{\alpha'} M' \xrightarrow{\beta'} N' \xrightarrow{\gamma'} \mathrm{T}(L')$$

are distinguished triangles, and $\phi : L \to L'$ and $\psi : M \to M'$ are morphisms that satisfy $\psi \circ \alpha = \alpha' \circ \phi$. Then there is a morphism $\chi : N \to N'$ such that

$$\begin{tikzcd}
L \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[r, "\beta"] \arrow[d, "\psi"'] & N \arrow[r, "\gamma"] \arrow[d, "\chi"'] & \mathrm{T}(L) \arrow[d, "\mathrm{T}(\phi)"'] \\
L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L')
\end{tikzcd}$$.

is a morphism of triangles.

(TR4) Suppose

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N' \to \mathrm{T}(L),$$

$$M \xrightarrow{\gamma} N \xrightarrow{\delta} L' \to \mathrm{T}(M)$$

and

$$L \xrightarrow{\gamma \circ \alpha} N \xrightarrow{\epsilon} M' \to \mathrm{T}(L),$$

are distinguished triangles. Then there is a distinguished triangle

$$N' \xrightarrow{\phi} M' \xrightarrow{\psi} L' \to \mathrm{T}(N')$$

making the diagram

$$\begin{tikzcd}
L \arrow[r, "\alpha"] \arrow[d, "1"'] & M \arrow[r, "\beta"] \arrow[d, "\gamma"'] & N' \arrow[r] \arrow[d, dashed, "\phi"'] & \mathrm{T}(L) \arrow[d, "1"'] \\
L \arrow[r, "\gamma\circ\alpha"] \arrow[d, "\alpha"'] & N \arrow[r, "\epsilon"] \arrow[d, "1"'] & M' \arrow[r] \arrow[d, dashed, "\psi"'] & \mathrm{T}(L) \arrow[d, "\mathrm{T}(\alpha)"'] \\
M \arrow[r, "\gamma"] \arrow[d, "\beta"'] & N \arrow[r, "\delta"] \arrow[d, "\epsilon"'] & L' \arrow[r] \arrow[d, "1"'] & \mathrm{T}(M) \arrow[d, "\mathrm{T}(\beta)"'] \\
N' \arrow[r, dashed, "\phi"] & M' \arrow[r, dashed, "\psi"] & L' \arrow[r, dashed] & \mathrm{T}(N')
\end{tikzcd}$$

commutative.

Here are a few remarks on this definition. The numbering of the axioms we use is taken from [RD]; the numbering in [Sc, KS1, KS2] is different.

In the situation that we care about, namely $\mathsf{K} = \mathbf{K}(\mathsf{M}, A)$, the distinguished triangles will be those triangles that are isomorphic, in $\mathbf{K}(\mathsf{M}, A)$, to the standard triangles in $\mathbf{C}(\mathsf{M}, A)$ from Definition 4.8.4.

The object N in item (b) of axiom (TR1) is referred to as a *cone* on $\alpha : L \to M$. We should think of the cone as something combining "the cokernel" and "the kernel" of α.

Axiom (TR2) says that if we "turn" a distinguished triangle we remain with a distinguished triangle.

Axiom (TR3) says that a commutative square induces a morphism on the cones of the horizontal morphisms, that fits into a morphism of distinguished triangles. Note however that the new morphism χ is *not unique*; in other words, *cones are not functorial.* This fact has some deep consequences in many applications. However, in the situations that will interest us, namely when $\mathsf{K} = \mathbf{K}(\mathsf{M}, A)$, the cones come from the standard cones in $\mathbf{C}(\mathsf{M}, A)$. And the cones in $\mathbf{C}(\mathsf{M}, A)$ are functorial (Definition 4.8.5).

Axiom (TR4) is called the *octahedron axiom.* It is very cumbersome, and will not be important in the situations that interest us. It is only needed for abstract triangulated categories. See the book [Ne] for a discussion of this axiom. It is not known whether the octahedron axiom is a consequence of the other axioms; there was a recent paper by Maccioca (arxiv:1506.00887) claiming that, but it had a fatal error in it.

To save us effort, we make the following convention:

Convention 6.2.2. An additive category with translation (K, T), equipped with a set of distinguished triangles, that satisfies axioms (TR1)-(TR3), will be referred to as a *triangulated*† *category.*

Some texts use the name "pretriangulated" instead of triangulated†. But this name is in conflict with the totally distinct notion of "pretriangulated DG category" from [BK].

6.3. **Triangulated Functors.** Suppose K and L are T-additive categories. (We are keeping the translation automorphisms implicit now.) The notion of T-additive functor $F : \mathsf{K} \to \mathsf{L}$ was defined in Definition 6.1.3 In that definition we also introduced the notion of morphism $\eta : F \to G$ between T-additive functors.

Definition 6.3.1. Let K and L be triangulated categories.

(1) A *triangulated functor* from K to L is a T-additive functor
$$(F, \xi) : \mathsf{K} \to \mathsf{L}$$
that sends distinguished triangles to distinguished triangles. Namely for any distinguished triangle
$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$
in K, the triangle
$$F(L) \xrightarrow{F(\alpha)} F(M) \xrightarrow{F(\beta)} F(N) \xrightarrow{\xi \circ F(\gamma)} \mathrm{T}(F(L))$$
is a distinguished triangle in L.

(2) Suppose $(G, \nu) : \mathsf{K} \to \mathsf{L}$ is another triangulated functor. A *morphism of triangulated functors* $\eta : (F, \xi) \to (G, \nu)$ is a morphism of T-additive functors.

We usually keep the isomorphism ξ implicit, and refer to F as a triangulated functor.

For a category K there is a canonical contravariant functor $\mathrm{op} : \mathsf{K} \to \mathsf{K}^{\mathrm{op}}$, that is the identity on objects, and reverses the arrows. In fact op is an anti-isomorphism of categories.

Proposition 6.3.2. *Let* K *be a triangulated category, and let* $\mathrm{op} : \mathsf{K} \to \mathsf{K}^{\mathrm{op}}$ *be the canonical contravariant isomorphism from* K *to its opposite category. Define a translation* T^{op} *on* K^{op} *by the formula* $\mathrm{T}^{\mathrm{op}} := \mathrm{op} \circ \mathrm{T}^{-1} \circ \mathrm{op}^{-1}$. *The distinguished triangles in* K^{op} *are defined to be the triangles*
$$N \xrightarrow{\mathrm{op}(\beta)} M \xrightarrow{\mathrm{op}(\alpha)} L \xrightarrow{\mathrm{op}(-\mathrm{T}^{-1}(\gamma))} \mathrm{T}^{\mathrm{op}}(N),$$
where $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$ *is any distinguished triangle in* K. *Then* $(\mathsf{K}^{\mathrm{op}}, T^{\mathrm{op}})$ *is a triangulated category.*

Proof. This is an exercise.

★ Please check that I got the formulas right! ★

(Hint: use the proof of the next proposition.) ☐

Definition 6.3.3. Let K be a triangulated† category, and let M be an abelian category. A *cohomological functor* $F : \mathsf{K} \to \mathsf{M}$ is an additive functor, such that for every distinguished triangle
$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$
in K, the sequence
$$F(L) \xrightarrow{F(\alpha)} F(M) \xrightarrow{F(\beta)} F(N)$$
is exact in M.

Proposition 6.3.4. *Let $F : \mathsf{K} \to \mathsf{M}$ be a cohomological functor, and let*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$$

be a distinguished triangle in K. Then the sequence

$$\cdots \to F(\mathrm{T}^i(L)) \xrightarrow{F(\mathrm{T}^i(\alpha))} F(\mathrm{T}^i(M)) \xrightarrow{F(\mathrm{T}^i(\beta))} F(\mathrm{T}^i(N)) \xrightarrow{F(\mathrm{T}^i(\gamma))} F(\mathrm{T}^{i+1}(L)) \xrightarrow{F(\mathrm{T}^{i+1}(\alpha))} F(\mathrm{T}^{i+1}(M)) \to \cdots$$

in M is exact.

Proof. By axiom (TR2) we have distinguished triangles

$$\mathrm{T}^i(L) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\alpha)} \mathrm{T}^i(M) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\beta)} \mathrm{T}^i(N) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\gamma)} \mathrm{T}^{i+1}(L),$$

$$\mathrm{T}^i(M) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\beta)} \mathrm{T}^i(N) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\gamma)} \mathrm{T}^{i+1}(L) \xrightarrow{(-1)^{i+1} \cdot \mathrm{T}^{i+1}(\alpha)} \mathrm{T}^{i+1}(M)$$

and

$$\mathrm{T}^i(N) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\gamma)} \mathrm{T}^{i+1}(L) \xrightarrow{(-1)^{i+1} \cdot \mathrm{T}^{i+1}(\alpha)} \mathrm{T}^{i+1}(M) \xrightarrow{(-1)^{i+1} \cdot \mathrm{T}^{i+1}(\beta)} \mathrm{T}^{i+1}(N).$$

Now use the definition, noting that multiplying morphisms in an exact sequence by -1 preserves exactness. □

Proposition 6.3.5. *Let K be a triangulated category.*

1. *If $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} T(L)$ is a distinguished triangle in K, then $\beta \circ \alpha = 0$.*
2. *For any $P \in \mathsf{K}$ the functors*

$$\mathrm{Hom}_{\mathsf{K}}(-, P) : \mathsf{K}^{\mathrm{op}} \to \mathsf{Ab}$$

and

$$\mathrm{Hom}_{\mathsf{K}}(P, -) : \mathsf{K} \to \mathsf{Ab}$$

are cohomological functors.

Proof. (1) By axioms (TR1) and (TR3) we have a commutative diagram

$$\begin{tikzcd}
L \arrow[r, "1_L"] \arrow[d, "1_L"'] & L \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d] & \mathrm{T}(L) \arrow[d, "\mathrm{T}(1_L)"'] \\
L \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & N \arrow[r, "\gamma"] & \mathrm{T}(L) .
\end{tikzcd}$$

We see that $\beta \circ \alpha$ factors through 0.

(2) We will prove the covariant statement; the contravariant statement is an immediate consequence, since

$$\mathrm{Hom}_{\mathsf{K}}(M, P) = \mathrm{Hom}_{\mathsf{K}^{\mathrm{op}}}(P, M),$$

and K^{op} is triangulated (with the correct triangulated structure to make this true).

Consider a distinguished triangle $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$. We have to prove that the sequence

$$\mathrm{Hom}_{\mathsf{K}}(P, L) \xrightarrow{\mathrm{Hom}(1_P, \alpha)} \mathrm{Hom}_{\mathsf{K}}(P, M) \xrightarrow{\mathrm{Hom}(1_P, \beta)} \mathrm{Hom}_{\mathsf{K}}(P, N)$$

is exact. In view of part (1), all we need to show is that for any $\psi : P \to M$ s.t. $\beta \circ \psi = 0$, there is some $\phi : P \to L$ s.t. $\psi = \alpha \circ \phi$. In a picture, we must show that the diagram below (solid arrows)

$$\begin{tikzcd}
P \arrow[r, "1"] \arrow[d, dashed, "\phi"'] & P \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \mathrm{T}(P) \arrow[d, dashed, "\phi"'] \\
L \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & N \arrow[r, "\gamma"] & \mathrm{T}(L) .
\end{tikzcd}$$

can be completed (dashed arrow). This is true by (TR) (=turning) and and (TR3) (=extending). ☐

Proposition 6.3.6. *Let* K *be a triangulated category, and let*

$$\begin{tikzcd}
L \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[r, "\beta"] \arrow[d, "\psi"'] & N \arrow[r, "\gamma"] \arrow[d, "\chi"'] & \mathrm{T}(L) \arrow[d, "\mathrm{T}(\phi)"'] \\
L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L') .
\end{tikzcd}$$

be a morphism of distinguished triangles. If ϕ *and* ψ *are isomorphisms, then* χ *is also an isomorphism.*

Proof. Take an arbitrary $P \in \mathsf{K}$, and let $F := \mathrm{Hom}_{\mathsf{K}}(P, -)$. We get a commutative diagram

$$\begin{tikzcd}
F(L) \arrow[r, "F(\alpha)"] \arrow[d, "F(\phi)"'] & F(M) \arrow[r, "F(\beta)"] \arrow[d, "F(\psi)"'] & F(N) \arrow[r, "F(\gamma)"] \arrow[d, "F(\chi)"'] & F(\mathrm{T}(L)) \arrow[r, "F(\mathrm{T}(\alpha))"] \arrow[d, "F(\mathrm{T}(\phi))"'] & F(\mathrm{T}(M)) \arrow[d, "F(\mathrm{T}(\psi))"'] \\
F(L') \arrow[r, "F(\alpha')"] & F(M') \arrow[r, "F(\beta')"] & F(N') \arrow[r, "F(\gamma')"] & F(\mathrm{T}(L')) \arrow[r, "F(\mathrm{T}(\alpha'))"] & F(\mathrm{T}(M'))
\end{tikzcd}$$

in Ab. By Proposition 6.3.5(2) the rows in the diagram are exact sequences. Since the other vertical arrows are isomorphisms, it follows that

$$F(\chi) : \mathrm{Hom}_{\mathsf{K}}(P, N) \to \mathrm{Hom}_{\mathsf{K}}(P, N')$$

is an isomorphism of abelian groups. By forgetting structure, we see that $F(\chi)$ is an isomorphism of sets.

We now use the Yoneda Lemma (Proposition 1.7.1). Let us write $Y_N := \mathrm{Hom}_{\mathsf{K}}(-, N)$ and $Y_{N'} := \mathrm{Hom}_{\mathsf{K}}(-, N')$, viewed as functors $\mathsf{K}^{\mathrm{op}} \to \mathsf{Set}$. For any object P we have isomorphisms of sets $Y_N(P) \cong F(N)$ and $Y_{N'}(P) \cong F(N')$. The calculation above shows that the morphism of functors $Y(\chi) : Y_N \to Y_{N'}$ is an isomorphism. Therefore the morphism $\chi : N \to N'$ in K is an isomorphism. ☐

6.4. **The Homotopy Category is Triangulated.** In this subsection we consider an additive category M and a DG ring A. These ingredients give rise to the DG category $\mathbf{C}(\mathsf{M}, A)$ of DG A-module in M, as in Subsection 4.6. The only change is that now our base ring is $\mathbb{K} = \mathbb{Z}$; but all can be stated in terms of any other commutative base ring.

The homotopy category $\mathbf{K}(\mathsf{M}, A)$ was introduced in Definition 4.6.5.

Proposition 6.4.1. *The categories* $\mathbf{C}(\mathsf{M}, A)$ *and* $\mathbf{K}(\mathsf{M}, A)$ *are additive.*

Proof. Let $M_1, \ldots, M_r$ be a finite collection in $\mathsf{C}(\mathsf{M}, A)$. Write $M_i = \{M_i^j\}_{j \in \mathbb{Z}}$. The direct sum of $M_1, \ldots, M_r$ exists in $\mathsf{C}(\mathsf{M})$: it is the complex $M = \{M^j\}_{j \in \mathbb{Z}}$ with $M^j := \bigoplus_{i=1}^r M_i^j$. The objects $M^j \in \mathsf{M}$ exist because this is assumed to be an additive category; and the differential $\mathrm{d}_M : M \to M$ exists by the universal property of direct sums. The DG A-module structure on M is defined similarly: for $a \in A^k$, there is an induced degree k morphism $f(a) : M \to M$ in $\mathsf{C}(\mathsf{M})$.

Now consider the finite collection $M_1, \ldots, M_r$ as living in $\mathsf{K}(\mathsf{M}, A)$. Because the functor $\mathsf{C}(\mathsf{M}, A) \to \mathsf{K}(\mathsf{M}, A)$ is additive, Proposition 2.4.2 says that M is also a direct sum of this collection in $\mathsf{K}(\mathsf{M}, A)$. □

According to Proposition 4.7.6(5), the translation T of $\mathsf{C}(\mathsf{M}, A)$ passes to $\mathsf{K}(\mathsf{M}, A)$, making it into an additive category with translation (or a T-additive category).

It will be convenient to have notation for the canonical full functor $\mathsf{C}(\mathsf{M}, A) \to \mathsf{K}(\mathsf{M}, A)$. Given a morphism $\phi : M \to N$ in $\mathsf{C}(\mathsf{M}, A)$, the corresponding morphism is $\bar{\phi} : M \to N$ in $\mathsf{K}(\mathsf{M}, A)$. Note that $\bar{\phi}$ is nothing but the homotopy class of the 0-cocycle ϕ. All morphisms in $\mathsf{K}(\mathsf{M}, A)$ are of this form.

Definition 6.4.2. A triangle

$$L \xrightarrow{\bar{\alpha}} M \xrightarrow{\bar{\beta}} N \xrightarrow{\bar{\gamma}} \mathrm{T}(L)$$

in $\mathsf{K}(\mathsf{M}, A)$ is called a *distinguished triangle* if there is a standard triangle

$$L' \xrightarrow{\alpha'} M' \xrightarrow{\beta'} N' \xrightarrow{\gamma'} \mathrm{T}(L')$$

in $\mathsf{C}(\mathsf{M}, A)$, as in Definition 4.8.4, and an isomorphism of triangles

$$\begin{tikzcd}
L' \arrow[r, "\bar{\alpha}'"] \arrow[d, "\bar{\phi}"'] & M' \arrow[r, "\bar{\beta}'"] \arrow[d, "\bar{\psi}"'] & N' \arrow[r, "\bar{\gamma}'"] \arrow[d, "\bar{\chi}"'] & \mathrm{T}(L') \arrow[d, "\mathrm{T}(\bar{\phi})"'] \\
L \arrow[r, "\bar{\alpha}"] & M \arrow[r, "\bar{\beta}"] & N \arrow[r, "\bar{\gamma}"] & \mathrm{T}(L) .
\end{tikzcd}$$

in $\mathsf{K}(\mathsf{M}, A)$.

Theorem 6.4.3. *The* T*-additive category* $\mathsf{K}(\mathsf{M}, A)$*, with the set of distinguished triangles defined above, is triangulated.*

The proof is after two lemmas.

★ The proof below is worked out using a hint in [RD], and a lemma in [KS1] – I hope it is correct! ★

Lemma 6.4.4. *Let* $M \in \mathsf{C}(\mathsf{M}, A)$*, and consider the cone* $N := \mathrm{Cone}(1_M)$*. Then the DG module* N *is null-homotopic, i.e.* $0 \to N$ *is an isomorphism in* $\mathsf{K}(\mathsf{M}, A)$*.*

Proof. We shall exhibit a homotopy θ from 0_N to 1_N. Define

$$\theta^i : N^i = M^{i+1} \oplus M^i \to N^{i-1} = M^i \oplus M^{i-1}$$

to be the matrix

$$\theta^i := \begin{bmatrix} 0 & 1_{M^i} \\ 0 & 0 \end{bmatrix}.$$

We have

$$\mathrm{d}_N^{i-1} \circ \theta^i + \theta^{i+1} \circ \mathrm{d}_N^i = \begin{bmatrix} 1_{M^{i+1}} & 0 \\ 0 & 1_{M^i} \end{bmatrix} = 1_{N^i}.$$

□

★ $\boxed{\text{fixed to here}}$ ★

Lemma 6.4.5 ([KS1, Lemma 1.4.2])**.** *Consider a morphism $\alpha : L \to M$ in $\mathbf{C}(\mathsf{M}, A)$, the standard triangle*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

associated to α, and the standard triangle

$$M \xrightarrow{\beta} N \xrightarrow{\phi} P \xrightarrow{\psi} M[1]$$

associated to β, all in $\mathbf{C}(\mathsf{M}, A)$. So $N = \operatorname{Cone}(\alpha)$ and and $P = \operatorname{Cone}(\beta)$. There is a morphism $\rho : L[1] \to P$ in $\mathbf{C}(\mathsf{M})$ s.t. $\bar{\rho}$ is an isomorphism in $\mathbf{K}(\mathsf{M})$, and the diagram

$$\begin{tikzcd}
M \arrow[r, "\bar{\beta}"] \arrow[d, "\bar{1}_M"'] & N \arrow[r, "\bar{\gamma}"] \arrow[d, "\bar{1}_N"'] & L[1] \arrow[r, "-\bar{\alpha}[1]"] \arrow[d, "\bar{\rho}"'] & M[1] \arrow[d, "\bar{1}_{M[1]}"'] \\
M \arrow[r, "\bar{\beta}"] & N \arrow[r, "\bar{\phi}"] & P \arrow[r, "\bar{\psi}"] & M[1]
\end{tikzcd}$$

commutes in $\mathbf{K}(\mathsf{M})$.

Proof. Note that $P^i = M^{i+1} \oplus L^{i+1} \oplus M^i$ and $L[1]^i = L^{i+1}$. Define morphisms $\rho^i : L^{i+1} \to P^i$ and $\chi^i : P^i \to L^{i+1}$ in M by the matrix representations

$$\rho^i := \begin{bmatrix} -\alpha^{i+1} \\ 1_{L^{i+1}} \\ 0 \end{bmatrix} , \ \chi^i := \begin{bmatrix} 0 & 1_{L^{i+1}} & 0 \end{bmatrix} .$$

We get morphisms of graded objects $\rho : L[1] \to P$ and $\chi : P \to L[1]$. Direct calculations (please verify!) show that:

- ρ and χ are morphisms in $\mathbf{C}(\mathsf{M})$.
- $\chi \circ \rho = 1_{L[1]}$.
- $\chi \circ \phi = \gamma$.
- $\psi \circ \rho = -\alpha[1]$.

It remains to prove that $\rho \circ \chi$ is homotopic to 1_P. Define a morphism $\theta^i : P^i \to P^{i-1}$ by the matrix

$$\theta^i := \begin{bmatrix} 0 & 0 & 1_{M^i} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

We get a morphism $\theta : P \to P[-1]$ of graded objects, and

$$1_P - \rho \circ \chi = \theta \circ \mathrm{d}_P + \mathrm{d}_P \circ \theta.$$

□

Proof of the Theorem. (TR1): The only nontrivial thing to show is that

$$M \xrightarrow{1_M} M \to 0 \to M[1]$$

is a distinguished triangle. But this follows from Lemma 6.4.4.

(TR2): This is an immediate consequence of Lemma 6.4.5, since the bottom triangle there (the one with P) is standard.

(TR3): Consider a commutative diagram (solid arrows) in $\mathsf{K}(\mathsf{M})$:

$$\begin{tikzcd} L \arrow[r, "\bar{\alpha}"] \arrow[d, "\bar{\phi}"'] & M \arrow[r, "\bar{\beta}"] \arrow[d, "\bar{\psi}"'] & N \arrow[r, "\bar{\gamma}"] \arrow[d, dashed, "\bar{\chi}"'] & T(L) \arrow[d, "T(\bar{\phi})"'] \\ L' \arrow[r, "\bar{\alpha}'"] & M' \arrow[r, "\bar{\beta}'"] & N' \arrow[r, "\bar{\gamma}'"] & T(L') \end{tikzcd}$$

where the horizontal triangles are distinguished. By definition this diagram is isomorphic to a diagram in $\mathsf{K}(\mathsf{M})$, that comes from a diagram

$$\begin{tikzcd} L \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[r, "\beta"] \arrow[d, "\psi"'] & N \arrow[r, "\gamma"] \arrow[d, dashed, "\chi"'] & T(L) \arrow[d, "T(\phi)"'] \\ L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & T(L') \end{tikzcd}$$

(solid arrows) in $\mathsf{C}(\mathsf{M})$, in which $N = \operatorname{cone}(\alpha)$, $N' = \operatorname{cone}(\alpha')$, and the horizontal triangles are the standard ones. However this diagram in $\mathsf{C}(\mathsf{M})$ in only commutative up to homotopy. This means that there is a degree -1 homomorphism $\theta : L \to M'$ s.t.

$$\alpha' \circ \phi = \psi \circ \alpha + \mathrm{d}(\theta).$$

For every i define the morphism

$$\chi^i : N^i = \begin{bmatrix} L^{i+1} \\ M^i \end{bmatrix} \to N'^i = \begin{bmatrix} L'^{i+1} \\ M'^i \end{bmatrix}$$

to be left multiplication with the matrix $\begin{bmatrix} \phi^{i+1} & 0 \\ -\theta^{i+1} & \psi^i \end{bmatrix}$. A matrix calculation shows that $\chi : N \to N'$ is a morphism in $\mathsf{C}(\mathsf{M})$. It is easy to see that $\chi \circ \beta = \beta' \circ \psi$ and $\phi[1] \circ \gamma = \gamma' \circ \chi$ in $\mathsf{C}(\mathsf{M})$. Hence passing to $\mathsf{K}(\mathsf{M})$ we have a morphism of triangles.

(TR4): I will not prove this axiom, since it looks as if we won't need it. □

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