BEN GURION UNIVERSITY OF THE NEGEV

אוניברסיטת בן גוריון בנגב

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המחלקה למתמטיקה
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A COURSE ON DERIVED CATEGORIES

BGU, 2015-16

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0. Introduction

These are notes for an advanced course given at Ben Gurion University in the academic year 2015-16. In this course I am following various sources, mostly the earlier course notes [Ye6] and the mini-course [Ye7]. However, this course contains a lot of new material. The main novelty is the use of DG categories and DG functors to describe the common structure of the various setups in which derived categories arise. More on this in the Synopsis (subsection 0.2 of the Introduction).

0.1. **A motivating discussion: duality.** By way of introduction to the subject, let us consider *duality*. Take a field K. Given a K-module M (i.e. a vector space), let

$$D(M) := \operatorname{Hom}_K(M, K),$$

be the dual module. There is a canonical homomorphism

$$\eta_M : M \to D(D(M)),$$

$\eta_M(m)(\phi) := \phi(m)$ for $m \in M$ and $\phi \in D(M)$. If M is finitely generated then η_M is an isomorphism (actually this is "if and only if").

To formalize this situation, let $\mathsf{Mod}\, K$ denote the category of K-modules. Then

$$D : \mathsf{Mod}\, K \to \mathsf{Mod}\, K$$

is a contravariant functor, and

$$\eta : \mathrm{id} \to D \circ D$$

is a natural transformation. Here id is the identity functor of $\mathsf{Mod}\, K$.

Now let us replace K by any nonzero commutative ring A. Again we can define a contravariant functor

$$D : \mathsf{Mod}\, A \to \mathsf{Mod}\, A, \quad D(M) := \operatorname{Hom}_A(M, A),$$

and a natural transformation $\eta : \mathrm{id} \to D \circ D$. It is easy to see that $\eta_M : M \to D(D(M))$ is an isomorphism if M is a finitely generated free module. Of course we can't expect reflexivity (i.e. η_M being an isomorphism) if M is not finitely generated; but what about a finitely generated module that is not free?

In order to understand this better, let us concentrate on the ring $A = \mathbb{Z}$. Since $\mathbb{Z}$-modules are just abelian groups, the category $\mathsf{Mod}\, \mathbb{Z}$ is often denoted by Ab. Let Ab_{f} be the full subcategory of finitely generated abelian groups. Any finitely generated abelian group is of the form $M \cong T \oplus F$, with F free and T finite (the letter "T" stands for "torsion".). It is important to note that this is *not a canonical isomorphism*. There is a canonical short exact sequence

$$0 \to T \xrightarrow{\phi} M \xrightarrow{\psi} F \to 0, \tag{0.1.1}$$

but the decomposition $M \cong T \oplus F$ comes from *choosing a splitting* $\sigma : F \to M$ of this sequence.

Exercise 0.1.2. Prove that the exact sequence (0.1.5) is functorial (i.e. natural); namely there are functors $T, F : \mathsf{Ab}_{\mathrm{f}} \to \mathsf{Ab}_{\mathrm{f}}$, and natural transformations $\phi : T \to \mathrm{id}$ and $\psi : \mathrm{id} \to F$, such that for any $M \in \mathsf{Ab}_{\mathrm{f}}$, the group $T(M)$ is finite; the group $F(M)$ is free; and the sequence of homomorphisms

$$0 \to T(M) \xrightarrow{\phi_M} M \xrightarrow{\psi_M} F(M) \to 0 \tag{0.1.3}$$

is exact.

Next, prove that there does not exist a *functorial decomposition* of a finitely generated abelian group into a free part and a finite part. Namely, there is no natural transformation $\sigma : F \to \mathrm{id}$, such that for every M, the homomorphism $\sigma_M : F(M) \to M$ splits the sequence (0.1.3). (Hint: find a counterexample.)

We know that for the free abelian group F there is reflexivity, i.e. $\eta_F : F \to D(D(F))$ is an isomorphism. But for the finite abelian group T we have

$$D(T) = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Z}) = 0.$$

Thus, whenever $T \neq 0$, reflexivity fails: $\eta_M : M \to D(D(M))$ is not an isomorphism.

On the other hand, for an abelian group M we can define another sort of dual:

$$D'(M) := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}).$$

There is a natural transformation $\eta' : \mathrm{id} \to D' \circ D'$. For a finite abelian group T the homomorphism $\eta'_T : T \to D'(D'(T))$ is an isomorphism; this can be seen by decomposing T into cyclic groups, and for a finite cyclic group it is clear. So D' is a duality for finite abelian groups. (We may view the abelian group $\mathbb{Q}/\mathbb{Z}$ as the group of roots of 1 in $\mathbb{C}$, via the exponential function; and then D' becomes *Pontryagin Duality.*)

But for a finitely generated free abelian group F we get $D'(D'(F)) = \widehat{F}$, the profinite completion of F. So once more this is not a good duality for all finitely generated abelian groups.

We could try to be more clever and "patch" the two dualities D and D', into something that we will call $D \oplus D'$. This looks pleasing at first – but then we recall that the decomposition $M \cong T \oplus F$ of a finitely generated group is not functorial, so that $D \oplus D'$ can't be a functor.

Here is where the *derived category* enters. For any commutative ring A there is the derived category $\mathbf{D}(\mathsf{Mod}\, A)$. Here is a very quick explanation of it.

Recall that a *complex* of A-modules is a diagram

$$M^{\cdot} = \big(\cdots \to M^{-1} \xrightarrow{\mathrm{d}_M^{-1}} M^0 \xrightarrow{\mathrm{d}_M^{0}} M^1 \to \cdots \big) \tag{0.1.4}$$

in $\mathsf{Mod}\, A$. Here M^i are A-modules, the differentials (or coboundary operators) $\mathrm{d}_M^i : M^i \to M^{i+1}$ are A-linear homomorphisms, and $\mathrm{d}_M^{i+1} \circ \mathrm{d}_M^i = 0$.

Given a second complex

$$N^{\cdot} = \big(\cdots \to N^{-1} \xrightarrow{\mathrm{d}_N^{-1}} N^0 \xrightarrow{\mathrm{d}_N^{0}} N^1 \to \cdots \big),$$

a *homomorphism of complexes* $\phi^{\cdot} : M^{\cdot} \to N^{\cdot}$ is a collection of homomorphisms $\phi^i : M^i \to N^i$ satisfying

$$\phi^{i+1} \circ \mathrm{d}_M^i = \mathrm{d}_N^i \circ \phi^i.$$

The resulting category is denoted by $\mathbf{C}(\mathsf{Mod}\, A)$.

The i-th *cohomology* of $M^{\cdot}$ is

$$\mathrm{H}^i(M^{\cdot}) := \frac{\mathrm{Ker}(\mathrm{d}_M^i)}{\mathrm{Im}(\mathrm{d}_M^{i-1})} \in \mathsf{Mod}\, A.$$

A homomorphism $\phi^{\cdot} : M^{\cdot} \to N^{\cdot}$ induces homomorphisms

$$\mathrm{H}^i(\phi^{\cdot}) : \mathrm{H}^i(M^{\cdot}) \to \mathrm{H}^i(N^{\cdot}).$$

We call $\phi^{\cdot}$ a *quasi-isomorphism* if all the homomorphisms $\mathrm{H}^i(\phi^{\cdot})$ are isomorphisms.

The derived category $\mathbf{D}(\mathsf{Mod}\, A)$ has the same objects are $\mathbf{C}(\mathsf{Mod}\, A)$, namely the complexes. There is a functor

$$\mathrm{Q} : \mathbf{C}(\mathsf{Mod}\, A) \to \mathbf{D}(\mathsf{Mod}\, A)$$

that is the identity of objects. If $\psi^{\cdot}$ is quasi-isomorphism in $\mathbf{C}(\mathsf{Mod}\, A)$, the morphism $\mathrm{Q}(\psi^{\cdot})$ is invertible in $\mathbf{D}(\mathsf{Mod}\, A)$, i.e. it is an isomorphisms. The morphisms in $\mathbf{D}(\mathsf{Mod}\, A)$ are all of the form

$$\mathrm{Q}(\phi^{\cdot}) \circ \mathrm{Q}(\psi^{\cdot})^{-1},$$

where $\phi^{\cdot}$ is any morphism in $\mathbf{C}(\mathsf{Mod}\, A)$, and $\psi^{\cdot}$ is any quasi-morphism in $\mathbf{C}(\mathsf{Mod}\, A)$ (and of course these are composable morphisms, re. source and target).

A single A-module M can be viewed as a complex concentrated in degree 0:

$$M^{\cdot} = \big(\cdots \to 0 \xrightarrow{0} M \xrightarrow{0} 0 \to \cdots \big). \tag{0.1.5}$$

In other words, $M^0 = M$ and the rest are 0. This turns out to be fully faithful embedding

$$\mathsf{Mod}\, A \to \mathbf{D}(\mathsf{Mod}\, A). \tag{0.1.6}$$

Moreover, any complex $M^{\cdot}$ whose cohomology is concentrated in degree 0, (i.e. $\mathrm{H}^i(M^{\cdot}) = 0$ for all $i \neq 0$) is isomorphic in $\mathbf{D}(\mathsf{Mod}\, A)$ to the module $\mathrm{H}^0(M)$. In this way we have *enlarged* the category of A-modules.

Here is a very important kind of quasi-isomorphism. Suppose

$$\cdots \to P^{-2} \xrightarrow{\mathrm{d}_P^{-2}} P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \xrightarrow{\epsilon^0} M \to 0 \tag{0.1.7}$$

is a free resolution of a module M. Let $M^{\cdot}$ be the complex from (0.1.5), and let $P^{\cdot}$ be the complex

$$P^{\cdot} = \big(\cdots \to P^{-2} \xrightarrow{\mathrm{d}_P^{-2}} P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \to 0 \to \cdots \big).$$

Then, letting $\epsilon^i := 0$ for $i \neq 0$, we get a quasi-isomorphism

$$\epsilon^{\cdot} : P^{\cdot} \to M^{\cdot}$$

in $\mathbf{C}(\mathsf{Mod}\, A)$, and thus an isomorphism

$$\mathrm{Q}(\epsilon^{\cdot}) : P^{\cdot} \to M^{\cdot}$$

in $\mathbf{D}(\mathsf{Mod}\, A)$.

Let us now return to $A = \mathbb{Z}$. The functor $D = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ from $\mathsf{Mod}\, \mathbb{Z}$ to itself has a *right derived functor*

$$\mathrm{R}D = \mathrm{RHom}_{\mathbb{Z}}(-, \mathbb{Z}),$$

which is a contravariant *triangulated functor*

$$\mathrm{R}D : \mathbf{D}(\mathsf{Mod}\, \mathbb{Z}) \to \mathbf{D}(\mathsf{Mod}\, \mathbb{Z}).$$

And there is a natural transformation of triangulated functors

$$\eta : \mathrm{id} \to \mathrm{R}D \circ \mathrm{R}D.$$

Here is the way to calculate the functor $\mathrm{R}D$, at least for a finitely generated abelian group M. Let us choose a free resolution of M like in (0.1.7). To be easy on ourselves, we take it to be like this:

$$P^{\cdot} = \big(\cdots \to 0 \to P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \to 0 \to \cdots \big) = \big(\cdots \to 0 \to \mathbb{Z}^{r_1} \xrightarrow{\mathrm{d}} \mathbb{Z}^{r_0} \to 0 \cdots \big),$$

where $r_0, r_1 \in \mathbb{N}$ and d is a matrix of integers. Because $\mathrm{Q}(\epsilon^{\cdot}) : P^{\cdot} \to M^{\cdot}$ is an isomorphism in $\mathsf{D}(\mathsf{Mod}\,\mathbb{Z})$, it suffices to calculate $\mathrm{R}D(P^{\cdot})$.

Now by construction,

$$\mathrm{R}D(P^{\cdot}) = D(P^{\cdot}) = \mathrm{Hom}_{\mathbb{Z}}(P^{\cdot}, \mathbb{Z}),$$

where the complex $\mathrm{Hom}_{\mathbb{Z}}(P^{\cdot}, \mathbb{Z})$ is

$$\mathrm{Hom}_{\mathbb{Z}}(P^{\cdot}, \mathbb{Z}) = \bigl(\cdots \to 0 \to \mathbb{Z}^{r_0} \xrightarrow{\mathrm{d}^*} \mathbb{Z}^{r_1} \to 0 \cdots\bigr),$$

concentrated in degrees 0 and 1, with the transpose matrix d^* are the differential.

Because $\mathrm{R}D(P^{\cdot}) = D(P^{\cdot})$ is itself a bounded complex of finite free modules, its derived dual is

$$\mathrm{R}D(\mathrm{R}D(P^{\cdot})) = D(D(P^{\cdot})) = \mathrm{Hom}_{\mathbb{Z}}\bigl(\mathrm{Hom}_{\mathbb{Z}}(P^{\cdot}, \mathbb{Z}), \mathbb{Z}\bigr).$$

The canonical morphism (in $\mathsf{C}(\mathsf{Mod}\,\mathbb{Z})$)

$$\eta_P : P^{\cdot} \to D(D(P^{\cdot}))$$

is an isomorphism; and therefore

$$\eta_M : M^{\cdot} \to \mathrm{R}D(\mathrm{R}D(M^{\cdot}))$$

is an isomorphism in $\mathsf{D}(\mathsf{Mod}\,\mathbb{Z})$.

We see that $\mathrm{R}D$ *is a duality that holds for all finitely generated* $\mathbb{Z}$*-modules* !

Here is the connection between the derived duality $\mathrm{R}D$ and the "classical" dualities D and D'. Take a finitely generated abelian group M, with short exact sequence (0.1.1). There are functorial isomorphisms

$$\mathrm{H}^0(\mathrm{R}D(M)) \cong \mathrm{Ext}^0_{\mathbb{Z}}(M, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \cong D(M)$$

and

$$\mathrm{H}^1(\mathrm{R}D(M)) \cong \mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z}) \cong D'(M).$$

The cohomologies $\mathrm{H}^i(\mathrm{R}D(M)) = 0$ for $i \neq 0, 1$.

Note that $D(M) \cong D(F)$ and $D'(M) \cong D'(T)$. We see that if M is neither free nor finite, then both $\mathrm{H}^0(\mathrm{R}D(M))$ and $\mathrm{H}^1(\mathrm{R}D(M))$ are both nonzero; so that the complex $D(M)$ is not isomorphic to an object of $\mathsf{Mod}\,\mathbb{Z}$, under the embedding (0.1.6).

This sort of duality holds for *many noetherian commutative rings* A. But the formula for the duality functor

$$\mathrm{R}D : \mathsf{D}(\mathsf{Mod}\,A) \to \mathsf{D}(\mathsf{Mod}\,A)$$

is somewhat different – it is

$$\mathrm{R}D(M) := \mathrm{RHom}_A(M, R),$$

where $R \in \mathsf{D}(\mathsf{Mod}\,A)$ is a *dualizing complex.* Such a dualizing complex is unique (up to shift and tensoring with an invertible module).

Interestingly, the structure of the dualizing complex R depends on the geometry of the ring A (i.e. of the scheme $\mathrm{Spec}\,A$). If A is a regular ring (like $\mathbb{Z}$) then $R = A$ is dualizing. If A is Cohen-Macaulay (and $\mathrm{Spec}\,A$ is connected) then R is a single A-module. But if A is a more complicated ring, then R must live in several degrees.

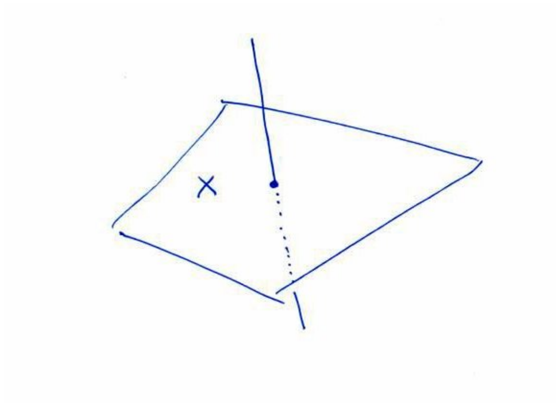

FIGURE 1. An algebraic variety that is connected but not equidimensional, and hence not Cohen-Macaulay.

Example 0.1.8. Consider the affine algebraic variety $X \subseteq \mathbf{A}^3_{\mathbb{R}}$ which is the union of a plane and a line, with coordinate ring

$$A = \mathbb{R}[t_1, t_2, t_3]/(t_3 \cdot t_1,\ t_3 \cdot t_2).$$

See figure 1. The dualizing complex R must live in two adjacent degrees; namely there is some i s.t. $\mathrm{H}^i(R)$ and $\mathrm{H}^{i+1}(R)$ are nonzero.

One can also talk about dualizing complexes over *noncommutative rings*. (This is a favorite topic of mine!)

0.2. **Synopsis.** Here is a section-by-section description of the material in the course.

Sections 1-2. These sections are pretty much a review of the standard material on categories and functors (especially abelian categories and additive functors) that is needed for the course. If the reader is familiar with this material, he can skip these sections. We do recommend looking at our notational convention, that are spelled out in Convention 1.2.1.

Section 3. A good understanding of *DG algebra* ("DG" is short for "differential graded") is essential in our approach to derived categories. We aim to study both the derived category $\mathbf{D}(\mathsf{M})$ of an abelian category M, and the derived category $\mathbf{D}(A)$ of DG modules over a DG ring A. In order to accomplish this, we introduce a new concept, that combines both these setups: the category $\mathbf{C}(A, \mathsf{M})$ of *DG A-modules in* M. See Subsection 3.6.

Actually, our methods can be expanded, without much effort, to handle the DG category $\mathbf{C}(\mathsf{A}, \mathsf{M})$ of DG A-modules in M, where A is a DG category (rather than a DG ring as above). This includes as a special case ($\mathsf{M} = \mathsf{Ab}$) the category $\mathbf{C}(\mathsf{A})$ of DG A-modules, in the sense of Keller; see Remark 3.6.7. We have decided to stick to the less general $\mathbf{C}(A, \mathsf{M})$, because the treatment is more streamlined.

There do not exist (to our knowledge) detailed textbook references for DG algebra (by which we mean DG rings, DG modules, DG categories, DG functors and related constructions). Therefore we have included a lot of basic material in this section. Moreover, we present a new treatment of translations and cones, using the "little t operator", following our paper [Ye9]. Among other things, we prove (in Theorem 3.7.7) that the translation functor T of $\mathbf{C}(A, \mathsf{M})$ is a DG functor, and $\mathrm{t} : \mathrm{id} \to \mathrm{T}$ is a degree -1 morphisms of DG functors from $\mathbf{C}(A, \mathsf{M})$ to itself.

Section 4. This section consists mostly of new material. We consider a DG functor

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N}), \tag{0.2.1}$$

where A and B are DG rings, and M and N are abelian categories. In Theorem 4.2.3 we show that there is a canonical isomorphism of DG functors

$$\zeta_F : F \circ \mathrm{T} \xrightarrow{\simeq} \mathrm{T} \circ F$$

called the *translation isomorphism.* Then, in Theorem 4.3.6, we prove that F sends *standard triangles* in the strict category $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ to standard triangles in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$.

We end this section with several examples of DG functors. These examples are prototypes – they can be easily extended to other setups.

Section 5. We start with the theory of *pretriangulated categories* and *triangulated functors*, following mainly [RD]. Because the *octahedron axiom* plays no role in our approach, we exclude it from the discussion, and this is the reason we do not talk about triangulated categories. In Subsection 5.4 we prove that the homotopy category $\mathbf{K}(A, \mathsf{M})$ is pretriangulated.

We conclude this section with Theorem 5.4.11. It says that given a DG functor F as in (0.2.1), with translation isomorphism ζ_F, the T-additive functor

$$(F, \zeta_F) : \mathbf{K}(A, \mathsf{M}) \to \mathbf{K}(B, \mathsf{N})$$

is triangulated. This could be a new result (unifying well-known yet disparate examples).

0.3. **Acknowledgments.** I want to thank the participants of the course in Spring 2012 for correcting many of my mistakes, both in real time during the lectures, and afterwards when writing the notes [Ye6]. Thanks also to J. Lipman, P. Schapira, A. Neeman and C. Weibel for helpful discussions on the material in [Ye6]. Thanks also to Vincent Beck, who corrected an error in Remark 1.2.2 of [Ye6] (that is Exercise 1.3.2 here).

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1. Basics Facts on Categories

1.1. **Set Theory.** In this course we will not try to be precise about issues of set theory. The blanket assumption is that we are given a *Grothendieck universe* U. This is a "large" infinite set. A *small set*, or a U-small set, is a set S that is an element of U. We want all the products $\prod_{i \in I} S_i$ and disjoint unions $\coprod_{i \in I} S_i$, with I and S_i small sets, to be small sets too. A U-*category* is a category C whose set of objects $\mathrm{Ob}(\mathsf{C})$ is a subset of U, and for every $C, D \in \mathrm{Ob}(\mathsf{C})$ the set of morphisms $\mathrm{Hom}_{\mathsf{C}}(C, D)$ is small. See [SGA 4] or [KS2, Section 1.1]. Another approach, involving "sets" vs "classes", can be found in [Ne].

We denote by Set the category of all small sets. So $\mathrm{Ob}(\mathsf{Set}) = \mathsf{U}$, and Set is a U-category. A group (or a ring, etc.) is called small if its underlying set is small. We denote by Grp, Ab, Ring and $\mathsf{Ring}_{\mathsf{c}}$ the categories of small groups, small abelian groups, small rings and small commutative rings respectively. For a small ring A we denote by $\mathsf{Mod}\, A$ the category of all small left A-modules.

By default we work with U-categories, and from now on U will remain implicit. The one exception is when we deal with localization of categories, where we shall briefly encounter a set theoretical issue; but for most interesting cases this issue has an easy solution.

1.2. **Notation.** Let C be a category. We often write $C \in \mathsf{C}$ as an abbreviation for $C \in \mathrm{Ob}(\mathsf{C})$. For an object C, its identity automorphism is denoted by id_C. The identity functor of C is denoted by id_{C}.

The opposite category of C is C^{op}. It has the same objects as C, but the morphism sets are

$$\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(C_0, C_1) := \mathrm{Hom}_{\mathsf{C}}(C_1, C_0),$$

and composition is reversed. The identity functor of C can be viewed as a contravariant functor $\mathrm{id}_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}^{\mathrm{op}}$. A contravariant functor $F : \mathsf{C} \to \mathsf{D}$ is the same as a covariant functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$. Since we prefer dealing only with covariant functors, we make the following convention:

Convention 1.2.1. By default all functors will be covariant, unless explicitly mentioned otherwise.

We will try to keep the following font and letter conventions:

- $f : C \to D$ is a morphism between objects in a category.
- $F : \mathsf{C} \to \mathsf{D}$ is a functor between categories.
- $\eta : F \to G$ is morphism of functors (i.e. a natural transformation) between functors $F, G : \mathsf{C} \to \mathsf{D}$.
- $f, \phi, \alpha : M \to N$ are morphisms between objects in an abelian category M.
- $F : \mathsf{M} \to \mathsf{N}$ is an additive functor between abelian categories.
- The derived category of an abelian category M is $\mathbf{D}(\mathsf{M})$.
- If M is a module category, and $M \in \mathrm{Ob}(\mathsf{M})$, then elements of M will be denoted by $m, n, m_i, \ldots$.

1.3. **Zero objects.** Let C be a category. Recall that a morphism $f : C \to D$ in C is called an *isomorphism* if there is a morphism $g : D \to C$ such that $f \circ g = \mathrm{id}_D$ and $g \circ f = \mathrm{id}_C$. The morphism g is called the *inverse* of f, it is unique (if it exists), and it is denoted by f^{-1}.

The morphism $f : C \to D$ in C is called an *epimorphism* if it has the right cancellation property: for any $g, g' : D \to E$, $g \circ f = g' \circ f$ implies $g = g'$. The morphism $f : C \to D$ is called a *monomorphism* if it has the left cancellation property: for any $g, g' : E \to C$, $f \circ g = f \circ g'$ implies $g = g'$.

Example 1.3.1. In Set the monomorphisms are the injections, and the epimorphisms are the surjections. A morphism $f : C \to D$ in Set that is both a monomorphism and an epimorphism is an isomorphism. The same holds in the category $\mathsf{Mod}\, A$ of left modules over a ring A.

This example could be misleading, because the property of being an epimorphism is often not preserved by forgetful functors, as the next exercise shows.

Exercise 1.3.2. Consider the category of rings Ring. (All rings have units, and ring homomorphisms are unital.) Show that the forgetful functor $\mathsf{Ring} \to \mathsf{Set}$ respects monomorphisms, but it does not respect epimorphisms. (Hint: show that the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in Ring.)

By a *subobject* of an object $C \in \mathsf{C}$ we mean a monomorphism $f : C' \to C$ in C. We sometimes write $C' \hookrightarrow C$ or $C' \subseteq C$ in this situation, but this is only notational (and does not mean inclusion of sets). Likewise, by a *quotient* of C we mean an epimorphism $g : C \to \bar{C}$ in C, and then sometimes write $C' \twoheadrightarrow C$.

An *initial object* in a category C is an object $C_0 \in \mathsf{C}$, such that for every object $C \in \mathsf{C}$ there is exactly one morphism $C_0 \to C$. Thus the set $\operatorname{Hom}_{\mathsf{C}}(C_0, C)$ is a singleton. A *terminal object* in C is an object $C_\infty \in \mathsf{C}$, such that for every object $C \in \mathsf{C}$ there is exactly one morphism $C \to C_\infty$.

Definition 1.3.3. A *zero object* in a category C is an object which is both initial and terminal.

Initial, terminal and zero objects are unique up to unique isomorphisms (but they need not exist).

Example 1.3.4. In Set, $\varnothing$ is an initial object, and any singleton is a terminal object. There is no zero object.

Example 1.3.5. In $\mathsf{Mod}\, A$, any trivial module (with only the zero element) is a zero object, and we denote this module by 0. This is allowed, since any other zero module is uniquely isomorphic to it.

1.4. **Products and Coproducts.** Let C be a category. By a *collection of objects of* C indexed by a (small) set I, we mean a function $I \to \operatorname{Ob}(\mathsf{C})$, $i \mapsto C_i$. We usually denote this collection like this: $\{C_i\}_{i \in I}$.

Given a a collection $\{C_i\}_{i \in I}$ of objects of C, its *product* is a pair $(C, \{p_i\}_{i \in I})$ consisting of an object $C \in \mathsf{C}$, and morphisms $p_i : C \to C_i$ for all $i \in I$, called *projections*. The pair $(C, \{p_i\}_{i \in I})$ must have this universal property: given any object D and morphisms $f_i : D \to C_i$, there is a unique morphism $f : D \to C$ s.t. $f_i = p_i \circ f$. Of course if a product $(C, \{p_i\}_{i \in I})$ exists, then it is unique up to a unique isomorphism; and we usually write $\prod_{i \in I} C_i := C$, leaving the projection morphisms implicit.

Example 1.4.1. In Set and $\mathsf{Mod}\, A$ all products exist, and they are the usual cartesian products.

For a collection $\{C_i\}_{i \in I}$ of objects of C, their *coproduct* is a pair $(C, \{e_i\}_{i \in I})$, consisting of an object C and morphisms $e_i : C_i \to C$, called *embeddings*. The pair

$(C, \{e_i\}_{i \in I})$ must have this universal property: given any object D and morphisms $f_i : C_i \to D$, there is a unique morphism $f : C \to D$ s.t. $f_i = f \circ e_i$. If a product $(C, \{e_i\}_{i \in I})$ exists, then it is unique up to a unique isomorphism; and we write $\coprod_{i \in I} C_i := C$, leaving the embeddings implicit.

Example 1.4.2. In Set the coproduct is the disjoint union. In $\mathsf{Mod}\,A$ the coproduct is the direct sum.

1.5. **Equivalence.** Recall that a functor $F : \mathsf{C} \to \mathsf{D}$ is an *equivalence* if there exists a functor $G : \mathsf{D} \to \mathsf{C}$, and isomorphisms of functors (i.e. natural isomorphisms) $G \circ F \xrightarrow{\simeq} \mathrm{id}_{\mathsf{C}}$ and $F \circ G \xrightarrow{\simeq} \mathrm{id}_{\mathsf{D}}$. Such a functor G is called a *quasi-inverse* of F, and it is unique up to isomorphism (if it exists), and it is denoted by F^{-1}.

The functor $F : \mathsf{C} \to \mathsf{D}$ is *full* (resp. *faithful*) if every $C_0, C_1 \in \mathsf{C}$ the function

$$F : \mathrm{Hom}_{\mathsf{C}}(C_0, C_1) \to \mathrm{Hom}_{\mathsf{D}}\big(F(C_0), F(C_1)\big)$$

is surjective (resp. injective).

We know that $F : \mathsf{C} \to \mathsf{D}$ is an equivalence iff these two conditions hold:

(i) F is essentially surjective on objects. This means that for every $D \in \mathsf{D}$ there is some $C \in \mathsf{C}$ and an isomorphism $F(C) \xrightarrow{\simeq} D$.
(ii) F is fully faithful (i.e. full and faithful).

Exercise 1.5.1. If you are not sure about the last claim (characterization of equivalences), then prove it. (Hint: use the axiom of choice to construct a quasi-inverse of F.)

Example 1.5.2. Let C and D be categories. A functor $F : \mathsf{C} \to \mathsf{D}$ is an *isomorphism* if it is bijective of sets of objects and on sets of morphisms. It is clear that an isomorphic of categories is an equivalence. In practice, it is very rare to find an isomorphism of categories.

1.6. **Bifunctors.** Let C and D be categories. Their product is the category $\mathsf{C} \times \mathsf{D}$ defined as follows: the set of objects is

$$\mathrm{Ob}(\mathsf{C} \times \mathsf{D}) := \mathrm{Ob}(\mathsf{C}) \times \mathrm{Ob}(\mathsf{D}).$$

The sets of morphisms are

$$\mathrm{Hom}_{\mathsf{C} \times \mathsf{D}}\big((C_0, D_0), (C_1, D_1)\big) := \mathrm{Hom}_{\mathsf{C}}(C_0, C_1) \times \mathrm{Hom}_{\mathsf{D}}(D_0, D_1).$$

The composition is

$$(f_1, g_1) \circ (f_0, g_0) := (f_1 \circ f_0, g_1 \circ g_0),$$

and the identity morphisms are $(\mathrm{id}_C, \mathrm{id}_D)$.

A *bifunctor*

$$F : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$$

is by definition a functor from the product category $\mathsf{C} \times \mathsf{D}$ to E. We say "bifunctor" because it is a functor of two arguments: $F(C, D) \in \mathsf{E}$. This will be especially useful when considering additive categories, because then we can talk about "bi-additive bifunctors".

1.7. **Representable Functors.** Let C be a category and $C \in \mathsf{C}$ an object. We get a functor

$$Y_C : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}, \quad Y_C := \mathrm{Hom}_{\mathsf{C}}(-, C),$$

called the *Yoneda functor*. This functor sends a morphism $\phi : C_0 \to C_1$ in C to

$$Y(\phi) := \mathrm{Hom}(\phi, \mathrm{id}_C) : Y_{C_1} \to Y_{C_0}.$$

Here is the first formulation of the *Yoneda Lemma*.

Proposition 1.7.1 (Yoneda Lemma v1). *Let C be a category, let $C_0, C_1 \in \mathsf{C}$ be objects, and let $\eta : Y_{C_1} \to Y_{C_0}$ be a morphism of functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$.*

(1) *There exists a unique morphism $\phi : C_0 \to C_1$ in C such that $Y(\phi) = \eta$.*
(2) *If $\eta : Y_{C_1} \to Y_{C_0}$ is an isomorphism of functors, then $\phi : C_0 \to C_1$ is an isomorphism in C.*

See [KS2] for a proof. The proof is not very hard, but it is confusing.

A functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is called *representable* if there is an isomorphism of functors $f : F \xrightarrow{\simeq} Y_C$ for some object $C \in \mathsf{C}$. By Proposition 1.7.1 the pair (C, f) is unique up to a unique isomorphism (if it exists). Note that the isomorphism of sets $f_C : F(C) \xrightarrow{\simeq} Y_C(C)$ gives a special element $\tilde{f} \in F(C)$ such that $f_C(\tilde{f}) = \mathrm{id}_C$.

Here is a fancier way to state this result. Consider the category $\mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$, whose objects are the functors $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$, and whose morphisms are the morphisms of functors (the natural transformations). There is a set-theoretic difficulty here: the sets of objects and morphisms of $\mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ are too big (unless C is a small category); so this is not a U-category, and we must enlarge the universe.

Proposition 1.7.2 (Yoneda Lemma v2). *The Yoneda functor*

$$Y : \mathsf{C} \to \mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$$

is fully faithful.

In other words, the Yoneda Lemma says that the functor Y is an equivalence from C to the category of representable functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$.

Dually, any $C \in \mathsf{C}$ gives rise to a functor

$$Y'_C : \mathsf{C} \to \mathsf{Set}, \quad Y'_C := \mathrm{Hom}_{\mathsf{C}}(C, -).$$

The identity automorphism id_C is a special element of the set $Y'_C(C)$.

A functor $F : \mathsf{C} \to \mathsf{Set}$ is called *corepresentable* if $F \cong Y'_C$ for some object $C \in \mathsf{C}$. The object C is said to corepresent the functor F. The dual Yoneda Lemma (v2) says that the functor Y' is an equivalence from C^{op} to the category of corepresentable functors $\mathsf{C} \to \mathsf{Set}$.

2. Abelian Categories

The concept of *abelian category* is an extremely useful abstraction of module categories, introduced by Grothendieck in 1957. Before defining it (in Definition 2.3.8), we need some preparation.

2.1. **Linear categories.**

Definition 2.1.1. Let $\mathbb{K}$ be a commutative ring. A $\mathbb{K}$-*linear category* is a category M, endowed with a $\mathbb{K}$-module structure on each of the sets of morphisms $\mathrm{Hom}_{\mathsf{M}}(M_0, M_1)$. The condition is this:

- For all $M_0, M_1, M_2 \in \mathsf{M}$ the composition function
$$\begin{aligned} &\mathrm{Hom}_{\mathsf{M}}(M_1, M_2) \times \mathrm{Hom}_{\mathsf{M}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{M}}(M_0, M_2) \\ &(\phi_2, \phi_1) \mapsto \phi_2 \circ \phi_1 \end{aligned}$$
is $\mathbb{K}$-bilinear.

If $\mathbb{K} = \mathbb{Z}$, we say that M is a *linear category*.

Let $\mathbb{K}$ be a commutative ring. By *central* $\mathbb{K}$-*ring* we mean a ring A, with a ring homomorphism $\mathbb{K} \to A$, such that the image of $\mathbb{K}$ is inside the center of A. (Many texts would call such A a "unital associative $\mathbb{K}$-algebra".)

Example 2.1.2. Let $\mathbb{K}$ be any nonzero commutative ring, and let n be a positive integer. Then the ring of matrices $A := \mathrm{Mat}_n(\mathbb{K})$ is a central $\mathbb{K}$-ring.

Proposition 2.1.3. *Let* M *be a* $\mathbb{K}$*-linear category.*

(1) *For any object* $M \in \mathsf{M}$*, the set*
$$\mathrm{End}_{\mathsf{M}}(M) := \mathrm{Hom}_{\mathsf{M}}(M, M),$$
with its given addition operation, and with the operation of composition, is a central $\mathbb{K}$*-ring.*

(2) *For any two objects* $M_0, M_1 \in \mathsf{M}$*, the set* $\mathrm{Hom}_{\mathsf{M}}(M_0, M_1)$*, with its given addition operation, and with the operations of composition, is a left module over the ring* $\mathrm{End}_{\mathsf{M}}(M_1)$*, and a right module over the ring* $\mathrm{End}_{\mathsf{M}}(M_0)$*. Furthermore, these left and right actions commute with each other.*

Proof. Exercise. □

This result can be reversed:

Example 2.1.4. Let A be a central $\mathbb{K}$-ring. Define a category M like this: there is a single object M, and its set of morphisms is $\mathrm{Hom}_{\mathsf{M}}(M, M) := A$. Composition in M is the multiplication of A. Then M is a $\mathbb{K}$-linear category.

Because of the above, in a linear category M, we often denote the identity automorphism of an object M by $1_M := \mathrm{id}_M \in \mathrm{End}_{\mathsf{M}}(M)$.

For a central $\mathbb{K}$-ring A, the opposite ring A^{op} has the same $\mathbb{K}$-module structure as A, but the multiplication is reversed.

Exercise 2.1.5. Let A be a nonzero ring. Let $P, Q \in \mathsf{Mod}\, A$ be distinct free A-modules of rank 1.

(1) Prove that there is a ring isomorphism $\mathrm{End}_{\mathsf{Mod}\, A}(P) \cong A^{\mathrm{op}}$. Is this ring isomorphism canonical?

(2) Let M be the full subcategory of $\mathsf{Mod}\, A$ on the set of objects $\{P, Q\}$. Compare the linear category M to the ring of matrices $\mathrm{Mat}_2(A^{\mathrm{op}})$.

2.2. **Additive categories.**

Definition 2.2.1. An *additive category* is a linear category M satisfying these conditions:

(i) M has a zero object 0.
(ii) M has finite coproducts.

Observe that $\operatorname{Hom}_{\mathsf{M}}(M, N) \neq \varnothing$ for any $M, N \in \mathsf{M}$, since this is an abelian group. Also

$$\operatorname{Hom}_{\mathsf{M}}(M, 0) = \operatorname{Hom}_{\mathsf{M}}(0, M) = 0,$$

the zero abelian group. We denote the unique arrows $0 \to M$ and $M \to 0$ also by 0. So the numeral 0 has a lot of meanings; but they are (hopefully) clear from the contexts. The coproduct in a linear category M is usually denoted by $\bigoplus$; cf. Example 1.4.2.

Example 2.2.2. Let A be a $\mathbb{K}$-central ring. The category $\mathsf{Mod}\, A$ is a $\mathbb{K}$-linear additive category. The full subcategory $\mathsf{F} \subseteq \mathsf{Mod}\, A$ on the free modules is also additive.

Proposition 2.2.3. *Let M be a linear category. Let $\{M_i\}_{i \in I}$ be a finite collection of objects of M, and assume the coproduct $M = \bigoplus_{i \in I} M_i$ exists, with embeddings $e_i : M_i \to M$.*

(1) *For any i let $p_i : M \to M_i$ be the unique morphism s.t. $p_i \circ e_i = 1_{M_i}$, and $p_i \circ e_j = 0$ for $j \neq i$. Then $(M, \{p_i\}_{i \in I})$ is a product of the collection $\{M_i\}_{i \in I}$.*
(2) $\sum_{i \in I} e_i \circ p_i = 1_M$.

Proof. Exercise. □

Part (1) directly implies:

Corollary 2.2.4. *An additive category has finite products.*

Definition 2.2.5. Let M be an additive category, and let N be a full subcategory of M. We say that N is a *full additive subcategory* of M if N is closed under finite direct sums.

Exercise 2.2.6. In the situation of Definition 2.2.5, the category N is itself additive.

Example 2.2.7. Consider the linear category M from Example 2.1.4, built from a ring A. It does not have a zero object (unless the ring A is the zero ring), so it is not additive.

A more puzzling question is this: Does M have finite direct sums? This turns out to be equivalent to whether or not $A \cong A \oplus A$ as right A-modules. To see why, choose a fully faithful additive functor $F : \mathsf{M} \to \mathsf{Mod}\, A^{\mathrm{op}}$, that sends the unique object $M \in \mathsf{M}$ to a rank 1 free right A-module P. (We identify right A-modules with left A^{op}-modules.) Compare to Exercise 2.1.5.

Let $I := \{1, 2\}$, and let $\{M_i\}_{i \in I}$ be the only possible collection in M indexed by I (i.e. $M_i = M$). If there is a coproduct in M, then it must be $M_1 \oplus M_2 \cong M$. According to Proposition 2.4.2, we get

$$P \oplus P \cong F(M_1) \oplus F(M_2) \cong F(M) \cong P$$

in $\mathsf{Mod}\, A^{\mathrm{op}}$.

We know that when A is nonzero and commutative, or nonzero and noetherian, then $A \ncong A \oplus A$ in $\mathsf{Mod}\, A^{\mathrm{op}}$. On the other hand, if we take a field $\mathbb{K}$, and a countable rank $\mathbb{K}$-module N, then $A := \operatorname{End}_{\mathbb{K}}(N)$ will satisfy $A \cong A \oplus A$.

Proposition 2.2.8. *Let* M *be a linear category, and* $N \in \mathsf{M}$. *The following conditions are equivalent*:

(i) *The ring* $\operatorname{End}_{\mathsf{M}}(N)$ *is trivial.*
(ii) N *is a zero object of* M.

Proof. (ii) $\Rightarrow$ (i): Since the set $\operatorname{End}_{\mathsf{M}}(N)$ is a singleton, it must be the trivial ring $(1 = 0)$.

(i) $\Rightarrow$ (ii): If the ring $\operatorname{End}_{\mathsf{M}}(N)$ is trivial, then all left and right modules over it must be trivial. Now use Proposition 2.1.3(2). □

2.3. **Abelian categories.**

Definition 2.3.1. Let M be an additive category, and let $f : M \to N$ be a morphism in M. A *kernel* of f is a pair (K, k), consisting of an object $K \in \mathsf{M}$ and a morphism $k : K \to M$, with these properties:

(i) $f \circ k = 0$.
(ii) If $k' : K' \to M$ is a morphism in M such that $f \circ k' = 0$, then there is a unique morphism $g : K' \to K$ such that $k' = k \circ g$.

In other words, the object K represents the functor $\mathsf{M}^{\mathrm{op}} \to \mathsf{Ab}$,

$$K' \mapsto \{k' \in \operatorname{Hom}_{\mathsf{M}}(K', M) \mid f \circ k' = 0\}.$$

The kernel of f is of course unique up to a unique isomorphism (if it exists), and we denote if by $\operatorname{Ker}(f)$. Sometimes $\operatorname{Ker}(f)$ refers only to the object K, and other times it refers only to the morphism k; as usual, this should be clear from the context.

Definition 2.3.2. Let M be an additive category, and let $f : M \to N$ be a morphism in M. A *cokernel* of f is a pair (C, c), consisting of an object $C \in \mathsf{M}$ and a morphism $c : N \to C$, with these properties:

(i) $c \circ f = 0$.
(ii) If $c' : N \to C'$ is a morphism in M such that $c' \circ f = 0$, then there is a unique morphism $g : C \to C'$ such that $c' = g \circ c$.

In other words, the object C corepresents the functor $\mathsf{M} \to \mathsf{Ab}$,

$$C' \mapsto \{c' \in \operatorname{Hom}_{\mathsf{M}}(M, C') \mid c' \circ f = 0\}.$$

The cokernel of f is of course unique up to a unique isomorphism (if it exists), and we denote if by $\operatorname{Coker}(f)$. Sometimes $\operatorname{Coker}(f)$ refers only to the object C, and other times it refers only to the morphism c; as usual, this should be clear from the context.

Example 2.3.3. In $\mathsf{Mod}\, A$ all kernels and cokernels exist. Given $f : M \to N$, the kernel is $k : K \to M$, where

$$K := \{m \in M \mid f(m) = 0\},$$

and the k is the inclusion. The cokernel is $c : N \to C$, where $C := N/f(M)$, and c is the canonical projection.

Proposition 2.3.4. *Let* M *be an additive category, and let* $f : M \to N$ *be a morphism in* M.

(1) *If $k : K \to M$ is a kernel of f, then k is a monomorphism.*
(2) *If $c : N \to C$ is a cokernel of f, then c is an epimorphism.*

Proof. Exercise. □

Definition 2.3.5. Assume the additive category M has kernels and cokernels. Let $f : M \to N$ be a morphism in M.

(1) Define the *image* of f to be
$$\mathrm{Im}(f) := \mathrm{Ker}(\mathrm{Coker}(f)).$$
(2) Define the *coimage* of f to be
$$\mathrm{Coim}(f) := \mathrm{Coker}(\mathrm{Ker}(f)).$$

The image is familiar, but the coimage is not. The next diagram should help. We start with a morphism $f : M \to N$ in M. The kernel and cokernel of f fit into this diagram:
$$K \xrightarrow{k} M \xrightarrow{f} N \xrightarrow{c} C.$$
Inserting $\alpha := \mathrm{Coker}(k) = \mathrm{Coim}(f)$ and $\beta := \mathrm{Ker}(c) = \mathrm{Im}(f)$ we get the following commutative diagram (solid arrows):

(2.3.6)
$$\begin{tikzcd} K \arrow[r, "k"] \arrow[dr, "0"'] & M \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, dashed, "\gamma"] & N \arrow[r, "c"] & C \\ & M' \arrow[r, dashed, "f'"'] & N' \arrow[u, "\beta"'] \arrow[ur, "0"'] & \end{tikzcd}$$

Since $c \circ f = 0$ there is a unique morphism γ making the diagram commutative. Now $\beta \circ \gamma \circ k = f \circ k = 0$; and β is a monomorphism; so $\gamma \circ k = 0$. Hence there is a unique morphism $f' : M' \to N'$ making the diagram commutative. We conclude that $f : M \to N$ induces a morphism

(2.3.7)
$$f' : \mathrm{Coim}(f) \to \mathrm{Im}(f).$$

Definition 2.3.8. An *abelian category* is an additive category M with these extra properties:

(i) All morphisms in M admit kernels and cokernels.
(ii) For any morphism $f : M \to N$ in M, the induced morphism f' in equation (2.3.7) is an isomorphism.

Here is a less precise but (maybe) easier to remember way to state property (ii). Because $M' = \mathrm{Coker}(\mathrm{Ker}(f))$ and $N' = \mathrm{Ker}(\mathrm{Coker}(f))$, we see that

(2.3.9)
$$\mathrm{Coker}(\mathrm{Ker}(f)) = \mathrm{Ker}(\mathrm{Coker}(f)).$$

From now on we forget all about the coimage.

Exercise 2.3.10. For any ring A, the category $\mathsf{Mod}\, A$ is abelian.

This includes the category $\mathsf{Ab} = \mathsf{Mod}\, \mathbb{Z}$, from which the name derives.

Definition 2.3.11. Let M be an abelian category, and let N be a full subcategory of M. We say that N is a *full abelian subcategory* of M if N is closed under finite direct sums, kernels and cokernels.

Exercise 2.3.12. In the situation of Definition 2.3.11, the category N is itself abelian.

Example 2.3.13. Let M_1 be the category of finitely generated abelian groups, and let M_0 be the category of finite abelian groups. Then M_1 is a full abelian subcategory of Ab, and M_0 is a full abelian subcategory of M_1.

Exercise 2.3.14. Let N be the full subcategory of Ab whose objects are the finitely generated free abelian groups. It is an additive subcategory of Ab (since it is closed under direct sums).

(1) Show that N is closed under kernels in Ab.
(2) Show that N is not closed under cokernels in Ab, so it is not a full abelian subcategory of Ab.
(3) Show that N has cokernels (not the same as those of Ab). Still, it fails to be an abelian category.

Exercise 2.3.15. Of course Grp is not an additive category. Still it has a zero object (the trivial group). Show that Grp has kernels and cokernels, but condition (ii) of Definition 2.3.8 fails.

Exercise 2.3.16. Let Hilb be the category of Hilbert spaces over $\mathbb{C}$. The morphisms are the $\mathbb{C}$-linear homomorphisms $f : M \to N$ that respect the inner products. Show that Hilb is a $\mathbb{C}$-linear additive category with kernels and cokernels, but it is not an abelian category.

Exercise 2.3.17. Let A be a ring. Show that A is *left noetherian* iff the category $\mathsf{Mod}_{\mathsf{f}}\, A$ of finitely generated left modules is a full abelian subcategory of $\mathsf{Mod}\, A$.

Example 2.3.18. Let $(X, \mathcal{A})$ be a ringed space; namely X is a topological space and $\mathcal{A}$ is a sheaf of rings on X. We denote by $\mathsf{PMod}\ \mathcal{A}$ the category of presheaves of left $\mathcal{A}$-modules on X. This is an abelian category. Given a morphism $f : \mathcal{M} \to \mathcal{N}$ in $\mathsf{PMod}\ \mathcal{A}$, its kernel is the presheaf $\mathcal{K}$ defined by

$$\Gamma(U, \mathcal{K}) := \operatorname{Ker}\big(f : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})\big)$$

on every open set $U \subseteq X$. The cokernel is the presheaf $\mathcal{C}$ defined by

$$\Gamma(U, \mathcal{C}) := \operatorname{Coker}\big(f : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})\big).$$

Now let $\mathsf{Mod}\, \mathcal{A}$ be the full subcategory of $\mathsf{PMod}\ \mathcal{A}$ consisting of sheaves. It is a full additive subcategory of $\mathsf{PMod}\ \mathcal{A}$, closed under kernels. We know that $\mathsf{Mod}\, \mathcal{A}$ is not closed under cokernels inside $\mathsf{PMod}\ \mathcal{A}$, and hence it is not a full abelian subcategory.

However $\mathsf{Mod}\, \mathcal{A}$ is itself an abelian category, but with different cokernels. Indeed, for a morphism $f : \mathcal{M} \to \mathcal{N}$ in $\mathsf{Mod}\, \mathcal{A}$, its cokernel $\operatorname{Coker}_{\mathsf{Mod}\, \mathcal{A}}(f)$ is the sheafification of the presheaf $\operatorname{Coker}_{\mathsf{PMod}\ \mathcal{A}}(f)$.

For educational purposes we state without proof:

Theorem 2.3.19 (Freyd & Mitchell)**.** *Let* M *be a small abelian category. Then* M *is equivalent to a full abelian subcategory of* $\mathsf{Mod}\, A$*, for a suitable ring* A*.*

This means that most of the time we can pretend that $\mathsf{M} \subseteq \mathsf{Mod}\, A$; this could be a helpful heuristic.

Proposition 2.3.20.
(1) *Let* M *be an additive category. Then the opposite category* M^{op} *is also additive.*
(2) *Let* M *be an abelian category. Then the opposite category* M^{op} *is also abelian.*

Proof. (1) First note that

$$\mathrm{Hom}_{\mathsf{M}^{\mathrm{op}}}(M, N) = \mathrm{Hom}_{\mathsf{M}}(N, M),$$

so this is an abelian group. The bilinearity of the composition in M^{op} is clear, and the zero objects are the same. Existence of finite coproducts in M^{op} is because of existence of finite products in M; see Proposition 2.2.3(1).

(2) M^{op} has kernels and cokernels, since $\mathrm{Ker}_{\mathsf{M}^{\mathrm{op}}}(f) = \mathrm{Coker}_{\mathsf{M}}(f)$ and vice versa. Also the symmetric condition (ii) of Definition 2.3.8 holds. □

Proposition 2.3.21. *Let $f : M \to N$ be a morphism in an abelian category M.*

1. *f is a monomorphism iff* $\mathrm{Ker}(f) = 0$.
2. *f is an epimorphism iff* $\mathrm{Coker}(f) = 0$.
3. *f is an isomorphism iff it is both a monomorphism and an epimorphism.*

Proof. Exercise. □

2.4. **Additive Functors.**

Definition 2.4.1. Let M and N be $\mathbb{K}$-linear categories. A functor $F : \mathsf{M} \to \mathsf{N}$ is called a $\mathbb{K}$-*linear functor* if for every $M_0, M_1 \in \mathsf{M}$ the function

$$F : \mathrm{Hom}_{\mathsf{M}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{N}}(F(M_0), F(M_1))$$

is a $\mathbb{K}$-linear homomorphism.

A $\mathbb{Z}$-linear functor is also called an *additive functor.*

Additive functors commute with finite direct sums. More precisely:

Proposition 2.4.2. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between linear categories, let $\{M_i\}_{i \in I}$ be a finite collection of objects of M, and assume that the direct sum $(M, \{e_i\}_{i \in I})$ of the collection $\{M_i\}_{i \in I}$ exists in M. Then $\big(F(M), \{F(e_i)\}_{i \in I}\big)$ is a direct sum of the collection $\{F(M_i)\}_{i \in I}$ in N.*

Proof. Exercise. (Hint: use Proposition 2.2.3.) □

Note that the proposition above also talks about finite products, because of Proposition 2.2.3.

Example 2.4.3. Let $A \to B$ be a ring homomorphism. The corresponding forgetful functor

$$F : \mathsf{Mod}\, B \to \mathsf{Mod}\, A$$

(also called restriction of scalars) is additive. The functor

$$G : \mathsf{Mod}\, A \to \mathsf{Mod}\, B$$

defined by $G(M) := B \otimes_A M$, called extension of scalars, is also additive.

Proposition 2.4.4. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between linear categories. Then*:

1. *For any $M \in \mathsf{M}$ the function*

$$F : \mathrm{End}_{\mathsf{M}}(M) \to \mathrm{End}_{\mathsf{N}}\big(F(M)\big)$$

is a ring homomorphism.

(2) *For any $M_0, M_1 \in \mathsf{M}$ the function*

$$F : \mathrm{Hom}_{\mathsf{M}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{N}}\big(F(M_0), F(M_1)\big)$$

is a homomorphism of left $\mathrm{End}_{\mathsf{M}}(M_1)$*-modules, and of right* $\mathrm{End}_{\mathsf{M}}(M_0)$*-modules.*

(3) *If M is a zero object of M, then $F(M)$ is a zero object of N.*

Proof. (1) By Definition 2.4.1 the function F respect addition. By definition of a functor, it respects multiplication and units.

(2) Immediate from the definitions, like (1).

(3) Combine part (1) with Proposition 2.2.8. □

Definition 2.4.5. Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between abelian categories.

(1) F is called *left exact* if it commutes with kernels. Namely for any morphism $\phi : M_0 \to M_1$ in M, with kernel $k : K \to M_0$, the morphism $F(k) : F(K) \to F(M_0)$ is a kernel of $F(\phi) : F(M_0) \to F(M_1)$.
(2) F is called *right exact* if it commutes with cokernels. Namely for any morphism $\phi : M_0 \to M_1$ in M, with cokernel $c : M_1 \to C$, the morphism $F(c) : F(M_1) \to F(C)$ is a cokernel of $F(\phi) : F(M_0) \to F(M_1)$.
(3) F is called *exact* if it both left exact and right exact.

This is illustrated in the following diagrams. Suppose $\phi : M_0 \to M_1$ is a morphism in M, with kernel K and cokernel C. Applying F to the diagram

$$K \xrightarrow{k} M_0 \xrightarrow{\phi} M_1 \xrightarrow{c} C$$

we get the solid arrows in

$$\begin{tikzcd}
F(K) \arrow[r, "F(k)"] \arrow[dr, dashed, "\psi"'] & F(M_0) \arrow[rr, "F(\phi)"] & & F(M_1) \arrow[r, "F(c)"] \arrow[d, dashed] & F(C) \\
& \mathrm{Ker}_{\mathsf{N}}(F(\phi)) \arrow[u, dashed] & & \mathrm{Coker}_{\mathsf{N}}(F(\phi)) \arrow[ur, dashed, "\chi"'] &
\end{tikzcd}$$

Because N is abelian, we get the vertical dashed arrows: the kernel and cokernel of $F(\phi)$. The slanted dashed arrows exist and are unique because $F(\phi) \circ F(k) = 0$ and $F(c) \circ F(\phi) = 0$. Left exactness requires ψ to be an isomorphism, and right exactness requires χ to be an isomorphism.

Definition 2.4.6. Let M be an abelian category. An *exact sequence* in M is a diagram

$$\cdots M_0 \xrightarrow{\phi_0} M_1 \xrightarrow{\phi_1} M_2 \cdots$$

(finite or infinite on either side) s.t. $\mathrm{Ker}(\phi_i) = \mathrm{Im}(\phi_{i-1})$ for all i (for which ϕ_i and ϕ_{i-1} are defined).

As usual, a *short exact sequence* is one of the form

$$0 \to M_0 \to M_1 \to M_2 \to 0. \tag{2.4.7}$$

Proposition 2.4.8. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between abelian categories.*

(1) *The functor F is left exact iff for every short exact sequence* (2.4.7) *in* M, *the sequence*
$$0 \to F(M_0) \to F(M_1) \to F(M_2)$$
is exact in N.

(2) *The functor F is right exact iff for every short exact sequence* (2.4.7) *in* M, *the sequence*
$$F(M_0) \to F(M_1) \to F(M_2) \to 0$$
is exact in N.

Proof. Exercise. (Hint: $M_0 \cong \operatorname{Ker}(M_1 \to M_2)$ etc.) □

Example 2.4.9. Let A be a commutative ring, and let M be a fixed A-module. Define functors $F, G : \mathsf{Mod}\,A \to \mathsf{Mod}\,A$ and $H : (\mathsf{Mod}\,A)^{\mathrm{op}} \to \mathsf{Mod}\,A$ like this: $F(N) := M \otimes_A N$, $G(N) := \operatorname{Hom}_A(M, N)$ and $H(N) := \operatorname{Hom}_A(N, M)$. Then F is right exact, and G and H are left exact.

Proposition 2.4.10. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between abelian categories. If F is an equivalence then it is exact.*

Proof. We will prove that F respects kernels; the proof for cokernels is similar. Take a morphism $\phi : M_0 \to M_1$ in M, with kernel K. We have this diagram (solid arrows):

$$\begin{tikzcd}
M \arrow[d, dashed, "\psi"'] \arrow[dr, dashed, "\theta"] & & \\
K \arrow[r, "k"] & M_0 \arrow[r, "\phi"] & M_1
\end{tikzcd}$$

Applying F we obtain this diagram (solid arrows):

$$\begin{tikzcd}
N = F(M) \arrow[d, dashed, "F(\psi)"'] \arrow[dr, dashed, "\bar{\theta}"] & & \\
F(K) \arrow[r, "F(k)"] & F(M_0) \arrow[r, "F(\phi)"] & F(M_1)
\end{tikzcd}$$

in N. Suppose $\bar{\theta} : N \to F(M_0)$ is a morphism in N s.t. $F(\phi) \circ \bar{\theta} = 0$. Since F is essentially surjective on objects, there is some $M \in \mathsf{M}$ with an isomorphism $\alpha : F(M) \xrightarrow{\simeq} N$. After replacing N with $F(M)$ and $\bar{\theta}$ with $\bar{\theta} \circ \alpha$, we can assume that $N = F(M)$.

Now since F is fully faithful, there is a unique $\theta : M \to M_0$ s.t. $F(\theta) = \bar{\theta}$; and $\phi \circ \theta = 0$. So there is a unique $\psi : M \to K$ s.t. $\theta = k \circ \psi$. It follows that $F(\psi) : F(M) \to F(M_0)$ is the unique morphism s.t. $\bar{\theta} = F(k) \circ F\psi)$. □

Here is a result that could afford another proof of the previous proposition.

Proposition 2.4.11. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between linear categories. Assume F is an equivalence, with quasi-inverse G. Then $G : \mathsf{N} \to \mathsf{M}$ is an additive functor.*

Proof. Exercise. □

2.5. **Projectives.** In this subsection M is an abelian category.

A *splitting* of an epimorphism $\psi : M \to M''$ in M is a morphism $\alpha : M'' \to M$ s.t. $\psi \circ \alpha = 1_{M''}$. A splitting of a monomorphism $\phi : M' \to M$ is a morphism $\beta : M \to M'$ s.t. $\beta \circ \phi = 1_{M'}$. A splitting of a short exact sequence

$$0 \to M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \to 0 \tag{2.5.1}$$

is a splitting of the epimorphism ψ, or equivalently a splitting of the monomorphism ϕ. The short exact sequence is said to be *split* if it has some splitting.

Exercise 2.5.2. Show how to get from a splitting of ϕ to a splitting of ψ, and vice versa. Show how any of those gives rise to an isomorphism $M \cong M' \oplus M''$.

Definition 2.5.3. An object $P \in \mathsf{M}$ is called a *projective object* if any diagram (solid arrows)

$$\begin{tikzcd} & P \arrow[dl, dashed, "\tilde{\gamma}"'] \arrow[d, "\gamma"] \\ M \arrow[r, "\psi"'] & N \end{tikzcd}$$

in M, in which ψ is an epimorphism, can be completed (dashed arrow) to a commutative diagram.

Proposition 2.5.4. *The following conditions are equivalent for $P \in \mathsf{M}$:*

(i) *P is projective.*

(ii) *The additive functor*

$$\operatorname{Hom}_{\mathsf{M}}(P, -) : \mathsf{M} \to \mathsf{Ab}$$

is exact.

Proof. Exercise. □

Definition 2.5.5. We say M *has enough projectives* if every $M \in \mathsf{M}$ admits an epimorphism $P \to M$ from a projective object P.

Example 2.5.6. Let A be a ring. An A-module P is projective iff it is a direct summand of a free module; i.e. $P \oplus P' \cong Q$ for some module P' and free module Q. The category $\mathsf{Mod}\, A$ has enough projectives.

Example 2.5.7. Let M be the category of finite abelian groups. The only projective object in M is 0. So M does not have enough projectives.

Example 2.5.8. Consider the scheme $X := \mathbf{P}^1_{\mathbb{K}}$, the projective line over a field $\mathbb{K}$. (If the reader prefers, he/she can assume $\mathbb{K}$ is algebraically closed, so X is a classical algebraic variety.) The structure sheaf (sheaf of functions) is $\mathcal{O}_X$. The category $\mathsf{Coh}\ \mathcal{O}_X$ of coherent $\mathcal{O}_X$-modules is abelian (it is a full abelian subcategory of $\mathsf{Mod}\,\mathcal{O}_X$, cf. Example 2.3.18). One can show that the only projective object of $\mathsf{Coh}\ \mathcal{O}_X$ is 0, but this is quite involved.

Let us only indicate why $\mathcal{O}_X$ is not projective. Denote by t_0, t_1 the homogeneous coordinates of X. These belong to $\Gamma(X, \mathcal{O}_X(1))$, so each determines a homomorphism of sheaves $t_j : \mathcal{O}_X(i) \to \mathcal{O}_X(i+1)$. We get a sequence

$$0 \to \mathcal{O}_X(-2) \xrightarrow{\left[\begin{smallmatrix} t_0 & t_1 \end{smallmatrix}\right]} \mathcal{O}_X(-1)^2 \xrightarrow{\left[\begin{smallmatrix} -t_1 \\ t_0 \end{smallmatrix}\right]} \mathcal{O}_X \to 0$$

in $\mathsf{Coh}\ \mathcal{O}_X$, which is known to be exact. Because $\Gamma(X, \mathcal{O}_X) = \mathbb{K}$, and $\Gamma(X, \mathcal{O}_X(-1)) = 0$, this sequence is not split.

2.6. **Injectives.** In this subsection M is an abelian category.

Definition 2.6.1. An object $I \in \mathsf{M}$ is called an *injective object* if any diagram (solid arrows)

$$\begin{tikzcd} I & \\ M \arrow[u, "\gamma"] \arrow[r, "\psi"'] & N \arrow[ul, dashed, "\tilde{\gamma}"'] \end{tikzcd}$$

in M, in which ψ is a monomorphism, can be completed (dashed arrow) to a commutative diagram.

Proposition 2.6.2. *The following conditions are equivalent for* $I \in \mathsf{M}$:

(i) I *is injective.*
(ii) *The additive functor*

$$\mathrm{Hom}_{\mathsf{M}}(-, I) : \mathsf{M}^{\mathrm{op}} \to \mathsf{Ab}$$

is exact.

Proof. Exercise. □

Example 2.6.3. Let A be a ring. Unlike projectives, the structure of injective objects in $\mathsf{Mod}\, A$ is very complicated, and not much is known (except that they exist). However if A is a commutative noetherian ring then we know this: every injective module I is a direct sum of indecomposable injective modules. And these indecomposables are parametrized by $\mathrm{Spec}\, A$, the set of prime ideals of A. These facts are due to Matlis; see [RD, pages 120-122] for details.

Definition 2.6.4. We say M *has enough injectives* if every $M \in \mathsf{M}$ admits a monomorphism $M \to I$ to an injective object I.

Here are a few results about injective objects. Recall that modules over a ring are always left modules by default.

Proposition 2.6.5. *Let* $f : A \to B$ *be a ring homomorphism, and let* I *be an injective* A*-module. Then* $J := \mathrm{Hom}_A(B, I)$ *is an injective* B*-module.*

Proof. Note that B is a left A-module via f, and a right B-module. This makes J into a left B-module. In a formula: for $\phi \in J$ and $b, b' \in B$ we have $(b \cdot \phi)(b') = \phi(b' \cdot b)$.

Now given any $N \in \mathsf{Mod}\, B$ there is an isomorphism

$$\mathrm{Hom}_B(N, J) = \mathrm{Hom}_B(N, \mathrm{Hom}_A(B, I)) \cong \mathrm{Hom}_A(N, I). \tag{2.6.6}$$

This is a natural isomorphism (of functors in N). So the functor $\mathrm{Hom}_B(-, J)$ is exact, and hence J is injective. □

Theorem 2.6.7 (Baer Criterion)**.** *Let* A *be a ring and* I *an* A*-module. Assume that every* A*-module homomorphism* $\mathfrak{a} \to I$ *from a left ideal* $\mathfrak{a} \subseteq A$ *extends to a homomorphism* $A \to I$*. Then the module* I *is injective.*

Proof. Consider an A-module M, a submodule $N \subseteq M$, and a homomorphism $\gamma : N \to I$. We have to prove that γ extends to a homomorphism $M \to I$. Look at the pairs (N', γ') consisting of a submodule $N' \subseteq M$ that contains N, and a homomorphism $\gamma' : N' \to I$ that extends γ. The set of all such pairs is ordered by inclusion, and it satisfies the conditions of Zorn's Lemma. Therefore there exists a maximal pair (N', γ'). We claim that $N' = M$.

Otherwise, there is an element $m \in M$ that does not belong to N'. Define $N'' := N' + A \cdot m \subseteq M$. Let

$$\mathfrak{a} := \{a \in A \mid a \cdot m \in N'\},$$

which is a left ideal of A. There is a short exact sequence

$$0 \to \mathfrak{a} \to N' \oplus A \to N'' \to 0$$

of A-modules. Let $\phi : \mathfrak{a} \to I$ be the homomorphism induced by $\gamma' : N' \to I$. By assumption, it extends to a homomorphism $\tilde{\phi} : A \to I$. We get a homomorphism

$$(\gamma', \tilde{\phi}) : N' \oplus A \to I$$

that agrees on $\mathfrak{a}$; and thus there is an induced homomorphism $\gamma'' : N'' \to I$. This contradicts the maximality of (N', γ'). □

Lemma 2.6.8. *The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is injective.*

Proof. By the Baer criterion, it is enough to consider a homomorphism $\gamma : \mathfrak{a} \to \mathbb{Q}/\mathbb{Z}$ for an ideal $\mathfrak{a} = n \cdot \mathbb{Z} \subseteq \mathbb{Z}$. We may assume that $n \neq 0$. Say $\gamma(n) = r + \mathbb{Z}$ with $r \in \mathbb{Q}$. Then we can extend γ to $\tilde{\gamma} : \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ with $\tilde{\gamma}(1) := r/n + \mathbb{Z}$. □

Lemma 2.6.9. *Let $\{I_x\}_{x \in X}$ be a collection of injective objects of M. If the product $\prod_{x \in X} I_x$ exists in M, then it is an injective object.*

Proof. Exercise. □

Theorem 2.6.10. *Let A be any ring. The category $\mathsf{Mod}\, A$ has enough injectives.*

Proof. Step 1. Here $A = \mathbb{Z}$. Take any nonzero $\mathbb{Z}$-module M and any nonzero $m \in M$. Consider the cyclic submodule $M' := \mathbb{Z} \cdot m \subseteq M$. There is a homomorphism $\gamma' : M' \to \mathbb{Q}/\mathbb{Z}$ s.t. $\gamma'(m) \neq 0$. Indeed, if $M' \cong \mathbb{Z}$, then we take any $r \in \mathbb{Q} - \mathbb{Z}$; and if $M' \cong \mathbb{Z}/(n)$ for some $n > 0$, then we take $r := 1/n$. In either case, we define $\gamma'(m) := r + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$. Since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, γ' extends to a homomorphism $\gamma : M \to \mathbb{Q}/\mathbb{Z}$. By construction we have $\gamma(m) \neq 0$.

Step 2. Now A is any ring, M is any nonzero A-module, and $m \in M$ a nonzero element. Define the A-module $I := \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$, which, according to Lemma 2.6.8 and Proposition 2.6.5, is an injective A-module. Let $\gamma : M \to \mathbb{Q}/\mathbb{Z}$ be a $\mathbb{Z}$-linear homomorphism such that $\gamma(m) \neq 0$. Such γ exists by step 1. Let $\tau : I \to \mathbb{Q}/\mathbb{Z}$ be the $\mathbb{Z}$-linear homomorphism that sends an element $\chi \in I$ to $\chi(1) \in \mathbb{Q}/\mathbb{Z}$. The adjunction formula (2.6.6) gives an A-module homomorphism $\psi : M \to I$ s.t. $\tau \circ \psi = \gamma$. We note that $(\tau \circ \psi)(m) = \gamma(m) \neq 0$, and hence $\psi(m) \neq 0$.

Step 3. Here A and M are arbitrary. Let I be as in step 2. For any nonzero $m \in M$ there is an A-linear homomorphism $\psi_m : M \to I$ such that $\psi_m(m) \neq 0$. For $m = 0$ let $\psi_0 : M \to I$ be an arbitrary homomorphism (e.g. $\psi_0 = 0$). Define the A-module $J := \prod_{m \in M} I$. There is a homomorphism $\psi := \prod_{m \in M} \psi_m : M \to J$, and it is easy to check that ψ is a monomorphism. By Lemma 2.6.9, J is an injective A-module. □

Exercise 2.6.11. At the price of getting a bigger injective module, we can make the construction of injective resolutions functorial. Let $I := \operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ as above. Given an A-module M, consider the set

$$X(M) := \operatorname{Hom}_A(M, I) \cong \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}).$$

Let $J(M) := \prod_{\psi \in X(M)} I$. There is a "tautological" homomorphism $\phi_M : M \to J(M)$. Show that ϕ_M is a monomorphism, $J : M \mapsto J(M)$ is a functor, and $\phi : \mathrm{id} \to J$ is a natural transformation.

Is the functor $J : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$ additive?

Example 2.6.12. Let N be the category of torsion abelian groups, and M the category of finite abelian groups. Then $\mathsf{N} \subseteq \mathsf{Ab}$ and $\mathsf{M} \subseteq \mathsf{N}$ are full abelian subcategories. M has no projectives nor injectives except 0. The only projective in N is 0. But N has enough injectives: this is because $\mathbb{Q}/\mathbb{Z} \in \mathsf{N}$, N is closed under infinite direct sums in Ab, and the next proposition.

Proposition 2.6.13. *If A is a left noetherian ring, then any direct sum of injective A-modules is an injective module.*

Proof. Exercise. (Hint: use the Baer criterion.) □

Remark 2.6.14. Actually, the converse of Proposition 2.6.13 is also true: if every direct sum of injective A-modules is injective, then A is left noetherian. But experience tells us that this fact is not very important...

Exercise 2.6.15. Here we study injectives in the category $\mathsf{Ab} = \mathsf{Mod}\, \mathbb{Z}$. By Lemma 2.6.8, the module $I := \mathbb{Q}/\mathbb{Z}$ is injective. For a (positive) prime number p, we denote by $\widehat{\mathbb{Z}}_p$ the ring of p-adic integers, and by $\widehat{\mathbb{Q}}_p$ its field of fractions (namely the p-adic completions of $\mathbb{Z}$ and $\mathbb{Q}$ respectively). Define the abelian group $I_p := \widehat{\mathbb{Q}}_p / \widehat{\mathbb{Z}}_p$.

(1) Show that I_p is an injective object of Ab.
(2) Show that I_p is indecomposable (i.e. it is not the direct sum of two nonzero objects).
(3) Show that $I \cong \bigoplus_p I_p$.
(4) The theory tells us that there is another indecomposable injective object in Ab, besides the I_p. Try to identify it.

Remark 2.6.16. Let $\mathbb{K}$ be a field and $A := \mathbb{K}[t]$, the polynomial ring in one variable. As we very well know, the categories $\mathsf{Mod}\, A$ and $\mathsf{Mod}\, \mathbb{Z}$ share many properties. Let $A^* := \operatorname{Hom}_{\mathbb{K}}(A, \mathbb{K})$, which is of course an injective A-module (because $\mathbb{K}$ is an injective $\mathbb{K}$-module). The structure of A^*, as a direct sum of indecomposable injectives, was used to cook up a counterexample in [Ye5, Section 6].

The abelian category $\mathsf{Mod}\, \mathcal{A}$ associated to a ringed space $(X, \mathcal{A})$ was introduced in Example 2.3.18.

Proposition 2.6.17. *Let $(X, \mathcal{A})$ be a ringed space. The category $\mathsf{Mod}\, \mathcal{A}$ has enough injectives.*

Proof. Let $\mathcal{M}$ be an $\mathcal{A}$-module. Take a point $x \in X$. The stalk $\mathcal{M}_x$ is a module over the ring $\mathcal{A}_x$, and by Theorem 2.6.10 we can find an embedding $\phi_x : \mathcal{M}_x \to I_x$ into an injective $\mathcal{A}_x$-module. Let $g_x : \{x\} \to X$ be the inclusion, which we may view as a map of ringed spaces from $(\{x\}, \mathcal{A}_x)$ to $(X, \mathcal{A})$. Define $\mathcal{I}_x := g_{x*}(I_x)$, which is an $\mathcal{A}$-module (in fact it is a constant sheaf supported on the closed set $\overline{\{x\}} \subseteq X$).

The adjunction formula gives rise to a sheaf homomorphism $\psi_x : \mathcal{M} \to \mathcal{I}_x$. Since the functor $g_x^* : \mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}_x$ is exact, the adjunction formula shows that $\mathcal{I}_x$ is an injective object of $\mathsf{Mod}\,\mathcal{A}$.

Finally let $\mathcal{J} := \prod_{x \in X} \mathcal{I}_x$. This is an injective $\mathcal{A}$-module. There is a homomorphism $\psi := \prod_{x \in X} \psi_x : \mathcal{M} \to \mathcal{J}$ in $\mathsf{Mod}\,\mathcal{A}$. This is a monomorphism, since for every point, letting $\mathcal{J}_x$ be the stalk of the sheaf $\mathcal{J}$, the composition $\mathcal{M}_x \xrightarrow{\psi_x} \mathcal{J}_x \xrightarrow{p_x} \mathcal{I}_x$ is the embedding $\phi_x : \mathcal{M}_x \to I_x$. $\square$

3. Differential Graded Algebra

In this section we fix a nonzero commutative base ring $\mathbb{K}$. (It seems more relaxing to have a base ring $\mathbb{K}$ around, rather than working with the universal base $\mathbb{K} = \mathbb{Z}$.) Throughout, "DG" stands for "differential graded".

3.1. DG $\mathbb{K}$-modules.

Definition 3.1.1. A *DG* $\mathbb{K}$*-module* is a graded $\mathbb{K}$-module $M = \bigoplus_{i \in \mathbb{Z}} M^i$, together with a $\mathbb{K}$-linear operator $\mathrm{d}_M : M \to M$ of degree 1, called the differential, satisfying $\mathrm{d}_M \circ \mathrm{d}_M = 0$.

When there is no danger of confusion, we may write d instead of d_M.

Definition 3.1.2. Let M and N be DG $\mathbb{K}$-modules. A *strict homomorphism of DG* $\mathbb{K}$*-modules* is a $\mathbb{K}$-linear homomorphism $\phi : M \to N$ that commutes with the differentials and respects the gradings. The resulting category is denoted by $\mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K}$.

Remark 3.1.3. The name "strict morphism of DG modules", and the corresponding notation $\mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K}$, are new. We introduced them to distinguish $\mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K}$ from the DG category $\mathsf{DGMod}\, \mathbb{K}$ that contains it; cf. Definitions 3.3.1 and 3.3.4.

Suppose M and N are DG $\mathbb{K}$-modules. For any integer i let

$$(M \otimes_{\mathbb{K}} N)^i := \bigoplus_{j \in \mathbb{Z}} (M^j \otimes_{\mathbb{K}} N^{i-j}).$$

Then

$$M \otimes_{\mathbb{K}} N = \bigoplus_{i \in \mathbb{Z}} (M \otimes_{\mathbb{K}} N)^i, \tag{3.1.4}$$

so it is a graded $\mathbb{K}$-module. We put on it the differential

$$\mathrm{d}(m \otimes n) := \mathrm{d}_M(m) \otimes n + (-1)^i \cdot m \otimes \mathrm{d}_N(n) \tag{3.1.5}$$

for $m \in M^i$ and $n \in N^j$. In this way $M \otimes_{\mathbb{K}} N$ becomes a DG $\mathbb{K}$-module.

For DG $\mathbb{K}$-modules M, N, we let $\mathrm{Hom}_{\mathbb{K}}(M, N)^i$ be the $\mathbb{K}$-module of degree i homomorphisms $\phi : M \to N$; namely

$$\mathrm{Hom}_{\mathbb{K}}(M, N)^i = \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{K}}(M^j, N^{j+i}).$$

We then define

$$\mathrm{Hom}_{\mathbb{K}}(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{K}}(M, N)^i. \tag{3.1.6}$$

This is also a DG $\mathbb{K}$-module. The differential is

$$\mathrm{d}(\phi) := \mathrm{d}_N \circ \phi - (-1)^i \cdot \phi \circ \mathrm{d}_M \tag{3.1.7}$$

for $\phi \in \mathrm{Hom}_{\mathbb{K}}(M, N)^i$. When we need to emphasize where d acts, we sometimes denote it by $\mathrm{d}_{\mathrm{Hom}}$ or $\mathrm{d}_{\mathrm{Hom}_{\mathbb{K}}(M,N)}$.

Let M be a DG $\mathbb{K}$-module. The module of degree i cocycles of M is

$$\mathrm{Z}^i(M) := \mathrm{Ker}(\mathrm{d}|_{M^i}) \subseteq M^i, \tag{3.1.8}$$

and the module of degree i coboundaries is

$$\mathrm{B}^i(M) := \mathrm{Im}(\mathrm{d}|_{M^{i-1}}) \subseteq M^i. \tag{3.1.9}$$

Since $\mathrm{d} \circ \mathrm{d} = 0$ we have $\mathrm{B}^i(N) \subseteq \mathrm{Z}^i(N)$. The i-th cohomology is

$$\mathrm{H}^i(M) := \mathrm{Z}^i(M) / \mathrm{B}^i(M). \tag{3.1.10}$$

These are all $\mathbb{K}$-modules, and in fact they are functors

$$\mathrm{Z}^i, \mathrm{B}^i, \mathrm{H}^i : \mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K} \to \mathsf{Mod}\, \mathbb{K}.$$

Let us record the following result, whose easy proof we leave out.

Proposition 3.1.11. *For DG $\mathbb{K}$-modules M and N, there is equality*

$$\mathrm{Hom}_{\mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K}}(M, N) = \mathrm{Z}^0\big(\mathrm{Hom}_{\mathbb{K}}(M, N)\big)$$

of these submodules of $\mathrm{Hom}_{\mathbb{K}}(M, N)$.

3.2. **DG Rings and Modules.**

Definition 3.2.1. A *DG ring* is a graded ring $A = \bigoplus_{i \in \mathbb{Z}} A^i$, together with an operator $\mathrm{d}_A : A \to A$ of degree 1 called the differential, satisfying the equation $\mathrm{d}_A \circ \mathrm{d}_A = 0$, and the graded Leibniz rule

$$\mathrm{d}_A(a \cdot b) = \mathrm{d}_A(a) \cdot b + (-1)^i \cdot a \cdot \mathrm{d}_A(b)$$

for all $a \in A^i$ and $b \in A^j$.

We sometimes write d instead of d_A.

Proposition 3.2.2. *Let A be a DG ring. The unit element 1_A of A is a 0-cocycle, namely $1_A \in \mathrm{Z}^0(A)$.*

Proof. Exercise. □

Definition 3.2.3. Let A and B be DG rings. A *homomorphism of DG rings* $f : A \to B$ is a ring homomorphism that commutes with the differentials and respects the gradings. The resulting category is denoted by DGRing.

As always for ring homomorphisms, f must preserve units, i.e. $f(1_A) = 1_B$.

Rings are viewed as DG rings concentrated in degree 0. Thus the category of rings Ring is a full subcategory of DGRing.

An element $a \in A^0$ is called *central* if $a \cdot a' = a' \cdot a$ for every $a' \in A$.

Definition 3.2.4. We say that A is a *central DG $\mathbb{K}$-ring* if there is a given DG ring homomorphism $\mathbb{K} \to A$, whose image is central in A.

We denote by $\mathsf{DGRing} /_{\mathrm{ce}}\, \mathbb{K}$ the category of central DG $\mathbb{K}$-rings, in which the morphisms $f : A \to B$ are the homomorphisms in DGRing that respect the given structural homomorphisms from $\mathbb{K}$.

Of course when $\mathbb{K} = \mathbb{Z}$ we have $\mathsf{DGRing} /_{\mathrm{ce}}\, \mathbb{K} = \mathsf{DGRing}$.

Here are few examples of DG rings. First a silly example.

Example 3.2.5. Let A be a central graded $\mathbb{K}$-ring. Then A is a central DG $\mathbb{K}$-ring, with trivial differential.

Example 3.2.6. Let X be a differentiable (i.e. of type C^∞) real manifold. The de Rham complex A of X is a central DG $\mathbb{R}$-ring, with the wedge product and the exterior differential.

The next example is the algebraic analogue of the previous one.

Example 3.2.7. Let C be a commutative $\mathbb{K}$-ring. Then the algebraic de Rham complex $A := \Omega_{C/\mathbb{K}} = \bigoplus_{p \geq 0} \Omega^p_{C/\mathbb{K}}$ is a central DG $\mathbb{K}$-ring.

Example 3.2.8. Let M be a DG $\mathbb{K}$-module. Consider the DG $\mathbb{K}$-module

$$\operatorname{End}_{\mathbb{K}}(M) := \operatorname{Hom}_{\mathbb{K}}(M, M)$$

from (3.1.6). Composition of endomorphisms is an associative multiplication on $\operatorname{End}_{\mathbb{K}}(M)$ that respects the grading, and the graded Leibniz rule holds. We see that $\operatorname{End}_{\mathbb{K}}(M)$ is a central DG $\mathbb{K}$-ring.

Example 3.2.9. Let C be a commutative ring and let $c \in C$ be an element. The *Koszul complex* of c is the DG C-module $\mathrm{K}(C; c)$ defined as follows. In degree 0 we let $\mathrm{K}^0(C; c) := C$. In degree -1, $\mathrm{K}^{-1}(C; c)$ is a free C-module of rank 1, with basis element x. All other homogeneous components are trivial. The differential d is determined by what it does to the basis element $x \in \mathrm{K}^{-1}(C; c)$, and we let $\mathrm{d}(x) := c \in \mathrm{K}^0(C; c)$.

To make $\mathrm{K}(C; c)$ into a DG ring, we treat x as an odd variable (in the sense of strongly commutative DG rings – see Remark 3.2.13). This dictates $x^2 = 0$. (Of course this is also dictated by the fact that $\mathrm{K}^{-2}(C; c) = 0$.) It is easy to verify that $\mathrm{K}(C; c)$ is a central DG C-ring.

Example 3.2.10. Let A and B be central DG $\mathbb{K}$-rings. The DG $\mathbb{K}$-module $A \otimes_{\mathbb{K}} B$ from 3.1.4 has a graded ring structure, with formula

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := (-1)^{k_2 \cdot l_1} \cdot (a_1 \cdot a_2) \otimes (b_1 \cdot b_2)$$

for $a_i \in A^{k_i}$ and $b_i \in B^{l_i}$. It is easy to verify that $A \otimes_{\mathbb{K}} B$ is a central DG $\mathbb{K}$-ring.

Example 3.2.11. Let C be a commutative ring and let $\boldsymbol{c} = (c_1, \ldots, c_n)$ be a sequence of elements in C. By combining Examples 3.2.9 and 3.2.10 we obtain the Koszul complex

$$\mathrm{K}(C; \boldsymbol{c}) := \mathrm{K}(C; c_1) \otimes_C \cdots \otimes_C \mathrm{K}(C; c_n).$$

This is a central DG C-ring.

Definition 3.2.12. Let A be a central DG $\mathbb{K}$-ring. The *opposite DG ring* A^{op} is the same DG $\mathbb{K}$-module as A, but the multiplication $\cdot^{\mathrm{op}}$ is reversed and twisted by signs:

$$a \cdot^{\mathrm{op}} b := (-1)^{ij} \cdot b \cdot a$$

for $a \in A^i$ and $b \in A^j$.

Remark 3.2.13. There are two variants of commutativity for DG rings. Following [Ye9], we say that a DG ring A is *weakly commutative* if $A = A^{\mathrm{op}}$. A is called *strongly commutative* if it is weakly commutative, and also $a^2 = 0$ for every homogeneous element a of odd degree.

Definition 3.2.14. Let A be a central DG $\mathbb{K}$-ring. A *left DG A-module* is a graded left A-module $M = \bigoplus_{i \in \mathbb{Z}} M^i$, with an operator $\mathrm{d}_M : M \to M$ of degree 1 called the differential, satisfying $\mathrm{d}_M \circ \mathrm{d}_M = 0$ and

$$\mathrm{d}_M(a \cdot m) = \mathrm{d}_A(a) \cdot m + (-1)^i \cdot a \cdot \mathrm{d}_M(m)$$

for $a \in A^i$ and $m \in M^j$.

Right DG A-modules are defined likewise, but we won't deal with them much. This is because right DG A-modules are left DG modules over the opposite DG ring A^{op}. More precisely, if M is a right DG A-module, then the formula

$$a \cdot m := (-1)^{ij} \cdot m \cdot a, \tag{3.2.15}$$

for $a \in A^i$ and $m \in M^j$, makes M in to a left DG A^{op}-module.

So we make this convention for the rest of the course (analogous to Convention 1.2.1):

Convention 3.2.16. By default, DG modules are *left DG modules*. In particular, a module over a ring is by default a left module.

Proposition 3.2.17. *Let A be a central DG $\mathbb{K}$-ring, and let M be a DG $\mathbb{K}$-module.*

(1) *Suppose $f : A \to \mathrm{End}_{\mathbb{K}}(M)$ is a DG $\mathbb{K}$-ring homomorphism. Then the formula $a \cdot m := f(a)(m)$, for $a \in A^i$ and $m \in M^j$, makes M into a DG A-module.*
(2) *Conversely, any DG A-module structure on M, that's compatible with the DG $\mathbb{K}$-module structure, arises in this way from a DG $\mathbb{K}$-ring homomorphism $f : A \to \mathrm{End}_{\mathbb{K}}(M)$.*

Proof. Exercise. □

Definition 3.2.18. Let M and N be DG A-modules. A *strict homomorphism of DG A-modules* is a $\mathbb{K}$-linear homomorphism $\phi : M \to N$ that commutes with the differentials, and respects the gradings and the action of A. The resulting category is denoted by $\mathsf{DGMod}_{\mathrm{str}}\, A$.

Exercise 3.2.19. Let A be a DG ring. Show that the cocycles $\mathrm{Z}(A) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Z}^i(A)$ are a graded subring of A, and the coboundaries $\mathrm{B}(A) := \bigoplus_{i \in \mathbb{Z}} \mathrm{B}^i(A)$ are a two-sided ideal of $\mathrm{Z}(A)$. Thus the cohomology $\mathrm{H}(A) := \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^i(A)$ is a graded ring. (Compare to Definition 3.3.4.)

Next show that given a DG A-module M, its cohomology $\mathrm{H}(M)$ is a graded $\mathrm{H}(A)$-module.

Lemma 3.2.20. *Let A be a central graded $\mathbb{K}$-ring, let M be a right graded A-module, and let N be a left graded A-module. Then*

$$M \otimes_A N = \bigoplus_{i \in \mathbb{Z}} (M \otimes_A N)^i,$$

where $(M \otimes_A N)^i$ is the $\mathbb{K}$-linear span of the tensors $m \otimes n$ with $m \in M^j$, $n \in N^k$ and $j + k = i$.

Proof. There is a canonical surjection of $\mathbb{K}$-modules

$$M \otimes_{\mathbb{K}} N \to M \otimes_A N.$$

Its kernel is the $\mathbb{K}$-submodule $L \subseteq M \otimes_{\mathbb{K}} N$ generated by the elements

$$(m \cdot a) \otimes n - m \otimes (a \cdot n),$$

for $m \in M^j$, $n \in N^k$ and $a \in A^l$. Since L is a graded submodule of $M \otimes_{\mathbb{K}} N$, so is the quotient. Finally we see that the i-th homogeneous component of $M \otimes_A N$ is precisely $(M \otimes_A N)^i$. □

Definition 3.2.21. Let A be a central DG $\mathbb{K}$-ring, let $M \in \mathsf{DGMod}\, A^{\mathrm{op}}$, and let $N \in \mathsf{DGMod}\, A$. By Lemma 3.2.20, $M \otimes_A N$ is a graded $\mathbb{K}$-module. We make it into a DG $\mathbb{K}$-module with the differential from formula (3.1.5).

Definition 3.2.22. Let A be a central DG $\mathbb{K}$-ring, and let $M, N \in \mathsf{DGMod}\, A$. For any $i \in \mathbb{Z}$, define $\mathrm{Hom}_A(M,N)^i$ to be the subset of $\mathrm{Hom}_{\mathbb{K}}(M,N)^i$ consisting of the homomorphisms $\phi : M \to N$ such that

$$\phi(a \cdot m) = (-1)^{ik} \cdot a \cdot \phi(m) \tag{3.2.23}$$

for all $a \in A^k$. Next let

$$\mathrm{Hom}_A(M,N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_A(M,N)^i.$$

This is a graded $\mathbb{K}$-module, and we endow it with the differential (3.1.7).

Remark 3.2.24. The reader might wonder why a sign occurs in formula (3.2.23). The reason is that we view $a \in A^k$ and $\phi \in \mathrm{Hom}_{\mathbb{K}}(M,N)^i$ as graded operators (on $M \oplus N$) that *commute in the graded sense*: $\phi \circ a = (-1)^{ik} \cdot a \circ \phi$. Note that $\mathrm{Hom}_A(M,N)$ is a subobject, in $\mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K}$, of $\mathrm{Hom}_{\mathbb{K}}(M,N)$.

Generalizing Proposition 3.1.11, for DG A-modules M and N there is equality

$$\mathrm{Hom}_{\mathsf{DGMod}_{\mathrm{str}}\, A}(M,N) = \mathrm{Z}^0\big(\mathrm{Hom}_A(M,N)\big).$$

3.3. **DG Categories.** Suppose C is a $\mathbb{K}$-linear category (Definition 2.1.1). Since the composition of morphisms is $\mathbb{K}$-bilinear, for any triple of objects $M_0, M_1, M_2 \in \mathsf{C}$, composition can be expressed as a $\mathbb{K}$-linear homomorphism

$$\begin{aligned} &\mathrm{Hom}_{\mathsf{C}}(M_1, M_2) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathsf{C}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{C}}(M_0, M_2) \\ &\phi_2 \otimes \phi_1 \mapsto \phi_2 \circ \phi_1. \end{aligned}$$

We refer to it as the composition homomorphism. It will be used in the following definition.

Definition 3.3.1. A $\mathbb{K}$-*linear DG category* is a $\mathbb{K}$-linear category C, endowed with a DG $\mathbb{K}$-module structure on each of the morphism $\mathbb{K}$-modules $\mathrm{Hom}_{\mathsf{C}}(M_0, M_1)$. The conditions are these:

(a) For any object M, the identity morphism 1_M is a degree 0 cocycle in $\mathrm{Hom}_{\mathsf{C}}(M, M)$.
(b) For any triple of objects $M_0, M_1, M_2 \in \mathsf{C}$, the composition homomorphism

$$\mathrm{Hom}_{\mathsf{C}}(M_1, M_2) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathsf{C}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{C}}(M_0, M_2)$$

is a strict homomorphism of DG $\mathbb{K}$-modules.

If $\mathbb{K} = \mathbb{Z}$, we say that C is a *DG category*.

In the condition (b) of the definition we use formula (3.1.4) for the DG module structure on a tensor product of DG $\mathbb{K}$-modules. It is possible that condition (a) is redundant (cf. Proposition 3.2.2 above).

Definition 3.3.2. Let C be a $\mathbb{K}$-linear DG category.

(1) A morphism $\phi : M \to N$ in C is called a *degree i morphism* if $\phi \in \mathrm{Hom}_{\mathsf{C}}(M,N)^i$.
(2) A morphism $\phi : M \to N$ in C is called a *cocycle* if $\mathrm{d}(\phi) = 0$.
(3) A morphism $\phi : M \to N$ in C is called a *strict morphism* if it is a degree 0 cocycle in $\mathrm{Hom}_{\mathsf{C}}(M,N)$.

Lemma 3.3.3. *Let C be a $\mathbb{K}$-linear DG category, and for $i = 1, 2, 3$ let $\phi_i : M_{i-1} \to M_i$ be a morphism in C of degree k_i.*

(1) *The morphism $\phi_2 \circ \phi_1$ has degree $k_1 + k_2$, and*
$$\mathrm{d}(\phi_2 \circ \phi_1) = \mathrm{d}(\phi_2) \circ \phi_1 + (-1)^{k_2} \cdot \phi_2 \circ \mathrm{d}(\phi_1).$$
(2) *If ϕ_1 and ϕ_2 are cocycles, then so is $\phi_2 \circ \phi_1$.*
(3) *If ϕ_2 is a coboundary, and ϕ_1 and ϕ_3 are cocycles, then $\phi_3 \circ \phi_2 \circ \phi_1$ is a coboundary.*

Proof. (1) This is just a rephrasing of item (b) in Definition 3.3.1.

(2) This is immediate from (1).

(3) Say $\phi_2 = \mathrm{d}(\psi_2)$ for some degree $k_2 - 1$ morphism $\psi_2 : M_1 \to M_2$. Then
$$\phi_3 \circ \phi_2 \circ \phi_1 = \mathrm{d}(\phi_3 \circ \psi_2 \circ \phi_1).$$
□

The lemma makes the next definition possible.

Definition 3.3.4. Let C be a $\mathbb{K}$-linear DG category.

(1) The *strict category* of C is the category $\mathrm{Str}(\mathsf{C})$, with the same objects as C, but with strict morphisms only. Thus
$$\mathrm{Hom}_{\mathsf{C}_{\mathrm{str}}}(M, N) = \mathrm{Z}^0\big(\mathrm{Hom}_{\mathsf{C}}(M, N)\big).$$
(2) The *homotopy category* of C is the category $\mathrm{Ho}(\mathsf{C})$, with the same objects as C, and with morphism sets
$$\mathrm{Hom}_{\mathrm{Ho}(\mathsf{C})}(M, N) := \mathrm{H}^0\big(\mathrm{Hom}_{\mathsf{C}}(M, N)\big).$$
(3) We denote by
$$\mathrm{P} : \mathrm{Str}(\mathsf{C}) \to \mathrm{Ho}(\mathsf{C})$$
the functor which is the identity on objects, and sends a strict morphism to its homotopy class.

The categories $\mathrm{Str}(\mathsf{C})$ and $\mathrm{Ho}(\mathsf{C})$ are $\mathbb{K}$-linear. The inclusion $\mathrm{Str}(\mathsf{C}) \to \mathsf{C}$ and $\mathrm{P} : \mathrm{Str}(\mathsf{C}) \to \mathrm{Ho}(\mathsf{C})$ are $\mathbb{K}$-linear functors. The first is faithful (injective on morphisms), and the second is full (surjective on morphisms).

Example 3.3.5. If A is a $\mathbb{K}$-linear DG category, then for every object $x \in \mathsf{A}$, its set of endomorphisms $A := \mathrm{End}_{\mathsf{A}}(x)$ is a $\mathbb{K}$-central DG ring. Conversely, any $\mathbb{K}$-central DG ring A can be viewed as a $\mathbb{K}$-linear DG category with a single object.

Example 3.3.6. Let A be a $\mathbb{K}$-central DG ring. The set of DG A-modules forms a $\mathbb{K}$-linear DG category $\mathsf{DGMod}\, A$, in which the morphism DG modules are
$$\mathrm{Hom}_{\mathsf{DGMod}\, A}(M, N) := \mathrm{Hom}_A(M, N)$$
from Definition 3.2.22. The strict category here is
$$\mathrm{Str}(\mathsf{DGMod}\, A) = \mathsf{DGMod}_{\mathrm{str}}\, A.$$

Remark 3.3.7. The fact that the concept of “DG category” includes both DG rings (Example 3.3.5) and DG modules over them (Example 3.3.6) is a source of confusion. See Remarks 3.3.9 and 3.6.7.

Here is the categorical version of Definition 3.2.12.

Definition 3.3.8. Let C be a $\mathbb{K}$-linear DG category. The *opposite DG category* C^{op} has the same set of objects. The morphism DG modules are

$$\operatorname{Hom}_{\mathsf{C}^{\mathrm{op}}}(M, N) := \operatorname{Hom}_{\mathsf{C}}(N, M).$$

The composition is reversed with signs:

$$\phi \circ^{\mathrm{op}} \psi := (-1)^{ij} \cdot \psi \circ \phi$$

for composable homogeneous morphisms ϕ and ψ in C of degrees i and j respectively.

Remark 3.3.9. There is a vast theory on DG categories. See the relatively old references [Ke], [BK]; for more modern accounts try to search the internet (there are numerous accounts, of many flavors). In this course we shall be exclusively concerned with the categories $\mathbf{C}(A, \mathsf{M})$, to be introduced in Subsection 3.6, that have a lot more structure than other DG categories. See Remark 3.6.7 regarding the DG category $\mathbf{C}(\mathsf{A}) = \mathbf{C}(\mathsf{A}, \mathsf{Mod}\,\mathbb{K})$ of left DG modules over a $\mathbb{K}$-linear DG category A, in the sense of [Ke].

Here is a useful result.

Proposition 3.3.10. *Let $\phi : M \to N$ be a degree i isomorphism in the $\mathbb{K}$-linear DG category C. Assume ϕ is a cocycle, namely $\mathrm{d}(\phi) = 0$. Then its inverse $\phi^{-1} : N \to M$ is also a cocycle.*

Proof. According the Leibniz rule (Lemma 3.3.3(1)), and the fact that 1_M is a cocycle, we have

$$0 = \mathrm{d}(1_M) = \mathrm{d}(\phi^{-1} \circ \phi) = \mathrm{d}(\phi^{-1}) \circ \phi - (-1)^{-i} \cdot \phi^{-1} \circ \mathrm{d}(\phi) = \mathrm{d}(\phi^{-1}) \circ \phi.$$

Because ϕ is an isomorphism, we conclude that $\mathrm{d}(\phi^{-1}) = 0$. $\square$

3.4. **DG Functors.** Here C and D are $\mathbb{K}$-linear DG categories (see Definition 3.3.1).

Definition 3.4.1. A functor $F : \mathsf{C} \to \mathsf{D}$ is called a $\mathbb{K}$-*linear graded functor* if it satisfies this condition:

- For any pair of objects $M_0, M_1 \in \mathsf{C}$, the function
$$F : \operatorname{Hom}_{\mathsf{C}}(M_0, M_1) \to \operatorname{Hom}_{\mathsf{D}}\big(F(M_0), F(M_1)\big)$$
is a degree 0 homomorphism of graded $\mathbb{K}$-modules.

Recall that "morphism of functors" is synonymous with "natural transformation".

Definition 3.4.2. Let $F, G : \mathsf{C} \to \mathsf{D}$ be $\mathbb{K}$-linear graded functors, and let $i \in \mathbb{Z}$. A *degree i morphism of graded functors* $\eta : F \to G$ is a collection $\eta = \{\eta_M\}_{M \in \mathsf{C}}$ of morphisms

$$\eta_M \in \operatorname{Hom}_{\mathsf{D}}\big(F(M), G(M)\big)^i,$$

such that for any morphism $\phi \in \operatorname{Hom}_{\mathsf{C}}(M_0, M_1)^j$, there is equality

$$G(\phi) \circ \eta_{M_0} = (-1)^{ij} \cdot \eta_{M_1} \circ F(\phi)$$

inside

$$\operatorname{Hom}_{\mathsf{D}}\big(F(M_0), G(M_1)\big)^{i+j}.$$

The differential of the DG $\mathbb{K}$-module $\operatorname{Hom}_{\mathsf{C}}(M_0, M_1)$, for objects $M_0, M_1 \in \mathsf{C}$, will be denoted by d_{C}. Likewise for objects of D.

Recall the meaning of a strict homomorphism of DG $\mathbb{K}$-modules: it has degree 0 and commutes with the differentials.

Definition 3.4.3. A functor $F : \mathsf{C} \to \mathsf{D}$ is called a *$\mathbb{K}$-linear DG functor* if it satisfies this condition:

- For any pair of objects $M_0, M_1 \in \mathsf{C}$, the function
$$F : \operatorname{Hom}_{\mathsf{C}}(M_0, M_1) \to \operatorname{Hom}_{\mathsf{D}}\big(F(M_0), F(M_1)\big)$$
is a strict homomorphism of DG $\mathbb{K}$-modules.

In other words, F is a DG functor if it is a graded functor, and

$$\mathrm{d}_{\mathsf{D}} \circ F = F \circ \mathrm{d}_{\mathsf{C}} \tag{3.4.4}$$

as degree 1 homomorphisms

$$\operatorname{Hom}_{\mathsf{C}}(M_0, M_1) \to \operatorname{Hom}_{\mathsf{D}}\big(F(M_0), F(M_1)\big).$$

Example 3.4.5. Let $f : A \to B$ be a homomorphism of $\mathbb{K}$-central DG rings. Define the DG categories C and D as follows: $\mathrm{Ob}(\mathsf{C}) := \{x\}$, $\operatorname{End}_{\mathsf{C}}(x) := A$, $\mathrm{Ob}(\mathsf{D}) := \{y\}$ and $\operatorname{End}_{\mathsf{D}}(y) := B$. Then f becomes a $\mathbb{K}$-linear DG functor $F : \mathsf{C} \to \mathsf{D}$.

Other examples of DG functors, more relevant to our study, will be given in Subsection 4.4.

Definition 3.4.6. Let $F, G : \mathsf{C} \to \mathsf{D}$ be $\mathbb{K}$-linear DG functors.

(1) A *degree i morphism of DG functors* $\eta : F \to G$ is a degree i morphism of graded functors, as in Definition 3.4.1.

(2) Let $\eta : F \to G$ be a degree i morphism of DG functors. For any object $M \in \mathsf{C}$ there is a degree $i + 1$ morphism
$$\mathrm{d}_{\mathsf{D}}(\eta_M) : F(M) \to G(M)$$
in D. We let
$$\mathrm{d}_{\mathsf{D}}(\eta) := \big\{\mathrm{d}_{\mathsf{D}}(\eta_M)\big\}_{M \in \mathsf{C}}.$$

(3) A *strict morphism of DG functors* is a degree 0 morphism of graded functors $\eta : F \to G$ such that $\mathrm{d}_{\mathsf{D}}(\eta) = 0$.

Proposition 3.4.7. *The collection of morphisms $\mathrm{d}_{\mathsf{D}}(\eta)$ defined above is a degree $i + 1$ morphism of DG functors $F \to G$.*

Proof. Exercise. □

The categories $\mathrm{Str}(\mathsf{C})$ and $\mathrm{Ho}(\mathsf{C})$ were introduced in Definition 3.3.4.

Proposition 3.4.8. *Let $F : \mathsf{C} \to \mathsf{D}$ be a $\mathbb{K}$-linear DG functor. Then F induces $\mathbb{K}$-linear functors*
$$\mathrm{Str}(F) : \mathrm{Str}(\mathsf{C}) \to \mathrm{Str}(\mathsf{D})$$
and
$$\mathrm{Ho}(F) : \mathrm{Ho}(\mathsf{C}) \to \mathrm{Ho}(\mathsf{D}).$$

Proof. Because F is a DG functor, it sends 0-cocycles in $\operatorname{Hom}_{\mathsf{C}}(M_0, M_1)$ to 0-cocycles in $\operatorname{Hom}_{\mathsf{D}}(F(M_0), F(M_1))$. The same for 0-coboundaries. □

By abuse of notation, when there is no danger for confusion, we will sometimes write F instead of $\mathrm{Str}(F)$ or $\mathrm{Ho}(F)$.

Exercise 3.4.9. Let A be a small $\mathbb{K}$-linear DG category. Define $\mathsf{DGFun}(\mathsf{A}, \mathsf{C})$ to be the set of $\mathbb{K}$-linear DG functors $F : \mathsf{A} \to \mathsf{C}$. Show that $\mathsf{DGFun}(\mathsf{A}, \mathsf{C})$ is a $\mathbb{K}$-linear DG category, where the morphisms are from Definition 3.4.6(1), and their differentials are from Definition 3.4.6(2).

3.5. **Complexes in Abelian Categories.** Here we recall facts about complexes from the classical homological theory, and place them within our context. In this subsection M is a $\mathbb{K}$-linear abelian category.

Remark 3.5.1. Actually, almost everything we do here makes sense when M is just a $\mathbb{K}$-linear category (not necessarily abelian). However, requiring it to be abelian eliminates the confusion between "ring-like" and "module-like" categories (only the latter are abelian). Cf. Remark 3.3.7.

A *complex* of objects of M, or a complex in M, is a diagram

$$\big(\cdots \to M^{-1} \xrightarrow{\mathrm{d}_M^{-1}} M^0 \xrightarrow{\mathrm{d}_M^{0}} M^1 \xrightarrow{\mathrm{d}_M^{1}} M^2 \to \cdots\big) \tag{3.5.2}$$

of objects and morphisms in M, such that $\mathrm{d}_M^{i+1} \circ \mathrm{d}_M^i = 0$. The collection of objects $M := \{M^i\}_{i\in\mathbb{Z}}$ is called a *graded object of* M. The collection of morphisms $\mathrm{d}_M := \{\mathrm{d}_M^i\}_{i\in\mathbb{Z}}$ is called a *differential*, or a *coboundary operator*. Thus a complex is a pair (M, d_M) made up of a graded object M and a differential d_M on it. We sometimes write d instead of d_M or d_M^i. At other times we leave the differential implicit, and just refer to the complex as M.

Let N be another complex in M. A *strict morphism of complexes* $\phi : M \to N$ is a collection $\phi = \{\phi^i\}_{i\in\mathbb{Z}}$ of morphisms $\phi^i : M^i \to N^i$ in M, such that

$$\mathrm{d}_N^i \circ \phi^i = \phi^{i+1} \circ \mathrm{d}_M^i. \tag{3.5.3}$$

Note that the strict morphism $\phi : M \to N$ can be viewed as a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & M^i \arrow[r, "\mathrm{d}_M^i"] \arrow[d, "\phi^i"'] & M^{i+1} \arrow[r] \arrow[d, "\phi^{i+1}"'] & \cdots \\
\cdots \arrow[r] & N^i \arrow[r, "\mathrm{d}_N^i"] & N^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Of course the identity morphism 1_M is a strict morphism.

Remark 3.5.4. In most textbooks, what we call "strict morphism of complexes" is simply called a "morphism of complexes". See Remark 3.1.3 for an explanation.

Let us denote by $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$ the category of complexes in M, with strict morphisms. This is a $\mathbb{K}$-linear abelian category. Indeed, a direct sum of complexes are degree-wise, i.e. $(M \oplus N)^i = M^i \oplus N^i$. The same for kernels and cokernels. If N is a full abelian subcategory of M, then $\mathbf{C}_{\mathrm{str}}(\mathsf{N})$ is a full subcategory of $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$.

Any single object $M \in \mathsf{M}$ can be viewed as a complex

$$M' := \big(\cdots \to 0 \to M \to 0 \to \cdots\big),$$

where M is in degree 0; the differential of this complex is of course zero. The assignment $M \mapsto M'$ is a fully faithful $\mathbb{K}$-linear functor $\mathsf{M} \to \mathbf{C}_{\mathrm{str}}(\mathsf{M})$.

Let M, N be complexes in M. For any integer i we define

$$\mathrm{Hom}_{\mathsf{M}}(M,N)^i := \prod_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{M}}(M^j, N^{j+i}) \in \mathsf{Mod}\,\mathbb{K}.$$

The graded $\mathbb{K}$-module

$$\mathrm{Hom}_{\mathsf{M}}(M,N) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathsf{M}}(M,N)^i \tag{3.5.5}$$

is a DG $\mathbb{K}$-module with differential

$$\mathrm{d} : \mathrm{Hom}_{\mathsf{M}}(M, N)^i \to \mathrm{Hom}_{\mathsf{M}}(M, N)^{i+1}$$

given by

$$\mathrm{d}(\phi) := \mathrm{d}_N \circ \phi - (-1)^i \cdot \phi \circ \mathrm{d}_M . \tag{3.5.6}$$

It is easy to check that $\mathrm{d} \circ \mathrm{d} = 0$. We sometimes denote this differential by $\mathrm{d}_{\mathrm{Hom}}$ or $\mathrm{d}_{\mathrm{Hom}_{\mathsf{M}}(M,N)}$.

Thus, an element $\phi \in \mathrm{Hom}_{\mathsf{M}}(M, N)^i$ is a collection $\phi = \{\phi^j\}_{j \in \mathbb{Z}}$ of morphisms $\phi^j : M^j \to N^{j+i}$. In a diagram, for $i = 2$, it looks like this:

$$\begin{tikzcd}
\cdots \arrow[r] & M^j \arrow[r, "\mathrm{d}"] \arrow[drr, "\phi^j"] & M^{j+1} \arrow[r, "\mathrm{d}"] \arrow[drr, "\phi^{j+1}"] & M^{j+2} \arrow[r, "\mathrm{d}"] & M^{j+3} \arrow[r] & \cdots \\
\cdots \arrow[r] & N^j \arrow[r, "\mathrm{d}"] & N^{j+1} \arrow[r, "\mathrm{d}"] & N^{j+2} \arrow[r, "\mathrm{d}"] & N^{j+3} \arrow[r] & {}
\end{tikzcd}$$

Since ϕ does not have to commute with the differentials, this is usually not a commutative diagram!

For a triple of complexes M_0, M_1, M_2, there are $\mathbb{K}$-linear homomorphisms

$$\mathrm{Hom}_{\mathsf{M}}(M_1, M_2)^{i_2} \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathsf{M}}(M_0, M_1)^{i_1} \to \mathrm{Hom}_{\mathsf{M}}(M_0, M_2)^{i_1+i_2},$$

$\phi_2 \otimes \phi_1 \mapsto \phi_2 \circ \phi_1$.

Lemma 3.5.7. *The composition homomorphism*

$$\mathrm{Hom}_{\mathsf{M}}(M_1, M_2) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathsf{M}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{M}}(M_0, M_2)$$

is a strict homomorphism of DG $\mathbb{K}$-modules.

Proof. This is a good exercise. □

The lemma justifies the next definition.

Definition 3.5.8. Let $\mathbf{C}(\mathsf{M})$ be the $\mathbb{K}$-linear DG category whose objects are the complexes in M, and the morphism DG $\mathbb{K}$-modules are $\mathrm{Hom}_{\mathsf{M}}(M, N)$ from formulas (3.5.5) and (3.5.6).

It is clear, from comparing formulas (3.5.6) and (3.5.3), that the strict morphisms of complexes defined at the top of this subsection are the same as those from Definition 3.3.4(1). In other words, $\mathrm{Str}(\mathbf{C}(\mathsf{M})) = \mathbf{C}_{\mathrm{str}}(\mathsf{M})$.

Remark 3.5.9. A possible ambiguity could arise in the meaning of $\mathrm{Hom}_{\mathsf{M}}(M, N)$ if $M, N \in \mathsf{M}$: does it mean the $\mathbb{K}$-module of morphisms in the category M ? Or, if we view M and N as complexes by the canonical embedding $\mathsf{M} \subseteq \mathbf{C}(\mathsf{M})$, does $\mathrm{Hom}_{\mathsf{M}}(M, N)$ mean the complex of $\mathbb{K}$-modules defined for complexes? It turns out that there is no actual difficulty: since the complex of $\mathbb{K}$-modules $\mathrm{Hom}_{\mathsf{M}}(M, N)$ is concentrated in degree 0, we may view it as a single $\mathbb{K}$-module, and this is precisely the $\mathbb{K}$-module of morphisms in the category M.

When $\mathsf{M} = \mathsf{Mod}\, A$ for a ring A, there is no essential distinction between complexes and DG modules:

Proposition 3.5.10. *Let A be a central $\mathbb{K}$-ring. Given a complex $M \in \mathbf{C}(\mathsf{Mod}\, A)$, with notation as in* (3.5.2)*, define the DG A-module*

$$F(M) := \bigoplus_{i \in \mathbb{Z}} M^i,$$

with differential $\mathrm{d} := \sum_{i \in \mathbb{Z}} \mathrm{d}_M^i$. *Then the functor*

$$F : \mathbf{C}(\mathsf{Mod}\, A) \to \mathsf{DGMod}\, A$$

is a $\mathbb{K}$-linear equivalence.

The proof is an exercise. The only hard part in it is to choose good notation. In fact, the functor F is an equivalence of DG categories; but we shall not try to make this notion precise (cf. Definition 3.4.6).

3.6. **DG A-modules in M.** We now combine material from previous subsections. The concept introduced in the definition below is new.

Definition 3.6.1. Let M be a $\mathbb{K}$-linear abelian category, and let A be a central DG $\mathbb{K}$-ring. A *DG A-module in* M is an object $M \in \mathbf{C}(\mathsf{M})$, together with DG $\mathbb{K}$-ring homomorphism $f : A \to \mathrm{End}_{\mathsf{M}}(M)$. The set of DG A-modules in M is denoted by $\mathbf{C}(A, \mathsf{M})$.

What the definition says is that any element $a \in A^i$ gives rise to a degree i endomorphism $f(a)$ of the complex M. In turn, this means that for every j, $f(a) : M^j \to M^{j+i}$ is a morphism in M. The operation f satisfies $f(1_A) = 1_M$, $f(a_1 \cdot a_2) = f(a_1) \circ f(a_2)$, and $f(\mathrm{d}(a)) = \mathrm{d}(f(a))$.

Example 3.6.2. If $A = \mathbb{K}$, then $\mathbf{C}(A, \mathsf{M}) = \mathbf{C}(\mathsf{M})$; and if $\mathsf{M} = \mathsf{Mod}\, \mathbb{K}$, then $\mathbf{C}(A, \mathsf{M}) = \mathsf{DGMod}\, A$. Because of this, we sometimes write $\mathbf{C}(A) := \mathsf{DGMod}\, A$.

It will be convenient to have notation for partial structures related to $\mathbf{C}(A, \mathsf{M})$. Let M be a $\mathbb{K}$-linear category. A *graded object of* M, or a *graded module in* M, is a collection $M = \{M^i\}_{i \in \mathbb{Z}}$ of objects of M. (It is like a complex, but without a differential.) The set of graded modules in M is denoted by $\mathbf{G}(\mathsf{M})$.

Now let A be a central graded $\mathbb{K}$-ring. A *graded A-module in* M is a graded module M in M, together with a graded $\mathbb{K}$-ring homomorphism $f : A \to \mathrm{End}_{\mathsf{M}}(M)$. The set of graded modules in M is denoted by $\mathbf{G}(A, \mathsf{M})$. It is a *graded category*, and there are there is a faithful functor $\mathbf{C}(A, \mathsf{M}) \to \mathbf{G}(A, \mathsf{M})$ that forgets the differentials.

The next definition is a variant of Definition 3.2.22.

Definition 3.6.3. Let M be a $\mathbb{K}$-linear abelian category, and let A be a central DG $\mathbb{K}$-ring. For $M, N \in \mathbf{C}(A, \mathsf{M})$ and $i \in \mathbb{Z}$ we define $\mathrm{Hom}_{A,\mathsf{M}}(M, N)^i$ to be the subset of $\mathrm{Hom}_{\mathsf{M}}(M, N)^i$ consisting of the degree i morphism $\phi : M \to N$ such that

$$\phi \circ f_M(a) = (-1)^{ik} \cdot f_N(a) \circ \phi$$

for all $a \in A^k$.

Next let

$$\mathrm{Hom}_{A,\mathsf{M}}(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{A,\mathsf{M}}(M, N)^i.$$

This is a graded $\mathbb{K}$-module, and we endow it with the differential

$$\mathrm{d} : \mathrm{Hom}_{A,\mathsf{M}}(M, N)^i \to \mathrm{Hom}_{A,\mathsf{M}}(M, N)^{i+1},$$

$$\mathrm{d}(\phi) := \mathrm{d}_N \circ \phi - (-1)^i \cdot \phi \circ \mathrm{d}_M,$$

making it into a DG $\mathbb{K}$-module. When we have to be specific, we denote the differential d by $\mathrm{d}_{\mathrm{Hom}}$, $\mathrm{d}_{A,\mathsf{M}}$, or $\mathrm{d}_{\mathrm{Hom}_{A,\mathsf{M}}(M,N)}$.

As we have seen before (in Lemma 3.5.7, and Conversely in Lemma 3.3.3), given $\phi_k \in \mathrm{Hom}_{A,\mathsf{M}}(M_{k-1}, M_k)^{i_k}$ for $k = 1, 2$, we have

$$\phi_2 \circ \phi_1 \in \mathrm{Hom}_{A,\mathsf{M}}(M_0, M_2)^{i_1+i_2},$$

and

$$\mathrm{d}(\phi_2 \circ \phi_1) = \mathrm{d}(\phi_2) \circ \phi_1 + (-1)^{i_2} \cdot \phi_2 \circ \mathrm{d}(\phi_1).$$

Also the identity 1_M belongs to $\mathrm{Hom}_{A,\mathsf{M}}(M, M)^0$, and $\mathrm{d}(1_M) = 0$. Therefore the next definition is legitimate.

Definition 3.6.4. Let M be a $\mathbb{K}$-linear abelian category, and let A be a central DG $\mathbb{K}$-ring. We put a $\mathbb{K}$-linear DG category structure on the set of objects $\mathbf{C}(A, \mathsf{M})$, with morphisms DG modules

$$\mathrm{Hom}_{\mathbf{C}(A,\mathsf{M})}(M_0, M_1) := \mathrm{Hom}_{A,\mathsf{M}}(M_0, M_1).$$

The composition is that of $\mathbf{C}(\mathsf{M})$.

Notice that forgetting the action of A gives a faithful $\mathbb{K}$-linear DG functor $\mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(\mathsf{M})$.

Definition 3.6.5. In the situation of Definition 3.6.4:

(1) The strict category of $\mathbf{C}(A, \mathsf{M})$ (see Definition 3.3.4(2)) is denoted by $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$.
(2) The homotopy category of $\mathbf{C}(A, \mathsf{M})$ (see Definition 3.3.4(3)) is denoted by $\mathbf{K}(A, \mathsf{M})$.

The next proposition is merely an interpretation of the definitions; but it is worthy of mentioning.

Proposition 3.6.6. *Let $\phi : M \to N$ be a degree 0 morphism in $\mathbf{C}(A, \mathsf{M})$. The next two conditions are equivalent*:

(i) *ϕ is strict.*
(ii) *$\phi \circ \mathrm{d}_M = \mathrm{d}_N \circ \phi$.*

Remark 3.6.7. Here is a generalization of Definition 3.6.4. Instead of a central DG $\mathbb{K}$-ring A we take a small $\mathbb{K}$-linear DG category A. We define the $\mathbb{K}$-linear DG category

$$\mathbf{C}(\mathsf{A}, \mathsf{M}) := \mathsf{DGFun}(\mathsf{A}, \mathbf{C}(\mathsf{M}))$$

as in Exercise 3.4.9.

This is indeed a generalization of Definition 3.6.4: when A has a single object x, and we write $A := \mathrm{End}_{\mathsf{A}}(x)$, then the functor $M \mapsto M(x)$ is an isomorphism of DG categories $\mathbf{C}(\mathsf{A}, \mathsf{M}) \xrightarrow{\simeq} \mathbf{C}(A, \mathsf{M})$.

In the special case of $\mathsf{M} = \mathsf{Mod}\,\mathbb{K}$, the DG category $\mathbf{C}(\mathsf{A}, \mathsf{M})$ is what Keller [Ke] calls the DG category of *left DG* A*-modules.*

Practically everything we do in this course for $\mathbf{C}(A, \mathsf{M})$ holds in the more general context of $\mathbf{C}(\mathsf{A}, \mathsf{M})$.

Definition 3.6.8. Let M be a $\mathbb{K}$-linear abelian category, and let A be a central DG $\mathbb{K}$-ring. For any integer i let

$$\mathrm{H}^i : \mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \to \mathsf{M}$$

be the $\mathbb{K}$-linear functor, that sends a complex M to its i-th cohomology $\mathrm{H}^i(M) \in \mathsf{M}$ as in (3.1.10), and that sends a strict morphism $\phi : M_0 \to M_1$ to the morphism

$$\mathrm{H}^i(\phi) : \mathrm{H}^i(M_0) \to \mathrm{H}^i(M_1).$$

3.7. **The Translation Functor.** As before, we fix a $\mathbb{K}$-linear abelian category M, and a central DG $\mathbb{K}$-ring A.

The translation goes back to the beginnings of derived categories. The treatment in this subsection (with the operator t) is taken from [Ye9, Section 1].

Definition 3.7.1. Let $M = \{M^i\}_{i \in \mathbb{Z}}$ be a graded module in M, i.e. an object of $\mathbf{G}(\mathsf{M})$. The *translation* of M is the object

$$\mathrm{T}(M) = \{\mathrm{T}(M)^i\}_{i \in \mathbb{Z}} \in \mathbf{G}(\mathsf{M})$$

defined as follows: the graded component of degree i of $\mathrm{T}(M)$ is $\mathrm{T}(M)^i := M^{i+1}$.

Definition 3.7.2 (The little t operator)**.** Let $M = \{M^i\}_{i \in \mathbb{Z}}$ be a graded module in M, i.e. an object of $\mathbf{G}(\mathsf{M})$. We define

$$\mathrm{t}_M : M \to \mathrm{T}(M)$$

to be the degree -1 morphism of graded objects of M, that in every degree $i+1$ is identity morphism

$$\mathrm{t}_M := 1_{M^{i+1}} : M^{i+1} \xrightarrow{\simeq} M^{i+1} = \mathrm{T}(M)^i$$

of the object M^{i+1} in M.

Note that the morphism

$$\mathrm{t}_M \in \mathrm{Hom}_{\mathbf{G}(\mathsf{M})}\big(M, \mathrm{T}(M)\big)^{-1}$$

is invertible, with inverse

$$\mathrm{t}_M^{-1} \in \mathrm{Hom}_{\mathbf{G}(\mathsf{M})}\big(\mathrm{T}(M), M\big)^{1}.$$

Definition 3.7.3. Let $M = \{M^i\}_{i \in \mathbb{Z}}$ be a DG A-module in M, i.e. an object of $\mathbf{C}(A, \mathsf{M})$. The *translation* of M is the object

$$\mathrm{T}(M) \in \mathbf{C}(A, \mathsf{M})$$

defined as follows.

(1) As graded object of M, it is as specified in Definition 3.7.1.
(2) The differential $\mathrm{d}_{\mathrm{T}(M)}$ is defined by the formula

$$\mathrm{d}_{\mathrm{T}(M)} := -\,\mathrm{t}_M \circ \mathrm{d}_M \circ \mathrm{t}_M^{-1}.$$

(3) Let $f_M : A \to \mathrm{End}_{\mathsf{M}}(M)$ be the DG ring homomorphism that determines the action of A on M. Then

$$f_{\mathrm{T}(M)} : A \to \mathrm{End}_{\mathsf{M}}(\mathrm{T}(M))$$

is defined by

$$f_{\mathrm{T}(M)}(a) := (-1)^j \cdot \mathrm{t}_M \circ f_M(a) \circ \mathrm{t}_M^{-1}$$

for $a \in A^j$.

Thus, the differential $\mathrm{d}_{\mathrm{T}(M)} = \{\mathrm{d}^i_{\mathrm{T}(M)}\}_{i \in \mathbb{Z}}$ makes this diagram in M commutative for every i :

$$\begin{tikzcd} \mathrm{T}(M)^i \arrow[r, "\mathrm{d}^i_{\mathrm{T}(M)}"] & \mathrm{T}(M)^{i+1} \\ M^{i+1} \arrow[u, "\mathrm{t}_M"] \arrow[r, "-\mathrm{d}^{i+1}_M"] & M^{i+2} \arrow[u, "\mathrm{t}_M"'] \end{tikzcd}$$

And the left A-module structure makes this diagram in M commutative for every i and every $a \in A^j$:

$$\begin{tikzcd} \mathrm{T}(M)^i \arrow[r, "f_{\mathrm{T}(M)}(a)"] & \mathrm{T}(M)^{i+j} \\ M^{i+1} \arrow[u, "\mathrm{t}_M"] \arrow[r, "(-1)^j \cdot f_M(a)"] & M^{i+j+1} \arrow[u, "\mathrm{t}_M"'] \end{tikzcd}$$

Warning: t_M is not a morphism in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$, because it has degree -1.

Proposition 3.7.4. *The morphisms t_M and t_M^{-1} are cocycles, in the DG $\mathbb{K}$-modules $\mathrm{Hom}_{A,\mathsf{M}}\big(M, \mathrm{T}(M)\big)$ and $\mathrm{Hom}_{A,\mathsf{M}}\big(\mathrm{T}(M), M\big)$ respectively.*

Proof. We use the notation $\mathrm{d}_{\mathrm{Hom}}$ for the differential in the DG module $\mathrm{Hom}_{A,\mathsf{M}}\big(M, \mathrm{T}(M)\big)$. Let us calculate. Because t_M has degree -1, we have

$$\begin{aligned} \mathrm{d}_{\mathrm{Hom}}(\mathrm{t}_M) &= \mathrm{d}_{\mathrm{T}(M)} \circ \mathrm{t}_M + \mathrm{t}_M \circ \mathrm{d}_M \\ &= (-\,\mathrm{t}_M \circ \mathrm{d}_M \circ \mathrm{t}_M^{-1}) \circ \mathrm{t}_M + \mathrm{t}_M \circ \mathrm{d}_M = 0. \end{aligned}$$

As for t_M^{-1} : this is done using the graded Leibniz rule, just like in the proof Proposition 3.3.10. □

Definition 3.7.5. Given a morphism

$$\phi \in \mathrm{Hom}_{A,\mathsf{M}}(M, N)^i,$$

we define the morphism

$$\mathrm{T}(\phi) \in \mathrm{Hom}_{A,\mathsf{M}}\big(\mathrm{T}(M), \mathrm{T}(N)\big)^i$$

to be

$$\mathrm{T}(\phi) := (-1)^i \cdot \mathrm{t}_N \circ \phi \circ \mathrm{t}_M^{-1} .$$

To clarify this definition, let us write $\phi = \{\phi^j\}_{j \in \mathbb{Z}}$, so that $\phi^j : M^j \to N^{j+i}$. Then

$$\mathrm{T}(\phi)^j : \mathrm{T}(M)^j \to \mathrm{T}(N)^{j+i}$$

is

$$\mathrm{T}(\phi)^j = (-1)^i \cdot \mathrm{t}_N \circ \phi^{j+1} \circ \mathrm{t}_M^{-1} .$$

The corresponding commutative diagram in M, for each i, j, is:

(3.7.6)
$$\begin{tikzcd} \mathrm{T}(M)^j \arrow[r, "\mathrm{T}(\phi)^j"] & \mathrm{T}(N)^{j+i} \\ M^{j+1} \arrow[u, "\mathrm{t}_M"] \arrow[r, "(-1)^i \cdot \phi^{j+1}"] & N^{j+i} \arrow[u, "\mathrm{t}_N"'] \end{tikzcd}$$

Theorem 3.7.7.

(1) *The assignments* $M \mapsto \mathrm{T}(M)$ *and* $\phi \mapsto \mathrm{T}(\phi)$ *are a* $\mathbb{K}$*-linear DG functor*
$$\mathrm{T} : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(A, \mathsf{M}).$$
(2) *The collection* $\mathrm{t} = \{\mathrm{t}_M\}_{M \in \mathbf{C}(A,\mathsf{M})}$ *is a degree* -1 *isomorphism*
$$\mathrm{t} : \mathrm{id} \to \mathrm{T}$$
of DG functors from $\mathbf{C}(A, \mathsf{M})$ *to itself.*

(3) *For any* $M \in \mathbf{C}(A, \mathsf{M})$ *there is equality*
$$\mathrm{T}(\mathrm{t}_M) = -\,\mathrm{t}_{\mathrm{T}(M)}$$
in $\mathrm{Hom}_{A,\mathsf{M}}\big(\mathrm{T}(M), \mathrm{T}^2(M)\big)^1$.

Proof. (1) Take morphisms $\phi_1 : M_0 \to M_1$ and $\phi_2 : M_1 \to M_2$, of degrees i_1 and i_2 respectively. Then
$$\begin{aligned}
\mathrm{T}(\phi_2 \circ \phi_1) &= (-1)^{i_1+i_2} \cdot \mathrm{t}_{M_2} \circ (\phi_2 \circ \phi_1) \circ \mathrm{t}_{M_0}^{-1} \\
&= (-1)^{i_1+i_2} \cdot \mathrm{t}_{M_2} \circ \phi_2 \circ (\mathrm{t}_{M_1}^{-1} \circ \mathrm{t}_{M_1}) \circ\ \phi_1 \circ \mathrm{t}_{M_0}^{-1} \\
&= \big((-1)^{i_2} \cdot \mathrm{t}_{M_2} \circ \phi_2 \circ \mathrm{t}_{M_1}^{-1}\big) \circ \big((-1)^{i_1} \cdot \mathrm{t}_{M_1} \circ \phi_1 \circ \mathrm{t}_{M_0}^{-1}\big) \\
&= \mathrm{T}(\phi_2) \circ \mathrm{T}(\phi_1).
\end{aligned}$$
Clearly $\mathrm{T}(1_M) = 1_M$, and
$$\mathrm{T}(\lambda \cdot \phi + \psi) = \lambda \cdot \mathrm{T}(\phi) + \mathrm{T}(\psi)$$
for all $\lambda \in \mathbb{K}$ and $\phi, \psi \in \mathrm{Hom}_{A,\mathsf{M}}(M_0, M_1)^i$. So T is a $\mathbb{K}$-linear graded functor.

By Proposition 3.7.4 we know that $\mathrm{d} \circ \mathrm{t} = -\,\mathrm{t} \circ \mathrm{d}$ and $\mathrm{d} \circ \mathrm{t}^{-1} = -\,\mathrm{t}^{-1} \circ \mathrm{d}$, This implies that for any morphism ϕ in $\mathbf{C}(A, \mathsf{M})$, we have $\mathrm{T}(\mathrm{d}(\phi)) = \mathrm{d}(\mathrm{T}(\phi))$. So T is a DG functor.

(2) Take any $\phi \in \mathrm{Hom}_{A,\mathsf{M}}(M_0, M_1)^i$. We have to prove that
$$\mathrm{t}_{M_1} \circ \phi = (-1)^i \cdot \mathrm{T}(\phi) \circ \mathrm{t}_{M_0}$$
as elements of $\mathrm{Hom}_{A,\mathsf{M}}(M_0, \mathrm{T}(M_1))^{i+1}$. But by Definition 3.7.5 we have
$$\mathrm{T}(\phi) \circ \mathrm{t}_{M_0} = \big((-1)^i \cdot \mathrm{t}_{M_1} \circ \phi \circ \mathrm{t}_{M_0}^{-1}\big) \circ \mathrm{t}_{M_0} = (-1)^i \cdot \mathrm{t}_{M_1} \circ \phi.$$

(3) This is an easy calculation:
$$\mathrm{T}(\mathrm{t}_M) = -\,\mathrm{t}_{\mathrm{T}(M)} \circ \mathrm{t}_M \circ \mathrm{t}_M^{-1} = -\,\mathrm{t}_{\mathrm{T}(M)}\,.$$
□

Definition 3.7.8. We call T the *translation functor* of the DG category $\mathbf{C}(A, \mathsf{M})$.

Note that T is actually an automorphism of the category $\mathbf{C}(A, \mathsf{M})$. Thus we have:

Corollary 3.7.9. *For any integer k, the k-th power of the translation is a DG functor*
$$\mathrm{T}^k : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(A, \mathsf{M}).$$

Remark 3.7.10. There are several names in the literature for the translation functor T : *twist*, *shift* and *suspension*. There are also several notations: $\mathrm{T}(M) = M[1] = \Sigma M$.

3.8. **The Cone of a Strict Morphism.** As before, we fix a $\mathbb{K}$-linear abelian category M, and a central DG $\mathbb{K}$-ring A. Here is the cone construction in $\mathbf{C}(A, \mathsf{M})$, as it looks using the operator t.

Definition 3.8.1. Let $\phi : M \to N$ be a strict morphism in $\mathbf{C}(A, \mathsf{M})$. The *cone of* ϕ is the object $\operatorname{Cone}(\phi) \in \mathbf{C}(A, \mathsf{M})$ defined as follows. As a graded A-module in M we let

$$\operatorname{Cone}(\phi) := N \oplus \mathrm{T}(M).$$

The differential $\mathrm{d}_{\mathrm{Cone}}$ is this: if we express the graded module as a column

$$\operatorname{Cone}(\phi) = \begin{bmatrix} N \\ \mathrm{T}(M) \end{bmatrix},$$

then $\mathrm{d}_{\mathrm{Cone}}$ is left multiplication by the matrix

$$\mathrm{d}_{\mathrm{Cone}} := \begin{bmatrix} \mathrm{d}_N & \phi \circ \mathrm{t}_M^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}(M)} \end{bmatrix}$$

of degree 1 morphisms of graded A-module in M.

In other words,

$$\mathrm{d}^i_{\mathrm{Cone}} : \operatorname{Cone}(\phi)^i \to \operatorname{Cone}(\phi)^{i+1}$$

is

$$\mathrm{d}^i_{\mathrm{Cone}} = \mathrm{d}^i_N + \mathrm{d}^i_{\mathrm{T}(M)} + \phi^{i+1} \circ \mathrm{t}_M^{-1},$$

where $\phi^{i+1} \circ \mathrm{t}_M^{-1}$ is the composed morphism

$$\mathrm{T}(M)^i \xrightarrow{\mathrm{t}_M^{-1}} M^{i+1} \xrightarrow{\phi^{i+i}} N^{i+1}.$$

Let us denote by

$$e_\phi : N \to N \oplus \mathrm{T}(M) \tag{3.8.2}$$

the embedding, and by

$$p_\phi : N \oplus \mathrm{T}(M) \to \mathrm{T}(M) \tag{3.8.3}$$

the projection. Thus, as matrices we have

$$e_\phi = \begin{bmatrix} 1_N \\ 0 \end{bmatrix} \quad \text{and} \quad p_\phi = \begin{bmatrix} 0 & 1_{\mathrm{T}(M)} \end{bmatrix}.$$

Definition 3.8.4. Let $\phi : M \to N$ be a morphism in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. The diagram

$$M \xrightarrow{\phi} N \xrightarrow{e_\phi} \operatorname{Cone}(\phi) \xrightarrow{p_\phi} \mathrm{T}(M)$$

in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ is called the *standard triangle* associated to ϕ.

The cone construction is functorial, in the following sense.

Proposition 3.8.5. *Let*

$$\begin{tikzcd} M_0 \arrow[r, "\phi_0"] \arrow[d, "\psi"'] & N_0 \arrow[d, "\chi"] \\ M_1 \arrow[r, "\phi_1"] & N_1 \end{tikzcd}$$

be a commutative diagram in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. *Then*

$$(\chi, \mathrm{T}(\psi)) : \operatorname{Cone}(\phi_0) \to \operatorname{Cone}(\phi_1) \tag{3.8.6}$$

is a morphism in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$*, and the diagram*

$$\begin{tikzcd} M_0 \arrow[r, "\phi_0"] \arrow[d, "\psi"'] & N_0 \arrow[r, "e_{\phi_0}"] \arrow[d, "\chi"'] & \mathrm{Cone}(\phi_0) \arrow[r, "p_{\phi_0}"] \arrow[d, "(\chi, \mathrm{T}(\psi))"'] & \mathrm{T}(M_0) \arrow[d, "\mathrm{T}(\psi)"'] \\ M_1 \arrow[r, "\phi_1"] & N_1 \arrow[r, "e_{\phi_1}"] & \mathrm{Cone}(\phi_1) \arrow[r, "p_{\phi_1}"] & \mathrm{T}(M_1) \end{tikzcd}$$

in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$ *is commutative.*

Proof. This is a simple consequence of the definitions. □

4. Properties of DG Functors

In this section we fix central DG $\mathbb{K}$-rings A and B, and $\mathbb{K}$-linear abelian categories M and N. The $\mathbb{K}$-linear DG categories $\mathbf{C}(A, \mathsf{M})$ and $\mathbf{C}(B, \mathsf{N})$ were introduced in Subsection 3.6. Graded functors, DG functors, and morphisms between them, were introduced in Subsection 3.4.

Some of the material in this section is new – such as the gauge in Definition 4.1.1, and its role in the the characterization of DG functors in Theorem 4.1.2. We think that Theorem 4.3.6, which says that a DG functor commutes with cones, is also a new result.

4.1. **The Gauge of a Graded Functor.** The next definition is new.

Definition 4.1.1. Let

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$$

be a $\mathbb{K}$-linear graded functor. For any object $M \in \mathbf{C}(A, \mathsf{M})$ let

$$\gamma_{F,M} := \mathrm{d}_{F(M)} - F(\mathrm{d}_M) \in \mathrm{Hom}_{B,\mathsf{N}}\big(F(M), F(M)\big)^1.$$

The collection of morphisms

$$\gamma_F := \{\gamma_{F,M}\}_{M \in \mathbf{C}(A,\mathsf{M})}$$

is called the *gauge of F*.

Theorem 4.1.2. [1] *The following two conditions are equivalent for a $\mathbb{K}$-linear graded functor*

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N}).$$

(i) *F is a DG functor.*
(ii) *The gauge γ_F is a degree 1 morphism of graded functors $\gamma_F : F \to F$.*

Proof. Recall that F is a DG functor (condition (i)) iff

$$(F \circ \mathrm{d}_{A,\mathsf{M}})(\phi) = (\mathrm{d}_{B,\mathsf{N}} \circ F)(\phi) \tag{4.1.3}$$

for every $\phi \in \mathrm{Hom}_{A,\mathsf{M}}(M_0, M_1)^i$. And γ_F is a degree 1 morphism of graded functors (condition (ii)) iff

$$\gamma_{F,M_1} \circ F(\phi) = (-1)^i \cdot F(\phi) \circ \gamma_{F,M_0} \tag{4.1.4}$$

for every such ϕ.

Here is the calculation. Because F is a graded functor, we get

$$\begin{aligned} F\big(\mathrm{d}_{A,\mathsf{M}}(\phi)\big) &= F\big(\mathrm{d}_{M_1} \circ \phi - (-1)^i \cdot \phi \circ \mathrm{d}_{M_0}\big) \\ &= F(\mathrm{d}_{M_1}) \circ F(\phi) - (-1)^i \cdot F(\phi) \circ F(\mathrm{d}_{M_0}) \end{aligned} \tag{4.1.5}$$

and

$$\mathrm{d}_{B,\mathsf{N}}\big(F(\phi)\big) = \mathrm{d}_{F(M_1)} \circ F(\phi) - (-1)^i \cdot F(\phi) \circ \mathrm{d}_{F(M_0)}. \tag{4.1.6}$$

Using equations (4.1.5) and (4.1.6), and the definition of γ_F, we obtain

$$\begin{aligned} &(F \circ \mathrm{d}_{A,\mathsf{M}} - \mathrm{d}_{B,\mathsf{N}} \circ F)(\phi) = F\big(\mathrm{d}_{A,\mathsf{M}}(\phi)\big) - \mathrm{d}_{B,\mathsf{N}}\big(F(\phi)\big) \\ &\quad = \big(F(\mathrm{d}_{M_1}) - \mathrm{d}_{F(M_1)}\big) \circ F(\phi) - (-1)^i \cdot F(\phi) \circ \big(F(\mathrm{d}_{M_0}) - \mathrm{d}_{F(M_0)}\big) \\ &\quad = \gamma_{F,M_1} \circ F(\phi) - (-1)^i \cdot F(\phi) \circ \gamma_{F,M_0}. \end{aligned} \tag{4.1.7}$$

[1] This theorem is due to Rishi Vyas.

Finally, the vanishing of the first expression in (4.1.7) is the same as equality in (4.1.3); whereas the vanishing of the last expression in (4.1.7) is the same as equality in (4.1.4). □

4.2. **The Translation Isomorphism of a DG Functor.** The translation functor of $\mathbf{C}(A, \mathsf{M})$ will be denoted here by $\mathrm{T}_{A,\mathsf{M}}$. Recall that for an object $M \in \mathbf{C}(A, \mathsf{M})$, we have the little t operator

$$\mathrm{t}_M \in \mathrm{Hom}_{A,\mathsf{M}}\big(M, \mathrm{T}_{A,\mathsf{M}}(M)\big)^{-1}.$$

This is an isomorphism in $\mathbf{C}(A, \mathsf{M})$. Likewise for the DG category $\mathbf{C}(B, \mathsf{N})$.

Definition 4.2.1. Let

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$$

be a $\mathbb{K}$-linear DG functor. For an object $M \in \mathbf{C}(A, \mathsf{M})$, let

$$\zeta_{F,M} : F(\mathrm{T}_{A,\mathsf{M}}(M)) \to \mathrm{T}_{B,\mathsf{N}}(F(M))$$

be the isomorphism

$$\zeta_{F,M} := \mathrm{t}_{F(M)} \circ F(\mathrm{t}_M)^{-1}$$

in $\mathbf{C}(B, \mathsf{N})$, called the *translation isomorphism* of the functor F at the object M.

The isomorphism $\zeta_{F,M}$ sits in the following commutative diagram

$$\begin{tikzcd}
F(\mathrm{T}_{A,\mathsf{M}}(M)) \arrow[r, "\zeta_{F,M}"] & \mathrm{T}_{B,\mathsf{N}}(F(M)) \\
F(M) \arrow[u, "F(\mathrm{t}_M)"] \arrow[ur, "\mathrm{t}_{F(M)}"'] &
\end{tikzcd}$$

of isomorphisms in the category $\mathbf{C}(B, \mathsf{N})$.

Proposition 4.2.2. *$\zeta_{F,M}$ is an isomorphism in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$.*

Proof. We know that $\zeta_{F,M}$ is an isomorphism in $\mathbf{C}(B, \mathsf{N})$. It suffices to prove that both $\zeta_{F,M}$ and its inverse $\zeta_{F,M}^{-1}$ are strict morphisms. Now by Proposition 3.7.4, t_M and t_M^{-1} are cocycles. Therefore, $F(\mathrm{t}_M)$ and $F(\mathrm{t}_M)^{-1} = F(\mathrm{t}_M^{-1})$ are cocycles. For the same reason, $\mathrm{t}_{F(M)}$ and $\mathrm{t}_{F(M)}^{-1}$ are cocycles. But $\zeta_{F,M} = \mathrm{t}_{F(M)} \circ F(\mathrm{t}_M)^{-1}$, and $\zeta_{F,M}^{-1} = F(\mathrm{t}_M) \circ \mathrm{t}_{F(M)}^{-1}$. □

Theorem 4.2.3. *Let*

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$$

be a $\mathbb{K}$-linear DG functor. Then the collection $\zeta_F := \{\zeta_{F,M}\}_{M \in \mathbf{C}(A,\mathsf{M})}$ is an isomorphism

$$\zeta_F : F \circ \mathrm{T}_{A,\mathsf{M}} \xrightarrow{\simeq} \mathrm{T}_{B,\mathsf{N}} \circ F$$

of functors

$$\mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \to \mathbf{C}_{\mathrm{str}}(B, \mathsf{N}).$$

The slogan summarizing this theorem is “A DG functor commutes with translations”.

Proof. In view of Proposition 4.2.2, all we need to prove is that $\zeta_{F,M}$ is a morphism of functors (i.e. it is a natural transformation).

Let $\phi : M_0 \to M_1$ be a morphism in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. We must prove that the diagram

$$\begin{tikzcd}
(F \circ \mathrm{T}_{A,\mathsf{M}})(M_0) \arrow[r, "\zeta_{F,M_0}"] \arrow[d, "(F\circ\mathrm{T}_{A,\mathsf{M}})(\phi)"'] & (\mathrm{T}_{B,\mathsf{N}} \circ F)(M_0) \arrow[d, "(\mathrm{T}_{B,\mathsf{N}} \circ F)(\phi)"] \\
(F \circ \mathrm{T}_{A,\mathsf{M}})(M_1) \arrow[r, "\zeta_{F,M_1}"] & (\mathrm{T}_{B,\mathsf{N}} \circ F)(M_1)
\end{tikzcd}$$

in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$ is commutative. This will be true if the next diagram

$$\begin{tikzcd}
(F \circ \mathrm{T}_{A,\mathsf{M}})(M_0) \arrow[d, "(F\circ\mathrm{T}_{A,\mathsf{M}})(\phi)"'] & F(M_0) \arrow[l, "F(\mathrm{t}_{M_0})"'] \arrow[r, "\mathrm{t}_{F(M_0)}"] \arrow[d, "F(\phi)"'] & (\mathrm{T}_{B,\mathsf{N}} \circ F)(M_0) \arrow[d, "(\mathrm{T}_{B,\mathsf{N}} \circ F)(\phi)"] \\
(F \circ \mathrm{T}_{A,\mathsf{M}})(M_1) & F(M_1) \arrow[l, "F(\mathrm{t}_{M_1})"'] \arrow[r, "\mathrm{t}_{F(M_1)}"] & (\mathrm{T}_{B,\mathsf{N}} \circ F)(M_1)
\end{tikzcd}$$

in $\mathbf{C}(B, \mathsf{N})$, whose horizontal arrows are isomorphisms, is commutative. For this to be true, it is enough to prove that both squares in this diagram are commutative. This is true by Theorem 3.7.7(2) □

Recall that the translation T and all its powers are DG functors. To finish this subsection, we calculate their translation isomorphisms.

Proposition 4.2.4. *For any integer k, the translation isomorphism of the DG functor T^k is*

$$\zeta_{\mathrm{T}^k} = (-1)^k \cdot 1_{\mathrm{T}^{k+1}},$$

where $1_{\mathrm{T}^{k+1}}$ is the identity automorphism of the functor T^{k+1}.

Proof. For $k = 1$ the formula is, by Definition 4.2.1 and item (3),

$$\zeta_{\mathrm{T},M} = \mathrm{t}_{\mathrm{T}(M)} \circ \mathrm{T}(\mathrm{t}_M)^{-1} = -1_{\mathrm{T}^2(M)},$$

the identity automorphism of the DG module $\mathrm{T}^2(M)$. Hence $\zeta_{\mathrm{T}} = -1_{\mathrm{T}^2}$. For other integers k the calculation is similar. □

4.3. **Cones and DG Functors.**

Definition 4.3.1. The subcategory $\mathbf{C}^0(A, \mathsf{M})$ of $\mathbf{C}(A, \mathsf{M})$ is defined to be the subcategory on all objects, but with degree 0 morphisms only.

There are inclusions of categories (faithful functors, identities on objects)

$$\mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \xrightarrow{\subseteq} \mathbf{C}^0(A, \mathsf{M}) \xrightarrow{\subseteq} \mathbf{C}(A, \mathsf{M}).$$

Let

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$$

be a $\mathbb{K}$-linear DG functor. Given a morphism $\phi : M_0 \to M_1$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$, we have a morphism

$$F(\phi) : F(M_0) \to F(M_1)$$

in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$, and objects $F(\mathrm{Cone}_{A,\mathsf{M}}(\phi))$ and $\mathrm{Cone}_{B,\mathsf{N}}(F(\phi))$ in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$. By definition

$$\mathrm{Cone}_{A,\mathsf{M}}(\phi) = M_1 \oplus \mathrm{T}_{A,\mathsf{M}}(M_0) \tag{4.3.2}$$

in $\mathbf{C}^0(A, \mathsf{M})$. Since F is an additive functor, it commutes with finite direct sums, and therefore there is a canonical isomorphism

$$F(\mathrm{Cone}_{A,\mathsf{M}}(\phi)) \cong F(M_1) \oplus F(\mathrm{T}_{A,\mathsf{M}}(M_0)) \tag{4.3.3}$$

in $\mathbf{C}^0(B, \mathsf{N})$. And by definition,

$$\mathrm{Cone}_{B,\mathsf{N}}(F(\phi)) = F(M_1) \oplus \mathrm{T}_{B,\mathsf{N}}(F(M_0)) \tag{4.3.4}$$

in $\mathbf{C}^0(B, \mathsf{N})$. Warning: the isomorphisms (4.3.2), (4.3.3) and (4.3.4) are usually not strict! Namely (see Proposition 3.6.6) they might not commute with the differentials. The differentials on the right sides are diagonal matrices, but not so on the left sides (see Definition 3.8.1).

Lemma 4.3.5. *Let*

$$F, G : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$$

be $\mathbb{K}$-linear graded functors, and let $\eta : F \to G$ be a degree j morphism of graded functors. Suppose $M \cong M_0 \oplus M_1$ in $\mathbf{C}^0(A, \mathsf{M})$, with embeddings $e_i : M_i \to M$ and projections $p_i : M \to M_i$. Then

$$\eta_M = \big(G(e_0), G(e_1)\big) \circ (\eta_{M_0}, \eta_{M_1}) \circ \big(F(p_0), F(p_1)\big),$$

as degree j morphisms $F(M) \to G(M)$ in $\mathbf{C}(B, \mathsf{N})$.

The lemma says that the diagram

$$\begin{tikzcd} F(M) \arrow[r, "\big(F(p_0), F(p_1)\big)"] \arrow[d, "\eta_M"'] & F(M_0) \oplus F(M_1) \arrow[d, "(\eta_{M_0}, \eta_{M_1})"] \\ G(M) & G(M_0) \oplus G(M_1) \arrow[l, "\big(G(e_0), G(e_1)\big)"'] \end{tikzcd}$$

in $\mathbf{C}(B, \mathsf{N})$ is commutative.

Proof. It suffice to prove that the diagram below is commutative for $i = 0, 1$:

$$\begin{tikzcd} F(M_i) \arrow[r, "F(e_i)"] \arrow[d, "\eta_{M_i}"'] \arrow[rr, bend left, "\mathrm{id}"] & F(M) \arrow[r, "F(p_i)"] \arrow[d, "\eta_M"'] & F(M_i) \arrow[d, "\eta_{M_i}"'] \\ G(M_i) \arrow[r, "G(e_i)"] \arrow[rr, bend right, "\mathrm{id}"'] & G(M) \arrow[r, "G(p_i)"] & G(M_i) \end{tikzcd}$$

This is true because η is a morphism of functors (a natural transformation). $\square$

Theorem 4.3.6. *Let*

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$$

be a $\mathbb{K}$-linear DG functor, and let $\phi : M_0 \to M_1$ be a morphism in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. Define the the isomorphism

$$\theta : F(\mathrm{Cone}_{A,\mathsf{M}}(\phi)) \to \mathrm{Cone}_{B,\mathsf{N}}(F(\phi))$$

in $\mathbf{C}^0(B, \mathsf{N})$ *to be*

$$\theta := (\mathrm{id}_{F(M_1)}, \zeta_{F,M_0}).$$

Then:

(1) *The isomorphism θ is strict; namely it commutes with the differentials.*
(2) *The diagram*

$$\begin{tikzcd}
F(M_0) \arrow[r, "F(\phi)"] \arrow[d, "="'] & F(M_1) \arrow[r, "F(e_\phi)"] \arrow[d, "="'] & F(\mathrm{Cone}_{A,\mathsf{M}}(\phi)) \arrow[r, "F(p_\phi)"] \arrow[d, "\theta"'] & F(\mathrm{T}_{A,\mathsf{M}}(M_0)) \arrow[d, "\zeta_{F,M_0}"'] \\
F(M_0) \arrow[r, "F(\phi)"] & F(M_1) \arrow[r, "e_{F(\phi)}"] & \mathrm{Cone}_{B,\mathsf{N}}(F(\phi)) \arrow[r, "p_{F(\phi)}"] & \mathrm{T}_{B,\mathsf{N}}(F(M_0))
\end{tikzcd}$$

in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$ *is commutative.*

When defining θ above, we are using the decompositions (4.3.3) and (4.3.4) in the category $\mathbf{C}^0(B, \mathsf{N})$, and the isomorphism ζ_{F,M_0} from Definition 4.2.1.

The slogan summarizing this theorem is "A DG functor sends standard triangles to standard triangles".

Proof. We have to prove that $\mathrm{d}_{B,\mathsf{N}}(\theta) = 0$. Let's write $P := \mathrm{Cone}_{A,\mathsf{M}}(\phi)$ and $Q := \mathrm{Cone}_{B,\mathsf{N}}(F(\phi))$. Recall that

$$\mathrm{d}_{B,\mathsf{N}}(\theta) = \mathrm{d}_Q \circ \theta - \theta \circ \mathrm{d}_{F(P)}.$$

We have to prove that this is the zero element in $\mathrm{Hom}_{B,\mathsf{N}}\big(F(P), Q\big)^1$.

Writing the cones as column modules:

$$P = \begin{bmatrix} M_1 \\ \mathrm{T}_{A,\mathsf{M}}(M_0) \end{bmatrix} \ , \ Q = \begin{bmatrix} F(M_1) \\ \mathrm{T}_{B,\mathsf{N}}(F(M_0)) \end{bmatrix} \ ,$$

the matrices representing the morphisms in question are

$$\theta = \begin{bmatrix} \mathrm{id}_{F(M_1)} & 0 \\ 0 & \zeta_{M_0} \end{bmatrix} \ , \ \mathrm{d}_P = \begin{bmatrix} \mathrm{d}_{M_1} & \phi \circ \mathrm{t}_{M_0}^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}_{A,\mathsf{M}}(M_0)} \end{bmatrix} \ , \ \mathrm{d}_Q = \begin{bmatrix} \mathrm{d}_{F(M_1)} & F(\phi) \circ \mathrm{t}_{F(M_0)}^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}_{B,\mathsf{N}}(F(M_0))} \end{bmatrix} .$$

Let us write $\gamma := \gamma_F$ for simplicity. According to Theorem 4.1.2, the gauge $\gamma : F \to F$ is a degree 1 morphism of functors $\mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$. Because the decomposition (4.3.2) is in the category $\mathbf{C}^0(A, \mathsf{M})$, Lemma 4.3.5 tells us that γ_P decomposes too, i.e.

$$\gamma_P = \begin{bmatrix} \gamma_{M_1} & 0 \\ 0 & \gamma_{\mathrm{T}_{A,\mathsf{M}}(M_0)} \end{bmatrix} .$$

By definition of γ_P we have

$$\mathrm{d}_{F(P)} = F(\mathrm{d}_P) + \gamma_P \in \mathrm{Hom}_{B,\mathsf{N}}\big(F(P), F(P)\big)^1.$$

It follows that

$$\begin{aligned}
\mathrm{d}_{F(P)} &= F(\mathrm{d}_P) + \gamma_P \\
&= \begin{bmatrix} F(\mathrm{d}_{M_1}) & F(\phi \circ \mathrm{t}_{M_0}^{-1}) \\ 0 & F(\mathrm{d}_{\mathrm{T}_{A,\mathsf{M}}(M_0)}) \end{bmatrix} + \begin{bmatrix} \gamma_{M_1} & 0 \\ 0 & \gamma_{\mathrm{T}_{A,\mathsf{M}}(M_0)} \end{bmatrix} \\
&= \begin{bmatrix} F(\mathrm{d}_{M_1}) + \gamma_{M_1} & F(\phi \circ \mathrm{t}_{M_0}^{-1}) \\ 0 & F(\mathrm{d}_{\mathrm{T}_{A,\mathsf{M}}(M_0)}) + \gamma_{\mathrm{T}_{A,\mathsf{M}}(M_0)} \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{d}_{F(M_1)} & F(\phi \circ \mathrm{t}_{M_0}^{-1}) \\ 0 & \mathrm{d}_{F(\mathrm{T}_{A,\mathsf{M}}(M_0))} \end{bmatrix}.
\end{aligned}$$

Finally we will check that $\theta \circ \mathrm{d}_{F(P)}$ and $\mathrm{d}_Q \circ \theta$ are equal as matrices of morphisms. We do that in each matrix position separately. The two left positions in the matrices $\theta \circ \mathrm{d}_{F(P)}$ and $\mathrm{d}_Q \circ \theta$ agree trivially. The bottom right positions in these matrices are $\zeta_{F,M_0} \circ \mathrm{d}_{F(\mathrm{T}_{A,\mathsf{M}}(M_0))}$ and $\mathrm{d}_{\mathrm{T}_{B,\mathsf{N}}(F(M_0))} \circ \zeta_{F,M_0}$ respectively; they are equal by Proposition 4.2.2. And in the top right positions we have $F(\phi \circ \mathrm{t}_{M_0}^{-1})$ and $F(\phi) \circ \mathrm{t}_{F(M_0)}^{-1} \circ \zeta_{F,M_0}$ respectively. Now $F(\phi \circ \mathrm{t}_{M_0}^{-1}) = F(\phi) \circ F(\mathrm{t}_{M_0}^{-1})$; so it suffices to prove that $F(\mathrm{t}_{M_0}^{-1}) = \mathrm{t}_{F(M_0)}^{-1} \circ \zeta_{F,M_0}$. This is immediate from the definition of ζ_{F,M_0}.

(2) By definition of θ, the diagram is commutative in $\mathbf{C}^0(B, \mathsf{N})$. But by part (1) we know that all morphisms in it lie in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$. □

4.4. **Examples of DG Functors.** Recall that M and N are $\mathbb{K}$-linear categories, and A and B are central DG $\mathbb{K}$-rings. Here are three examples of DG functors, of various types. These examples should serve as templates for constructing other DG functors.

Example 4.4.1. Here $A = B = \mathbb{K}$, so $\mathbf{C}(A, \mathsf{M}) = \mathbf{C}(\mathsf{M})$ and $\mathbf{C}(B, \mathsf{N}) = \mathbf{C}(\mathsf{N})$. Let $F : \mathsf{M} \to \mathsf{N}$ be a $\mathbb{K}$-linear functor. It extends to a functor

$$\mathbf{C}(F) : \mathbf{C}(\mathsf{M}) \to \mathbf{C}(\mathsf{N})$$

as follows: on objects, a complex

$$M = \big(\{M^i\}_{i \in \mathbb{Z}}, \{\mathrm{d}_M^i\}_{i \in \mathbb{Z}}\big) \in \mathbf{C}(\mathsf{M})$$

goes to the complex

$$\mathbf{C}(F)(M) := \big(\{F(M^i)\}, \{F(\mathrm{d}_M^i)\}\big) \in \mathbf{C}(\mathsf{N}).$$

A morphism $\phi = \{\phi^j\}$ in $\mathbf{C}(\mathsf{M})$ goes to the morphism $\mathbf{C}(\phi) := \{F(\phi^j)\}$ in $\mathbf{C}(\mathsf{N})$. A slightly tedious calculation shows that $\mathbf{C}(F)$ is a $\mathbb{K}$-linear DG functor.

Given a complex $M \in \mathbf{C}(\mathsf{M})$, let $N := \mathbf{C}(F)(M) \in \mathbf{C}(\mathsf{N})$. Then the translations are

$$\mathrm{T}_{\mathsf{N}}(N) = \mathbf{C}(F)\big(\mathrm{T}_{\mathsf{M}}(M)\big);$$

and $\mathbf{C}(F)(\mathrm{t}_M) = \mathrm{t}_N$. So the translation isomorphism

$$\zeta_{\mathbf{C}(F)} : \mathbf{C}(F) \circ \mathrm{T}_{\mathsf{M}} \xrightarrow{\simeq} \mathrm{T}_{\mathsf{N}} \circ \mathbf{C}(F)$$

of functors $\mathbf{C}_{\mathrm{str}}(\mathsf{M}) \to \mathbf{C}_{\mathrm{str}}(\mathsf{N})$ is equality.

Let $\phi : M_0 \to M_1$ be a morphism in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, whose image under $\mathbf{C}(F)$ is the morphism $\psi : N_0 \to N_1$ in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$. Then

$$\mathrm{Cone}(\psi) = N_1 \oplus \mathrm{T}_{\mathsf{N}}(N_0) = \mathbf{C}(F)\big(\mathrm{Cone}(\phi)\big)$$

as graded objects of N, with differential

$$\mathrm{d}_{\mathrm{Cone}(\psi)} = \begin{bmatrix} \mathrm{d}_{N_1} & \psi \circ \mathrm{t}_{N_0}^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}(N_0)} \end{bmatrix} = \mathbf{C}(F)\Big(\begin{bmatrix} \mathrm{d}_{M_1} & \phi \circ \mathrm{t}_{M_0}^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}(M_0)} \end{bmatrix} \Big) = \mathbf{C}(F)\big(\mathrm{d}_{\mathrm{Cone}(\phi)}\big).$$

We see that the cone isomorphism θ is equality, and the gauge $\gamma_{\mathbf{C}(F)}$ is zero.

The next example is much more complicated, and we work out the full details (only once – later on, such details will be left to the reader).

Example 4.4.2. Let A and B be central DG $\mathbb{K}$-rings, and fix some

$$N \in \mathsf{DGMod}(B \otimes_{\mathbb{K}} A^{\mathrm{op}}).$$

In other words, N is a DG B-A-bimodule. For any $M \in \mathsf{DGMod}\, A$ we have a DG $\mathbb{K}$-module

$$F(M) := N \otimes_A M,$$

as in Definition 3.2.21. The differential of $F(M)$ is

$$\mathrm{d}_{F(M)} = \mathrm{d}_N \otimes \mathrm{id}_M + \mathrm{id}_N \otimes \mathrm{d}_M. \tag{4.4.3}$$

But $F(M)$ has a structure of a DG B-module: for any $b \in B$, $n \in N$ and $m \in M$, the action is

$$b \cdot (n \otimes m) := (b \cdot n) \otimes m.$$

Clearly

$$F : \mathbf{C}(A) = \mathsf{DGMod}\, A \to \mathbf{C}(B) = \mathsf{DGMod}\, B$$

is a $\mathbb{K}$-linear functor. We will show that it is actually a DG functor.

Let $M_0, M_1 \in \mathbf{C}(A)$, and consider the $\mathbb{K}$-linear homomorphism

$$F : \mathrm{Hom}_A(M_0, M_1) \to \mathrm{Hom}_B(N \otimes_A M_0, N \otimes_A M_1). \tag{4.4.4}$$

Take any $\phi \in \mathrm{Hom}_A(M_0, M_1)^i$. Then

$$F(\phi) \in \mathrm{Hom}_B(N \otimes_A M_0, N \otimes_A M_1)$$

is the homomorphism that on a homogeneous tensor $n \otimes m \in (N \otimes_A M_0)^{k+j}$, with $n \in N^k$ and $m \in M_0^j$, has the value

$$F(\phi)(n \otimes m) = (-1)^{ik} \cdot n \otimes \phi(m) \in (N \otimes_A M_1)^{k+j+i}.$$

In other words,

$$F(\phi) = \mathrm{id}_N \otimes \phi. \tag{4.4.5}$$

We see that the homomorphism $F(\phi)$ has degree i. So F is a graded functor.

Let us figure out what is its translation isomorphism ζ_F. Take $M \in \mathbf{C}(A)$. Then

$$\mathrm{T}_B(F(M)) = \mathrm{T}_B(N \otimes_A M) = N \otimes_A \mathrm{T}_A(M) = F(\mathrm{T}_A(M))$$

as DG B-modules. The little t operators

$$\mathrm{t}_{F(M)}, F(\mathrm{t}_M) : F(M) \to \mathrm{T}_B(F(M)) = F(\mathrm{T}_A(M))$$

are

$$\mathrm{t}_{F(M)}(n \otimes m) = (-1)^k \cdot n \otimes \mathrm{t}_M(m) = F(\mathrm{t}_M)(n \otimes m)$$

for $n \in N^k$ and $m \in M^j$. We see that $\mathrm{t}_{F(M)} = F(\mathrm{t}_M)$. Therefore

$$\zeta_{F,M} : F(\mathrm{T}_A(M)) \xrightarrow{\simeq} \mathrm{T}_B(F(M))$$

is the identity automorphism.

Finally let us calculate γ_F, the gauge of F. From (4.4.5) and (4.4.3) we get

$$\gamma_{F,M} = \mathrm{d}_N \otimes \mathrm{id}_M,$$

which is often nonzero! Still, take any degree i morphism $\phi : M_0 \to M_1$ in $\mathbf{C}(A)$. Then

$$\begin{aligned}\gamma_{M_1} \circ F(\phi) &= (\mathrm{d}_N \otimes \mathrm{id}_M) \circ (\mathrm{id}_N \otimes \phi)\\ &= \mathrm{d}_N \otimes \phi = (-1)^i \cdot (\mathrm{id}_N \otimes \phi) \circ (\mathrm{d}_N \otimes \mathrm{id}_M) = (-1)^i \cdot F(\phi) \circ \gamma_{M_0}.\end{aligned}$$

We see that γ satisfies the condition of Definition 3.4.6(1), which is really Definition 3.4.2. By Theorem 4.1.2, F is a DG functor. (It is possible to calculate directly that F is a DG functor, but this takes more work.)

Example 4.4.6. Let A and B be central DG $\mathbb{K}$-rings, and fix some

$$N \in \mathsf{DGMod}(A \otimes_{\mathbb{K}} B^{\mathrm{op}}).$$

For any $M \in \mathsf{DGMod}\, A$ we define

$$F(M) := \mathrm{Hom}_A(N, M).$$

This is a DG B-module: for any $b \in B^i$ and $\phi \in \mathrm{Hom}_A(N, M)^j$, the homomorphism $b \cdot \phi \in \mathrm{Hom}_A(N, M)^{i+j}$ is

$$(b \cdot \phi)(n) := (-1)^{i \cdot (j+k)} \cdot \phi(n \cdot b) \in M$$

on $n \in N^k$. As in the previous example,

$$F : \mathbf{C}(A) = \mathsf{DGMod}\, A \to \mathbf{C}(B) = \mathsf{DGMod}\, B$$

is a $\mathbb{K}$-linear graded functor. The value of the gauge γ_F at $M \in \mathbf{C}(A)$ is

$$\gamma_{F,M} = \mathrm{Hom}(\mathrm{d}_N, \mathrm{id}_M).$$

Namely for

$$\psi \in F(M)^j = \mathrm{Hom}_A(N, M)^j$$

we have

$$\gamma_{F,M}(\psi) = (-1)^j \cdot \psi \circ \mathrm{d}_N.$$

It is not too hard to check that γ_F is a degree 1 morphism of functors. Hence, by Theorem 4.1.2, F is a DG functor.

5. Pretriangulated Categories and Triangulated Functors

In this section we introduce pretriangulated categories and triangulated functors, following [RD]. "Pretriangulated" means that the octahedron axiom is not required to hold. There is one result here that seems to be new: Theorem 5.4.11, which asserts that a DG functor between DG module categories induces a triangulated functor between the associated homotopy categories.

As in previous sections, we fix a base commutative ring $\mathbb{K}$. All linear categories and linear functors here are implicitly assumed to be $\mathbb{K}$-linear. In particular, this assumption says that all DG rings are central $\mathbb{K}$-rings, and all DG ring homomorphisms are $\mathbb{K}$-linear.

5.1. **T-Additive Categories.**

Definition 5.1.1. Let K be an additive category. A *translation* on K is an additive automorphism T of K, called the *translation functor*. The pair (K, T) is called a *T-additive category.*

Remark 5.1.2. Some texts give a more relaxed definition: T is only required to be an additive auto-equivalence of K. The resulting theory is somewhat more complicated.

Later we will write $M[k] := \mathrm{T}^k(M)$, the k-th translation of an object M.

Definition 5.1.3. Suppose $(\mathsf{K}, \mathrm{T}_{\mathsf{K}})$ and $(\mathsf{L}, \mathrm{T}_{\mathsf{L}})$ are T-additive categories. A *T-additive functor* between them is a pair (F, ξ), consisting of an additive functor $F : \mathsf{K} \to \mathsf{L}$, together with an isomorphism

$$\xi : F \circ \mathrm{T}_{\mathsf{K}} \xrightarrow{\simeq} \mathrm{T}_{\mathsf{L}} \circ F$$

of functors $\mathsf{K} \to \mathsf{L}$.

Definition 5.1.4. Suppose $(\mathsf{K}, \mathrm{T}_{\mathsf{K}})$ and $(\mathsf{L}, \mathrm{T}_{\mathsf{L}})$ are T-additive categories, and

$$(F, \xi), (G, \nu) : (\mathsf{K}, \mathrm{T}_{\mathsf{K}}) \to (\mathsf{L}, \mathrm{T}_{\mathsf{L}})$$

are T-additive functors. A *morphism of T-additive functors*

$$\eta : (F, \xi) \to (G, \nu)$$

is a morphism of functors $\eta : F \to G$, such that this diagram of morphisms of functors is commutative:

$$\begin{tikzcd} F \circ \mathrm{T}_{\mathsf{K}} \arrow[r, "\xi"] \arrow[d, "\eta \circ \mathrm{id}"'] & \mathrm{T}_{\mathsf{L}} \circ F \arrow[d, "\mathrm{id} \circ \eta"] \\ G \circ \mathrm{T}_{\mathsf{K}} \arrow[r, "\nu"] & \mathrm{T}_{\mathsf{L}} \circ G \ . \end{tikzcd}$$

Definition 5.1.5. Let (K, T) be an additive category with translation. A *triangle* in (K, T) is a diagram

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

in K.

Definition 5.1.6. Let (K, T) be a T-additive category. Suppose

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

and
$$L' \xrightarrow{\alpha'} M' \xrightarrow{\beta'} N' \xrightarrow{\gamma'} \mathrm{T}(L')$$
are triangles in (K, T). A *morphism of triangles* between them is a commutative diagram
$$\begin{tikzcd}
L \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[r, "\beta"] \arrow[d, "\psi"'] & N \arrow[r, "\gamma"] \arrow[d, "\chi"'] & \mathrm{T}(L) \arrow[d, "\mathrm{T}(\phi)"'] \\
L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L')
\end{tikzcd}$$
in K.

The morphism of triangles (ϕ, ψ, χ) is called an isomorphism if ϕ, ψ and χ are all isomorphisms.

Remark 5.1.7. Why "triangle"? This is because sometimes a triangle
$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$
is written as a diagram
$$\begin{tikzcd}
& N \arrow[dl, "\gamma"'] & \\
L \arrow[rr, "\alpha"] & & M \arrow[ul, "\beta"']
\end{tikzcd}$$
But here γ is a morphism "of degree 1".

5.2. **Pretriangulated Categories.**

Definition 5.2.1. A *pretriangulated category* is a T-additive category (K, T), equipped with a set of triangles called *distinguished triangles*. The following axioms have to be satisfied:

(TR1) (a) Any triangle that is isomorphic to a distinguished triangle is also a distinguished triangle.

(b) For every morphism $\alpha : L \to M$ in K there is a distinguished triangle
$$L \xrightarrow{\alpha} M \to N \to \mathrm{T}(L).$$

(c) For every object M the triangle
$$M \xrightarrow{1_M} M \to 0 \to \mathrm{T}(M)$$
is distinguished.

(TR2) A triangle
$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$
is distinguished iff the triangle
$$M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L) \xrightarrow{-\mathrm{T}(\alpha)} \mathrm{T}(M)$$
is distinguished.

(TR3) Suppose
$$\begin{tikzcd}
L \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[r, "\beta"] \arrow[d, "\psi"'] & N \arrow[r, "\gamma"] & \mathrm{T}(L) \\
L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L')
\end{tikzcd}$$

is a commutative diagram in K in which the rows are distinguished triangles. Then there exists a morphism $\chi : N \to N'$ such that the diagram

$$\begin{tikzcd} L \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[r, "\beta"] \arrow[d, "\psi"'] & N \arrow[r, "\gamma"] \arrow[d, "\chi"'] & \mathrm{T}(L) \arrow[d, "\mathrm{T}(\phi)"'] \\ L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L') \end{tikzcd} .$$

is a morphism of triangles.

Remark 5.2.2. The numbering of the axioms we use is taken from [RD]; the numbering in [Sc], [KS1] [KS2] and [Ne] is different.

In the situation that we care about, namely $\mathsf{K} = \mathbf{K}(A, \mathsf{M})$, the distinguished triangles will be those triangles that are isomorphic, in $\mathbf{K}(A, \mathsf{M})$, to the standard triangles in $\mathbf{C}(A, \mathsf{M})$ from Definition 3.8.4. See Definition 5.4.3 below for the precise statement.

The object N in item (b) of axiom (TR1) is referred to as a *cone* on $\alpha : L \to M$. We should think of the cone as something combining "the cokernel" and "the kernel" of α.

Axiom (TR2) says that if we "turn" a distinguished triangle we remain with a distinguished triangle.

Axiom (TR3) says that a commutative square (ϕ, ψ) induces a morphism χ on the cones of the horizontal morphisms, that fits into a morphism of distinguished triangles (ϕ, ψ, χ). Note however that the new morphism χ is *not unique*; in other words, *cones are not functorial*. This fact has some deep consequences in many applications. However, in the situations that will interest us, namely when $\mathsf{K} = \mathbf{K}(A, \mathsf{M})$, the cones come from the standard cones in $\mathbf{C}(A, \mathsf{M})$; and the standard cones in $\mathbf{C}(A, \mathsf{M})$ are functorial (Definition 3.8.5).

Remark 5.2.3. There is a fourth axiom in the literature, called the *octahedron axiom*; it is axiom (TR4) in [RD]. Keeping with the traditional usage, the name *triangulated category* is reserved for a pretriangulated category that also satisfies this extra axiom.

Because the octahedron axiom is extremely cumbersome to state (and prove), and also it does not play any role in the study of derived categories, we have decided to ignore it completely in our course.

For the role of the octahedron axiom in the structure of abstract triangulated categories, see the book [Ne]. It is not known whether the octahedron axiom is a consequence of the other axioms; there was a recent paper by Maccioca (arxiv:1506.00887) claiming that, but it had a fatal error in it.

The reader should not confuse the meaning of the name "pretriangulated category", as used here, with the "pretriangulated DG category" from [BK]. See Remark 5.4.13.

For a category K there is a canonical contravariant functor $\mathrm{op} : \mathsf{K} \to \mathsf{K}^{\mathrm{op}}$, that is the identity on objects, and reverses the arrows. Note that op is an anti-isomorphism of categories, so its inverse op^{-1} is uniquely defined (not only up to isomorphism).

Proposition 5.2.4. *Let K be a pretriangulated category, and let $\mathrm{op} : \mathsf{K} \to \mathsf{K}^{\mathrm{op}}$ be the canonical contravariant functor from K to its opposite category. Define a* *translation* T^{op} *on* K^{op} *by the formula* $\mathrm{T}^{\mathrm{op}} := \mathrm{op} \circ \mathrm{T}^{-1} \circ \mathrm{op}^{-1}$. *The distinguished triangles in* K^{op} *are defined to be the triangles*

$$N \xrightarrow{\mathrm{op}(\beta)} M \xrightarrow{\mathrm{op}(\alpha)} L \xrightarrow{\mathrm{op}(-\mathrm{T}^{-1}(\gamma))} \mathrm{T}^{\mathrm{op}}(N),$$

where $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$ *is any distinguished triangle in* K. *Then* $(\mathsf{K}^{\mathrm{op}}, \mathrm{T}^{\mathrm{op}})$ *is a pretriangulated category.*

Proof. This is an exercise. (Hint: use the proof of Proposition 5.3.3 below.) □

Proposition 5.2.5. *Let* K *be a pretriangulated category. If*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

i s a distinguished triangle in K, *then* $\beta \circ \alpha = 0$.

Proof. By axioms (TR1) and (TR3) we have a commutative diagram

$$\begin{tikzcd}
L \arrow[r, "1_L"] \arrow[d, "1_L"'] & L \arrow[r] \arrow[d, "\alpha"'] & 0 \arrow[r] \arrow[d] & \mathrm{T}(L) \arrow[d, "\mathrm{T}(1_L)"'] \\
L \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & N \arrow[r, "\gamma"] & \mathrm{T}(L)\ .
\end{tikzcd}$$

We see that $\beta \circ \alpha$ factors through 0. □

5.3. **Triangulated and Cohomological Functors.** Suppose K and L are T-additive categories, with translation functors T_{K} and T_{L} respectively. The notion of T-additive functor $F : \mathsf{K} \to \mathsf{L}$ was defined in Definition 5.1.3 In that definition we also introduced the notion of morphism $\eta : F \to G$ between T-additive functors.

Definition 5.3.1. Let K and L be pretriangulated categories.

(1) A *triangulated functor* from K to L is a T-additive functor

$$(F, \xi) : \mathsf{K} \to \mathsf{L}$$

that satisfies this condition: for any distinguished triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}_{\mathsf{K}}(L)$$

in K, the triangle

$$F(L) \xrightarrow{F(\alpha)} F(M) \xrightarrow{F(\beta)} F(N) \xrightarrow{\xi_L \circ F(\gamma)} \mathrm{T}_{\mathsf{L}}(F(L))$$

is a distinguished triangle in L.

(2) Suppose $(G, \nu) : \mathsf{K} \to \mathsf{L}$ is another triangulated functor. A *morphism of triangulated functors* $\eta : (F, \xi) \to (G, \nu)$ is a morphism of T-additive functors.

Sometimes we keep the isomorphism ξ implicit, and refer to F as a triangulated functor.

Definition 5.3.2. Let K be a pretriangulated category, and let M be an abelian category. A *cohomological functor* $F : \mathsf{K} \to \mathsf{M}$ is an additive functor, such that for every distinguished triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

in K, the sequence

$$F(L) \xrightarrow{F(\alpha)} F(M) \xrightarrow{F(\beta)} F(N)$$

is exact in M.

Proposition 5.3.3. *Let $F : \mathsf{K} \to \mathsf{M}$ be a cohomological functor, and let*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

be a distinguished triangle in K. Then the sequence

$$\cdots \to F(\mathrm{T}^i(L)) \xrightarrow{F(\mathrm{T}^i(\alpha))} F(\mathrm{T}^i(M)) \xrightarrow{F(\mathrm{T}^i(\beta))} F(\mathrm{T}^i(N)) \xrightarrow{F(\mathrm{T}^i(\gamma))} F(\mathrm{T}^{i+1}(L)) \xrightarrow{F(\mathrm{T}^{i+1}(\alpha))} F(\mathrm{T}^{i+1}(M)) \to \cdots$$

in M is exact.

Proof. By axiom (TR2) we have distinguished triangles

$$\mathrm{T}^i(L) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\alpha)} \mathrm{T}^i(M) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\beta)} \mathrm{T}^i(N) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\gamma)} \mathrm{T}^{i+1}(L),$$

$$\mathrm{T}^i(M) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\beta)} \mathrm{T}^i(N) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\gamma)} \mathrm{T}^{i+1}(L) \xrightarrow{(-1)^{i+1} \cdot \mathrm{T}^{i+1}(\alpha)} \mathrm{T}^{i+1}(M)$$

and

$$\mathrm{T}^i(N) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\gamma)} \mathrm{T}^{i+1}(L) \xrightarrow{(-1)^{i+1} \cdot \mathrm{T}^{i+1}(\alpha)} \mathrm{T}^{i+1}(M) \xrightarrow{(-1)^{i+1} \cdot \mathrm{T}^{i+1}(\beta)} \mathrm{T}^{i+1}(N).$$

Now use the definition, noting that multiplying morphisms in an exact sequence by -1 preserves exactness. □

Proposition 5.3.4. *Let K be a pretriangulated category. For any $P \in \mathsf{K}$ the functors*

$$\mathrm{Hom}_{\mathsf{K}}(-, P) : \mathsf{K}^{\mathrm{op}} \to \mathsf{Ab}$$

and

$$\mathrm{Hom}_{\mathsf{K}}(P, -) : \mathsf{K} \to \mathsf{Ab}$$

are cohomological functors.

Proof. We will prove the covariant statement; the contravariant statement is an immediate consequence, since

$$\mathrm{Hom}_{\mathsf{K}}(M, P) = \mathrm{Hom}_{\mathsf{K}^{\mathrm{op}}}(P, M),$$

and K^{op} is pretriangulated (with the correct pretriangulated structure to make this true).

Consider a distinguished triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

in K. We have to prove that the sequence

$$\mathrm{Hom}_{\mathsf{K}}(P, L) \xrightarrow{\mathrm{Hom}(1_P, \alpha)} \mathrm{Hom}_{\mathsf{K}}(P, M) \xrightarrow{\mathrm{Hom}(1_P, \beta)} \mathrm{Hom}_{\mathsf{K}}(P, N)$$

is exact. In view of Proposition 5.2.5, all we need to show is that for any $\psi : P \to M$ s.t. $\beta \circ \psi = 0$, there is some $\phi : P \to L$ s.t. $\psi = \alpha \circ \phi$. In a picture, we must show that the diagram below (solid arrows)

$$\begin{tikzcd} P \arrow[r, "1"] \arrow[d, dashed, "\phi"'] & P \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \mathrm{T}(P) \arrow[d, dashed, "\phi"'] \\ L \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & N \arrow[r, "\gamma"] & \mathrm{T}(L) \end{tikzcd} .$$

can be completed (dashed arrow). This is true by (TR2) (= turning) and and (TR3) (= extending). $\square$

Proposition 5.3.5. *Let* K *be a pretriangulated category, and let*

$$\begin{tikzcd} L \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[r, "\beta"] \arrow[d, "\psi"'] & N \arrow[r, "\gamma"] \arrow[d, "\chi"'] & \mathrm{T}(L) \arrow[d, "\mathrm{T}(\phi)"'] \\ L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L') \end{tikzcd} .$$

be a morphism of distinguished triangles. If ϕ *and* ψ *are isomorphisms, then* χ *is also an isomorphism.*

Proof. Take an arbitrary $P \in \mathsf{K}$, and let $F := \operatorname{Hom}_{\mathsf{K}}(P, -)$. We get a commutative diagram

$$\begin{tikzcd} F(L) \arrow[r, "F(\alpha)"] \arrow[d, "F(\phi)"'] & F(M) \arrow[r, "F(\beta)"] \arrow[d, "F(\psi)"'] & F(N) \arrow[r, "F(\gamma)"] \arrow[d, "F(\chi)"'] & F(\mathrm{T}(L)) \arrow[r, "F(\mathrm{T}(\alpha))"] \arrow[d, "F(\mathrm{T}(\phi))"'] & F(\mathrm{T}(M)) \arrow[d, "F(\mathrm{T}(\psi))"'] \\ F(L') \arrow[r, "F(\alpha')"] & F(M') \arrow[r, "F(\beta')"] & F(N') \arrow[r, "F(\gamma')"] & F(\mathrm{T}(L')) \arrow[r, "F(\mathrm{T}(\alpha'))"] & F(\mathrm{T}(M')) \end{tikzcd}$$

in Ab. By Proposition 5.3.4(2) the rows in the diagram are exact sequences. Since the other vertical arrows are isomorphisms, it follows that

$$F(\chi) : \operatorname{Hom}_{\mathsf{K}}(P, N) \to \operatorname{Hom}_{\mathsf{K}}(P, N')$$

is an isomorphism of abelian groups. By forgetting structure, we see that $F(\chi)$ is an isomorphism of sets.

We now use the Yoneda Lemma. Let us write $Y_N := \operatorname{Hom}_{\mathsf{K}}(-, N)$ and $Y_{N'} := \operatorname{Hom}_{\mathsf{K}}(-, N')$, viewed as functors $\mathsf{K}^{\mathrm{op}} \to \mathsf{Set}$. For any object $P \in \mathsf{K}$ we have isomorphisms of sets $Y_N(P) \cong F(N)$ and $Y_{N'}(P) \cong F(N')$. The calculation above shows that the morphism of functors $Y(\chi) : Y_N \to Y_{N'}$ is an isomorphism. According to Proposition 1.7.1(2), the morphism $\chi : N \to N'$ in K is an isomorphism. $\square$

Proposition 5.3.6. *Let* K *be a pretriangulated category, and let*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

be a distinguished triangle in it. The two conditions below are equivalent:

(i) $\alpha : L \to M$ *is an isomorphism.*
(ii) $N \cong 0$.

Proof. Exercise. (Hint: use Proposition 5.3.5.) $\square$

Question 5.3.7. Let K and L be pretriangulated categories, and let $F : \mathsf{K} \to \mathsf{L}$ be an additive functor. Is it true that there is at most one isomorphism of functors $\xi : F \circ \mathrm{T}_{\mathsf{K}} \xrightarrow{\simeq} \mathrm{T}_{\mathsf{L}} \circ F$ such that the pair (F, ξ) is a triangulated functor?

5.4. **The Homotopy Category is Pretriangulated.** In this subsection we consider an abelian category M and a DG ring A (everything over the base ring $\mathbb{K}$). These ingredients give rise to the $\mathbb{K}$-linear DG category $\mathbf{C}(A, \mathsf{M})$ of DG A-module in M, as in Subsection 3.6.

The strict category $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ and the homotopy category $\mathbf{K}(A, \mathsf{M})$ was introduced in Definition 3.6.5. Recall that these linear categories have the same objects as $\mathbf{C}(A, \mathsf{M})$. The morphisms groups are

$$\mathrm{Hom}_{\mathbf{C}_{\mathrm{str}}(A,\mathsf{M})}(M_0, M_1) = \mathrm{Z}^0\big(\mathrm{Hom}_{\mathbf{C}(A,\mathsf{M})}(M_0, M_1)\big)$$

and

$$\mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(M_0, M_1) = \mathrm{H}^0\big(\mathrm{Hom}_{\mathbf{C}(A,\mathsf{M})}(M_0, M_1)\big).$$

Thus the morphisms $M_0 \to M_1$ in $\mathbf{K}(A, \mathsf{M})$ are the homotopy classes $\bar{\phi} : M_0 \to M_1$ of the morphisms $\phi : M_0 \to M_1$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$.

Recall the full additive functor

$$\mathrm{P} : \mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \to \mathbf{K}(A, \mathsf{M}) \tag{5.4.1}$$

from Definition 3.3.4, that is the identity on objects, and on morphisms it is $\mathrm{P}(\phi) := \bar{\phi}$.

Consider the translation functor T from Definition 3.7.8. Since T is a DG functor from $\mathbf{C}(A, \mathsf{M})$ to itself (see Corollary 3.7.9), it restricts to a linear functor from $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ to itself, and it induces a linear functor $\bar{\mathrm{T}}$ from $\mathbf{K}(A, \mathsf{M})$ to itself, such that $P \circ \mathrm{T} = \bar{\mathrm{T}} \circ P$.

Proposition 5.4.2.

(1) *The category $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$, equipped with the translation functor T, is a T-additive category.*
(2) *The category $\mathbf{K}(A, \mathsf{M})$, equipped with the translation functor $\bar{\mathrm{T}}$, is a T-additive category.*
(3) *Let $\xi : \mathrm{P} \circ \mathrm{T} \xrightarrow{\simeq} \bar{\mathrm{T}} \circ \mathrm{P}$ be equality. Then the pair*
$$(\mathrm{P}, \xi) : \mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \to \mathbf{K}(A, \mathsf{M})$$
is a T-additive functor.

Proof. (1) We need to prove that $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ is additive. Of course the zero complex is a zero object. Next we consider finite direct sums. Let $M_1, \ldots, M_r$ be a finite collection of objects in $\mathbf{C}(A, \mathsf{M})$. Each M_i is a DG A-module in M, and we write it as $M_i = \{M_i^j\}_{j \in \mathbb{Z}}$. In each degree j the direct sum $M^j := \bigoplus_{i=1}^r M_i^j$ exists in M. Let $M := \{M^j\}_{j \in \mathbb{Z}}$ be the resulting graded object in M. The differential $\mathrm{d}_M : M^j \to M^{j+1}$ exists by the universal property of direct sums; so we obtain a complex $M \in \mathbf{C}(\mathsf{M})$. The DG A-module structure on M is defined similarly: for $a \in A^k$, there is an induced degree k morphism $f(a) : M \to M$ in $\mathbf{C}(\mathsf{M})$. Thus M becomes an object of $\mathbf{C}(A, \mathsf{M})$. But the embeddings $e_i : M_i \to M$ are strict morphisms, so $(M, \{e_i\})$ is a coproduct of the collection $\{M_i\}$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$.

(2) Now consider the category $\mathbf{K}(A, \mathsf{M})$. Because the functor $\mathrm{P} : \mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \to \mathbf{K}(A, \mathsf{M})$ is additive, and is bijective on objects, part (1) above and Proposition 2.4.2 say that $\mathbf{K}(A, \mathsf{M})$ is an additive category.

(3) Clear. $\square$

From now on we denote by T, instead of by $\bar{\mathrm{T}}$, the translation functor of $\mathbf{K}(A, \mathsf{M})$.

Definition 5.4.3. A triangle

$$L \xrightarrow{\bar{\alpha}} M \xrightarrow{\bar{\beta}} N \xrightarrow{\bar{\gamma}} \mathrm{T}(L)$$

in $\mathbf{K}(A, \mathsf{M})$ is said to be a *distinguished triangle* if there is a standard triangle

$$L' \xrightarrow{\alpha'} M' \xrightarrow{\beta'} N' \xrightarrow{\gamma'} \mathrm{T}(L')$$

in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$, as in Definition 3.8.4, and an isomorphism of triangles

$$\begin{tikzcd}
L' \arrow[r, "\mathrm{P}(\alpha')"] \arrow[d, "\bar{\phi}"'] & M' \arrow[r, "\mathrm{P}(\beta')"] \arrow[d, "\bar{\psi}"'] & N' \arrow[r, "\mathrm{P}(\gamma')"] \arrow[d, "\bar{\chi}"'] & \mathrm{T}(L') \arrow[d, "\mathrm{T}(\bar{\phi})"'] \\
L \arrow[r, "\bar{\alpha}"] & M \arrow[r, "\bar{\beta}"] & N \arrow[r, "\bar{\gamma}"] & \mathrm{T}(L)
\end{tikzcd} .$$

in $\mathbf{K}(A, \mathsf{M})$.

Theorem 5.4.4. *The T-additive category $\mathbf{K}(A, \mathsf{M})$, with the set of distinguished triangles defined above, is a pretriangulated category.*

The proof is after three lemmas.

Lemma 5.4.5. *Let $M \in \mathbf{C}(A, \mathsf{M})$, and consider the cone $N := \operatorname{Cone}(1_M)$. Then the DG module N is null-homotopic, i.e. $0 \to N$ is an isomorphism in $\mathbf{K}(A, \mathsf{M})$.*

Proof. We shall exhibit a homotopy θ from 0_N to 1_N. Recall from Subsection 3.8 that

$$N = \operatorname{Cone}(1_M) = M \oplus \mathrm{T}(M) = \begin{bmatrix} M \\ \mathrm{T}(M) \end{bmatrix}$$

as graded modules, with differential whose matrix presentation is

$$\mathrm{d}_N = \begin{bmatrix} \mathrm{d}_M & \mathrm{t}_M^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}(M)} \end{bmatrix} .$$

And by the definition in Subsection 3.7 we have

$$\mathrm{d}_{\mathrm{T}(M)} = -\,\mathrm{t}_M \circ \mathrm{d}_M \circ \mathrm{t}_M^{-1} .$$

Define $\theta : N \to N$ to be the degree -1 morphism with matrix presentation

$$\theta := \begin{bmatrix} 0 & 0 \\ \mathrm{t}_M & 0 \end{bmatrix} .$$

Then, using the formulas above for d_N and $\mathrm{d}_{\mathrm{T}(M)}$, we get

$$\mathrm{d}_N \circ \theta + \theta \circ \mathrm{d}_N = \begin{bmatrix} 1_M & 0 \\ 0 & 1_{\mathrm{T}(M)} \end{bmatrix} = 1_N .$$

$\square$

Exercise 5.4.6. Here is a generalization of Lemma 5.4.5. Consider a morphism $\phi : M_0 \to M_1$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. Show that the three conditions below are equivalent:

(i) ϕ is a homotopy equivalence.

(ii) $\bar{\phi}$ is an isomorphism in $\mathbf{K}(A, \mathsf{M})$.
(iii) The DG module $\operatorname{Cone}(\phi)$ is null-homotopic.

Try to do this directly, not using Proposition 5.3.4(2) and Theorem 5.4.4.

The next lemma is based on [KS1, Lemma 1.4.2].

Lemma 5.4.7. *Consider a morphism $\alpha : L \to M$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$, the standard triangle*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

associated to α, and the standard triangle

$$M \xrightarrow{\beta} N \xrightarrow{\phi} P \xrightarrow{\psi} \mathrm{T}(M)$$

associated to β, all in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. So $N = \operatorname{Cone}(\alpha)$ and and $P = \operatorname{Cone}(\beta)$. There is a morphism $\rho : \mathrm{T}(L) \to P$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ s.t. $\bar{\rho}$ is an isomorphism in $\mathbf{K}(A, \mathsf{M})$, and the diagram

$$\begin{tikzcd}
M \arrow[r, "\bar{\beta}"] \arrow[d, "\bar{1}_M"'] & N \arrow[r, "\bar{\gamma}"] \arrow[d, "\bar{1}_N"'] & \mathrm{T}(L) \arrow[r, "-\mathrm{T}(\bar{\alpha})"] \arrow[d, "\bar{\rho}"'] & \mathrm{T}(M) \arrow[d, "\bar{1}_{\mathrm{T}(M)}"'] \\
M \arrow[r, "\bar{\beta}"] & N \arrow[r, "\bar{\phi}"] & P \arrow[r, "\bar{\psi}"] & \mathrm{T}(M)
\end{tikzcd}$$

commutes in $\mathbf{K}(A, \mathsf{M})$.

Proof. Note that $N = M \oplus \mathrm{T}(L)$ and $P = N \oplus \mathrm{T}(M) = M \oplus \mathrm{T}(L) \oplus \mathrm{T}(M)$ as graded module. Thus P and d_P have the following matrix presentations:

$$P = \begin{bmatrix} M \\ \mathrm{T}(L) \\ \mathrm{T}(M) \end{bmatrix} , \quad \mathrm{d}_P = \begin{bmatrix} \mathrm{d}_M & \alpha \circ \mathrm{t}_L^{-1} & \mathrm{t}_M^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}(L)} & 0 \\ 0 & 0 & \mathrm{d}_{\mathrm{T}(M)} \end{bmatrix} .$$

Define morphisms $\rho : \mathrm{T}(L) \to P$ and $\chi : P \to \mathrm{T}(L)$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ by the matrix presentations

$$\rho := \begin{bmatrix} 0 \\ 1_{\mathrm{T}(L)} \\ -\mathrm{T}(\alpha) \end{bmatrix} , \quad \chi := \begin{bmatrix} 0 & 1_{\mathrm{T}(L)} & 0 \end{bmatrix} .$$

Direct calculations show that:

- $\chi \circ \rho = 1_{\mathrm{T}(L)}$.
- $\rho \circ \gamma = \rho \circ \chi \circ \phi$.
- $\psi \circ \rho = -\mathrm{T}(\alpha)$.

It remains to prove that $\rho \circ \chi$ is homotopic to 1_P. Define a degree -1 morphism $\theta : P \to P$ by the matrix

$$\theta := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{t}_M & 0 & 0 \end{bmatrix} .$$

Then a direct calculation, using the equalities

$$\mathrm{t}_M \circ \mathrm{d}_M + \mathrm{d}_{\mathrm{T}(M)} \circ \mathrm{t}_M = 0$$

and

$$\mathrm{T}(\alpha) = \mathrm{t}_M \circ \alpha \circ \mathrm{t}_L^{-1}$$

gives

$$\theta \circ \mathrm{d}_P + \mathrm{d}_P \circ \theta = 1_P - \rho \circ \chi.$$

□

Lemma 5.4.8. *Consider a standard triangle*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$. *For any integer* k, *the triangle*

$$\mathrm{T}^k(L) \xrightarrow{\mathrm{T}^k(\alpha)} \mathrm{T}^k(M) \xrightarrow{\mathrm{T}^k(\beta)} \mathrm{T}^k(N) \xrightarrow{(-1)^k \cdot \mathrm{T}^k(\gamma)} \mathrm{T}^{k+1}(L)$$

is isomorphic, in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$, *to a standard triangle.*

Proof. Combine Corollary 3.7.9, Theorem 4.3.6 with $F = \mathrm{T}$, and Proposition 4.2.4. □

Proof of Theorem 5.4.4. (TR1): By definition the set of distinguished triangles in $\mathsf{K}(A, \mathsf{M})$ is closed under isomorphisms. This establishes item (a).

As for item (b): consider any morphism $\bar{\alpha} : L \to M$ in $\mathsf{K}(A, \mathsf{M})$. It is represented by a morphism $\alpha : L \to M$ in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$. Take the standard triangle on α in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$. Its image in $\mathsf{K}(A, \mathsf{M})$ has the desired property.

Finally, Lemma 5.4.5 shows that the triangle

$$M \xrightarrow{\bar{1}_M} M \to 0 \to \mathrm{T}(M)$$

is isomorphic in $\mathsf{K}(A, \mathsf{M})$ to the triangle

$$M \xrightarrow{\bar{1}_M} M \xrightarrow{\bar{e}} \mathrm{Cone}(1_M) \xrightarrow{\bar{p}} \mathrm{T}(M).$$

The latter is the image of a standard triangle, and so it is distinguished.

(TR2): Consider the triangles

$$L \xrightarrow{\bar{\alpha}} M \xrightarrow{\bar{\beta}} N \xrightarrow{\bar{\gamma}} \mathrm{T}(L) \tag{5.4.9}$$

and

$$M \xrightarrow{\bar{\beta}} N \xrightarrow{\bar{\gamma}} \mathrm{T}(L) \xrightarrow{-\mathrm{T}(\bar{\alpha})} \mathrm{T}(M) \tag{5.4.10}$$

in $\mathsf{K}(A, \mathsf{M})$. If (5.4.9) is distinguished, then by Lemma 5.4.7 so is (5.4.10).

Conversely, if (5.4.10) is distinguished, then by turning it 5 times, and using the previous step (namely by Lemma 5.4.7), we see that the triangle

$$\mathrm{T}^2(L) \xrightarrow{\mathrm{T}^2(\bar{\alpha})} \mathrm{T}^2(M) \xrightarrow{\mathrm{T}^2(\bar{\beta})} \mathrm{T}^2(N) \xrightarrow{\mathrm{T}^2(\bar{\gamma})} \mathrm{T}^3(L)$$

is distinguished. According to Lemma 5.4.8 (with $k = -2$), the triangle gotten by applying T^{-2} to this is distinguished. But this is just the triangle (5.4.9).

(TR3): Consider a commutative diagram in $\mathsf{K}(A, \mathsf{M})$:

$$\begin{tikzcd}
L \arrow[r, "\bar{\alpha}"] \arrow[d, "\bar{\phi}"'] & M \arrow[r, "\bar{\beta}"] \arrow[d, "\bar{\psi}"'] & N \arrow[r, "\bar{\gamma}"] & T(L) \\
L' \arrow[r, "\bar{\alpha}'"] & M' \arrow[r, "\bar{\beta}'"] & N' \arrow[r, "\bar{\gamma}'"] & T(L')
\end{tikzcd}$$

where the horizontal triangles are distinguished. By definition this diagram is isomorphic to a diagram in $\mathsf{K}(A, \mathsf{M})$, that is gotten by applying the functor P to a diagram

$$\begin{tikzcd} L \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[r, "\beta"] \arrow[d, "\psi"'] & N \arrow[r, "\gamma"] & T(L) \\ L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & T(L') \end{tikzcd}$$

in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$, in which $N = \mathrm{Cone}(\alpha)$, $N' = \mathrm{Cone}(\alpha')$, and the horizontal triangles are the standard ones (see Definition 3.8.4). However this diagram in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$ in only commutative up to homotopy. This means that there is a degree -1 morphism $\theta : L \to M'$ in $\mathsf{C}(A, \mathsf{M})$ s.t.

$$\alpha' \circ \phi = \psi \circ \alpha + \mathrm{d}(\theta).$$

Define the morphism

$$\chi : N = \begin{bmatrix} M \\ \mathrm{T}(L) \end{bmatrix} \to N' = \begin{bmatrix} M' \\ \mathrm{T}(L') \end{bmatrix}$$

by the matrix presentation

$$\chi := \begin{bmatrix} \psi & \theta \circ \mathrm{t}_L^{-1} \\ 0 & \mathrm{T}(\phi) \end{bmatrix}.$$

An easy calculation shows that χ is a morphism in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$, and that there are equalities $\mathrm{T}(\phi) \circ \gamma = \gamma' \circ \chi$ and $\chi \circ \beta = \beta' \circ \psi$. Therefore, when we apply the functor P we have a commutative diagram

$$\begin{tikzcd} L \arrow[r, "\bar{\alpha}"] \arrow[d, "\bar{\phi}"'] & M \arrow[r, "\bar{\beta}"] \arrow[d, "\bar{\psi}"'] & N \arrow[r, "\bar{\gamma}"] \arrow[d, "\bar{\chi}"'] & T(L) \arrow[d, "T(\bar{\phi})"'] \\ L' \arrow[r, "\bar{\alpha}'"] & M' \arrow[r, "\bar{\beta}'"] & N' \arrow[r, "\bar{\gamma}'"] & T(L') \end{tikzcd}$$

in $\mathsf{K}(A, \mathsf{M})$, where $\bar{\chi} := \mathrm{P}(\chi)$. □

We now add a second DG ring B, and a second additive category N. DG functors were introduced in Subsection 3.4.

Consider a DG functor

$$F : \mathsf{C}(A, \mathsf{M}) \to \mathsf{C}(B, \mathsf{N}).$$

From Theorem 4.2.3 we know that the translation isomorphism is an isomorphism of DG functors

$$\zeta_F : F \circ \mathrm{T}_{A,\mathsf{M}} \xrightarrow{\simeq} \mathrm{T}_{B,\mathsf{N}} \circ F.$$

Therefore, when we pass to the homotopy categories, and writing $\bar{F} := \mathrm{Ho}(F)$, we get a T-additive functor

$$(\bar{F}, \bar{\zeta}_F) : \mathsf{K}(A, \mathsf{M}) \to \mathsf{K}(B, \mathsf{N}).$$

Theorem 5.4.11. *Let*

$$F : \mathsf{C}(A, \mathsf{M}) \to \mathsf{C}(B, \mathsf{N})$$

be a DG functor, with translation isomorphism ζ_F. Then the T-additive functor

$$(\bar{F}, \bar{\zeta}_F) : \mathsf{K}(A, \mathsf{M}) \to \mathsf{K}(B, \mathsf{N})$$

is a triangulated functor.

Proof. Take a distinguished triangle

$$L \xrightarrow{\bar{\alpha}} M \xrightarrow{\bar{\beta}} N \xrightarrow{\bar{\gamma}} \mathrm{T}(L)$$

in $\mathbf{K}(A, \mathsf{M})$. Since we are only interested in triangles up to isomorphism, we can assume that this is the image under the functor P of a standard triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. According to Theorem 4.3.6, there is a standard triangle

$$L' \xrightarrow{\alpha'} M' \xrightarrow{\beta'} N' \xrightarrow{\gamma'} \mathrm{T}(L')$$

in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$, and a commutative diagram

$$\begin{tikzcd}
F(L) \arrow[r, "F(\alpha)"] \arrow[d, "\phi"'] & F(M) \arrow[r, "F(\beta)"] \arrow[d, "\psi"'] & F(N) \arrow[r, "\zeta_{F,L} \circ F(\gamma)"] \arrow[d, "\chi"'] & F(\mathrm{T}(L)) \arrow[d, "\mathrm{T}(\phi)"'] \\
L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L')
\end{tikzcd}$$

in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$, in which the vertical arrows are isomorphisms. (Actually, we can take $L' = F(L)$, $\phi = \mathrm{id}_{F(L)}$, etc.) After applying the functor P to this diagram, we see that the condition in Definition 5.3.1(1) is satisfied. □

Corollary 5.4.12. *For any integer k, the pair*

$$\big(\mathrm{T}^k, (-1)^k \cdot 1_{\mathrm{T}^{k+1}}\big)$$

is a triangulated functor from $\mathbf{K}(A, \mathsf{M})$ to itself.

Proof. Combine Theorems 5.4.11 and 3.7.7(3). □

Remark 5.4.13. In [BK], Bondal and Kapranov introduce the concept of *pretriangulated DG category*. This is a DG category C for which the homotopy category $\mathrm{Ho}(\mathsf{C})$ is canonically triangulated (the details of the definition are too complicated to mention here). Our DG categories $\mathbf{C}(A, \mathsf{M})$ are pretriangulated in the sense of [BK]; but they have a lot more structure (e.g. the objects have cohomologies too).

Suppose C and C' are pretriangulated DG categories. In [BK] there is a (rather complicated) definition of *pre-exact DG functor* $F : \mathsf{C} \to \mathsf{C}'$. It is stated there that if F is a pre-exact DG functor, then $\mathrm{Ho}(F) : \mathrm{Ho}(\mathsf{C}) \to \mathrm{Ho}(\mathsf{C}')$ is a triangulated functor. This is analogous to our Theorem 5.4.11. Presumably, Theorems 4.2.3 and 4.3.6 imply that any DG functor $F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(A', \mathsf{M}')$ is pre-exact in the sense of [BK]; but we did not verify this.

6. Preview of Next Semester

★ This section will be omitted in final version. Some of it will be incorporated in the Intro. ★

Let $\mathbb{K}$ be a nonzero commutative base ring. We consider $\mathbb{K}$-linear abelian categories M and N, and $\mathbb{K}$-central DG rings A and B. This data gives rise to $\mathbb{K}$-linear DG categories $\mathbf{C}(A, \mathsf{M})$ and $\mathbf{C}(B, \mathsf{N})$, and to $\mathbb{K}$-linear triangulated categories $\mathbf{K}(A, \mathsf{M})$ and $\mathbf{K}(B, \mathsf{N})$.

We know that if

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$$

is a DG functor, then there is an induced triangulated functor

$$F : \mathbf{K}(A, \mathsf{M}) \to \mathbf{K}(B, \mathsf{N}).$$

So we have a rich source of examples of triangulated functors (since we know how to build DG functors).

A morphism $\phi : M_0 \to M_1$ in $\mathbf{K}(A, \mathsf{M})$ is called a *quasi-isomorphism* if the morphisms $\mathrm{H}^i(\phi) : \mathrm{H}^i(M_0) \to \mathrm{H}^i(M_1)$ in M are isomorphisms for all i.

Because quasi-isomorphisms are closed under composition, they form a subcategory $\mathbf{S}(A, \mathsf{M})$ of $\mathbf{K}(A, \mathsf{M})$, on the same set of objects.

Note that for any object M, the set $S := \mathrm{End}_{\mathbf{S}(A,\mathsf{M})}(M)$ is a multiplicatively closed subset of the ring $A := \mathrm{End}_{\mathbf{K}(A,\mathsf{M})}(M)$.

We will prove that the category $\mathbf{S}(A, \mathsf{M})$ satisfies the *left and right denominator conditions*. Just as in ring theory, there is an Ore localization: a triangulated category

$$\mathbf{D}(A, \mathsf{M}) := \mathbf{K}(A, \mathsf{M})_{\mathbf{S}(A,\mathsf{M})}$$

whose objects are the same as those of $\mathbf{K}(\mathsf{M})$, with a triangulated functor

$$\mathrm{Q} : \mathbf{K}(A, \mathsf{M}) \to \mathbf{D}(A, \mathsf{M})$$

that is the identity on objects. For every isomorphism $\psi \in \mathbf{S}(A, \mathsf{M})$, the morphism $\mathrm{Q}(\psi)$ is invertible. Every morphism χ in $\mathbf{D}(A, \mathsf{M})$ can be written (of course not uniquely) as

$$\chi = \mathrm{Q}(\phi_1) \circ \mathrm{Q}(\psi_1)^{-1} = \mathrm{Q}(\psi_2)^{-1} \circ \mathrm{Q}(\phi_2)$$

for some $\phi_i \in \mathbf{K}(A, \mathsf{M})$ and $\psi_i \in \mathbf{S}(A, \mathsf{M})$.

There is also a universal property: given any triangulated category E, and any triangulated functor $F : \mathbf{K}(A, \mathsf{M}) \to \mathsf{E}$ such that $F(\psi)$ is invertible for every $\psi \in \mathbf{S}(A, \mathsf{M})$, there is a unique triangulated functor $F_{\mathbf{S}} : \mathbf{D}(A, \mathsf{M}) \to \mathsf{E}$ such that $F = F_{\mathbf{S}} \circ \mathrm{Q}$.

Let E a triangulated category, and $F : \mathbf{K}(A, \mathsf{M}) \to \mathsf{E}$ a triangulated functor. A *right derived functor* of F is a triangulated functor

$$\mathrm{R}F : \mathbf{D}(A, \mathsf{M}) \to \mathsf{E},$$

together with a morphism

$$\eta : F \to \mathrm{R}F \circ Q$$

of triangulated functors $\mathbf{K}(A, \mathsf{M}) \to \mathsf{E}$. The pair $(\mathrm{R}F, \eta)$ must have this universal property:

($*$) Given any triangulated functor $G : \mathbf{D}(A, \mathsf{M}) \to \mathsf{E}$, and a morphism of triangulated functors $\eta' : F \to G \circ Q$, there is a unique morphism of triangulated functors $\theta : \mathrm{R}F \to G$ s.t. $\eta' = \theta \circ \eta$.

Pictorially: there is a diagram

$$\begin{tikzcd} \mathbf{K}(A, \mathsf{M}) \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathsf{E} \\ \mathbf{D}(A, \mathsf{M}) \arrow[ur, "\mathrm{R}F"'] & \end{tikzcd}$$

where the plain arrows are triangulated functors, and the double arrow is a morphism of triangulated functors. For any other pair (G, η') there is a unique morphism θ

$$\begin{tikzcd} \mathbf{K}(A, \mathsf{M}) \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[dr, phantom, "\Downarrow\eta \quad \Downarrow\eta'"] & \mathsf{E} \\ \mathbf{D}(A, \mathsf{M}) \arrow[ur, "\mathrm{R}F"'] \arrow[ur, bend right=60, "G"'] \arrow[ur, phantom, bend right=30, "\Downarrow\theta"] & \end{tikzcd}$$

s.t. $\eta' = \theta \circ \eta$.

It is not hard to see that if a right derived functor $(\mathrm{R}F, \eta)$ exists, then it is unique, up to a unique isomorphism of triangulated functors.

In complete symmetry we have this: Let E be a triangulated category, and let $F : \mathbf{K}(A, \mathsf{M}) \to \mathsf{E}$ a triangulated functor. A *left derived functor* of F is a triangulated functor

$$\mathrm{L}F : \mathbf{D}(A, \mathsf{M}) \to \mathsf{E},$$

together with a morphism

$$\eta : \mathrm{L}F \circ \mathrm{Q} \to F$$

of triangulated functors $\mathbf{K}(A, \mathsf{M}) \to \mathsf{E}$. The pair $(\mathrm{L}F, \eta)$ must have this universal property:

($*$) Given any triangulated functor $G : \mathbf{D}(A, \mathsf{M}) \to \mathsf{E}$, and a morphism of triangulated functors $\eta' : G \circ \mathrm{Q} \to F$, there is a unique morphism of triangulated functors $\theta : G \to \mathrm{L}F$ s.t. $\eta' = \eta \circ \theta$.

Again, if a left derived functor $(\mathrm{L}F, \eta)$ exists, then it is unique, up to a unique isomorphism of triangulated functors.

The question of *existence of derived functors* rests on *resolutions*.

We will prove that under suitable assumptions on (A, M), the category $\mathbf{C}(A, \mathsf{M})$ has *enough K-injective modules*. Examples are:

- $\mathbf{C}(\mathsf{Mod}\,\mathcal{A})$ for a ringed space $(X, \mathcal{A})$.
- $\mathbf{C}(A)$ for a DG ring A.

When this happens, all triangulated functors can be right derived. The formula is

$$\mathrm{R}F(M) := F(I),$$

where $M \to I$ is any K-injective resolution.

Another useful fact is that in this situation, letting $\mathsf{K}(A, \mathsf{M})_{\mathrm{inj}}$ be the full subcategory of $\mathsf{K}(A, \mathsf{M})$ on the K-injective objects, the functor

$$\mathrm{Q} : \mathsf{K}(A, \mathsf{M})_{\mathrm{inj}} \to \mathsf{D}(A, \mathsf{M})$$

is an equivalence. So we do not need denominators!

Similarly, if $\mathsf{C}(A, \mathsf{M})$ has *enough K-projective modules*, then all triangulated functors can be left derived. The formula is

$$\mathrm{L}F(M) := F(P),$$

where $P \to M$ is any K-projective resolution.

Again, in this situation, letting $\mathsf{K}(A, \mathsf{M})_{\mathrm{prj}}$ be the full subcategory of $\mathsf{K}(A, \mathsf{M})$ on the K-projective objects, the functor

$$\mathrm{Q} : \mathsf{K}(A, \mathsf{M})_{\mathrm{prj}} \to \mathsf{D}(A, \mathsf{M})$$

is an equivalence.

An example is:

- $\mathsf{C}(A)$ for a DG ring A.

After that we shall consider *bi-DG functors*, such as

$$- \otimes_A - : \mathsf{C}(A) \times \mathsf{C}(A^{\mathrm{op}}) \to \mathsf{C}(\mathbb{K})$$

and

$$\mathrm{Hom}_A(-, -) : \mathsf{C}(A)^{\mathrm{op}} \times \mathsf{C}(A) \to \mathsf{C}(\mathbb{K}).$$

These will induce *bitriangulated bifunctors*

$$- \otimes_A - : \mathsf{K}(A) \times \mathsf{K}(A^{\mathrm{op}}) \to \mathsf{K}(\mathbb{K})$$

and

$$\mathrm{Hom}_A(-, -) : \mathsf{K}(A)^{\mathrm{op}} \times \mathsf{K}(A) \to \mathsf{K}(\mathbb{K}).$$

The resolutions mentioned above will also serve to obtain a left derived functor

$$- \otimes^{\mathrm{L}}_A - : \mathsf{D}(A) \times \mathsf{D}(A^{\mathrm{op}}) \to \mathsf{D}(\mathbb{K})$$

and a right derived functor

$$\mathrm{RHom}_A(-, -) : \mathsf{D}(A)^{\mathrm{op}} \times \mathsf{D}(A) \to \mathsf{D}(\mathbb{K}).$$

This will end the study of the general theory. After that, we will work on applications, from the list below.

(1) Commutative algebra via derived categories. Dualizing complexes, local duality, MGM equivalence, rigid dualizing complexes.
(2) Geometric derived categories (of sheaves on spaces). Direct and inverse image functors, Grothendieck duality, Poicaré-Verdier duality, perverse sheaves.
(3) Derived categories associated to noncommutative rings. Dualizing complexes, tilting complexes, the derived Picard group, derived Morita theory.
(4) Survey of derived categories in modern algebraic geometry and mathematical physics. Survey of derived algebraic geometry.

7. Review of First Semester

★ This section will be omitted in final version. Some of it will be incorporated in the Intro. ★

Here is a quick recollection of what we did in the first part of the course.

We always had a base ring $\mathbb{K}$, which is some nonzero commutative ring. The universal choice is $\mathbb{K} = \mathbb{Z}$; but sometimes we might want to work over a base field $\mathbb{K}$, and so on. Much of the time $\mathbb{K}$ was implicit.

We talked about abelian categories. Our notation for an abelian category is M. The primary example is the category $\mathsf{Mod}\, A$ of left modules over a ring A. If A is a central $\mathbb{K}$-ring, then $\mathsf{Mod}\, A$ is a $\mathbb{K}$-linear abelian category.

We then talked about $\mathbb{K}$-linear DG categories, and $\mathbb{K}$-linear DG functors between them.

The primary example of a DG category is the category $\mathbf{C}(\mathsf{M})$ of complexes in an abelian category M. For the DG structure on $\mathbf{C}(\mathsf{M})$, the DG module of homomorphisms $M \to N$ is $\mathrm{Hom}_{\mathsf{M}}(M, N)$, that we already worked with in classical homological algebra, but then we had no proper understanding of its role. The "classical" homomorphisms of complexes are the strict morphisms in $\mathbf{C}(\mathsf{M})$, that are by definition the elements of

$$\mathrm{Z}^0\big(\mathrm{Hom}_{\mathsf{M}}(M, N)\big).$$

DG rings arise in many contexts. There is the Koszul complex in commutative algebra; the de Rham complex in differential geometry; and for any object M in a $\mathbb{K}$-linear DG category C, the set of self-morphisms

$$\mathrm{End}_{\mathsf{C}}(M) := \mathrm{Hom}_{\mathsf{C}}(M, M)$$

is a $\mathbb{K}$-central DG ring. A DG $\mathbb{K}$-central ring A has left DG modules, and they form the $\mathbb{K}$-linear DG category $\mathbf{C}(A)$.

In order to treat both $\mathbf{C}(A)$ and $\mathbf{C}(\mathsf{M})$ at once, we introduced the $\mathbb{K}$-linear DG category $\mathbf{C}(A, \mathsf{M})$ of DG A-modules in M. Recall that an object in it is a complex $M \in \mathbf{C}(\mathsf{M})$, together with a DG ring homomorphism $A \to \mathrm{End}_{\mathsf{M}}(M)$. The morphisms $M \to N$ are the morphisms in $\mathbf{C}(\mathsf{M})$ that commute (in the graded sense) with the action of the elements of A.

We were mostly interested in $\mathbb{K}$-linear DG functors

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N}). \tag{7.0.1}$$

There were a few prototypical examples of the formation of such DG functors.

The next topic was pretriangulated categories. A pretriangulated category K is an additive category, equipped with a translation automorphism T, and a set of distinguished triangles, satisfying axioms (TR1), (TR2) and (TR3). We exclude the octahedron axiom on purpose – it is satisfied in all cases we consider, but we never use it.

Our primary example of pretriangulated category is the homotopy category $\mathbf{K}(A, \mathsf{M})$, whose objects are the DG A-modules in M, and the morphisms $M \to N$ are

$$\mathrm{H}^0\big(\mathrm{Hom}_{A,\mathsf{M}}(M, N)\big).$$

The translation T and the distinguished triangles are determined by the DG structure of $\mathbf{C}(A, \mathsf{M})$. Our last theorem was that a $\mathbb{K}$-linear DG functor F as in (7.0.1) induces a $\mathbb{K}$-linear triangulated functor

$$F : \mathsf{K}(A, \mathsf{M}) \to \mathsf{K}(B, \mathsf{N}).$$

8. Localization of Categories

Our goal in this section is this: we have the triangulated category $\mathsf{K}(A, \mathsf{M})$ that's associated to an abelian category M and a DG ring A. Inside $\mathsf{K}(A, \mathsf{M})$ we have the set of quasi-isomorphisms $\mathsf{S}(A, \mathsf{M})$. We will prove that it is possible to localize with respect to $\mathsf{S}(A, \mathsf{M})$ – namely, there is a triangulated category

$$\mathsf{D}(A, \mathsf{M}) := \mathsf{K}(A, \mathsf{M})_{\mathsf{S}(A,\mathsf{M})}, \tag{8.0.1}$$

in which the quasi-isomorphisms are invertible, and there is a useful calculus of fractions. The triangulated category $\mathsf{D}(A, \mathsf{M})$ is the *derived category* of (A, M).

8.1. The Formalism of Localization.

We will start with a category A, without even assuming it is linear. Still we use the notation A, because it will be suggestive to think about a linear category A with a single object, which is just a ring A. The reason is that our localization procedure is the same as that in noncommutative ring theory (the only change being that we allow multiple objects).

The emphasis will be on morphisms rather than on objects. Thus it will be convenient to write

$$\mathsf{A}(M, N) := \mathrm{Hom}_{\mathsf{A}}(M, N)$$

for $M, N \in \mathrm{Ob}(\mathsf{A})$. We use sometimes use the notation $a \in \mathsf{A}$ for a morphism $a \in \mathsf{A}(M, N)$, leaving the objects implicit. When we write $b \circ a$ for $a, b \in \mathsf{A}$, we implicitly mean that these morphisms are composable.

For heuristic purposes, we can think of A as a linear category (e.g. living inside some category of modules), with objects $M, N, \ldots$. For any given object M, we then have a genuine ring $\mathsf{A}(M) := \mathsf{A}(M, M)$.

Definition 8.1.1. Let A be a category. A *multiplicatively closed set of morphisms* in A is a subcategory $\mathsf{S} \subseteq \mathsf{A}$ such that $\mathrm{Ob}(\mathsf{S}) = \mathrm{Ob}(\mathsf{A})$.

In other words, for any pair of objects $M, N \in \mathsf{A}$ there is a subset $\mathsf{S}(M, N) \subseteq \mathsf{A}(M, N)$, such that $1_M \in \mathsf{S}(M, M)$, and such that for any $s \in \mathsf{S}(L, M)$ and $t \in \mathsf{S}(M, N)$, the composition $t \circ s \in \mathsf{S}(L, N)$.

Using our shorthand, we can write the definition like this: $1_M \in \mathsf{S}$, and $s, t \in \mathsf{S}$ implies $t \circ s \in \mathsf{S}$.

If $A = \mathsf{A}$ is a single object linear category, namely a ring, then $S = \mathsf{S}$ is a multiplicatively closed set in the sense of ring theory.

There are various notions of localization in the literature. We restrict attention to two of them.

Definition 8.1.2. Let S be a multiplicatively closed set of morphisms in a category A. A *localization*[2] of A with respect to S is a pair $(\mathsf{A}_{\mathsf{S}}, \mathrm{Q})$, consisting of a category A_{S} and a functor $\mathrm{Q} : \mathsf{A} \to \mathsf{A}_{\mathsf{S}}$, called the localization functor, having the following properties:

- (L1) There is equality $\mathrm{Ob}(\mathsf{A}_{\mathsf{S}}) = \mathrm{Ob}(\mathsf{A})$, and Q is the identity on objects.
- (L2) For every $s \in \mathsf{S}$, the morphism $\mathrm{Q}(s) \in \mathsf{A}_{\mathsf{S}}$ is invertible (i.e. it is an isomorphism).
- (L3) Suppose B is a category, and $F : \mathsf{A} \to \mathsf{B}$ is a functor such that $F(s)$ is invertible for every $s \in \mathsf{S}$. Then there is a unique functor $F_{\mathsf{S}} : \mathsf{A}_{\mathsf{S}} \to \mathsf{B}$ such that $F_{\mathsf{S}} \circ \mathrm{Q} = F$ as functors $\mathsf{A} \to \mathsf{B}$.

[2] In [Ye6] this was called a "strict localization".

In a commutative diagram:

$$\begin{tikzcd} \mathsf{S} \arrow[r, "\mathrm{inc}"] & \mathsf{A} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] & \mathsf{B} \\ & \mathsf{A}_{\mathsf{S}} \arrow[ur, "F_{\mathsf{S}}"'] & \end{tikzcd}$$

In the ring case, $F : A \to B$ is a ring homomorphism, etc.

Proposition 8.1.3. *A localization* (*in the sense of Definition* 8.1.2) *is unique up to a unique isomorphism. Namely if* $(\mathsf{A}'_{\mathsf{S}}, \mathrm{Q}')$ *is another localization, then there is a unique functor* $G : \mathsf{A}_{\mathsf{S}} \to \mathsf{A}'_{\mathsf{S}}$ *which is the identity on objects, bijective on morphisms, and* $G \circ \mathrm{Q} = \mathrm{Q}'$.

Proof. Exercise. □

A localization in this general sense always exists, but often it is of little value, because there is no practical way to describe the morphisms in it.

8.2. **Ore Localization.** There is a better notion of localization. The references here are [RD], [GZ], [Wei], [KS1], [Ste] and [Row]. The first four references talk about localization of categories; and the last two talk about noncommutative rings. It seems that historically, this *noncommutative calculus of fractions* was discovered by Ore and Asano in ring theory, around 1930. There was progress in the categorical side, notably by Gabriel around 1960.

In this subsection we mostly follow the treatment of [Ste]; but we sometimes use diagrams instead of formulas involving letters[3].

Definition 8.2.1. Let S be a multiplicatively closed set of morphisms in a category A. A *right Ore localization* of A with respect to S is a pair $(\mathsf{A}_{\mathsf{S}}, \mathrm{Q})$, consisting of a category A_{S} and a functor $\mathrm{Q} : \mathsf{A} \to \mathsf{A}_{\mathsf{S}}$, having the following properties:

(RO1) There is equality $\mathrm{Ob}(\mathsf{A}_{\mathsf{S}}) = \mathrm{Ob}(\mathsf{A})$, and Q is the identity on objects.
(RO2) For every $s \in \mathsf{S}$, the morphism $\mathrm{Q}(s) \in \mathsf{A}_{\mathsf{S}}$ is an isomorphism.
(RO3) Every morphism $q \in \mathsf{A}_{\mathsf{S}}$ can be written as $q = \mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1}$ for some $a \in \mathsf{A}$ and $s \in \mathsf{S}$.
(RO4) Suppose $a, b \in \mathsf{A}$ satisfy $\mathrm{Q}(a) = \mathrm{Q}(b)$. Then $a \circ s = b \circ s$ for some $s \in \mathsf{S}$.

The letters "RO" stand for "right Ore". We refer to the expression $q = \mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1}$ as a *right fraction representation of* q.

There is an obvious notion of *left Ore localization*, with properties (LO1)-(LO4) that are identical to (RO1)-(RO4) respectively, except that in the last two the compositions are reversed: $q = \mathrm{Q}(s)^{-1} \circ \mathrm{Q}(a)$ and $s \circ a = s \circ b$. The results to follow in this section all have "left" versions, with identical proofs (just a matter of reversing some arrows or compositions), and so they will be omitted.

Lemma 8.2.2. *Let* $(\mathsf{A}_{\mathsf{S}}, \mathrm{Q})$ *be a right Ore localization, let* $a_1, a_2 \in \mathsf{A}$ *and* $s_1, s_2 \in \mathsf{S}$. *TFAE:*

(i) $\mathrm{Q}(a_1) \circ \mathrm{Q}(s_1)^{-1} = \mathrm{Q}(a_2) \circ \mathrm{Q}(s_2)^{-1}$ *in* A_{S}.
(ii) *There are* $b_1, b_2 \in \mathsf{A}$ *s.t.* $a_1 \circ b_1 = a_2 \circ b_2$, *and* $s_1 \circ b_1 = s_2 \circ b_2 \in \mathsf{S}$.

[3] This is the only way the author was able to understand the proofs.

Proof. (ii) $\Rightarrow$ (i): Since $\mathrm{Q}(s_i)$ and $\mathrm{Q}(s_i \circ b_i)$ are invertible, it follows that $\mathrm{Q}(b_i)$ are invertible. So

$$\begin{aligned} \mathrm{Q}(a_1) \circ \mathrm{Q}(s_1)^{-1} &= \mathrm{Q}(a_1) \circ \mathrm{Q}(b_1) \circ \mathrm{Q}(b_1)^{-1} \circ \mathrm{Q}(s_1)^{-1} \\ &= \mathrm{Q}(a_2) \circ \mathrm{Q}(b_2) \circ \mathrm{Q}(b_2)^{-1} \circ \mathrm{Q}(s_2)^{-1} = \mathrm{Q}(a_2) \circ \mathrm{Q}(s_2)^{-1}. \end{aligned}$$

(i) $\Rightarrow$ (ii): By property (RO3) there are $c \in \mathsf{A}$ and $u \in \mathsf{S}$ s.t.

$$\mathrm{Q}(s_2)^{-1} \circ \mathrm{Q}(s_1) = \mathrm{Q}(c) \circ \mathrm{Q}(u)^{-1}. \tag{8.2.3}$$

Rewriting this equation we get

$$\mathrm{Q}(s_1 \circ u) = \mathrm{Q}(s_2 \circ c). \tag{8.2.4}$$

It is given that

$$\mathrm{Q}(a_1) = \mathrm{Q}(a_2) \circ \mathrm{Q}(s_2)^{-1} \circ \mathrm{Q}(s_1).$$

Plugging (8.2.3) into it we obtain

$$\mathrm{Q}(a_1) = \mathrm{Q}(a_2) \circ \mathrm{Q}(c) \circ \mathrm{Q}(u)^{-1}.$$

Rearranging this equation we get

$$\mathrm{Q}(a_1 \circ u) = \mathrm{Q}(a_2 \circ c). \tag{8.2.5}$$

By property (RO4) there is $v \in \mathsf{S}$ s.t.

$$a_1 \circ u \circ v = a_2 \circ c \circ v.$$

Likewise, from equation (8.2.4) and property (RO4), there is $v' \in \mathsf{S}$ s.t.

$$s_1 \circ u \circ v' = s_2 \circ c \circ v'.$$

Again using property (RO3), there are $d \in \mathsf{A}$ and $w \in \mathsf{S}$ s.t.

$$\mathrm{Q}(v)^{-1} \circ \mathrm{Q}(v') = \mathrm{Q}(d) \circ \mathrm{Q}(w)^{-1}.$$

Rearranging we get

$$\mathrm{Q}(v' \circ w) = \mathrm{Q}(v \circ d).$$

By property (RO4) there is $w' \in \mathsf{S}$ s.t.

$$v' \circ w \circ w' = v \circ d \circ w'.$$

Define

$$b_1 := u \circ v \circ d \circ w' \ , \ b_2 := c \circ v \circ d \circ w' \ .$$

Then

$$\begin{aligned} s_1 \circ b_1 &= s_1 \circ u \circ v \circ d \circ w' = s_1 \circ u \circ v' \circ w \circ w' \\ &= s_2 \circ c \circ v' \circ w \circ w' = s_2 \circ b_2, \end{aligned}$$

and it is in S. Also

$$a_1 \circ b_1 = a_1 \circ u \circ v \circ d \circ w' = a_2 \circ c \circ v \circ d \circ w' = a_2 \circ b_2.$$

□

Proposition 8.2.6. *A right Ore localization* $(\mathsf{A}_\mathsf{S}, \mathrm{Q})$ *is a localization in the sense of Definition* 8.1.2.

Proof. Say B is a category, and $F : \mathsf{A} \to \mathsf{B}$ is a functor such that $F(s)$ is an isomorphism for every $s \in \mathsf{S}$.

The uniqueness of a functor $F_{\mathsf{S}} : \mathsf{A}_{\mathsf{S}} \to \mathsf{B}$ satisfying $F_{\mathsf{S}} \circ \mathrm{Q} = F$ is clear from property (RO3). We have to prove existence.

Define F_{S} to be F on objects, and

$$F_{\mathsf{S}}(q) := F(a_1) \circ F(s_1)^{-1},$$

where

$$q = \mathrm{Q}(a_1) \circ \mathrm{Q}(s_1)^{-1} \in \mathsf{A}_{\mathsf{S}},\ a_1 \in \mathsf{A},\ s_1 \in \mathsf{S}$$

is any presentation of q as a right fraction, that exists by (RO3). We have to prove that this is well defined. So suppose that $q = \mathrm{Q}(a_2) \circ \mathrm{Q}(s_2)^{-1}$ is another presentation of q. Let $b_1, b_2 \in \mathsf{A}$ be as in Lemma 8.2.2. Since $F(s_i)$ and $F(s_i \circ b_i)$ are invertible, then so is $F(b_i)$. We get

$$F(a_2) = F(a_1) \circ F(b_1) \circ F(b_2)^{-1}$$

and

$$F(s_2) = F(s_1) \circ F(b_1) \circ F(b_2)^{-1}.$$

Hence

$$F(a_2) \circ F(s_2)^{-1} = F(a_1) \circ F(s_1)^{-1}.$$

It remains to prove that F_{S} is a functor. Since the identity 1_M of the object M in A_{S} can be presented as $1_M = \mathrm{Q}(1_M) \circ \mathrm{Q}(1_M)^{-1}$, we see that $F_{\mathsf{S}}(1_M) = 1_{F(M)}$.

Next let q_1 and q_2 be morphisms in A_{S}, such that composition $q_2 \circ q_1$ exists (i.e. the target of q_1 is the source of q_2). We have to show that $F_{\mathsf{S}}(q_2 \circ q_1)$ equals $F_{\mathsf{S}}(q_2) \circ F_{\mathsf{S}}(q_1)$. Choose presentations $q_i = \mathrm{Q}(a_i) \circ \mathrm{Q}(s_i)^{-1}$, so that

$$F_{\mathsf{S}}(q_2) \circ F_{\mathsf{S}}(q_1) = F(a_2) \circ F(s_2)^{-1} \circ F(a_1) \circ F(s_1)^{-1}. \tag{8.2.7}$$

By property (R03) there is a right fraction presentation

$$\mathrm{Q}(s_2)^{-1} \circ \mathrm{Q}(a_1) = \mathrm{Q}(b) \circ \mathrm{Q}(t)^{-1} \tag{8.2.8}$$

for some $b \in \mathsf{A}$ and $t \in \mathsf{S}$. Because

$$\mathrm{Q}(a_1 \circ t) = \mathrm{Q}(s_2 \circ b),$$

by (RO4) there is some $r \in \mathsf{S}$ such that

$$a_1 \circ t \circ r = s_2 \circ b \circ r.$$

Therefore

$$F(a_1 \circ t \circ r) = F(s_2 \circ b \circ r),$$

which implies, by canceling the invertible morphism $F(r)$ and rearranging, that

$$F(s_2)^{-1} \circ F(a_1) = F(b) \circ F(t)^{-1} \tag{8.2.9}$$

in B.

Let us continue. Using equation (8.2.8) we have

$$\begin{aligned} q_2 \circ q_1 &= \mathrm{Q}(a_2) \circ \mathrm{Q}(s_2)^{-1} \circ \mathrm{Q}(a_1) \circ \mathrm{Q}(s_1)^{-1} \\ &= \mathrm{Q}(a_2) \circ \mathrm{Q}(b) \circ \mathrm{Q}(t)^{-1} \circ \mathrm{Q}(s_1)^{-1} = \mathrm{Q}(a_2 \circ b) \circ \mathrm{Q}(s_1 \circ t)^{-1}. \end{aligned}$$

Using this presentation of $q_2 \circ q_1$, and the equality (8.2.9), we obtain

$$\begin{aligned} F_{\mathsf{S}}(q_2 \circ q_1) &= F(a_2 \circ b) \circ F(s_1 \circ t)^{-1} = F(a_2) \circ F(b) \circ F(t)^{-1} \circ F(s_1)^{-1} \\ &= F(a_2) \circ F(s_2)^{-1} \circ F(a_1) \circ F(s_1)^{-1}. \end{aligned}$$

This is the same as (8.2.7). □

Corollary 8.2.10. *Let* S *be a multiplicatively closed set of morphisms in a category* A. *Assume that* $(\mathsf{A}_{\mathsf{S}}, \mathrm{Q})$ *and* $(\mathsf{A}'_{\mathsf{S}}, \mathrm{Q}')$ *are either right Ore localizations or left Ore localizations of* A *with respect to* S. *Then there is a unique isomorphism of localizations*

$$(\mathsf{A}_{\mathsf{S}}, \mathrm{Q}) \cong (\mathsf{A}'_{\mathsf{S}}, \mathrm{Q}'),$$

as in Proposition 8.1.3.

Proof. By Proposition 8.2.6 (in its right or left versions, as the case may be), both $(\mathsf{A}_{\mathsf{S}}, \mathrm{Q})$ and $(\mathsf{A}'_{\mathsf{S}}, \mathrm{Q}')$ are localizations in the sense of Definition 8.1.2. Hence, by Proposition 8.1.3, there is a unique isomorphism $(\mathsf{A}_{\mathsf{S}}, \mathrm{Q}) \cong (\mathsf{A}'_{\mathsf{S}}, \mathrm{Q}')$. □

Definition 8.2.11. Let S be multiplicatively closed set of morphisms in a category A. We say that S is a *right denominator set* if it satisfies these two conditions:

(RD1) Given $a \in \mathsf{A}$ and $s \in \mathsf{S}$, there exist $b \in \mathsf{A}$ and $t \in \mathsf{S}$ s.t. $a \circ t = s \circ b$.
(RD2) Given $a, b \in \mathsf{A}$ and $s \in \mathsf{S}$ s.t. $s \circ a = s \circ b$, there exists $t \in \mathsf{S}$ s.t. $a \circ t = b \circ t$.

In commutative diagrams:

$$\begin{tikzcd} & K \arrow[dl, dashed, "t"'] \arrow[dr, dashed, "b"] & \\ M \arrow[dr, "a"'] & & N \arrow[dl, "s"] \\ & L & \end{tikzcd} \qquad \begin{tikzcd} K \arrow[d, dashed, "t"'] \\ M \arrow[d, bend right, "a"'] \arrow[d, bend left, "b"] \\ N \arrow[d, "s"'] \\ L \end{tikzcd}$$

There is a similar left version of this definition, with conditions (LD1) and (LD2). Here is the main theorem regarding Ore localization.

Theorem 8.2.12. *The following conditions are equivalent for a category* A *and a multiplicatively closed set of morphisms* $\mathsf{S} \subseteq \mathsf{A}$.

(i) *The right Ore localization* $(\mathsf{A}_{\mathsf{S}}, \mathrm{Q})$ *exists.*
(ii) S *is a right denominator set.*

The proof of Theorem 8.2.12 is after some preparation. The hard part is proving that (ii) $\Rightarrow$ (i). The general idea is the same as in commutative localization: we consider the set of pairs of morphisms $\mathsf{A} \times \mathsf{S}$, and define a relation $\sim$ on it, with the hope that this is an equivalence relation, and that the quotient set A_{S} will be a category, and it will have the desired properties.

Let's assume that S is a right denominator set. For any $M, N \in \mathrm{Ob}(A)$ consider the set

$$(\mathsf{A} \times \mathsf{S})(M, N) := \coprod_{L \in \mathrm{Ob}(A)} \mathsf{A}(L, N) \times \mathsf{S}(L, M).$$

Remark 8.2.13. The set $(\mathsf{A} \times \mathsf{S})(M, N)$ could be a big, namely not an element of the initial universe U. This would require the introduction of a larger universe, say V, in which U is an element. And the resulting category A_{S} will be a V-category.

We will ignore this issue. Moreover, in many cases of interest (derived categories where there are DG enhancements, such as the K-injective enhancement), there will be an alternative presentation of A_{S} as a U-category. We will refer to this when we get to it.

An element $(a, s) \in (\mathsf{A} \times \mathsf{S})(M, N)$ can be pictured as a diagram

(8.2.14)
$$\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "a"] & \\ M & & N \end{tikzcd}$$

in A. This diagram will eventually represent the right fraction

$$\mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1} : M \to N.$$

Definition 8.2.15. We define a relation $\sim$ on the set $\mathsf{A} \times \mathsf{S}$ like this:

$$(a_1, s_1) \sim (a_2, s_2)$$

if there exist $b_1, b_2 \in \mathsf{A}$ s.t.

$$a_1 \circ b_1 = a_2 \circ b_2 \text{ and } s_1 \circ b_1 = s_2 \circ b_2 \in \mathsf{S}\,.$$

Note that the relation $\sim$ imposes condition (ii) of Lemma 8.2.2.

Here it is in a commutative diagram, in which we have made the objects explicit:

(8.2.16)
$$\begin{tikzcd} & K \arrow[dl, "b_1"'] \arrow[dr, "b_2"] & \\ L_1 \arrow[d, "s_1"'] \arrow[drr, "a_1"' near end] & & L_2 \arrow[d, "a_2"] \arrow[dll, "s_2" near end] \\ M & & N \end{tikzcd}$$

The arrows ending at M are in S.

Lemma 8.2.17. *If the right Ore condition holds then the relation $\sim$ is an equivalence.*

Proof. Reflexivity: take $K := L$ and $b_i := 1_L : L \to L$. Symmetry is trivial.

Now to prove transitivity. Suppose we are given $(a_1, s_1) \sim (a_2, s_2)$ and $(a_2, s_2) \sim (a_3, s_3)$. So we have the solid part of the first diagram in Figure 2, and it is commutative. The arrows ending at M are in S.

By condition (RD1) applied to $K \to M \leftarrow J$ there are $t \in \mathsf{S}$ and $d \in \mathsf{A}$ s.t.

$$(s_3 \circ c_3) \circ d = (s_1 \circ b_1) \circ t.$$

Comparing arrows $I \to M$ in this diagram, we see that

$$s_2 \circ (b_2 \circ t) = s_1 \circ b_1 \circ t = s_3 \circ c_3 \circ d = s_2 \circ (c_2 \circ d).$$

By (RD2) there is $u \in \mathsf{S}$ s.t.

$$(b_2 \circ t) \circ u = (c_2 \circ d) \circ u.$$

So all paths $H \to M$ are equal and belong to S, and all paths $H \to N$ are equal. Now delete the object L_2 and the arrows going through it. Then delete the objects I, J, K, but keep the paths going through them. We get the second diagram in

$$
\begin{tikzcd}
& H \arrow[d, dashed, "u"] & \\
& I \arrow[dl, dashed, "t"'] \arrow[dr, dashed, "d"] & \\
K \arrow[d, "b_1"'] \arrow[dr, "b_2"] & & J \arrow[dl, "c_2"'] \arrow[d, "c_3"] \\
L_1 \arrow[d, "s_1"'] \arrow[drr, "a_1"] & L_2 \arrow[dl, "s_2"] \arrow[dr, "a_2"'] & L_3 \arrow[dll, "s_3"] \arrow[d, "a_3"] \\
M & & N
\end{tikzcd}
\qquad
\begin{tikzcd}
& H \arrow[dl, "b_1 \circ t \circ u"'] \arrow[dr, "c_3 \circ d \circ u"] & \\
L_1 \arrow[d, "s_1"'] \arrow[drr, "a_1"] & & L_3 \arrow[dll, "s_3"] \arrow[d, "a_3"] \\
M & & N
\end{tikzcd}
$$

FIGURE 2.

Figure 2. It is commutative, and all arrows ending at M are in S. This is evidence for $(a_1, s_1) \sim (a_3, s_3)$. $\square$

Proof of Theorem 8.2.12.

Step 1. In this step we prove (i) $\Rightarrow$ (ii). Take $a \in \mathsf{A}$ and $s \in \mathsf{S}$. Consider $q := \mathrm{Q}(s)^{-1} \circ \mathrm{Q}(a)$. By (RO3) there are $b \in \mathsf{A}$ and $t \in \mathsf{S}$ s.t. $q = \mathrm{Q}(b) \circ \mathrm{Q}(t)^{-1}$. So

$$\mathrm{Q}(s \circ b) = \mathrm{Q}(a \circ t).$$

By (RO4) there is $u \in \mathsf{S}$ s.t.

$$(s \circ b) \circ u = (a \circ t) \circ u.$$

We read this as

$$s \circ (b \circ u) = a \circ (t \circ u),$$

and note that $t \circ u \in \mathsf{S}$. So (RD1) holds.

Next $a, b \in \mathsf{A}$ and $s \in \mathsf{S}$ s.t. $s \circ a = s \circ b$. Then $\mathrm{Q}(s \circ a) = \mathrm{Q}(s \circ b)$. But $\mathrm{Q}(s)$ is invertible, so $\mathrm{Q}(a) = \mathrm{Q}(b)$. By (RO4) there is $t \in \mathsf{S}$ s.t. $a \circ t = b \circ t$. We have proved (RD2).

Step 2. Now we assume that condition (ii) holds, and we define the morphism sets $\mathsf{A}_{\mathsf{S}}(M, N)$, composition between them, and the identity morphisms.

For any $M, N \in \mathrm{Ob}(\mathsf{A})$ let

$$\mathsf{A}_{\mathsf{S}}(M, N) := \frac{(\mathsf{A} \times \mathsf{S})(M, N)}{\sim},$$

where $\sim$ is the relation from Definition 8.2.15, which is an equivalence relation by Lemma 8.2.17.

$$\begin{tikzcd}
 & & K \arrow[dl, "u"'] \arrow[dr, "c"] & & \\
 & L_1 \arrow[dl, "s_1"'] \arrow[dr, "a_1"] & & L_2 \arrow[dl, "s_2"'] \arrow[dr, "a_2"] & \\
M_0 & & M_1 & & M_2
\end{tikzcd}$$

FIGURE 3.

$$\begin{tikzcd}
J_1 \arrow[d, dashed, "b_1"'] \arrow[dr, dashed, "b_1'"] & K \arrow[dl, "u"'] \arrow[drr, "c"] & & K' \arrow[dll, "u'"'] \arrow[dr, "c'"] & J_2 \arrow[dl, dashed, "b_2"'] \arrow[d, dashed, "b_2'"] \\
L_1 \arrow[d, "s_1"'] \arrow[drr, "a_1"] & L_1' \arrow[dl, "s_1'"] \arrow[dr, "a_1'"] & & L_2 \arrow[dl, "s_2"'] \arrow[dr, "a_2"] & L_2' \arrow[dll, "s_2'"] \arrow[d, "a_2'"] \\
M_0 & & M_1 & & M_2
\end{tikzcd}$$

FIGURE 4.

We define composition like this. Given $q_1 \in \mathsf{A}_{\mathsf{S}}(M_0, M_1)$ and $q_2 \in \mathsf{A}_{\mathsf{S}}(M_1, M_2)$, choose representatives $(a_i, s_i) \in (\mathsf{A} \times \mathsf{S})(M_{i-1}, M_i))$. We use the notation $q_i = \overline{(a_i, s_i)}$ to indicate this. By (RD1) there are $c \in \mathsf{A}$ and $u \in \mathsf{S}$ s.t. $s_2 \circ c = a_1 \circ u$. The composition

$$q_2 \circ q_1 \in \mathsf{A}_{\mathsf{S}}(M_0, M_2)$$

is defined to be

$$q_2 \circ q_1 := \overline{(a_2 \circ c, s_1 \circ u)} \in (\mathsf{A} \times \mathsf{S})(M_0, M_2)).$$

The idea behind the formula can be seen in the diagram in Figure 3.

We have to verify that this definition is independent of the representatives. So suppose we take other representatives $q_i = \overline{(a'_i, s'_i)}$, and we choose morphisms u', c' to construct the composition. This is the solid part of the diagram in Figure 4, and it is a commutative diagram. We must prove that

$$\overline{(a_2 \circ c, s_1 \circ u)} = \overline{(a'_2 \circ c', s'_1 \circ u')}.$$

There are morphisms b_i, b'_i the are evidence for $(a_i, s_i) \sim (a'_i, s'_i)$. They are depicted as the dashed arrows in Figure 4. That whole diagram is commutative. The morphisms $J_1 \to M_0$, $K \to M_0$, $K' \to M_0$ and $J_2 \to M_1$ are all in S.

Choose $v_1 \in \mathsf{S}$ and $d_1 \in \mathsf{A}$ s.t. the first diagram in Figure 5 is commutative. This can be done by (RD1).

Consider the solid part of the middle diagram in Figure 5. Since $J_2 \to M_1$ is in S, by (RD1) there are $\tilde{v}_2 \in \mathsf{S}$ and $\tilde{d}_2 \in \mathsf{A}$ s.t. the two paths $\tilde{I}_2 \to M_1$ are equal. By (RD2) there is $\tilde{v} \in \mathsf{S}$ s.t. the two paths $I_2 \to L'_2$ are equal. We get the commutative

$$
\begin{tikzcd}
 & I_1 \arrow[dl, "v_1"'] \arrow[dr, "d_1"] & \\
J_1 \arrow[dr, "b_1"'] & & K \arrow[dl, "u"] \\
 & L_1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & I_2 \arrow[d, dashed, "\tilde{v}"] & \\
 & \tilde{I}_2 \arrow[dl, dashed, "\tilde{v}_2"'] \arrow[dr, dashed, "\tilde{d}_2"] & \\
K' \arrow[dr, "c'"'] & & J_2 \arrow[dl, "b_2'"] \\
 & L_2' \arrow[d, "s_2'"'] & \\
 & M_1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & I_2 \arrow[dl, "v_2"'] \arrow[dr, "d_2"] & \\
K' \arrow[dr, "c'"'] & & J_2 \arrow[dl, "b_2'"] \\
 & L_2' &
\end{tikzcd}
$$

FIGURE 5.

$$
\begin{tikzcd}
H'' \arrow[r, dashed, "w''"] & H' \arrow[r, dashed, "w'"] & H \arrow[dl, dashed, "w"'] \arrow[dr, dashed, "e"] & & \\
 & I_1 \arrow[dl, "v_1"'] \arrow[d, "d_1"] & & I_2 \arrow[d, "v_2"'] \arrow[dr, "d_2"] & \\
J_1 \arrow[d, "b_1"'] \arrow[dr, "b_1'"] & K \arrow[dl, "u"] \arrow[drr, "c"] & & K' \arrow[dll, "u'"'] \arrow[dr, "c'"] & J_2 \arrow[dl, "b_2"'] \arrow[d, "b_2'"] \\
L_1 \arrow[d, "s_1"'] \arrow[drr, "a_1"] & L_1' \arrow[dl, "s_1'"] \arrow[dr, "a_1'"] & & L_2 \arrow[dl, "s_2"'] \arrow[dr, "a_2"] & L_2' \arrow[dll, "s_2'"'] \arrow[d, "a_2'"] \\
M_0 & & M_1 & & M_2
\end{tikzcd}
$$

FIGURE 6.

diagram in the middle of Figure 5. Next, defining $d_2 := \tilde{d}_2 \circ \tilde{v}$ and $v_2 := \tilde{v}_2 \circ \tilde{v} \in \mathsf{S}$, we obtain the third commutative diagram in Figure 5.

We now embed the first and third diagrams from Figure 5 into the diagram in Figure 4. This gives us the solid diagram in Figure 6, and it is commutative. The morphisms $I_1 \to M_0$ belong to S.

Choose $w \in \mathsf{S}$ and $e \in \mathsf{A}$, starting at an object H, to fill the diagram $I_1 \to M_0 \leftarrow I_2$, using (RD1). The path $H \to I_1 \to M_0$ is in S, and all the paths $H \to M_0$

$$
\begin{tikzcd}
 & & H'' \arrow[dl, "w \circ w' \circ w''"'] \arrow[dr, "e \circ w' \circ w''"] & & \\
 & I_1 \arrow[d, "d_1"] & & I_2 \arrow[d, "v_2"'] & \\
 & K \arrow[dl, "u"] \arrow[drr, "c"] & & K' \arrow[dll, "u'"'] \arrow[dr, "c'"'] & \\
L_1 \arrow[d, "s_1"'] & L_1' \arrow[dl, "s_1'"] & & L_2 \arrow[dr, "a_2"'] & L_2' \arrow[d, "a_2'"] \\
M_0 & & & & M_2
\end{tikzcd}
$$

FIGURE 7.

are equal. But we could have failure of commutativity in the paths $H \to L_1'$ and $H \to L_2$.

The two paths $H \to L_1'$ in Figure 6 satisfy

$$s_1' \circ (b_1' \circ v_1 \circ w) = s_1' \circ (u' \circ v_2 \circ e).$$

Therefore there is $w' \in \mathsf{S}$ s.t.

$$(b_1' \circ v_1 \circ w) \circ w' = (u' \circ v_2 \circ e) \circ w'.$$

Next, the two paths $H' \to L_2$ satisfy

$$s_2 \circ (c \circ d_1 \circ w \circ w') = s_2 \circ (b_2 \circ d_2 \circ e \circ w');$$

this is because we can take a detour through L_1'. Therefore there is $w'' \in \mathsf{S}$ s.t.

$$(c \circ d_1 \circ w \circ w') \circ w'' = (b_2 \circ d_2 \circ e \circ w') \circ w''.$$

Now all paths $H'' \to M_2$ in Figure 6 are equal. All paths $H'' \to M_0$ are equal and are in S.

Erase the objects M_1, J_1, J_2 and all arrows touching them from Figure 6. Then erase H, H', but keep the paths through them. We obtain the commutative diagram in Figure 7. This is evidence for

$$(a_2 \circ c, s_1 \circ u) \sim (a_2' \circ c', s_1' \circ u').$$

The proof that composition is well-defined is done.

The identity morphism 1_M of an object M is $\overline{(1_M, 1_M)}$.

Step 3. We have to verify the associativity and the identity properties of composition in A_S. Namely that A_S is a category. This seems to be not too hard, given Step 2, and we leave it as an exercise!

Step 4. The functor $\mathsf{Q} : \mathsf{A} \to \mathsf{A}_{\mathsf{S}}$ is defined to be $\mathrm{Q}(M) := M$ on objects, and $\mathrm{Q}(a) := \overline{(a, 1_M)}$ for $a : M \to N$ in A. We have to verify this is a functor... Again, an exercise.

Step 5. Finally we verify properties (RO1)-(RO4). (RO1) is clear. The inverse of $\mathrm{Q}(s)$ is $\overline{(1, s)}$, so (RO2) holds.

It is not hard to see that

$$\overline{(a, s)} = \overline{(a, 1)} \circ \overline{(1, s)};$$

this is (RO3).

If $\mathrm{Q}(a_1) = \mathrm{Q}(a_2)$, then $(a_1, 1_M) \sim (a_2, 1_M)$; so there are $b_1, b_2 \in \mathsf{A}$ s.t. $a_1 \circ b_1 = a_2 \circ b_2$ and $1 \circ b_1 = 1 \circ b_2 \in \mathsf{S}$. Writing $s := b_1 \in \mathsf{S}$, we get $a_1 \circ s = a_2 \circ s$. This proves (RO4). $\square$

Proposition 8.2.18. *Let* A *be a category, let* S *be a right denominator set in* A*, and let* $(\mathsf{A}_{\mathsf{S}}, \mathrm{Q})$ *be the right Ore localization. For any two morphisms* $q_1, q_2 : M \to N$ *in* A_{S} *there is a common denominator. Namely we can write*

$$q_i = \mathrm{Q}(a_i) \circ \mathrm{Q}(s)^{-1}$$

for suitable $a_i \in \mathsf{A}$ *and* $s \in \mathsf{S}$*.*

Proof. Choose representatives $q_i = \mathrm{Q}(a'_i) \circ \mathrm{Q}(s'_i)^{-1}$. By (RD1) applied to $L_1 \to M \leftarrow L_2$, there are $b \in \mathsf{A}$ and $t \in \mathsf{S}$ s.t. the diagram above M commutes:

$$\begin{tikzcd}
& L \arrow[dl, "t"'] \arrow[dr, "b"] & \\
L_1 \arrow[d, "s'_1"'] \arrow[drr, "a'_1"' near end] & & L_2 \arrow[d, "a'_2"] \arrow[dll, "s'_2"' near start] \\
M & & N
\end{tikzcd}$$

Write $s := s'_1 \circ t = s'_2 \circ b$, $a_1 := a'_1 \circ t$ and $a_2 := a'_2 \circ b$. By Lemma 8.2.2 we get $q_i = \mathrm{Q}(a_i) \circ \mathrm{Q}(s)^{-1}$. $\square$

Exercise 8.2.19. Let A be a category, let S be a right denominator set in A. Let Y be a subset of $\mathrm{Ob}(\mathsf{A})$, and let B and T be the full subcategories of A and S respectively on the set of objects Y.

1. Is T a right denominator set in B ?
2. Show that if T a right denominator set in B, then the inclusion functor $F : \mathsf{B} \to \mathsf{A}$ extends uniquely to a functor $F_{\mathsf{T}} : \mathsf{B}_{\mathsf{T}} \to \mathsf{A}_{\mathsf{S}}$.
3. Assume that T a right denominator set in B. Is the functor F_{T} full or faithful?

We will return to these questions later.

8.3. **Localization of Linear Categories.** Until now in this section we dealt with arbitrary categories. In this and the subsequent subsection, our categories will be linear over some commutative base ring $\mathbb{K}$ (that will be implicit in everything).

Theorem 8.3.1. *Let* A *be a* $\mathbb{K}$*-linear category, let* S *be a right denominator set in* A*, and let* $(\mathsf{A}_{\mathsf{S}}, \mathrm{Q})$ *be the right Ore localization.*

(1) *The category* A_S *has a unique* $\mathbb{K}$*-linear structure, such that* $\mathrm{Q} : \mathsf{A} \to \mathsf{A}_\mathsf{S}$ *is a* $\mathbb{K}$*-linear functor.*
(2) *Suppose* B *is another* $\mathbb{K}$*-linear category, and* $F : \mathsf{A} \to \mathsf{B}$ *is a* $\mathbb{K}$*-linear functor s.t.* $F(s)$ *is invertible for every* $s \in \mathsf{S}$*. Let* $F_\mathsf{S} : \mathsf{A}_\mathsf{S} \to \mathsf{B}$ *be the localization of* F*. Then* F_S *is a* $\mathbb{K}$*-linear functor.*
(3) *If* A *is an additive category, then so is* A_S*.*

Proof. (1) Let $q_i : M \to N$ be morphisms in A_S. Choose common denominator presentations $q_i = \mathrm{Q}(a_i) \circ \mathrm{Q}(s)^{-1}$. Since Q must be an additive functor, we have to define

$$\mathrm{Q}(a_1) + \mathrm{Q}(a_2) := \mathrm{Q}(a_1 + a_2). \tag{8.3.2}$$

By the distributive law (bilinearity of composition) we must define

$$q_1 + q_2 := \mathrm{Q}(a_1 + a_2) \circ \mathrm{Q}(s)^{-1}.$$

For $\lambda \in \mathbb{K}$ we must define

$$\lambda \cdot q_i := \mathrm{Q}(\lambda \cdot a_i) \circ \mathrm{Q}(s)^{-1}.$$

The usual tricks are then used to prove independence of representatives. For instance, to prove that (8.3.2) is independent of choices, suppose that $\mathrm{Q}(a_1) = \mathrm{Q}(a_1')$ and $\mathrm{Q}(a_2) = \mathrm{Q}(a_2')$. Then, by (RO4), there are $t_1, t_2 \in \mathsf{S}$ such that $a_1 \circ t_1 = a_1' \circ t_1$ and $a_2 \circ t_2 = a_2' \circ t_2$. By (RD1) there exist $b \in \mathsf{A}$ and $v \in \mathsf{S}$ s.t. $t_1 \circ b = t_2 \circ v$. Let $t_3 := t_2 \circ v \in \mathsf{S}$. Then

$$(a_1 + a_2) \circ t_3 = (a_1' + a_2') \circ t_3,$$

and hence

$$\mathrm{Q}(a_1 + a_2) = \mathrm{Q}(a_1' + a_2').$$

In this way A_S is a $\mathbb{K}$-linear category, and Q is a $\mathbb{K}$-linear functor.

(2) The only option for F_S is $F_\mathsf{S}(q_i) := F(a_i) \circ F(s)^{-1}$. The usual tricks are used to prove independence of representatives.

(3) Clear from Propositions 2.4.2 and 2.4.4. □

This includes the case $\mathbb{K} = \mathbb{Z}$ of course. There is a left version of Theorem 8.3.1.

Example 8.3.3. Let A be a ring, which we can think of as a one object linear category A. In this context, Theorem 8.3.1 is one of the most important results in ring theory. See [Row, Ste].

Example 8.3.4. Suppose A is a commutative ring, and S is a multiplicatively closed set in it. Because A is commutative, the denominator conditions hold automatically. The localized category A_S is the single object category, with endomorphism set A_S. This is simply the usual commutative localization.

Note that if S contains a nilpotent element, then the ring A_S is trivial.

The observation above should serve as a warning: localization can sometimes kill everything. This is the singularity effect: dividing by zero!

Fortunately, the localization procedure (8.0.1), that gives rise to the derived category, does not cause any catastrophe, as we shall see in Proposition 8.4.9.

Remark 8.3.5. Suppose A is a ring and S is a right denominator set in it. Then the right Ore localization A_S is *flat* as left A-module. See [Row, Theorem 3.1.20]. I have no idea if something like this is true for linear categories with more than one object.

8.4. **Localization of Pretriangulated Categories.** Let K be a pretriangulated category, with translation functor T.

Proposition 8.4.1. *Suppose $H : \mathsf{K} \to \mathsf{M}$ is a cohomological functor, where M is some abelian category. Let*

$$\mathsf{S} := \{s \in \mathsf{K} \mid H(\mathrm{T}^i(s)) \text{ is invertible for all } i \in \mathbb{Z}\}.$$

Then S is a left and right denominator set in K.

Proof. It is clear that S is closed under composition and contains the identity morphisms. So it is a multiplicatively closed set.

Let's prove that condition (RD1) of Definition 8.2.11 holds. Suppose we are given morphisms $L \xrightarrow{a} N \xleftarrow{s} M$ with $s \in \mathsf{S}$. We need to find morphisms $L \xleftarrow{t} K \xrightarrow{b} M$ with $t \in \mathsf{S}$ and such that $a \circ t = s \circ b$.

Consider the solid commutative diagram

$$\begin{tikzcd}
K \arrow[r, "t"] \arrow[d, dashed, "b"'] & L \arrow[r, "c \circ a"] \arrow[d, "a"'] & P \arrow[r] \arrow[d, "="'] & \mathrm{T}(K) \arrow[d, dashed, "\mathrm{T}(b)"'] \\
M \arrow[r, "s"] & N \arrow[r, "c"] & P \arrow[r] & \mathrm{T}(M)
\end{tikzcd}$$

where the bottom row is a distinguished triangle built on $M \xrightarrow{s} N$, and and the top row is a distinguished triangle built on $L \xrightarrow{c \circ a} P$, then turned $120°$ to the right. By axiom (TR3) there is a morphism b making the diagram commutative. Thus $a \circ t = s \circ b$. Since $H(\mathrm{T}^i(s))$ are invertible for all $i \in \mathbb{Z}$, it follows that $H(\mathrm{T}^i(P)) = 0$. But then $H(\mathrm{T}^i(t))$ are invertible for all $i \in \mathbb{Z}$, so $t \in \mathsf{S}$.

Next we prove condition (RD2) of Definition 8.2.11. Because we are in an additive category, this condition is simplified: given $a \in \mathsf{K}$ and $s \in \mathsf{S}$ satisfying $s \circ a = 0$, we have to find $t \in \mathsf{S}$ satisfying $a \circ t = 0$.

Say the objects involved are $L \xrightarrow{a} M \xrightarrow{s} N$. Take a distinguished triangle built on s and then turned:

$$P \xrightarrow{b} M \xrightarrow{s} N \to \mathrm{T}(P).$$

We get an exact sequence

$$\mathrm{Hom}_{\mathsf{K}}(L, P) \xrightarrow{b \circ -} \mathrm{Hom}_{\mathsf{K}}(L, M) \xrightarrow{s \circ -} \mathrm{Hom}_{\mathsf{K}}(L, N).$$

Since $s \circ a = 0$, there is $c : L \to P$ s.t. $a = b \circ c$. Now look at a distinguished triangle built on c, and then turned:

$$K \xrightarrow{t} L \xrightarrow{c} P \to \mathrm{T}(K).$$

We know that $c \circ t = 0$; hence $a \circ t = b \circ c \circ t = 0$. But $(s \in \mathsf{S}) \Rightarrow (H(\mathrm{T}^i(P)) = 0$ for all $i) \Rightarrow (t \in \mathsf{S})$.

The left versions (LD1) and (LD2) are proved the same way. $\square$

Theorem 8.4.2. *Let K be a pretriangulated category. Let S be the denominator set in K associated to a cohomological functor, as in Proposition* 8.4.1*, and let $(\mathsf{K}_{\mathsf{S}}, \mathrm{Q})$ be the Ore localization. The additive category K_{S} has a unique pretriangulated structure such that these two properties hold*:

(i) *The functor $\mathrm{Q} : \mathsf{K} \to \mathsf{K}_{\mathsf{S}}$ is triangulated.*

(ii) *Suppose* E *is another pretriangulated category, and* $F : \mathsf{K} \to \mathsf{E}$ *is a triangulated functor such that* $F(s)$ *is invertible for every* $s \in \mathsf{S}$*. Let* $F_{\mathsf{S}} : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$ *be the localization of* F*. Then* F_{S} *is a triangulated functor.*

Proof. Step 1. We define the translation functor T_{S} on K_{S}. For objects we take $\mathrm{T}_{\mathsf{S}}(M) := \mathrm{T}(M)$ of course. And for a morphism $q \in \mathsf{K}_{\mathsf{S}}$ we choose a presentation $q = \mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1}$, and define $\mathrm{T}_{\mathsf{S}}(q) := \mathrm{Q}(\mathrm{T}(a)) \circ \mathrm{Q}((\mathrm{T}(s))^{-1}$. We must prove independence of presentation; but this is standard.

Step 2. The distinguished triangles in K_{S} are defined to be those triangles that are isomorphic to the images under Q of distinguished triangles in K. Let us verify the axioms of pretriangulated category.

(TR1). By definition every triangle that's isomorphic to a distinguished triangle is distinguished; and the triangle

$$M \xrightarrow{1_M} M \to 0 \to \mathrm{T}(M)$$

in K_{S} is clearly distinguished.

Suppose we are given a morphism $\alpha : L \to M$ in K_{S}. We have to build a distinguished triangle on it. Choose a fraction presentation $\alpha = \mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1}$. Using condition (LD1) we can find $b \in \mathsf{K}$ and $t \in \mathsf{S}$ such that $t \circ a = b \circ s$. These fit into the solid commutative diagram

$$\begin{tikzcd}
L \arrow[rr, bend left, dashed, "\alpha"] \arrow[dr, "b"'] & K \arrow[l, "s"'] \arrow[r, "a"] & M \arrow[dl, "t"] \\
& \tilde{K} &
\end{tikzcd}$$

in K. (The dashed arrow α is in K_{S}.)

Consider the solid commutative diagram below, where the rows are distinguished triangles built on a and b respectively.

$$\begin{tikzcd}
K \arrow[r, "a"] \arrow[d, "s"'] & M \arrow[r, "e"] \arrow[d, "t"'] & N \arrow[r, "c"] \arrow[d, dashed, "u"'] & \mathrm{T}(K) \arrow[d, "\mathrm{T}(s)"'] \\
L \arrow[r, "b"] & \tilde{K} \arrow[r] & P \arrow[r, "d"] & \mathrm{T}(L)
\end{tikzcd} \tag{8.4.3}$$

By (TR3) there is a morphism u that makes the whole diagram commutative. Since $s, t \in \mathsf{S}$ and H is a cohomological functor, it follows that $u \in \mathsf{S}$. Applying the functor Q to (8.4.3), and using the isomorphism $\mathrm{Q}(t) : M \to \tilde{K}$ to replace $\tilde{K}$ with M, we get the commutative diagram

$$\begin{tikzcd}
K \arrow[r, "\mathrm{Q}(a)"] \arrow[d, "\mathrm{Q}(s)"'] & M \arrow[r, "\mathrm{Q}(e)"] \arrow[d, "\mathrm{Q}(1_M)"'] & N \arrow[r, "\mathrm{Q}(c)"] \arrow[d, "\mathrm{Q}(u)"'] & \mathrm{T}(K) \arrow[d, "\mathrm{T}(\mathrm{Q}(s))"'] \\
L \arrow[r, "\alpha"] & M \arrow[r, "\mathrm{Q}(u \circ e)"] & P \arrow[r, "\mathrm{Q}(d)"] & \mathrm{T}(L)
\end{tikzcd}$$

in K_{S}. The top row is a distinguished triangle, and the vertical arrows are isomorphisms. So the bottom row is a distinguished triangle. This is the triangle we were looking for.

(TR2). Turning: this is trivial.

(TR3). We are given the solid commutative diagram in K_S, where the rows are distinguished triangles:

(8.4.4)
$$\begin{tikzcd} L \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[r, "\beta"] \arrow[d, "\psi"'] & N \arrow[r, "\gamma"] \arrow[d, dashed, "\chi"'] & \mathrm{T}(L) \arrow[d, "\mathrm{T}(\phi)"'] \\ L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L') \end{tikzcd}$$

and we have to find χ to complete the diagram.

By replacing the rows with isomorphic triangles, we can assume they come from K. Thus we can replace (8.4.4) with this diagram:

(8.4.5)
$$\begin{tikzcd} L \arrow[r, "\mathrm{Q}(\alpha)"] \arrow[d, "\phi"'] & M \arrow[r, "\mathrm{Q}(\beta)"] \arrow[d, "\psi"'] & N \arrow[r, "\mathrm{Q}(\gamma)"] \arrow[d, dashed, "\chi"'] & \mathrm{T}(L) \arrow[d, "\mathrm{T}(\phi)"'] \\ L' \arrow[r, "\mathrm{Q}(\alpha')"] & M' \arrow[r, "\mathrm{Q}(\beta')"] & N' \arrow[r, "\mathrm{Q}(\gamma')"] & \mathrm{T}(L') \end{tikzcd}$$

in which $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are morphisms in K. It is a commutative diagram. Let us choose fraction presentations $\phi = \mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1}$ and $\psi = \mathrm{Q}(b) \circ \mathrm{Q}(t)^{-1}$. Then the solid diagram (8.4.5) comes from applying Q to the diagram

(8.4.6)
$$\begin{tikzcd} L \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & N \arrow[r, "\gamma"] & \mathrm{T}(L) \\ \tilde{L} \arrow[u, "s"] \arrow[d, "a"'] & \tilde{M} \arrow[u, "t"] \arrow[d, "b"'] & & \mathrm{T}(\tilde{L}) \arrow[u, "\mathrm{T}(s)"'] \arrow[d, "\mathrm{T}(a)"'] \\ L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L') \end{tikzcd}$$

in K. Here the rows are distinguished triangles in K; but the diagram might fail to be commutative.

By axiom (RO3) we can find $c \in \mathsf{K}$ and $u \in \mathsf{S}$ s.t.

$$\mathrm{Q}(t)^{-1} \circ \mathrm{Q}(\alpha) \circ \mathrm{Q}(s) = \mathrm{Q}(c) \circ \mathrm{Q}(u)^{-1}.$$

This is the solid diagram:

$$\begin{tikzcd} & & L \arrow[r, "\alpha"] & M \\ \tilde{L}'' \arrow[r, dashed, "u'"] & \tilde{L}' \arrow[r, "u"] \arrow[rr, bend right, "c"'] & \tilde{L} \arrow[u, "s"] \arrow[d, "a"] & \tilde{M} \arrow[u, "t"'] \arrow[d, "b"'] \\ & & L' \arrow[r, "\alpha'"] & M' \end{tikzcd}$$

Thus

$$\mathrm{Q}(\alpha \circ s \circ u) = \mathrm{Q}(t \circ c).$$

By (RO4) there is $u' \in \mathsf{S}$ s.t.

$$(\alpha \circ s \circ u) \circ u' = (t \circ c) \circ u'.$$

We get

$$\phi = \mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1} = \mathrm{Q}(a \circ u \circ u') \circ \mathrm{Q}(s \circ u \circ u')^{-1}$$

in K_{S}. Thus, after substituting $\tilde{L} := \tilde{L}''$, $s := s \circ u \circ u'$, $a := a \circ u \circ u'$ and $c := c \circ u'$, we get a new diagram

$$\begin{tikzcd}
L \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & N \arrow[r, "\gamma"] & \mathrm{T}(L) \\
\tilde{L} \arrow[u, "s"] \arrow[r, "c"] \arrow[d, "a"'] & \tilde{M} \arrow[u, "t"] \arrow[d, "b"'] & & \mathrm{T}(\tilde{L}) \arrow[u, "\mathrm{T}(s)"'] \arrow[d, "\mathrm{T}(a)"'] \\
L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L')
\end{tikzcd} \tag{8.4.7}$$

in K instead of (8.4.6). In this new diagram the top left square is commutative; but maybe the bottom left square is not commutative.

When we apply Q to the diagram (8.4.7), the whole diagram, including the bottom left square, becomes commutative, since (8.4.5) is commutative. Again using condition (RO4), there is $v \in \mathsf{S}$ s.t.

$$(\alpha' \circ a) \circ v = (b \circ c) \circ v.$$

In a diagram:

$$\begin{tikzcd}
 & L \arrow[r, "\alpha"] & M \\
\tilde{L}' \arrow[r, "v"] & \tilde{L} \arrow[u, "s"] \arrow[r, "c"] \arrow[d, "a"'] & \tilde{M} \arrow[u, "t"'] \arrow[d, "b"'] \\
 & L' \arrow[r, "\alpha'"] & M'
\end{tikzcd}$$

Performing the replacements $\tilde{L} := \tilde{L}'$, $s := s \circ v$, $c := c \circ v$ and $a := a \circ v$ we now have a commutative square also at the bottom left of (8.4.7). Since $\gamma \circ \beta = 0$ and $\gamma' \circ \beta' = 0$, in fact the whole diagram (8.4.7) in K is now commutative.

Now by (TR1) we can embed the morphism c in a distinguished triangle. We get the solid diagram

$$\begin{tikzcd}
L \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & N \arrow[r, "\gamma"] & \mathrm{T}(L) \\
\tilde{L} \arrow[u, "s"] \arrow[r, "c"] \arrow[d, "a"'] & \tilde{M} \arrow[u, "t"] \arrow[r, "\tilde{\beta}"] \arrow[d, "b"'] & \tilde{N} \arrow[u, dashed, "w"] \arrow[r, "\tilde{\gamma}"] \arrow[d, dashed, "d"'] & \mathrm{T}(\tilde{L}) \arrow[u, "\mathrm{T}(s)"'] \arrow[d, "\mathrm{T}(a)"'] \\
L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L')
\end{tikzcd} \tag{8.4.8}$$

in K. The rows are distinguished triangles. Since $\tilde{\gamma} \circ \tilde{\beta} = 0$, the solid diagram is commutative. By (TR3) there are morphisms w and d that make the whole diagram commutative. Now the morphism $w \in \mathsf{S}$ by the usual long exact sequence argument. The morphism

$$\chi := \mathrm{Q}(d) \circ \mathrm{Q}(w)^{-1} : N \to N'$$

solves the problem.

Step 3. Suppose F is a triangulated functor as in condition (ii). The functor F_{S} exists by the universal property of localization. By Theorem 8.3.1 F_{S} is an additive functor. The construction of the pretriangulated structure on K_{S} in the previous steps show that F_{S} is a triangulated functor.

Step 4. At this point K_{S} is a pretriangulated category, and conditions (i)-(ii) of the theorem are satisfied. We need to prove the uniqueness of the pretriangulated structure on K_{S}. Condition (i) fixes the translation functor on K_{S}, and it says that we can't have less distinguished triangles than those we declared. We can't have more distinguished triangles, because of condition (ii). ☐

Proposition 8.4.9. *Consider the situation of Proposition* 8.4.1 *and Theorem* 8.4.2*.*

(1) *The cohomological functor $H : \mathsf{K} \to \mathsf{M}$ factors into $H = H_{\mathsf{S}} \circ \mathsf{Q}$, where $H_{\mathsf{S}} : \mathsf{K}_{\mathsf{S}} \to \mathsf{M}$ is a cohomological functor.*
(2) *Let M be an object of K. The object $\mathrm{Q}(M)$ is zero in K_{S} iff the objects $H(\mathrm{T}^i((M))$ are zero in M for all i.*

Proof. (1) On objects we define $H_{\mathsf{S}}(M) := H(M)$. On morphisms we define $H_{\mathsf{S}}(q) := H(a) \circ H(s)^{-1}$ for any fraction presentation $q = \mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1}$. This makes sense because $H(s)$ is an isomorphism in M. The usual tricks are used to prove that $H_{\mathsf{S}}(q)$ is well-defined. We leave it as an exercise to show that H_{S} is a cohomological functor.

(2) Since H_{S} is an additive functor, if $\mathrm{Q}(M) = 0$, then so is $H(M) = H_{\mathsf{S}}(\mathrm{Q}(M))$. And of course $\mathrm{Q}(M) = 0$ iff $\mathrm{Q}(\mathrm{T}^i(M)) = 0$ for all i.

For the converse, let $\phi : 0 \to M$ be the zero morphism in K. If $H(\mathrm{T}^i(M)) = 0$ for all i, then $H(\mathrm{T}^i(\phi)) : 0 \to H(\mathrm{T}^i(M))$ are isomorphisms for all i. Then $\phi \in \mathsf{S}$, and so $\mathrm{Q}(\phi) : 0 \to \mathrm{Q}(M)$ is an isomorphism in K_{S}. ☐

8.5. **The Derived Category.**

Proposition 8.5.1. *Let M be an abelian category and let A be a DG ring. The functor*

$$\mathrm{H}^0 : \mathbf{K}(A, \mathsf{M}) \to \mathsf{M}$$

is cohomological.

Proof. Clearly H^0 is additive. Consider a distinguished triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L) \tag{8.5.2}$$

in $\mathbf{K}(A, \mathsf{M})$. We can assume that it is the image of a standard triangle in $\mathbf{C}(A, \mathsf{M})$, namely that N is the cone associated to α, as in Definition 3.8.4, $\beta = e_\alpha$ and $\gamma = p_\alpha$. By construction, the cone N sits in an exact sequence of complexes

$$0 \to M \xrightarrow{e_\alpha} N \xrightarrow{p_\alpha} \mathrm{T}(L) \to 0. \tag{8.5.3}$$

Consider the diagram

$$\begin{tikzcd}
\mathrm{H}^{-1}(\mathrm{T}(L)) \arrow[r, "\mathrm{conn}"] \arrow[d, "\mathrm{H}(\mathrm{t}_L^{-1})"'] & \mathrm{H}^0(M) \arrow[r, "\mathrm{H}^0(e_\alpha)"] \arrow[d, "="'] & \mathrm{H}^0(N) \arrow[d, "="'] \\
\mathrm{H}^0(L) \arrow[r, "\mathrm{H}^0(\alpha)"] & \mathrm{H}^0(M) \arrow[r, "\mathrm{H}^0(\beta)"] & \mathrm{H}^0(N)
\end{tikzcd}$$

in M, where the first row is part of the long exact cohomology sequence for (8.5.3), and the second row comes from (8.5.2). The first square is commutative because any lifting represents the connecting homomorphism. The second square is also commutative. It follows that the diagram is commutative, and that the bottom row is exact. □

Definition 8.5.4. A morphism ϕ in $\mathbf{K}(A, \mathsf{M})$ is called a *quasi-isomorphism* if the morphisms $\mathrm{H}^i(\phi)$ in M are isomorphisms for all i.

The set quasi-isomorphisms in $\mathbf{K}(A, \mathsf{M})$ is denoted by $\mathbf{S}(A, \mathsf{M})$.

Note that $\mathrm{H}^i = \mathrm{H}^0 \circ \mathrm{T}^i$. By Proposition 8.5.1 the functor H^0 is cohomological. Therefore Theorem 8.4.2 applies to the set of morphisms $\mathbf{S}(A, \mathsf{M})$, and the next definition makes sense.

Definition 8.5.5. Let M be a $\mathbb{K}$-linear abelian category and A a central DG $\mathbb{K}$-ring. The *derived category* of (A, M) is the $\mathbb{K}$-linear pretriangulated category

$$\mathbf{D}(A, \mathsf{M}) := \mathbf{K}(A, \mathsf{M})_{\mathbf{S}(A, \mathsf{M})}.$$

In our situation we have additive functors

$$\mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \xrightarrow{\mathrm{P}} \mathbf{K}(A, \mathsf{M}) \xrightarrow{\mathrm{Q}} \mathbf{D}(A, \mathsf{M}),$$

that are the identity on objects. Recall that the functor P sends a strict morphism of DG modules to its homotopy class; and Q is the localization functor with respect to quasi-isomorphisms.

Definition 8.5.6. Let M be an abelian category and let A be a DG ring. Define the functor

$$\bar{\mathrm{Q}} := \mathrm{Q} \circ \mathrm{P} : \mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \to \mathbf{D}(A, \mathsf{M}).$$

It is sometimes convenient to describe morphisms in $\mathbf{D}(A, \mathsf{M})$ in terms of the functor $\bar{\mathrm{Q}}$. A morphism $s \in \mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ is called a quasi-isomorphism if $\mathrm{P}(s)$ is a quasi-isomorphism in $\mathbf{K}(A, \mathsf{M})$; i.e. if all the $\mathrm{H}^i(s)$ are isomorphisms.

Proposition 8.5.7.

(1) *Any morphism ϕ in $\mathbf{D}(A, \mathsf{M})$ can be written as a right fraction*

$$\phi = \bar{\mathrm{Q}}(a) \circ \bar{\mathrm{Q}}(s)^{-1}$$

where $a, s \in \mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ and s is a quasi-isomorphism.

(2) *The kernel of $\bar{\mathrm{Q}}$ is this: $\bar{\mathrm{Q}}(a) = 0$ in $\mathbf{D}(A, \mathsf{M})$ iff there exists a quasi-isomorphism s in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ such that $a \circ s$ is a coboundary in $\mathbf{C}(A, \mathsf{M})$.*

Proof. (1) This is because of property (RO3) of Definition 8.2.1 and the fact that P is full.

(2) Property (RO4) of Definition 8.2.1 tells us what the kernel of Q is; and by definition the kernel of P is the 0-coboundaries. □

Of course there is a left version of this proposition.

9. Full Subcategories of the Homotopy Category

In this section there is a commutative base ring $\mathbb{K}$, that shall remain implicit. We fix a central DG $\mathbb{K}$-ring A, and a $\mathbb{K}$-linear abelian category M. The DG category $\mathbf{C}(A, \mathsf{M})$ was introduced in Subsection 3.6, and the pretriangulated categories $\mathbf{K}(A, \mathsf{M})$ and $\mathbf{D}(A, \mathsf{M})$ were introduced in 5.4 and 8.5 respectively. The category of quasi-isomorphisms in $\mathbf{K}(A, \mathsf{M})$ is denoted by $\mathbf{S}(A, \mathsf{M})$. Recall that

$$\mathbf{D}(A, \mathsf{M}) = \mathbf{K}(A, \mathsf{M})_{\mathbf{S}(A, \mathsf{M})},$$

and there is a localization functor

$$\mathrm{Q} : \mathbf{K}(A, \mathsf{M}) \to \mathbf{D}(A, \mathsf{M}).$$

Sometimes we use

9.1. **Localization of Full Subcategories.**

Definition 9.1.1. Let K be a pretriangulated category. A *full pretriangulated subcategory* of K is a full subcategory $\mathsf{L} \subseteq \mathsf{K}$, s.t. these conditions hold:

(a) L is closed under translations, i.e. $L \in \mathsf{L}$ iff $\mathrm{T}(L) \in \mathsf{L}$.
(b) L is closed under distinguished triangles, i.e. if

$$L' \to L \to L'' \to \mathrm{T}(L)$$

is a distinguished triangle in K s.t. $L', L \in \mathsf{L}$, then also $L'' \in \mathsf{L}$.

Observe that these conditions imply that L is closed under finite direct sums. Thus L itself is pretriangulated, and the inclusion $\mathsf{L} \to \mathsf{K}$ is a triangulated functor.

Definition 9.1.2. Let K be a pretriangulated category, with translation functor T. A *denominator set of cohomological origin* in K is a set of morphisms S in K satisfying this condition: there is a cohomological functor $H : \mathsf{K} \to \mathsf{M}$, and

$$\mathsf{S} = \{s \in \mathsf{K} \mid H(\mathrm{T}^i(s)) \text{ is invertible for all } i \in \mathbb{Z}\}.$$

The morphisms in S are called *quasi-isomorphisms*.

Note that by Proposition 8.4.1, the subcategory S is indeed a denominator set. Moreover, we know from Theorem 8.4.2 that the localization K_{S} is a pretriangulated category.

Example 9.1.3. This is the most important example for us: $\mathsf{K} = \mathbf{K}(A, \mathsf{M})$, $H = \mathrm{H}^0 : \mathbf{K}(A, \mathsf{M}) \to \mathsf{M}$ and $\mathsf{S} = \mathbf{S}(A, \mathsf{M})$. Here $\mathsf{K}_{\mathsf{S}} = \mathbf{D}(A, \mathsf{M})$, the derived category.

Proposition 9.1.4. *Let K be a pretriangulated category, let S be a denominator set of cohomological origin in K, and let K' be a full pretriangulated subcategory of K. Then $\mathsf{S}' := \mathsf{K}' \cap \mathsf{S}$ is a denominator set of cohomological origin in K', the Ore localization $\mathsf{K}'_{\mathsf{S}'}$ exists, and $\mathsf{K}'_{\mathsf{S}'}$ is a pretriangulated category.*

Proof. The functor $H|_{\mathsf{K}'} : \mathsf{K}' \to \mathsf{M}$ is also cohomological, and the set of morphisms S' satisfies

$$\mathsf{S}' = \{s \in \mathsf{K}' \mid H|_{\mathsf{K}'}(\mathrm{T}^i(s)) \text{ is an isomorphism for all } i\}.$$

Hence Proposition 8.4.1 and Theorem 8.4.2 apply. □

In the situation of the proposition, the localization functor is denoted by $\mathrm{Q}' : \mathsf{K}' \to \mathsf{K}'_{\mathsf{S}'}$.

Proposition 9.1.5. *In the situation of Proposition* 9.1.4*, let* $F : \mathsf{K}' \to \mathsf{E}$ *be a triangulated functor into some pretriangulated category* E*. Assume that for every* $s \in \mathsf{S}'$*, the morphism* $F(s)$ *is an isomorphism in* E*. Then there is a unique triangulated functor* $F_{\mathsf{S}'} : \mathsf{K}'_{\mathsf{S}'} \to \mathsf{E}$ *that extends* F*; Namely* $F_{\mathsf{S}'} \circ \mathrm{Q}' = F$ *as functors* $\mathsf{K}' \to \mathsf{E}$*.*

Proof. This is part of Theorem 8.4.2. □

In particular we can look at the functor $F : \mathsf{K}' \xrightarrow{\mathrm{inc}} \mathsf{K} \xrightarrow{\mathrm{Q}} \mathsf{K}_{\mathsf{S}}$, and its extension $F_{\mathsf{S}'} : \mathsf{K}'_{\mathsf{S}'} \to \mathsf{K}_{\mathsf{S}}$. We are interested in sufficient conditions for the functor $F_{\mathsf{S}'}$ to be fully faithful.

Proposition 9.1.6. *Let* K *be a pretriangulated category, let* S *be a denominator set of cohomological origin in* K*, and let* $\mathsf{K}'' \subseteq \mathsf{K}'$ *be full pretriangulated subcategories of* K*. Define* $\mathsf{S}' := \mathsf{K}' \cap \mathsf{S}$ *and* $\mathsf{S}'' := \mathsf{K}'' \cap \mathsf{S}$*. Assume either of these conditions holds*:

(r) *Let* $M \in \mathrm{Ob}(\mathsf{K}')$*. If there exists a morphism* $s : M \to L$ *in* S' *with* $L \in \mathrm{Ob}(\mathsf{K}'')$*, there exists a morphism* $t : K \to M$ *in* S' *with* $K \in \mathrm{Ob}(\mathsf{K}'')$*.*

(l) *The same, but with arrows reversed.*

Then the functor $F_{\mathsf{S}''} : \mathsf{K}''_{\mathsf{S}''} \to \mathsf{K}'_{\mathsf{S}'}$ *is fully faithful.*

Proof. We will prove the proposition under condition (r); the other condition is done the same way.

Let $L_1, L_2 \in \mathrm{Ob}(\mathsf{K}'')$, and let $q : L_1 \to L_2$ be a morphism in $\mathsf{K}'_{\mathsf{S}'}$. Choose a presentation $q = \mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1}$ with $s : M \to L_1$ a morphism in S' and $a : M \to L_2$ a morphism in K'. By condition (r) we can find a morphism $t : K \to M$ in S' with $K \in \mathrm{Ob}(\mathsf{K}'')$.

$$\begin{tikzcd} & K \arrow[d, "t"'] & \\ & M \arrow[dl, "s"'] \arrow[dr, "a"] & \\ L_1 \arrow[rr, "q"] & & L_2 \end{tikzcd}$$

Then $q = \mathrm{Q}(a \circ t) \circ \mathrm{Q}(s \circ t)^{-1}$. But $s \circ t \in \mathsf{S}''$ and $a \circ t \in \mathsf{K}''$, so q is in the image of the functor $F_{\mathsf{S}''}$. We see that $F_{\mathsf{S}''}$ is full.

Now let $q : L_1 \to L_2$ be a morphism in $\mathsf{K}''_{\mathsf{S}''}$ such that $F_{\mathsf{S}''}(q) = 0$. Choose a presentation $q = \mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1}$ with $s : N \to L_1$ a morphism in S'' and $a : N \to L_2$ a morphism in K''. Because $F_{\mathsf{S}''}(q) = 0$, and using Lemma 8.2.2, there is a morphism $u : M \to N$ in K' such that $a \circ u = 0$ and $s \circ u \in \mathsf{S}'$. Note that $u \in \mathsf{S}'$. By condition (r), applied to $u : M \to N$, there is a morphism $t : K \to M$ in S' such that $K \in \operatorname{Ob}(\mathsf{K}'')$.

$$\begin{tikzcd}
 & K \arrow[d, "t"'] & \\
 & M \arrow[d, "u"'] & \\
 & N \arrow[dl, "s"'] \arrow[dr, "a"] & \\
L_1 \arrow[rr, "q"] & & L_2
\end{tikzcd}$$

Then we have

$$q = \mathrm{Q}(a \circ u \circ t) \circ \mathrm{Q}(s \circ u \circ t)^{-1} = 0.$$

This proves that $F_{\mathsf{S}''}$ is faithful. □

9.2. **Boundedness Conditions.** A graded object $M = \{M^i\}_{i \in \mathbb{Z}}$ of M is said to be *bounded above* if the set $\{i \mid M^i \neq 0\}$ is bounded above. Likewise we define *bounded below* and *bounded* graded objects.

Definition 9.2.1. We define $\mathbf{C}^-(A, \mathsf{M})$, $\mathbf{C}^+(A, \mathsf{M})$ and $\mathbf{C}^{\mathrm{b}}(A, \mathsf{M})$ to be full subcategories of $\mathbf{C}(A, \mathsf{M})$ consisting of bounded above, bounded below and bounded complexes respectively.

Likewise we define $\mathbf{K}^-(A, \mathsf{M})$, $\mathbf{K}^+(A, \mathsf{M})$ and $\mathbf{K}^{\mathrm{b}}(A, \mathsf{M})$ to be the corresponding full subcategories of $\mathbf{K}(A, \mathsf{M})$.

Of course

$$\mathbf{C}^{\mathrm{b}}(A, \mathsf{M}) = \mathbf{C}^-(A, \mathsf{M}) \cap \mathbf{C}^+(A, \mathsf{M}),$$

and the same for $\mathbf{K}^{\mathrm{b}}(A, \mathsf{M})$. The subcategories $\mathbf{K}^\star(A, \mathsf{M})$, for $\star \in \{-, +, \mathrm{b}\}$, are full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$; this is because the operations of translation and cone preserve the various boundedness conditions.

As the next example shows, sometimes the category $\mathbf{K}^\star(A, \mathsf{M})$ can be very degenerate.

Example 9.2.2. Let A be the DG ring $\mathbb{K}[t, t^{-1}]$, the ring of Laurent polynomials in the variable t of degree 1, with the zero differential. If $M = \{M^i\}_{i \in \mathbb{Z}}$ is a nonzero object of $\mathbf{C}(A, \mathsf{M})$, then $M^i \neq 0$ for all i. Therefore the categories $\mathbf{C}^\star(A, \mathsf{M})$ and $\mathbf{K}^\star(A, \mathsf{M})$ are zero for $\star \in \{-, +, \mathrm{b}\}$.

Let

$$\mathbf{S}^\star(A, \mathsf{M}) := \mathbf{K}^\star(A, \mathsf{M}) \cap \mathbf{S}(A, \mathsf{M}),$$

the category of quasi-isomorphisms in $\mathbf{K}^\star(A, \mathsf{M})$. As already mentioned, Theorem 8.4.2 applies here, so we can localize.

Definition 9.2.3. For $\star \in \{-, +, \mathrm{b}\}$ we define

$$\mathbf{D}^\star(A, \mathsf{M}) := \mathbf{K}^\star(A, \mathsf{M})_{\mathbf{S}^\star(A, \mathsf{M})},$$

the Ore localization of $\mathbf{K}^\star(A, \mathsf{M})$ with respect to $\mathbf{S}^\star(A, \mathsf{M})$.

Here is another kind of boundedness condition.

Definition 9.2.4. For $\star \in \{-,+,\mathrm{b}\}$ we define $\mathsf{D}(A, \mathsf{M})^{\star}$ to be the full subcategory of $\mathsf{D}(A, \mathsf{M})$ on the complexes M whose cohomology $\mathrm{H}(M)$ is of boundedness type $\star$.

Of course $\mathsf{D}(A, \mathsf{M})^{\star}$ is a full pretriangulated subcategory of $\mathsf{D}(A, \mathsf{M})$.

The next proposition refers to the abelian case only – namely to $\mathsf{D}(\mathsf{M}) = \mathsf{D}(\mathbb{K}, \mathsf{M})$. See Remark 9.2.10 for a generalization.

Proposition 9.2.5. *For $\star \in \{-,+,\mathrm{b}\}$ the canonical functor $\mathsf{D}^{\star}(\mathsf{M}) \to \mathsf{D}(\mathsf{M})^{\star}$ is an equivalence.*

Proof. Step 1. Here we prove that $F^{-} : \mathsf{D}^{-}(\mathsf{M}) \to \mathsf{D}(\mathsf{M})$ is fully faithful. Let $s : M \to L$ be a quasi-isomorphism with $L \in \mathsf{K}^{-}(\mathsf{M})$. Say L is concentrated in degrees $\leq i$. Then $\mathrm{H}^j(M) = \mathrm{H}^j(L) = 0$ for all $j > i$. Consider the *smart truncation* of M at i:

$$\operatorname{smt}^{\leq i}(M) := \big(\cdots \to M^{i-2} \xrightarrow{\mathrm{d}} M^{i-1} \xrightarrow{\mathrm{d}} \mathrm{Z}^i(M) \to 0 \to \cdots \big) \tag{9.2.6}$$

where $\mathrm{Z}^i(M) := \operatorname{Ker}(\mathrm{d} : M^i \to M^{i+1})$, the object of i-cocycles, is in degree i. Then $\operatorname{smt}^{\leq i}(M)$ is a subcomplex of M, $\operatorname{smt}^{\leq i}(M) \in \mathsf{K}^{-}(\mathsf{M})$, and the inclusion $t : \operatorname{smt}^{\leq i}(M) \to M$ is a quasi-isomorphism. According to Proposition 9.1.6, with $\mathsf{K} = \mathsf{K}' = \mathsf{K}(\mathsf{M})$ and $\mathsf{K}'' = \mathsf{K}^{-}(\mathsf{M})$, and with condition (r), we see that $F^{-} : \mathsf{D}^{-}(\mathsf{M}) \to \mathsf{D}(\mathsf{M})$ is fully faithful.

Step 2. Here we prove that $F^{+} : \mathsf{D}^{+}(A, \mathsf{M}) \to \mathsf{D}(A, \mathsf{M})$ is fully faithful. Let $s : L \to M$ be a quasi-isomorphism with $L \in \mathsf{K}^{+}(\mathsf{M})$. Say L is concentrated in degrees $\geq i$. Then $\mathrm{H}^j(M) = \mathrm{H}^j(L) = 0$ for all $j < i$. Consider the other smart truncation of M at i:

$$\operatorname{smt}^{\geq i}(M) := \big(\cdots \to 0 \to \mathrm{Y}^i(M) \xrightarrow{\mathrm{d}} M^{i+1} \xrightarrow{\mathrm{d}} M^{i+2} \to \cdots \big) \tag{9.2.7}$$

where

$$\mathrm{Y}^i(M) := \operatorname{Coker}(\mathrm{d} : M^{i-1} \to M^i) \tag{9.2.8}$$

is in degree i. Then $\operatorname{smt}^{\geq i}(M)$ is a quotient complex of M, $\operatorname{smt}^{\geq i}(M) \in \mathsf{K}^{+}(\mathsf{M})$, and the projection $t : M \to \operatorname{smt}^{\geq i}(M)$ is a quasi-isomorphism. According to Proposition 9.1.6, with condition (l), we see that $F^{+} : \mathsf{D}^{-}(\mathsf{M}) \to \mathsf{D}(\mathsf{M})$ is fully faithful.

Step 3. The arguments in step 1 we show that $\mathsf{D}^{\mathrm{b}}(\mathsf{M}) \to \mathsf{D}^{+}(\mathsf{M})$ is fully faithful. And by step 2, $\mathsf{D}^{+}(\mathsf{M}) \to \mathsf{D}(\mathsf{M})$ is fully faithful. Therefore $\mathsf{D}^{\mathrm{b}}(\mathsf{M}) \to \mathsf{D}(\mathsf{M})$ is fully faithful.

Step 4. Smart truncation shows that the functor $\mathsf{D}^{\star}(\mathsf{M}) \to \mathsf{D}(\mathsf{M})^{\star}$ is essentially surjective on objects. □

Remark 9.2.9. Most advanced texts write $\mathsf{D}^{\star}(\mathsf{M})$ instead of $\mathsf{D}(\mathsf{M})^{\star}$, and do not use the notation $\mathsf{D}(\mathsf{M})^{\star}$ at all. This is harmless by Proposition 9.2.5. We might do this too later in the course... Perhaps when we start writing $M[i]$ instead of $\mathrm{T}^i(M)$ for the i-translation.

Remark 9.2.10. Proposition 9.2.5 actually applies to $\mathsf{D}^{\star}(A, \mathsf{M})$ when A is a *nonpositive DG ring*, i.e. $A = \bigoplus_{p \leq 0} A^p$. The reason is that when A is nonpositive, the differential on any DG A-module is A^0-linear, and this implies that the smart truncations exist. We will not discuss this variant here, and the interested reader can turn to [Ye8] for information and usage.

9.3. **Thick Subcategories of M.** Let M be an abelian category. A *thick abelian subcategory* of M is a full abelian subcategory N that is closed under extensions. Namely if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence in M with $M', M'' \in \mathsf{N}$, then $M \in \mathsf{N}$ too.

Let $\mathbf{D}_{\mathsf{N}}(\mathsf{M})$ be the full subcategory of $\mathbf{D}(\mathsf{M})$ consisting of complexes M such that $\mathrm{H}^i(M) \in \mathsf{N}$ for every i.

Proposition 9.3.1. *If N is a thick abelian subcategory of M then $\mathbf{D}_{\mathsf{N}}(\mathsf{M})$ is a pretriangulated subcategory of $\mathbf{D}(\mathsf{M})$.*

Proof. Clearly $\mathbf{D}_{\mathsf{N}}(\mathsf{M})$ is closed under translations. Now suppose

$$M' \to M \to M'' \to M[1]$$

is a distinguished triangle in $\mathbf{D}(\mathsf{M})$ such that $M', M \in \mathbf{D}_{\mathsf{N}}(\mathsf{M})$; we have to show that M'' is also in $\mathbf{D}_{\mathsf{N}}(\mathsf{M})$. Consider the exact sequence

$$\mathrm{H}^i(M') \to \mathrm{H}^i(M) \to \mathrm{H}^i(M'') \to \mathrm{H}^{i+1}(M') \to \mathrm{H}^{i+1}(M).$$

The four outer objects belong to N. Since N is a thick abelian subcategory of M it follows that $\mathrm{H}^i(M'') \in \mathsf{N}$. $\square$

Example 9.3.2. Let A be a noetherian commutative ring. The category $\mathsf{Mod}_{\mathrm{f}} A$ of finitely generated modules is a thick abelian subcategory of $\mathsf{Mod}\, A$.

Example 9.3.3. Consider $\mathsf{Mod}\, \mathbb{Z} = \mathsf{Ab}$. As above we have the thick abelian subcategory $\mathsf{Ab}_{\mathrm{fg}} = \mathsf{Mod}_{\mathrm{f}} \mathbb{Z}$ of finitely generated abelian groups. There is also the thick abelian subcategory $\mathsf{Ab}_{\mathrm{tors}}$ of torsion abelian groups (every element has a finite order). The intersection of $\mathsf{Ab}_{\mathrm{tors}}$ and $\mathsf{Ab}_{\mathrm{fg}}$ is the category $\mathsf{Ab}_{\mathrm{fin}}$ of finite abelian groups. This is also thick.

Example 9.3.4. Let X be a noetherian scheme (e.g. an algebraic variety over an algebraically closed field). Consider the abelian category $\mathsf{Mod}\, \mathcal{O}_X$ of $\mathcal{O}_X$-modules. In it there the thick abelian subcategory $\mathsf{QCoh}\, \mathcal{O}_X$ of quasi-coherent sheaves, and in that there the thick abelian subcategory $\mathsf{Coh}\, \mathcal{O}_X$ of coherent sheaves.

For a left noetherian ring A we write

$$\mathbf{D}_{\mathrm{f}}(\mathsf{Mod}\, A) := \mathbf{D}_{\mathsf{Mod}_{\mathrm{f}} A}(\mathsf{Mod}\, A).$$

Proposition 9.3.5. *Let A be a left noetherian ring and $\star \in \{-, \mathrm{b}\}$. Then the canonical functor*

$$\mathbf{D}^{\star}(\mathsf{Mod}_{\mathrm{f}}\, A) \to \mathbf{D}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\star}$$

is an equivalence.

Proof. Consider the functor

$$F : \mathbf{D}^{-}(\mathsf{Mod}_{\mathrm{f}}\, A) \to \mathbf{D}(\mathsf{Mod}\, A).$$

Suppose $s : M \to L$ is a quasi-isomorphism in $\mathbf{K}(\mathsf{Mod}\, A)$, such that $L \in \mathbf{K}^{-}(\mathsf{Mod}_{\mathrm{f}}\, A)$. Then $M \in \mathbf{D}_{\mathrm{f}}(\mathsf{Mod}\, A)^{-}$. A bit later (Theorem ????) we will prove that M admits a free resolution $P \to M$, where P is a bounded above complex of finitely generated free modules. Thus we get a quasi-isomorphism $t : P \to M$ with $P \in \mathbf{K}^{-}(\mathsf{Mod}_{\mathrm{f}}\, A)$. By Proposition 9.1.6 with condition (r) we conclude that F is fully faithful. This also shows that the essential image of F is $\mathbf{D}_{\mathrm{f}}(\mathsf{Mod}\, A)^{-}$.

Next consider the functor

$$G : \mathbf{D}^{\mathrm{b}}(\mathsf{Mod}_{\mathrm{f}} A) \to \mathbf{D}^{-}(\mathsf{Mod}_{\mathrm{f}} A).$$

Suppose $s : L \to M$ is a quasi-isomorphism in $\mathbf{K}^{-}(\mathsf{Mod}_{\mathrm{f}} A)$ with $L \in \mathbf{K}^{\mathrm{b}}(\mathsf{Mod}_{\mathrm{f}} A)$. Say $\mathrm{H}(L)$ is concentrated in the integer interval $[d_0, d_1]$. Then $t : M \to \mathrm{smt}^{\geq d_0}(M)$ is a quasi-isomorphism, and $\mathrm{smt}^{\geq d_0}(M) \in \mathbf{K}^{\mathrm{b}}(\mathsf{Mod}_{\mathrm{f}} A)$. By Proposition 9.1.6 with condition (l) we conclude that G is fully faithful. Therefore the composition

$$F \circ G : \mathbf{D}^{\mathrm{b}}(\mathsf{Mod}_{\mathrm{f}} A) \to \mathbf{D}(\mathsf{Mod}\, A)$$

is fully faithful. Suitable truncations ($\mathrm{smt}^{\geq d_0}$ and $\mathrm{smt}^{\leq d_1}$) show that the essential image of $F \circ G$ is $\mathbf{D}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\mathrm{b}}$. □

9.4. **The Embedding of M in $\mathbf{D}(\mathsf{M})$.** Here again we only consider an abelian category M.

For $M, N \in \mathsf{M}$ there is no difference between $\mathrm{Hom}_{\mathsf{M}}(M,N)$, $\mathrm{Hom}_{\mathbf{C}(\mathsf{M})}(M,N)$ and $\mathrm{Hom}_{\mathbf{K}(\mathsf{M})}(M,N)$. Thus the canonical functors $\mathsf{M} \to \mathbf{C}(\mathsf{M})$ and $\mathsf{M} \to \mathbf{K}(\mathsf{M})$ are fully faithful. The same is true for $\mathbf{D}(\mathsf{M})$, but this requires a proof.

Let $\mathbf{D}(\mathsf{M})^0$ be the full subcategory of $\mathbf{D}(\mathsf{M})$ consisting of complexes whose cohomology is concentrated in degree 0. This is an additive subcategory of $\mathbf{D}(\mathsf{M})$.

Proposition 9.4.1. *The canonical functor $\mathsf{M} \to \mathbf{D}(\mathsf{M})^0$ is an equivalence.*

Proof. Let's denote the canonical functor $\mathsf{M} \to \mathbf{D}(\mathsf{M})^0$ by F. Under the fully faithful embedding $\mathsf{M} \subseteq \mathbf{C}_{\mathrm{str}}(\mathsf{M})$, F is just the restriction of $\bar{\mathrm{Q}}$.

Since the functor $\mathrm{H}^0 : \mathbf{D}(\mathsf{M}) \to \mathsf{M}$ satisfies $\mathrm{H}^0 \circ F = \mathrm{id}_{\mathsf{M}}$. This implies that F is faithful.

Next we prove that F is full. Take any objects $M, N \in \mathsf{M}$ and a morphism $q : M \to N$ in $\mathbf{D}(\mathsf{M})$. By Proposition 8.5.7 we know that $q = \bar{\mathrm{Q}}(a) \circ \bar{\mathrm{Q}}(s)^{-1}$ for some morphisms $a : L \to N$ and $s : L \to M$ in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, with s a quasi-isomorphism. Let $L' := \mathrm{smt}^{\leq 0}(L)$, as in (9.2.6); so there is a quasi-isomorphism $u : L' \to L$ in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$. Writing $a' := a \circ u$ and $s' := s \circ u$, we see that s' is a quasi-isomorphism, and $q = \bar{\mathrm{Q}}(a') \circ \bar{\mathrm{Q}}(s')^{-1}$.

Next let $L'' := \mathrm{smt}^{\geq 0}(L')$, as in (9.2.8); so there is a surjective quasi-isomorphism $v : L' \to L''$ in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$. Because L'' is a complex concentrated in degree 0, we can view it as an object of M. The morphisms a' and s' factor as $a' = a'' \circ v$ and $s' = s'' \circ v$, where $a'' : L'' \to N$ and $s'' : L'' \to M$ are morphisms in M. But s'' is a quasi-isomorphism in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, and so it is actually an isomorphism in M. Therefore we have a morphism $a'' \circ (s'')^{-1}$ in M, and

$$\bar{\mathrm{Q}}(a'' \circ (s'')^{-1}) = \bar{\mathrm{Q}}(a'') \circ \bar{\mathrm{Q}}(s'')^{-1} = \bar{\mathrm{Q}}(a') \circ \bar{\mathrm{Q}}(s')^{-1} = q.$$

Finally we have to prove that any $L \in \mathbf{D}(\mathsf{M})^0$ is isomorphic, in $\mathbf{D}(\mathsf{M})$, to a complex L'' that's concentrated in degree 0. But we already showed it in the previous paragraphs. □

Proposition 9.4.2. *Let M be an abelian category. Let*

$$0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$$

be a diagram in M. The following conditions are equivalent:

(i) *The diagram is an exact sequence.*

(ii) *There is a distinguished triangle*

$$L \xrightarrow{\phi} M \xrightarrow{\psi} N \xrightarrow{\theta} \mathrm{T}(L)$$

in $\mathbf{D}(\mathsf{M})$.

Proof. Exercise. $\square$

The last two propositions say that the abelian category M can be recovered from the pretriangulated category $\mathbf{D}(\mathsf{M})$.

10. Derived Functors

As before, $\mathbb{K}$ is a commutative base ring, that shall remain implicit. Let A be a central DG $\mathbb{K}$-ring, and M a $\mathbb{K}$-linear abelian category. The category $\mathbf{C}(A, \mathsf{M})$ of DG A-modules in M was introduced in Subsection 3.6. It is a DG category. The pretriangulated categories $\mathbf{K}(A, \mathsf{M})$ and $\mathbf{D}(A, \mathsf{M})$ were introduced in 5.4 and 8.5 respectively. There is a triangulated localization functor

$$\mathrm{Q} : \mathbf{K}(A, \mathsf{M}) \to \mathbf{D}(A, \mathsf{M}).$$

Let (B, N) be another pair of DG ring and abelian category. Suppose we are given a DG functor

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N}).$$

Then, according to Theorem 5.4.11, there is an induced triangulated functor

$$(\bar{F}, \bar{\zeta}_F) : \mathbf{K}(A, \mathsf{M}) \to \mathbf{K}(B, \mathsf{N})$$

Most triangulated functors that we shall encounter arise this way. For convenience of notation, let us suppress mentioning the translation isomorphism $\bar{\zeta}_F$, and let us write F instead of $\bar{F}$.

By postcomposing with the localization functor of $\mathbf{K}(B, \mathsf{N})$ we obtain a triangulated functor

$$\mathrm{Q} \circ F : \mathbf{K}(A, \mathsf{M}) \to \mathbf{D}(B, \mathsf{N}). \tag{10.0.1}$$

Again we denote this triangulated functor by F.

Our goal in this section is to extend F to triangulated functors

$$\mathrm{R}F, \mathrm{L}F : \mathbf{D}(A, \mathsf{M}) \to \mathbf{D}(B, \mathsf{N}).$$

These are the right and left derived functors of F, respectively.

It will be easier to state matters more generally. Thus we shall mostly work in the setup below.

Setup 10.0.2. The following are given:

(1) Pretriangulated categories K and E.
(2) A triangulated functor $F : \mathsf{K} \to \mathsf{E}$.
(3) A denominator set of cohomological origin $\mathsf{S} \subseteq \mathsf{K}$ (see Definition 9.1.2).

Recall that the morphisms in S are called quasi-isomorphisms.

By Proposition 8.4.1 and Theorem 8.4.2, the localization K_S exists, and it is a pretriangulated category. The triangulated localization functor is $\mathrm{Q} : \mathsf{K} \to \mathsf{K}_\mathsf{S}$.

This setup specializes to (10.0.1) when we take $\mathsf{K} = \mathbf{K}(A, \mathsf{M})$, $\mathsf{S} = \mathbf{S}(A, \mathsf{M})$ and $\mathsf{E} = \mathbf{D}(B, \mathsf{N})$.

Remark 10.0.3. As far as we know, all previous textbooks only consider the special case of the derived functors

$$\mathrm{R}F, \mathrm{L}F : \mathbf{D}(\mathsf{M}) \to \mathbf{D}(\mathsf{N})$$

of a triangulated functor

$$F : \mathbf{K}(\mathsf{M}) \to \mathbf{K}(\mathsf{N}),$$

where M and N are abelian categories. The DG variant is not mentioned at all. However, the definitions and the main existence results, as stated in this section, are virtually the same.

Furthermore, previous textbooks avoid the 2-categorical notation, and that (in our opinion) is a cause for undue difficulties in the presentation.

10.1. **2-Categorical Notation.** In this section are going to do a lot of work with morphisms of functors (i.e. natural transformations). The language and notation of ordinary category theory that we used so far is not adequate for this purpose. Therefore we will now introduce notation from the theory of 2-*categories*. (We will not give a definition of a 2-category here; but it is basically the data mentioned below, satisfying a few conditions, most of which will be mentioned below too.) In the subsequent sections we will revert to the usual (i.e. 1-categorical) language. For more details on 2-categories the reader can look at [ML] or [Ye10, Section 1].

Consider the set **Cat** of all categories. The set theoretical aspects are neglected, as explained in Subsection 1.1. (Briefly, the precise solution is this: **Cat** is the set of all U-categories; so **Cat** is a subset of a bigger Grothendieck universe, say V, and it is a V-category.)

The set **Cat** is the set of objects of a 2-category. This means that in **Cat** there are two kinds of morphisms: 1-*morphisms* between objects, and 2-*morphisms* between 1-morphisms. There are several kinds of compositions, and these have several properties. All this will be explained below.

Suppose C_0, $\mathsf{C}_1, \ldots$ are categories, namely objects of **Cat**. The 1-morphisms between them are the functors. The notation is as usual: $F : \mathsf{C}_0 \to \mathsf{C}_1$ denotes a functor.

Suppose $F, G : \mathsf{C}_0 \to \mathsf{C}_1$ are functors (with the same source and target objects). The 2-morphisms from F to G are the morphisms of functors (i.e. the natural transformations), and the notation is $\eta : F \Rightarrow G$. The double arrow is the distinguishing notation for 2-morphisms. When specializing to an object $M \in \mathsf{C}_0$ we revert to the single arrow notation, namely $\eta_M : F(M) \to G(M)$ is the corresponding morphism in C_1. The diagram depicting this is

$$\begin{tikzcd} \mathsf{C}_0 \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\eta\Downarrow"] & \mathsf{C}_1 \end{tikzcd}$$

We shall refer to such a diagram as a 2-*diagram*.

Each object (category) C has its identity 1-morphism (functor) $\mathrm{id}_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}$. Each 1-morphism F has its identity 2-morphism (natural transformation) $\mathrm{id}_F : F \Rightarrow F$.

Now we consider compositions. For functors there is nothing new: given functors $F_i : \mathsf{C}_{i-1} \to \mathsf{C}_i$, the composition, that we now call *horizontal composition*, is the functor $F_2 \circ F_1 : \mathsf{C}_0 \to \mathsf{C}_2$. The diagram is

$$\begin{tikzcd} \mathsf{C}_0 \arrow[r, "F_1"] \arrow[rr, bend right, "F_2 \circ F_1"'] & \mathsf{C}_1 \arrow[r, "F_2"] & \mathsf{C}_2 \end{tikzcd}$$

This can be viewed as a commutative 1-diagram, or as a shorthand for the 2-diagram

$$\begin{tikzcd} \mathsf{C}_0 \arrow[r, "F_1"] \arrow[rr, bend right=40, "F_2 \circ F_1"'] & \mathsf{C}_1 \arrow[r, "F_2"] \arrow[d, phantom, "\mathrm{id}\Downarrow"] & \mathsf{C}_2 \\ & {} & \end{tikzcd}$$

in which id is the identity 2-morphism of $F_2 \circ F_1$.

The complication begins with compositions of 2-morphisms. Suppose we are given 1-morphisms $F_i, G_i : \mathsf{C}_{i-1} \to \mathsf{C}_i$ and 2-morphisms $\eta_i : F_i \Rightarrow G_i$. In a diagram:

$$\begin{tikzcd}
\mathsf{C}_0 \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\eta_1 \Downarrow"] & \mathsf{C}_1 \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\eta_2 \Downarrow"] & \mathsf{C}_2
\end{tikzcd}$$

The *horizontal composition* is the morphism of functors

$$\eta_2 \circ \eta_1 : F_2 \circ F_1 \Rightarrow G_2 \circ G_1.$$

The diagram is

$$\begin{tikzcd}
\mathsf{C}_0 \arrow[r, bend left, "F_2 \circ F_1"] \arrow[r, bend right, "G_2 \circ G_1"'] \arrow[r, phantom, "\eta_2 \circ \eta_1 \Downarrow"] & \mathsf{C}_2
\end{tikzcd}$$

Exercise 10.1.1. For an object $M \in \mathsf{C}_0$, give an explicit formula for the morphism

$$(\eta_2 \circ \eta_1)_M : (F_2 \circ F_1)(M) \to (G_2 \circ G_1)(M)$$

in the category C_2.

Suppose we are given 1-morphisms $E, F, G : \mathsf{C}_0 \to \mathsf{C}_1$, and 2-morphisms $\zeta : E \Rightarrow F$ and $\eta : F \Rightarrow G$. The diagram depicting this is

$$\begin{tikzcd}
\mathsf{C}_0 \arrow[r, bend left=60, "E"] \arrow[r, "F"'] \arrow[r, bend right=60, "G"'] \arrow[r, phantom, shift left=3, "\zeta \Downarrow"] \arrow[r, phantom, shift right=3, "\eta \Downarrow"] & \mathsf{C}_1
\end{tikzcd}$$

The *vertical composition* of ζ and η is the 2-morphism

$$\eta * \zeta : E \to G.$$

Notice the new symbol for this operation. The corresponding diagram is

$$\begin{tikzcd}
\mathsf{C}_0 \arrow[r, bend left, "E"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\eta_2 * \eta_1 \Downarrow"] & \mathsf{C}_1
\end{tikzcd}$$

Exercise 10.1.2. For an object $M \in \mathsf{C}_0$, give an explicit formula for the morphism

$$(\eta * \zeta)_M : E(M) \to G(M)$$

in the category C_1.

Something intricate occurs in the situation shown in the next diagram.

$$
\begin{tikzcd}[column sep=huge]
\mathsf{C}_0 \arrow[r, bend left=60, "E_1", ""{name=E1, below}] \arrow[r, "F_1"' near end, ""{name=F1a, above}, ""{name=F1b, below}] \arrow[r, bend right=60, "G_1"', ""{name=G1, above}] & \mathsf{C}_1 \arrow[r, bend left=60, "E_2", ""{name=E2, below}] \arrow[r, "F_2"' near end, ""{name=F2a, above}, ""{name=F2b, below}] \arrow[r, bend right=60, "G_2"', ""{name=G2, above}] & \mathsf{C}_2
\arrow[Rightarrow, from=E1, to=F1a, "\zeta_1"'] \arrow[Rightarrow, from=F1b, to=G1, "\eta_1"'] \arrow[Rightarrow, from=E2, to=F2a, "\zeta_2"'] \arrow[Rightarrow, from=F2b, to=G2, "\eta_2"']
\end{tikzcd}
$$

It turns out that

$$(\eta_2 * \zeta_2) \circ (\eta_1 * \zeta_1) = (\eta_2 \circ \eta_1) * (\zeta_2 \circ \zeta_1)$$

as morphisms $E_2 \circ E_1 \Rightarrow G_2 \circ G_1$. This is called the *exchange property.*

Exercise 10.1.3. Prove the exchange property.

Just like general categories, we can talk about pretriangulated categories. There is the 2-category $\mathsf{\mathbf{PTrCat}}$ of all pretriangulated categories (over $\mathbb{K}$). The objects here are the pretriangulated categories (K, T); the 1-morphisms are the triangulated functors (F, ξ); and the 2-morphisms are the morphisms of triangulated functors η. This is what we are going to use.

10.2. **Some Preliminaries on Triangulated Functors.**

Proposition 10.2.1. *Let $(F, \xi) : \mathsf{K} \to \mathsf{L}$ be a triangulated functor between pretriangulated categories. Assume F is an equivalence (of abstract categories), with quasi-inverse $G : \mathsf{L} \to \mathsf{K}$, and with adjunction isomorphisms $\alpha : G \circ F \xRightarrow{\simeq} \mathrm{id}_{\mathsf{K}}$ and $\beta : F \circ G \xRightarrow{\simeq} \mathrm{id}_{\mathsf{L}}$.*

Then there is an isomorphism of functors

$$\nu : G \circ \mathrm{T}_{\mathsf{L}} \xRightarrow{\simeq} \mathrm{T}_{\mathsf{K}} \circ G$$

such that $(G, \nu) : \mathsf{L} \to \mathsf{K}$ is a triangulated functor, and α and β are isomorphisms of triangulated functors.

Proof. It is well-known that G is additive (or in our case, $\mathbb{K}$-linear); but since the proof is so easy, we shall reproduce it. Take any pair of objects $M, N \in \mathsf{L}$. We have to prove that the bijection

$$G_{M,N} : \mathrm{Hom}_{\mathsf{L}}(M, N) \to \mathrm{Hom}_{\mathsf{K}}\big(G(M), G(N)\big)$$

is linear. But

$$G_{M,N} = \alpha^{-1}_{M,N} \circ F_{G(M),G(N)}$$

as bijections (of sets) between these modules. Since $\alpha^{-1}_{M,N}$ and $F_{G(M),G(N)}$ are $\mathbb{K}$-linear, then so is $G_{M,N}$.

We define the isomorphism of triangulated functors ν by the formula

$$\nu := (\alpha \circ \mathrm{id}_{\mathrm{T}_{\mathsf{K}} \circ G}) * (\mathrm{id}_G \circ \xi \circ \mathrm{id}_G)^{-1} * (\mathrm{id}_{G \circ \mathrm{T}_{\mathsf{L}}} \circ \beta)^{-1},$$

in terms of the 2-categorical notation. This gives rise to a commutative diagram of isomorphisms

$$
\begin{tikzcd}[column sep=large]
G \circ \mathrm{T}_{\mathsf{L}} \circ F \circ G \arrow[d, Rightarrow, "\mathrm{id} \circ \beta"'] & G \circ F \circ \mathrm{T}_{\mathsf{K}} \circ G \arrow[l, Rightarrow, "\mathrm{id} \circ \xi \circ \mathrm{id}"'] \arrow[d, Rightarrow, "\alpha \circ \mathrm{id}"] \\
G \circ \mathrm{T}_{\mathsf{L}} \arrow[r, Rightarrow, "\nu"] & \mathrm{T}_{\mathsf{K}} \circ G
\end{tikzcd}
$$

of additive functors $\mathsf{L} \to \mathsf{K}$. So the pair (G, ν) is a T-additive functor.

The verification that (G, ν) preserves triangles (in the sense of Definition 5.3.1(1)) is done like the proof of the additivity of G, but now using axiom (TR1.a) from Definition 5.2.1 . We leave this as an exercise. □

Exercise 10.2.2. Finish the proof above (the last assertion).

10.3. **Right Derived Functors.**

Definition 10.3.1. Assume Setup 10.0.2. A *right derived functor* of F is a triangulated functor

$$\mathrm{R}F : \mathsf{K}_{\mathsf{S}} \to \mathsf{E},$$

together with a morphism

$$\eta : F \Rightarrow \mathrm{R}F \circ \mathrm{Q}$$

of triangulated functors $\mathsf{K} \to \mathsf{E}$. The pair $(\mathrm{R}F, \eta)$ must have this universal property:

($\Diamond$) Given any pair (G, θ), consisting of a triangulated functor $G : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$ and a morphism of triangulated functors $\theta : F \Rightarrow G \circ \mathrm{Q}$, there is a unique morphism of triangulated functors $\mu : \mathrm{R}F \Rightarrow G$ such that $\theta = (\mu \circ \mathrm{id}_{\mathrm{Q}}) * \eta$.

Pictorially: there is a 2-diagram

$$\begin{tikzcd}
\mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[dr, phantom, "\Downarrow \eta"] & \mathsf{E} \\
\mathsf{K}_{\mathsf{S}} \arrow[ur, "\mathrm{R}F"'] &
\end{tikzcd}$$

For any other pair (G, θ) there is a unique morphism μ that sits in this 2-diagram:

$$\begin{tikzcd}
\mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[dr, phantom, "\Downarrow \eta \ \Downarrow \theta"] & \mathsf{E} \\
\mathsf{K}_{\mathsf{S}} \arrow[ur, "\mathrm{R}F"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, bend right=20, "\Downarrow \mu"] &
\end{tikzcd}$$

The 1-morphisms in this 2-diagram do not (necessarily) commute; but the diagram of 2-morphisms (with $*$ composition)

$$\begin{tikzcd}
F \arrow[d, Rightarrow, "\eta"'] \arrow[dr, Rightarrow, "\theta"] & \\
\mathrm{R}F \circ \mathrm{Q} \arrow[r, Rightarrow, "\mu \circ \mathrm{id}_{\mathrm{Q}}"'] & G \circ \mathrm{Q}
\end{tikzcd}$$

is commutative.

Proposition 10.3.2. *If a right derived functor* $(\mathrm{R}F, \eta)$ *exists, then it is unique, up to a unique isomorphism. Namely, if* (G, θ) *is another right derived functor of* F*, then there is a unique isomorphism of triangulated functors* $\mu : \mathrm{R}F \xRightarrow{\simeq} G$ *such that* $\theta = (\mu \circ \mathrm{id}_{\mathrm{Q}}) * \eta$.

Proof. Despite the apparent complication of the situation, the usual argument for uniqueness of universals (here it is a universal 1-morphism) applies. It shows that the morphism μ from condition ($\Diamond$) is an isomorphism. $\square$

Existence is much harder. Here is a sufficient condition. It is a rephrasing of [RD, Theorem I.5.1], and the proof is basically the same (but we give many more details).

Theorem 10.3.3. *Given Setup* 10.0.2*, assume there is a full pretriangulated subcategory* $\mathsf{J} \subseteq \mathsf{K}$ *with these two properties*:

(a) *If* $\phi : I \to I'$ *is a quasi-isomorphism in* J*, then* $F(\phi) : F(I) \to F(I')$ *is an isomorphism in* E*.*
(b) *Every object* $M \in \mathsf{K}$ *admits a quasi-isomorphism* $\rho : M \to I$ *to some object* $I \in \mathsf{J}$*.*

Then the right derived functor

$$(\mathrm{R}F, \eta) : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$$

exists. Moreover, for any object $I \in \mathsf{J}$ *the morphism*

$$\eta_I : F(I) \to (\mathrm{R}F \circ \mathrm{Q})(I)$$

in E *is an isomorphism.*

Remark 10.3.4. A quasi-isomorphism $\rho : M \to I$ as in condition (b) is supposed to be viewed as a "generalized injective resolution" of M. See Example 10.3.22, where this is made concrete.

We use the letter J for the category of "generalized injective complexes" because the letter I, in this particular font, is too ambiguous.

The proof of the theorem follows some preparation. We we will sometimes suppress the localization functors Q and Q', for the sake of clarity. For instance, given a morphism $s \in \mathsf{S}$, we might say that s is invertible in K_{S}.

Let us denote by $U : \mathsf{J} \to \mathsf{K}$ the inclusion functor, so $I \circ U$ is the identity on the set $\mathrm{Ob}(\mathsf{J})$. Let us define $F' := F \circ U : \mathsf{J} \to \mathsf{E}$ and $\mathsf{S}' := \mathsf{J} \cap \mathsf{S}$. The localization functor of J is denoted by $\mathrm{Q}' : \mathsf{J} \to \mathsf{J}_{\mathsf{S}'}$. There is a triangulated functor $U_{\mathsf{S}'} : \mathsf{J}_{\mathsf{S}'} \to \mathsf{K}_{\mathsf{S}}$ extending U, and there is equality $\mathrm{Q} \circ U = U_{\mathsf{S}'} \circ \mathrm{Q}'$. These sit in a commutative diagram

$$\begin{tikzcd} \mathsf{J} \arrow[r, "U"] \arrow[d, "\mathrm{Q}'"'] & \mathsf{K} \arrow[d, "\mathrm{Q}"] \\ \mathsf{J}_{\mathsf{S}'} \arrow[r, "U_{\mathsf{S}'}"] & \mathsf{K}_{\mathsf{S}} \end{tikzcd}$$

We know (from Theorem 8.4.2) that the functor F' extends uniquely to a triangulated functor $F'_{\mathsf{S}'} : \mathsf{J}_{\mathsf{S}'} \to \mathsf{E}$. Let $\eta' := \mathrm{id}_{F'}$, which is a 2-morphism

$$\eta' : F' \Rightarrow F'_{\mathsf{S}'} \circ \mathrm{Q}' . \tag{10.3.5}$$

The 2-diagram is:

$$\begin{tikzcd} \mathsf{J} \arrow[r, "F'"] \arrow[d, "\mathrm{Q}'"'] \arrow[dr, phantom, "\Downarrow \eta'"] & \mathsf{E} \\ \mathsf{J}_{\mathsf{S}'} \arrow[ur, "F'_{\mathsf{S}'}"'] & \end{tikzcd} \tag{10.3.6}$$

Lemma 10.3.7. *The pair $(F'_{\mathsf{S}'}, \eta')$ is a right derived functor of F'.*

Proof. We need to verify condition $(\Diamond)$ of Definition 10.3.1. Say a triangulated functor $G' : \mathsf{J}_{\mathsf{S}'} \to \mathsf{E}$ is given. Because Q' is the identity on objects, the data of a morphism of triangulated functors $\mu' : F'_{\mathsf{S}'} \Rightarrow G'$, namely a collection of morphisms $\mu'_I : F'(I) \to G'(I)$ in E for all $I \in \mathsf{J}$, is the same data of a morphism of triangulated functors

$$\theta' := \mu' \circ \mathrm{id}_{\mathrm{Q}'} = (\mu' \circ \mathrm{id}_{\mathrm{Q}'}) * \eta' : F' \Rightarrow G' \circ \mathrm{Q}' . \tag{10.3.8}$$

This implies that the function $\mu' \mapsto \theta'$ is injective. Here is the relevant 2-diagram:

$$\begin{tikzcd} \mathsf{J} \arrow[r, "F'"] \arrow[d, "\mathrm{Q}'"'] \arrow[dr, phantom, "\Downarrow \eta' \;\; \Downarrow \theta'"] & \mathsf{E} \\ \mathsf{J}_{\mathsf{S}'} \arrow[ur, "F'_{\mathsf{S}'}"] \arrow[ur, bend right, "G'"'] \arrow[ur, phantom, bend right=20, "\Downarrow \mu'"] & \end{tikzcd} \tag{10.3.9}$$

We have to prove that the function $\mu' \mapsto \theta'$ is surjective. This amounts to showing that for any morphism $q : I \to J$ in $\mathsf{J}_{\mathsf{S}'}$ there is equality

$$\theta'_J \circ F'_{\mathsf{S}'}(q) = G'(q) \circ \theta'_I$$

of morphisms in E. Let us choose a right fraction presentation $q = a \circ s^{-1}$, with $a : K \to J$ in J and $s : K \to I$ in S'. Because $\theta' : F' \Rightarrow G' \circ \mathrm{Q}'$ is a morphism of functors $\mathsf{J} \to \mathsf{E}$, the solid diagram below

$$\begin{tikzcd} F'(I) \arrow[rr, dashed, bend left, "F'_{\mathsf{S}'}(q)"] \arrow[d, "\theta'_I"'] & F'(K) \arrow[l, "F'(s)"] \arrow[r, "F'(a)"'] \arrow[d, "\theta'_K"'] & F'(J) \arrow[d, "\theta'_J"'] \\ G'(I) \arrow[rr, dashed, bend right, "G'(q)"'] & G'(K) \arrow[l, "G'(s)"'] \arrow[r, "G'(a)"] & G'(J) \end{tikzcd}$$

is commutative. But then, since $F'(s)$ and $G'(s)$ are invertible in E, the whole diagram is commutative. □

Lemma 10.3.10. *The functor $U_{\mathsf{S}'} : \mathsf{J}_{\mathsf{S}'} \to \mathsf{K}_{\mathsf{S}}$ is an equivalence of pretriangulated categories.*

Proof. By the proof of Proposition 9.1.6, with condition (l). □

Definition 10.3.11. In the situation of Theorem 10.3.3, by a *system of right* J*-resolutions* we mean a pair (I, ρ), where $I : \mathrm{Ob}(\mathsf{K}) \to \mathrm{Ob}(\mathsf{J})$ is a function, and $\rho = \{\rho_M\}_{M \in \mathrm{Ob}(\mathsf{K})}$ is a collection of quasi-isomorphisms $\rho_M : M \to I(M)$ in K. Moreover, if $M \in \mathrm{Ob}(\mathsf{J})$, then $I(M) = M$ and $\rho_M = \mathrm{id}_M$.

Property (b) of Theorem 10.3.3 guarantees that a system of right J-resolutions (I, ρ) exists.

Lemma 10.3.12. *Suppose a system of right* J*-resolutions* (I, ρ) *has been chosen. Then the function* I *extends uniquely to a triangulated functor* $I : \mathsf{K}_{\mathsf{S}} \to \mathsf{J}_{\mathsf{S}'}$*, such that* $\mathrm{id}_{\mathsf{J}_{\mathsf{S}'}} = I \circ U_{\mathsf{S}'}$*, and* $\rho : \mathrm{id}_{\mathsf{K}_{\mathsf{S}}} \Rightarrow U_{\mathsf{S}'} \circ I$ *is an isomorphism of triangulated functors.*

In other words, the triangulated functor I is a quasi-inverse of $U_{\mathsf{S}'}$. The relevant 2-diagram is this:

$$\begin{tikzcd}
\mathsf{J} \arrow[r, "\mathrm{Q}'"] \arrow[d, "U"'] & \mathsf{J}_{\mathsf{S}'} \arrow[r, "\mathrm{id}"] \arrow[dr, "U_{\mathsf{S}'}"] & \mathsf{J}_{\mathsf{S}'} \\
\mathsf{K} \arrow[r, "\mathrm{Q}"] & \mathsf{K}_{\mathsf{S}} \arrow[u, "I"] \arrow[r, "\mathrm{id}"'] \arrow[ur, phantom, "\Uparrow\rho"] & \mathsf{K}_{\mathsf{S}} \arrow[u, "I"']
\end{tikzcd}$$

Proof. By Lemma 10.3.10 the functor $U_{\mathsf{S}'}$ is an equivalence. Take any pair of objects $M, N \in \mathsf{K}$. There is a bijection

$$U_{\mathsf{S}'} : \mathrm{Hom}_{\mathsf{J}_{\mathsf{S}'}}\big(I(M), I(N)\big) \to \mathrm{Hom}_{\mathsf{K}_{\mathsf{S}}}\big(I(M), I(N)\big),$$

and another bijection

$$\mathrm{Hom}(\rho_M^{-1}, \rho_N) : \mathrm{Hom}_{\mathsf{K}_{\mathsf{S}}}(M, N) \to \mathrm{Hom}_{\mathsf{K}_{\mathsf{S}}}\big(I(M), I(N)\big).$$

These bijections say that to any morphism $\psi : M \to N$ in K_{S} there corresponds a unique morphism $I(\psi) : I(M) \to I(N)$ in $\mathsf{J}_{\mathsf{S}'}$, such that

$$U_{\mathsf{S}'}(I(\psi)) \circ \rho_M = \rho_N \circ \psi.$$

An easy calculation shows that $I : \mathsf{K}_{\mathsf{S}} \to \mathsf{J}_{\mathsf{S}'}$ is a functor. Morever, there is equality of functors $I \circ U_{\mathsf{S}'} = \mathrm{id}$, and an isomorphism of functors $\rho : \mathrm{id} \xRightarrow{\simeq} U_{\mathsf{S}'} \circ I$. This says that I is a quasi-inverse of $U_{\mathsf{S}'}$. Therefore, by Proposition 10.2.1, I is a triangulated functor, and ρ is an isomorphism of triangulated functors. □

Lemma 10.3.13. *Under the assumptions of the theorem, let* $G : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$ *be triangulated functor, and define* $G' := G \circ U_{\mathsf{S}'}$*. Suppose* $\eta' : F' \Rightarrow G' \circ \mathrm{Q}'$ *is a morphism of triangulated functors* $\mathsf{J} \to \mathsf{E}$*. Then there is a unique morphism* $\eta : F \Rightarrow G \circ \mathrm{Q}$ *of triangulated functors* $\mathsf{K} \to \mathsf{E}$ *that extends* η'*, namely such that* $\eta \circ \mathrm{id}_U = \eta'$*.*

Here are the corresponding 2-diagrams:

$$\begin{tikzcd}
\mathsf{J} \arrow[r, "F'"] \arrow[d, "\mathrm{Q}'"'] \arrow[dr, phantom, "\Downarrow\eta'"] & \mathsf{E} \\
\mathsf{J}_{\mathsf{S}'} \arrow[ur, "G'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{J} \arrow[r, "U"] \arrow[d, "\mathrm{Q}'"'] & \mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"] \arrow[dr, phantom, "\Downarrow\eta"] & \mathsf{E} \\
\mathsf{J}_{\mathsf{S}'} \arrow[r, "U_{\mathsf{S}'}"'] & \mathsf{K}_{\mathsf{S}} \arrow[ur, "G"'] &
\end{tikzcd}$$

Here is another way to state the lemma. Let us denote by $\mathrm{Hom}^2_{\mathsf{PTrCat}}(-,-)$ the set of 2-morphisms (morphisms of triangulated functors). Then the operation $\eta \mapsto \eta \circ \mathrm{id}_U$ is a function

$$- \circ \mathrm{id}_U : \mathrm{Hom}^2_{\mathsf{PTrCat}}(F, G \circ \mathrm{Q}) \to \mathrm{Hom}^2_{\mathsf{PTrCat}}(F', G' \circ \mathrm{Q}'),$$

and the lemma asserts that this is a bijection.

Proof. Choose a system of right J-resolutions (I, ρ). For any object $M \in \mathsf{K}$ the morphism ρ_M is invertible in K_S. Hence the morphism

$$G(\rho_M) : G(M) \to G(I(M))$$

is invertible in E. We are given the morphism

$$\eta'_{I(M)} : F'(I(M)) \to G'(I(M))$$

in E. Recall that $F'(I(M)) = F(I(M))$ and $G'(I(M)) = G(I(M))$. Let us define

$$\eta_M := G(\rho_M)^{-1} \circ \eta'_{I(M)} \circ F(\rho_M), \tag{10.3.14}$$

which is a morphism $F(M) \to G(M)$ in E. We get a commutative diagram

$$\begin{tikzcd} F(M) \arrow[r, "\eta_M"] \arrow[d, "F(\rho_M)"'] & G(M) \arrow[d, "G(\rho_M)"] \\ F'(I(M)) \arrow[r, "\eta'_{I(M)}"] & G'(I(M)) \end{tikzcd} \tag{10.3.15}$$

in E.

It is now routine to check that η is a morphism of triangulated functors $F \Rightarrow G \circ \mathrm{Q}$. By construction η extends η'. The uniqueness of η follows from the fact that the diagram (10.3.15) must commute, and thus formula (10.3.14) must hold. □

Proof of Theorem 10.3.3.

Step 1. We choose a system of right J-resolutions (I, ρ). For any object $M \in \mathsf{K}$ we define the object

$$\mathrm{R}F(M) := F(I(M)) \in \mathsf{E} \tag{10.3.16}$$

and the morphism

$$\eta_M := F(\rho_M) : F(M) \to \mathrm{R}F(M) \tag{10.3.17}$$

in E. We still did not say what $\mathrm{R}F$ does to morphisms.

Step 2. For any object $M \in \mathsf{K}$ we have, by construction, $\mathrm{R}F(M) = F'(I(M))$. This means that $\mathrm{R}F = F'_{\mathsf{S}'} \circ I$ on objects. The definition

$$\mathrm{R}F := F'_{\mathsf{S}'} \circ I : \mathsf{K}_\mathsf{S} \to \mathsf{E}\,. \tag{10.3.18}$$

upgrades $\mathrm{R}F$ to a triangulated functor. And there is a commutative diagram of triangulated functors

$$\begin{tikzcd} \mathsf{J} \arrow[r, "\mathrm{Q}'"] \arrow[d, "U"'] \arrow[rr, bend left, "F'"] & \mathsf{J}_{\mathsf{S}'} \arrow[r, "F'_{\mathsf{S}'}"] & \mathsf{E} \\ \mathsf{K} \arrow[r, "\mathrm{Q}"] & \mathsf{K}_\mathsf{S} \arrow[u, "I"'] \arrow[ur, "\mathrm{R}F"'] & \end{tikzcd}$$

Step 3. Recall that we already defined $\eta_M = F(\rho_M)$. In this step we prove that η is a morphism of triangulated functors $\eta : F \to \mathrm{R}F \circ \mathrm{Q}$.

According to Lemma 10.3.13, the morphism of triangulated functors $\eta' : F' \Rightarrow F'_{\mathsf{S}'} \circ \mathrm{Q}'$ from (10.3.5) extends uniquely to a morphism of triangulated functors $\tilde{\eta} : F \Rightarrow \mathrm{R}F \circ \mathrm{Q}$. The 2-diagram is

$$\begin{tikzcd} \mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[dr, phantom, "\Downarrow \tilde{\eta}"] & \mathsf{E} \\ \mathsf{K}_{\mathsf{S}} \arrow[ur, "\mathrm{R}F"'] & \end{tikzcd}$$

We know that $\eta'_{I(M)} = \mathrm{id}_{F(I(M))}$ and $\mathrm{R}F = F'_{\mathsf{S}'} \circ I$. By construction of the functor I we have $I(\rho_M) = \mathrm{id}_{I(M)}$ in $\mathsf{J}_{\mathsf{S}'}$. Plugging this and $G = \mathrm{R}F$ into formula (10.3.14) we obtain

$$\begin{aligned} \tilde{\eta}_M &= (F'_{\mathsf{S}'}(I(\rho_M))^{-1} \circ \eta'_{I(M)} \circ F(\rho_M) \\ &= (\mathrm{id}_{F(I(M))})^{-1} \circ \mathrm{id}_{F(I(M))} \circ F(\rho_M) = F(\rho_M). \end{aligned}$$

So the morphism $\tilde{\eta}_M$ coincides with η_M. As M varies we get $\tilde{\eta} = \eta$.

Step 4. It remains to verify condition ($\Diamond$) of Definition 10.3.1. Say a pair (G, θ) is given. Define $G' := G \circ U_{\mathsf{S}'}$ and $\theta' := \theta \circ \mathrm{id}_U$. In Lemma 10.3.7 we proved that $(F'_{\mathsf{S}'}, \eta')$ is the right derived functor of F'. Therefore there is a unique morphism $\mu' : F'_{\mathsf{S}'} \Rightarrow G'$ of triangulated functors $\mathsf{J}_{\mathsf{S}'} \to \mathsf{E}$ such that $\mu' \circ \mathrm{id}_{\mathrm{Q}'} = \theta'$. In terms of vertical composition, and using the equality $\eta' = \mathrm{id}_{F'}$, this is

$$(\mu' \circ \mathrm{id}_{\mathrm{Q}'}) * \eta' = \theta'. \tag{10.3.19}$$

In a 2-diagram:

$$\begin{tikzcd} \mathsf{J} \arrow[r, "F'"] \arrow[d, "\mathrm{Q}'"'] \arrow[dr, phantom, "\Downarrow \eta' \quad \Downarrow \theta'"] & \mathsf{E} \\ \mathsf{J}_{\mathsf{S}'} \arrow[ur, "F'_{\mathsf{S}'}"] \arrow[ur, bend right, "G'"'] \arrow[ur, phantom, "\Downarrow \mu'"] & \end{tikzcd}$$

Recall that $F'_{\mathsf{S}'} = \mathrm{R}F \circ U_{\mathsf{S}'}$. The functor $U_{\mathsf{S}'}$ is an equivalence. Hence (like Lemma 10.3.13 but much easier) there is a unique morphism $\mu : \mathrm{R}F \to G$ such that $\mu \circ \mathrm{id}_{U_{\mathsf{S}'}} = \mu'$. We get this 2-diagram:

$$\begin{tikzcd} \mathsf{J} \arrow[r, "U"] \arrow[d, "\mathrm{Q}'"'] & \mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[dr, phantom, "\Downarrow \eta \quad \Downarrow \theta"] & \mathsf{E} \\ \mathsf{J}_{\mathsf{S}'} \arrow[r, "U_{\mathsf{S}'}"'] & \mathsf{K}_{\mathsf{S}} \arrow[ur, "\mathrm{R}F"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, "\Downarrow \mu"] & \end{tikzcd}$$

The exchange condition says that

$$\mathrm{id}_{\mathrm{Q}} \circ \mathrm{id}_U = \mathrm{id}_{U_{\mathsf{S}'}} \circ \mathrm{id}_{\mathrm{Q}'} .$$

Hence

$$(\mu \circ \mathrm{id}_{\mathrm{Q}} \circ \mathrm{id}_U) * (\eta \circ \mathrm{id}_U) = (\mu \circ \mathrm{id}_{U_{\mathsf{S}'}} \circ \mathrm{id}_{\mathrm{Q}'}) * \eta' = (\mu' \circ \mathrm{id}'_{\mathrm{Q}}) * \eta'.$$

Taking this with formula (10.3.19), and using the exchange condition once more, we deduce that

$$((\mu \circ \mathrm{id}_{\mathrm{Q}}) * \eta) \circ \mathrm{id}_U = \theta'.$$

The uniqueness in Lemma 10.3.13 now implies that

$$(\mu \circ \mathrm{id}_{\mathrm{Q}}) * \eta = \theta. \tag{10.3.20}$$

Finally we have to establish the uniqueness of μ. Suppose $\tilde{\mu}$ is another morphism $\mathrm{R}F \Rightarrow G$ satisfying (10.3.20). Then $\tilde{\mu}' := \tilde{\mu} \circ \mathrm{id}_{U_{\mathsf{S}'}}$ satisfies (10.3.19). But then, by the uniqueness of μ', we have $\tilde{\mu}' = \mu'$. Therefore (because $U_{\mathsf{S}'}$ is an equivalence) we see that $\tilde{\mu} = \mu$. □

Definition 10.3.21. The construction of the right derived functor $(\mathrm{R}F, \eta)$ in the proof of the theorem above, and specifically formulas (10.3.16) and (10.3.17), is called a *presentation of* $(\mathrm{R}F, \eta)$ *by the system of right* J*-resolutions* (I, ρ).

Of course any other right derived functor of F (perhaps presented by another system of right J-resolutions) is uniquely isomorphic to $(\mathrm{R}F, \eta)$. This is according to Proposition 10.3.2.

In Section 11 we shall give several existence results for the right derived functor

$$(\mathrm{R}F, \eta) : \mathbf{D}^{\star}(A, \mathsf{M}) \to \mathsf{E}$$

of a triangulated functor

$$F : \mathbf{K}^{\star}(A, \mathsf{M}) \to \mathsf{E},$$

under various assumptions on F, A, M and $\star$. These existence results will be based on Theorem 10.3.3: we will prove existence of suitable resolving subcategories $\mathsf{J} \subseteq \mathbf{K}^{\star}(A, \mathsf{M})$. The example below is one such case.

Example 10.3.22. Suppose we start from an additive functor $F : \mathsf{M} \to \mathsf{N}$. We know how to extend it to a DG functor $F : \mathbf{C}^+(\mathsf{M}) \to \mathbf{C}^+(\mathsf{N})$, and then to a triangulated functor $F : \mathbf{K}^+(\mathsf{M}) \to \mathbf{K}^+(\mathsf{N})$. By composing with Q we get a triangulated functor $\mathrm{Q} \circ F : \mathbf{K}^+(\mathsf{M}) \to \mathbf{D}^+(\mathsf{N})$, that we also denote by F for simplicity.

Assume that the abelian category M has enough injectives (this means that any object $M \in \mathsf{M}$ admits an injective resolution). Define J to be the full subcategory of $\mathsf{K} := \mathbf{K}^+(\mathsf{M})$ on the bounded below complexes of injective objects; and let $\mathsf{E} := \mathbf{D}^+(\mathsf{N})$. We will prove later that properties (a) and (b) of Theorem 10.3.3 hold in this situation. Therefore we have a right derived functor

$$\mathrm{R}F : \mathbf{D}^+(\mathsf{M}) \to \mathbf{D}^+(\mathsf{N}).$$

In case the functor F is left exact, it has the classical right derived functors $\mathrm{R}^q F : \mathsf{M} \to \mathsf{N}$, $q \geq 0$. Formula (10.3.16) shows that for any $M \in \mathsf{M}$ there is equality $\mathrm{R}^q F(M) = \mathrm{H}^q(\mathrm{R}F(M))$ as objects of N. We will prove that more is true:

$$\mathrm{R}^q F = \mathrm{H}^q \circ \mathrm{R}F$$

as functors $\mathsf{M} \to \mathsf{N}$.

In the situation of Theorem 10.3.3, let $\mathsf{K}^{\dagger}$ be a full pretriangulated subcategory of K. Define $\mathsf{S}^{\dagger} := \mathsf{K}^{\dagger} \cap \mathsf{S}$ and $\mathsf{J}^{\dagger} := \mathsf{K}^{\dagger} \cap \mathsf{J}$. Denote by $V : \mathsf{K}^{\dagger} \to \mathsf{K}$ the inclusion functor, and by $V_{\mathsf{S}^{\dagger}} : \mathsf{K}^{\dagger}_{\mathsf{S}^{\dagger}} \to \mathsf{K}_{\mathsf{S}}$ its localization. Warning: the functor $V_{\mathsf{S}^{\dagger}}$ is not necessarily fully faithful; cf. Proposition 9.1.6.

Proposition 10.3.23. *Assume that every $M \in \mathsf{K}^{\dagger}$ admits a quasi-isomorphism $M \to I$ where $I \in \mathsf{J}^{\dagger}$. Then the pair*

$$(\mathrm{R}F \circ V_{\mathsf{S}^{\dagger}}, \eta \circ \mathrm{id}_V)$$

is a right derived functor of $F \circ V : \mathsf{K}^{\dagger} \to \mathsf{E}$.

Loosely speaking, the proposition says that

$$\mathrm{R}(F \circ V) = \mathrm{R}F \circ V_{\mathsf{S}^{\dagger}}.$$

The proof is an exercise.

Exercise 10.3.24. Proof the last proposition. (Hint: Start by choosing a system of right $\mathsf{J}^{\dagger}$-resolutions of $\mathsf{K}^{\dagger}$. Then extend it to a system of right J-resolutions of K. Now follow the proof of the theorem.)

10.4. **Left Derived Functors.** Left derived functors behave just like right derived functors, except for a change of sides in the target category. Because of this our treatment will be brief: we will state the definitions and the main results, but won't give proofs, beyond a hint here and there.

Definition 10.4.1. Assume Setup 10.0.2. A *left derived functor* of F is a triangulated functor

$$\mathrm{L}F : \mathsf{K}_{\mathsf{S}} \to \mathsf{E},$$

together with a morphism

$$\eta : \mathrm{L}F \circ \mathrm{Q} \Rightarrow F$$

of triangulated functors $\mathsf{K} \to \mathsf{E}$. The pair $(\mathrm{L}F, \eta)$ must have this universal property:

($\Diamond$) Given any pair (G, θ), consisting of a triangulated functor $G : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$ and a morphism of triangulated functors $\theta : G \circ \mathrm{Q} \Rightarrow F$, there is a unique morphism of triangulated functors $\mu : G \Rightarrow \mathrm{L}F$ such that $\theta = \eta * (\mu \circ \mathrm{id}_{\mathrm{Q}})$.

Pictorially: there is a 2-diagram

$$\begin{tikzcd}
\mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[dr, phantom, "\Uparrow \eta"] & \mathsf{E} \\
\mathsf{K}_{\mathsf{S}} \arrow[ur, "\mathrm{L}F"'] &
\end{tikzcd}$$

For any other pair (G, θ) there is a unique morphism μ that sits in this 2-diagram:

$$\begin{tikzcd}
\mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[dr, phantom, "\Uparrow \eta \quad \Uparrow \theta"] & \mathsf{E} \\
\mathsf{K}_{\mathsf{S}} \arrow[ur, "\mathrm{L}F"'] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, bend right=20, "\Uparrow \mu"] &
\end{tikzcd}$$

The 1-morphisms in this 2-diagram do not (necessarily) commute; but the diagram of 2-morphisms (with $*$ composition)

$$\begin{tikzcd} F & \\ \mathrm{L}F \circ \mathrm{Q} \arrow[u, Rightarrow, "\eta"] & G \circ \mathrm{Q} \arrow[l, Rightarrow, "\mu \circ \mathrm{id}_{\mathrm{Q}}"] \arrow[ul, Rightarrow, "\theta"'] \end{tikzcd}$$

is commutative.

Proposition 10.4.2. *If a left derived functor* $(\mathrm{L}F, \eta)$ *exists, then it is unique, up to a unique isomorphism. Namely, if* (G, θ) *is another left derived functor of* F*, then there is a unique isomorphism of triangulated functors* $\mu : G \xRightarrow{\simeq} \mathrm{L}F$ *such that* $\theta = \eta * (\mu \circ \mathrm{id}_{\mathsf{Q}})$.

The proof is the same as that of Proposition 10.3.2, with direction of arrows in E reversed.

Theorem 10.4.3. *Given Setup* 10.0.2*, assume there is a full pretriangulated subcategory* $\mathsf{P} \subseteq \mathsf{K}$ *with these two properties*:

(a) *If* $\phi : P \to P'$ *is a quasi-isomorphism in* P*, then* $F(\phi) : F(P) \to F(P')$ *is an isomorphism in* E.

(b) *Every object* $M \in \mathsf{K}$ *admits a quasi-isomorphism* $\rho : P \to M$ *from some object* $P \in \mathsf{P}$.

Then the right derived functor

$$(\mathrm{L}F, \eta) : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$$

exists. Moreover, for any object $P \in \mathsf{P}$ *the morphism*

$$\eta_P : (\mathrm{L}F \circ \mathrm{Q})(P) \to F(P)$$

in E *is an isomorphism.*

The category P is a “generalized category of projectives”.

The proof is the same as that of Theorem 10.3.3, with direction of arrows in E reversed.

Definition 10.4.4. In the situation of Theorem 10.4.3, by a *system of left* P*-resolutions* we mean a pair (P, ρ), where $P : \mathrm{Ob}(\mathsf{K}) \to \mathrm{Ob}(\mathsf{P})$ is a function, and $\rho = \{\rho_M\}_{M \in \mathrm{Ob}(\mathsf{K})}$ is a collection of quasi-isomorphisms $\rho_M : P(M) \to M$ in K. Moreover, if $M \in \mathrm{Ob}(\mathsf{P})$, then $P(M) = M$ and $\rho_M = \mathrm{id}_M$.

Property (b) of Theorem 10.4.3 guarantees that a system of left P-resolutions (P, ρ) exists.

Definition 10.4.5. The construction of the left derived functor $(\mathrm{L}F, \eta)$, when proving Theorem 10.4.3 along the lines of Theorem 10.3.3, and specifically the formulas

$$\mathrm{L}F(M) := F(P(M)) \tag{10.4.6}$$

and

$$\eta_M := F(\rho_M) : \mathrm{L}F(M) \to F(M),$$

is called a *presentation of* $(\mathrm{L}F, \eta)$ *by the system of left* P*-resolutions* (P, ρ).

In Section 11 we shall give several existence results for the left derived functor

$$(\mathrm{L}F, \eta) : \mathsf{D}^{\star}(A, \mathsf{M}) \to \mathsf{E}$$

of a triangulated functor

$$F : \mathsf{K}^{\star}(A, \mathsf{M}) \to \mathsf{E},$$

under various assumptions on F, A, M and $\star$. These existence results will be based on Theorem 10.4.3: we will prove existence of suitable resolving subcategories $\mathsf{P} \subseteq \mathsf{K}^{\star}(A, \mathsf{M})$. The example below is one such case.

Example 10.4.7. Suppose we start from an additive functor $F : \mathsf{M} \to \mathsf{N}$. We know how to extend it to a DG functor $F : \mathsf{C}^{-}(\mathsf{M}) \to \mathsf{C}^{-}(\mathsf{N})$, and then to a triangulated functor $F : \mathsf{K}^{-}(\mathsf{M}) \to \mathsf{K}^{-}(\mathsf{N})$. By composing with Q we get a triangulated functor $\mathrm{Q} \circ F : \mathsf{K}^{-}(\mathsf{M}) \to \mathsf{D}^{-}(\mathsf{N})$, that we also denote by F for simplicity.

Assume that the abelian category M has enough projectives (this means that any object $M \in \mathsf{M}$ admits a projective resolution). Define P to be the full subcategory of $\mathsf{K} := \mathsf{K}^{-}(\mathsf{M})$ on the bounded above complexes of projective objects; and let $\mathsf{E} := \mathsf{D}^{-}(\mathsf{N})$. We will prove later that properties (a) and (b) of Theorem 10.4.3 hold in this situation. Therefore we have a left derived functor

$$\mathrm{L}F : \mathsf{D}^{-}(\mathsf{M}) \to \mathsf{D}^{-}(\mathsf{N}).$$

In case the functor F is right exact, it has the classical left derived functors $\mathrm{L}_q F : \mathsf{M} \to \mathsf{N}$, $q \geq 0$. Formula (10.4.6) shows that for any $M \in \mathsf{M}$ there is equality $\mathrm{L}_q F(M) = \mathrm{H}^{-q}(\mathrm{L}F(M))$ as objects of N. We will prove that more is true:

$$\mathrm{L}_q F = \mathrm{H}^{-q} \circ \mathrm{L}F$$

as functors $\mathsf{M} \to \mathsf{N}$.

In the situation of Theorem 10.4.3, let $\mathsf{K}^{\dagger}$ be a full pretriangulated subcategory of K. Define $\mathsf{S}^{\dagger} := \mathsf{K}^{\dagger} \cap \mathsf{S}$ and $\mathsf{P}^{\dagger} := \mathsf{K}^{\dagger} \cap \mathsf{P}$. Denote by $V : \mathsf{K}^{\dagger} \to \mathsf{K}$ the inclusion functor, and by $V_{\mathsf{S}^{\dagger}} : \mathsf{K}^{\dagger}_{\mathsf{S}^{\dagger}} \to \mathsf{K}_{\mathsf{S}}$ its localization. Warning: the functor $V_{\mathsf{S}^{\dagger}}$ is not necessarily fully faithful; cf. Proposition 9.1.6.

Proposition 10.4.8. *Assume that every $M \in \mathsf{K}^{\dagger}$ admits a quasi-isomorphism $P \to M$ where $P \in \mathsf{P}^{\dagger}$. Then the pair*

$$(\mathrm{L}F \circ V_{\mathsf{S}^{\dagger}}, \eta \circ \mathrm{id}_V)$$

is a left derived functor of $F \circ V : \mathsf{K}^{\dagger} \to \mathsf{E}$.

The proof is just like that of Proposition 10.3.23 (which was an exercise...).

11. Resolutions of DG Modules

In this section we are back to the more concrete setting: A is a DG ring, and M is an abelian category (both over a base ring $\mathbb{K}$). We will define *K-projective* and *K-injective* DG modules in $\mathbf{K}(A, \mathsf{M})$. These DG modules form full pretriangulated subcategories of $\mathbf{K}(A, \mathsf{M})$, and are concrete versions of the abstract categories J and P, that played imporant roles in Subsections 10.3 and 10.4 respectively. For $\mathbf{K}(A)$ we also define *K-flat DG modules*.

11.1. **K-Injective DG Modules.**

For any i we have an additive functor

$$\mathrm{H}^i : \mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \to \mathsf{M}\,.$$

There is equality $\mathrm{H}^i = \mathrm{H}^0 \circ \mathrm{T}^i$. The functors H^i pass to the homotopy category, and

$$\mathrm{H}^0 : \mathbf{K}(A, \mathsf{M}) \to \mathsf{M}$$

is a cohomological functor in the sense of Definition 5.3.2.

Definition 11.1.1. A DG module $N \in \mathbf{C}(A, \mathsf{M})$ is called *acyclic* if $\mathrm{H}^i(N) = 0$ for all i.

Definition 11.1.2. A DG module $I \in \mathbf{C}(A, \mathsf{M})$ is called *K-injective* if for every acyclic DG module $N \in \mathbf{C}(A, \mathsf{M})$, the DG $\mathbb{K}$-module $\mathrm{Hom}_{A,\mathsf{M}}(N, I)$ is acyclic.

The definition above characterizes K-injectives as objects of $\mathbf{C}(A, \mathsf{M})$. The next proposition shows that being K-injective is intrinsic to the pretriangulated category $\mathbf{K}(A, \mathsf{M})$, with the cohomological functor H^0 (that tells us which are the acyclic objects).

Proposition 11.1.3. *A DG module $I \in \mathbf{K}(A, \mathsf{M})$ is K-injective iff $\mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(N, I) = 0$ for every acyclic DG module $N \in \mathbf{K}(A, \mathsf{M})$.*

Proof. This is because for any integer p we have

$$\mathrm{H}^p\big(\mathrm{Hom}_{A,\mathsf{M}}(N, I)\big) \cong \mathrm{H}^0\big(\mathrm{Hom}_{A,\mathsf{M}}(\mathrm{T}^{-p}(N), I)\big) \cong \mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(\mathrm{T}^{-p}(N), I),$$

and N is acyclic iff $\mathrm{T}^{-p}(N)$ is acyclic. □

The concept of K-injective complex (i.e. a K-injective object of $\mathbf{K}(\mathsf{M})$) was introduced by Spaltenstein [Sp] in 1988. At about the same time other authors (Keller [Ke], Bockstedt-Neeman [BN], Bernstein-Lunts [BL], ...) discovered this concept independently, with other names (such as *homotopically injective complex*). The texts [BL] and [Ke] already talk about DG modules over DG rings.

Remark 11.1.4. When the smart truncation functors exist (e.g. when A is a nonpositive DG ring), it is enough to check for K-injectivity of a DG module $I \in \mathbf{K}^{\star}(A, \mathsf{M})$ against acyclic DG modules $N \in \mathbf{K}^{\star}(A, \mathsf{M})$. We will not prove that here. Cf. Proposition 9.2.5 and Remark 9.2.10.

Definition 11.1.5. Let $M \in \mathbf{K}(A, \mathsf{M})$. A *K-injective resolution* of M is a quasi-isomorphism $\rho : M \to I$ in $\mathbf{K}(A, \mathsf{M})$, where I is a K-injective DG module.

Remark 11.1.6. In some other texts

★ and in our Sec 12... ★

"resolution" refers to a quasi-isomorphism $\rho : M \to I$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. It usually makes no difference which meaning is used (as long as we know what we are talking about).

In the next section we will prove existence of K-injectives in several contexts. Here is an easy one.

Exercise 11.1.7. Let $I \in \mathbf{K}(\mathsf{M})$ be a complex of injective objects of M, with zero differential. Prove that I is K-injective.

Definition 11.1.8. Let K be a full subcategory of $\mathbf{K}(A, \mathsf{M})$. The full subcategory of K on the K-injective DG modules in it is denoted by $\mathsf{K}_{\mathrm{inj}}$. In other words,

$$\mathsf{K}_{\mathrm{inj}} = \mathbf{K}(A, \mathsf{M})_{\mathrm{inj}} \cap \mathsf{K}\,.$$

Proposition 11.1.9. *If K is a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$, then $\mathsf{K}_{\mathrm{inj}}$ is a full pretriangulated subcategory of K.*

Proof. It suffcies to prove that $\mathbf{K}(A, \mathsf{M})_{\mathrm{inj}}$ is a pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$. It is easy to see that $\mathbf{K}(A, \mathsf{M})_{\mathrm{inj}}$ is closed under translations. Suppose

$$I \to J \to K \to \mathrm{T}(I)$$

is a distinguished triangle in $\mathbf{K}(A, \mathsf{M})$, with I, J being K-injective DG modules. We have to show that K is also K-injective. Take any acyclic DG module $N \in \mathbf{K}(A, \mathsf{M})$. For every p there is an exact sequence

$$\mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(N, \mathrm{T}^p(J)) \to \mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(N, \mathrm{T}^p(K)) \to \mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(N, \mathrm{T}^{p+1}(I))$$

in $\mathsf{Mod}\,\mathbb{K}$. Because $\mathrm{T}^p(J)$ and $\mathrm{T}^{p+1}(I)$ are K-injectives, Proposition 11.1.3 says that

$$\mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(N, \mathrm{T}^p(J)) = 0 = \mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(N, \mathrm{T}^{p+1}(I)).$$

Therefore $\mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(N, \mathrm{T}^p(K)) = 0$. But N is an arbitrary acyclic DG module, so $\mathrm{T}^p(K)$ is K-injective. Hence K is K-injective too. □

Example 11.1.10. Let $\star$ be some boundedness condition (namely b, $+$, $-$ or nothing). We know that $\mathbf{K}^{\star}(A, \mathsf{M})$ is a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$. Hence $\mathbf{K}^{\star}(A, \mathsf{M})_{\mathrm{inj}}$ is a pretriangulated subcategory too.

Definition 11.1.11. Let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$. We say that K *has enough K-injectives* if any DG module $M \in \mathsf{K}$ admits a K-injective resolution inside K. I.e. there is a quasi-isomorphism $\rho : M \to I$ where $I \in \mathsf{K}_{\mathrm{inj}}$.

Here is the crucial fact regarding K-injectives.

Lemma 11.1.12. *Let K be a full subcategory of $\mathbf{K}(A, \mathsf{M})$. Let $s : I \to M$ be a quasi-isomorphism in K, and assume I is K-injective. Then s has a left inverse, namely there is a morphism $t : M \to I$ in K such that $t \circ s = \mathrm{id}_I$.*

Proof. Since K is a full subcategory of $\mathbf{K}(A, \mathsf{M})$, we can assume that $\mathsf{K} = \mathbf{K}(A, \mathsf{M})$. Consider a distinguished triangle

$$I \xrightarrow{s} M \to N \to \mathrm{T}(I)$$

in $\mathbf{K}(A, \mathsf{M})$ that's built on s. The long exact cohomology sequence tells us that N is an acyclic DG module. So

$$\mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(\mathrm{T}^p(N), I) = 0$$

for all p. The exact sequence

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(N, I) \to \mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(M, I) \\ &\qquad \to \mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(I, I) \to \mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(\mathrm{T}^{-1}(N), I) \end{aligned}$$

shows that $\phi \mapsto \phi \circ s$ is a bijection

$$\operatorname{Hom}_{\mathsf{K}(A,\mathsf{M})}(M, I) \xrightarrow{\simeq} \operatorname{Hom}_{\mathsf{K}(A,\mathsf{M})}(I, I).$$

We take $t : M \to I$ to be the unique morphism in $\mathbf{K}(A, \mathsf{M})$ such that $t \circ s = \mathrm{id}_I$. □

Theorem 11.1.13. *Let A be a DG ring, let M be an abelian category, and let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$. Denote by S the set of quasi-isomorphisms in K. Then the localization functor*

$$\mathrm{Q} : \mathsf{K}_{\mathrm{inj}} \to \mathsf{K}_{\mathsf{S}}$$

is fully faithful.

Proof. Consider any pair of objects $I, J \in \mathsf{K}_{\mathrm{inj}}$. We must prove that the $\mathbb{K}$-module homomorphism

$$\mathrm{Q} : \operatorname{Hom}_{\mathsf{K}}(I, J) \to \operatorname{Hom}_{\mathsf{K}_{\mathsf{S}}}(I, J) \tag{11.1.14}$$

is bijective.

Suppose $q : I \to J$ is a morphism in K_{S}. Let us present q as a left fraction: $q = \mathrm{Q}(s)^{-1} \circ \mathrm{Q}(a)$, where $a : I \to N$ and $s : J \to N$ are morphisms in K, and s is a quasi-isomorphism. By Lemma 11.1.12 s has a left inverse t. We get a morphism $t \circ a : I \to J$ in K, and an easy calculation shows that $\mathrm{Q}(t \circ a) = q$ in K_{S}. This proves surjectivity of (11.1.14).

Now let's prove injectivity of (11.1.14). If $a : I \to J$ is a morphism in K such that $\mathrm{Q}(a) = 0$, then by axiom (LO4) of Ore localization (the left version of axiom (RO4) in Definition 8.2.1), there is a quasi-isomorphism $s : J \to L$ in K such that $s \circ a = 0$ in K. Let t be the left inverse of s. Then $a = t \circ s \circ a = 0$ in K. □

Corollary 11.1.15. *Let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$. If K has enough K-injectives, then the localization functor*

$$\mathrm{Q} : \mathsf{K}_{\mathrm{inj}} \to \mathsf{K}_{\mathsf{S}}$$

is an equivalence.

Proof. By the theorem the functor Q is fully faithful. The extra condition guarantees that Q is essentially surjective on objects. □

Corollary 11.1.16. *Let $\star$ be any boundedness condition. If $\mathbf{K}^{\star}(A, \mathsf{M})$ has enough K-injectives, then the triangulated functor*

$$\mathrm{Q} : \mathbf{K}^{\star}(A, \mathsf{M})_{\mathrm{inj}} \to \mathbf{D}^{\star}(A, \mathsf{M})$$

is an equivalence.

Proof. Since $\mathbf{K}^{\star}(A, \mathsf{M})$ is a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$, this is a special case of the previous corollary. □

Remark 11.1.17. This result is of tremendous importance, both theoretically and practically. In the theory, it shows that the localized category $\mathbf{D}^{\star}(A, \mathsf{M})$, which is too big to lie inside the original universe U (see Remark 8.2.13), is equivalent to a U-category. On the practical side, it means that among K-injective objects we do not need fractions to represent morphisms.

Corollary 11.1.18. *Let $\star$ and $\dagger$ be boundedness conditions such that*

$$\mathsf{K}^{\star}(A, \mathsf{M}) \subseteq \mathsf{K}^{\dagger}(A, \mathsf{M}).$$

Assume these categories have enough K-injectives. Then the canonical functor

$$\mathsf{D}^{\star}(A, \mathsf{M}) \to \mathsf{D}^{\dagger}(A, \mathsf{M})$$

is fully faithful.

Proof. Combine Corollary 11.1.16 with the fact that $\mathsf{K}^{\star}(A, \mathsf{M}) \to \mathsf{K}^{\dagger}(A, \mathsf{M})$ is fully faithful. $\square$

Remark 11.1.19. Earlier we only proved that $\mathsf{D}^{\star}(A, \mathsf{M}) \to \mathsf{D}(A, \mathsf{M})$ is fully faithful in special cases (see Proposition 9.2.5 and Remark 9.2.10).

Here is another useful definition and result. The definition is a variant of Definition 10.3.11.

Definition 11.1.20. Let K be a full pretriangulated subcategory of $\mathsf{K}(A, \mathsf{M})$, and assume K has enough K-injectives. A *system of K-injective resolutions* in K is a pair (I, ρ), where $I : \mathrm{Ob}(\mathsf{K}) \to \mathrm{Ob}(\mathsf{K}_{\mathrm{inj}})$ is a function, and $\rho = \{\rho_M\}_{M \in \mathrm{Ob}(\mathsf{K})}$ is a collection of quasi-isomorphisms $\rho_M : M \to I(M)$ in K. Moreover, if $M \in \mathrm{Ob}(\mathsf{K}_{\mathrm{inj}})$, then $I(M) = M$ and $\rho_M = \mathrm{id}_M$.

The proposition is a variant of Lemma 10.3.12.

Proposition 11.1.21. *Suppose a system of K-iniective resolutions (I, ρ) has been chosen. Then the function I extends uniquely to a triangulated functor $I : \mathsf{K}_{\mathsf{S}} \to \mathsf{K}_{\mathrm{inj}}$, such that $\mathrm{id}_{\mathsf{K}_{\mathrm{inj}}} = I \circ \mathrm{Q}|_{\mathsf{K}_{\mathrm{inj}}}$, and $\rho : \mathrm{id}_{\mathsf{K}_{\mathsf{S}}} \Rightarrow \mathrm{Q} \circ I$ is an isomorphism of triangulated functors.*

Proof. The proof is the same as that of Lemma 10.3.12, except that here we use Corollary 11.1.15. $\square$

Theorem 11.1.22. *Let K be a full pretriangulated subcategory of $\mathsf{K}(A, \mathsf{M})$, and denote by S the set of quasi-isomorphisms in K. Assume K has enough K-injectives. Let E be any pretriangulated category, and let*

$$F : \mathsf{K} \to \mathsf{E}$$

be any triangulated functor. Then F has a right derived functor

$$(\mathrm{R}F, \eta) : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}\,.$$

Furthermore, for any $I \in \mathsf{K}_{\mathrm{inj}}$ the morphism $\eta_I : F(I) \to \mathrm{R}F(I)$ in E is an isomorphism.

Proof. We will use Theorem 10.3.3. In the notation of that theorem, let $\mathsf{J} := \mathsf{K}_{\mathrm{inj}}$. Condition (b) of that theorem holds (this is the "enough K-injectives" assertion). Next, Theorem 11.1.13 implies that any quasi-isomorphism $\phi : I \to J$ in $\mathsf{K}_{\mathrm{inj}}$ is actually an isomorphism. Therefore $F(\phi)$ is an isomorphism in E, and this is condition (a) of Theorem 10.3.3. $\square$

Example 11.1.23. Let A be any DG ring. We will prove later that $\mathsf{K}(A)$ has enough K-injectives. Therefore, given any triangulated functor $F : \mathsf{K}(A) \to \mathsf{E}$ into any pretriangulated category E, the right derived functor

$$(\mathrm{R}F, \eta) : \mathsf{D}(A) \to \mathsf{E}$$

exists.

Suppose we choose a system of K-injective resolutions (I, ρ) in $\mathsf{K}(A)$. Then we get a presentation of $(\mathrm{R}F, \eta)$ as follows: $\mathrm{R}F(M) = F(I(M))$ and $\eta_M = F(\rho_M)$.

11.2. **K-Projective DG Modules.** This subsection is dual to the previous one, and so we will be brief.

Definition 11.2.1. A DG module $P \in \mathsf{C}(A, \mathsf{M})$ is called *K-projective* if for every acyclic DG module $N \in \mathsf{C}(A, \mathsf{M})$, the DG $\mathbb{K}$-module $\mathrm{Hom}_{A,\mathsf{M}}(P, N)$ is acyclic.

Proposition 11.2.2. *A DG module $P \in \mathsf{K}(A, \mathsf{M})$ is K-projective if and only if $\mathrm{Hom}_{\mathsf{K}(A,\mathsf{M})}(P, N) = 0$ for every acyclic DG module $N \in \mathsf{K}(A, \mathsf{M})$.*

The proof is like that of Proposition 11.1.3.

Definition 11.2.3. Let $M \in \mathsf{K}(A, \mathsf{M})$. A *K-projective resolution* of M is a quasi-isomorphism $\rho : P \to M$ in $\mathsf{K}(A, \mathsf{M})$, where P is a K-projective DG module.

Definition 11.2.4. Let K be a full subcategory of $\mathsf{K}(A, \mathsf{M})$. The full subcategory of K on the K-projective DG modules in it is denoted by $\mathsf{K}_{\mathrm{prj}}$. In other words,

$$\mathsf{K}_{\mathrm{prj}} = \mathsf{K}(A, \mathsf{M})_{\mathrm{prj}} \cap \mathsf{K} .$$

Proposition 11.2.5. *If K is a full pretriangulated subcategory of $\mathsf{K}(A, \mathsf{M})$, then $\mathsf{K}_{\mathrm{prj}}$ is a full pretriangulated subcategory of K.*

The proof is like that of Proposition 11.1.9.

Example 11.2.6. Let $\star$ be some boundedness condition (namely b, $+$, $-$ or nothing). Since $\mathsf{K}^{\star}(A, \mathsf{M})$ is a full pretriangulated subcategory of $\mathsf{K}(A, \mathsf{M})$, we see that $\mathsf{K}^{\star}(A, \mathsf{M})_{\mathrm{prj}}$ is a pretriangulated subcategory too.

Definition 11.2.7. Let K be a full pretriangulated subcategory of $\mathsf{K}(A, \mathsf{M})$. We say that K *has enough K-projectives* if any DG module $M \in \mathsf{K}$ admits a K-projective resolution inside K. I.e. there is a quasi-isomorphism $\rho : P \to M$ where $P \in \mathsf{K}_{\mathrm{prj}}$.

Lemma 11.2.8. *Let K be a full subcategory of $\mathsf{K}(A, \mathsf{M})$. Let $s : M \to P$ be a quasi-isomorphism in K, and assume P is K-projective. Then s has a right inverse; namely there is a morphism $t : P \to M$ in K such that $s \circ t = \mathrm{id}_P$.*

Same proof as that of Lemma 11.1.12.

Theorem 11.2.9. *Let A be a DG ring, let M be an abelian category, and let K be a full pretriangulated subcategory of $\mathsf{K}(A, \mathsf{M})$. Denote by S the set of quasi-isomorphisms in K. Then the localization functor*

$$\mathrm{Q} : \mathsf{K}_{\mathrm{prj}} \to \mathsf{K}_{\mathsf{S}}$$

is fully faithful.

The proof is the same as that of Theorem 11.1.13. The next corollaries and Proposition are also proved like their K-injective counterparts.

Corollary 11.2.10. *Let K be a full pretriangulated subcategory of $\mathsf{K}(A, \mathsf{M})$. If K has enough K-projectives, then the localization functor*

$$\mathrm{Q} : \mathsf{K}_{\mathrm{prj}} \to \mathsf{K}_{\mathsf{S}}$$

is an equivalence.

Corollary 11.2.11. *Let $\star$ and $\dagger$ be boundedness conditions such that*

$$\mathsf{K}^{\star}(A, \mathsf{M}) \subseteq \mathsf{K}^{\dagger}(A, \mathsf{M}).$$

Assume these categories have enough K-projectives. Then the canonical functor

$$\mathsf{D}^{\star}(A, \mathsf{M}) \to \mathsf{D}^{\dagger}(A, \mathsf{M})$$

is fully faithful.

Definition 11.2.12. Let K be a full pretriangulated subcategory of $\mathsf{K}(A, \mathsf{M})$, and assume K has enough K-projectives. A *system of K-projective resolutions* in K is a pair (P, ρ), where $P : \operatorname{Ob}(\mathsf{K}) \to \operatorname{Ob}(\mathsf{K}_{\mathrm{prj}})$ is a function, and $\rho = \{\rho_M\}_{M \in \operatorname{Ob}(\mathsf{K})}$ is a collection of quasi-isomorphisms $\rho_M : P(M) \to M$ in K. Moreover, if $M \in \operatorname{Ob}(\mathsf{K}_{\mathrm{prj}})$, then $P(M) = M$ and $\rho_M = \mathrm{id}_M$.

Proposition 11.2.13. *Suppose a system of K-projective resolutions (P, ρ) has been chosen. Then the function P extends uniquely to a triangulated functor $P : \mathsf{K}_{\mathsf{S}} \to \mathsf{K}_{\mathrm{prj}}$, such that $\mathrm{id}_{\mathsf{K}_{\mathrm{prj}}} = P \circ \mathrm{Q}|_{\mathsf{K}_{\mathrm{prj}}}$, and $\rho : \mathrm{Q} \circ P \Rightarrow \mathrm{id}_{\mathsf{K}_{\mathsf{S}}}$ is an isomorphism of triangulated functors.*

Theorem 11.2.14. *Let K be a full pretriangulated subcategory of $\mathsf{K}(A, \mathsf{M})$, and denote by S the set of quasi-isomorphisms in K. Assume K has enough K-projectives. Let E be any pretriangulated category, and let*

$$F : \mathsf{K} \to \mathsf{E}$$

be any triangulated functor. Then F has a left derived functor

$$(\mathrm{L}F, \eta) : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}\,.$$

Furthermore, for any $P \in \mathsf{K}_{\mathrm{prj}}$ the morphism $\eta_P : \mathrm{L}F(P) \to F(P)$ in E is an isomorphism.

The proof is like that of Theorem 11.1.22.

Example 11.2.15. Let A be any DG ring. We will prove later that $\mathsf{K}(A)$ has enough K-projectives. Therefore, given any triangulated functor $F : \mathsf{K}(A) \to \mathsf{E}$ into any pretriangulated category E, the left derived functor

$$(\mathrm{L}F, \eta) : \mathsf{D}(A) \to \mathsf{E}$$

exists.

Suppose we choose a system of K-projective resolutions (P, ρ) in $\mathsf{K}(A)$. Then we get a presentation of $(\mathrm{L}F, \eta)$ as follows: $\mathrm{L}F(M) = F(P(M))$ and $\eta_M = F(\rho_M)$.

11.3. **K-Flat DG Modules.** Recall that A^{op} is the opposite DG ring. The objects of $\mathsf{C}(A^{\mathrm{op}})$ are the right DG A-modules.

Definition 11.3.1. A DG module $P \in \mathsf{C}(A)$ is called *K-flat* if for every acyclic DG module $N \in \mathsf{C}(A^{\mathrm{op}})$, the DG $\mathbb{K}$-module $N \otimes_A P$ is acyclic.

Proposition 11.3.2. *If $P \in \mathsf{C}(A)$ is K-projective then it is K-flat.*

Proof. Let J be an injective cogenerator of $\mathsf{M}(\mathbb{K}) = \mathsf{Mod}\,\mathbb{K}$. This means that J is an injective $\mathbb{K}$-module, such that any nonzero $\mathbb{K}$-module L admits a nonzero homomorphism $L \to J$. A universal choice is $J = \operatorname{Hom}_{\mathbb{Z}}(\mathbb{K}, \mathbb{Q}/\mathbb{Z})$.

Take an acyclic complex $N \in \mathsf{C}(A^{\mathrm{op}})$. Then by Hom-tensor adjunction there is an isomorphism of DG $\mathbb{K}$-modules

$$\operatorname{Hom}_{\mathbb{K}}(N \otimes_A P, J) \cong \operatorname{Hom}_A\big(P, \operatorname{Hom}_{\mathbb{K}}(N, J)\big).$$

The right side is acyclic by our assumptions. Hence so is the left side. It follows that $N \otimes_A P$ is acyclic. $\square$

The proof above also gives a hint to the next proposition.

Proposition 11.3.3. *A DG module $P \in \mathsf{K}(A)$ is K-flat iff*

$$\operatorname{Hom}_{\mathsf{K}(A)}\big(P, \operatorname{Hom}_{\mathbb{K}}(N, J)\big) = 0$$

for every acyclic $N \in \mathsf{C}(A^{\mathrm{op}})$ and every injective $J \in \mathsf{Mod}\,\mathbb{K}$.

Exercise 11.3.4. Prove Proposition 11.3.3.

Remark 11.3.5. In view of Proposition 11.3.2, the reader might wonder why we bother with K-flat DG modules. The reason is that on a ringed space $(X, \mathcal{A})$ there are usually very few projective $\mathcal{A}$-modules. But, as we shall prove, there are enough K-flat complexes in $\mathsf{C}(\mathcal{A}) = \mathsf{C}(\mathsf{Mod}\,\mathcal{A})$.

12. Existence of Resolutions

In this section we continue in the more concrete setting: A is a DG ring, and M is an abelian category (both over a commutative base ring $\mathbb{K}$). We will prove existence of K-projective, K-flat and K-injective reolutions in several contexts.

12.1. **Direct and Inverse Limits of Complexes.**

We shall have to work with limits in this section. Limits in abstract abelian and DG categories (not to mention pretriangulated categories) are a very delicate issue. We will try to be as concrete as possible, in order to avoid pitfalls and confusion.

Let C be a category. A *direct system* in C is data

$$\big(\{M_k\}_{k\in\mathbb{N}}, \{\mu_k\}_{k\in\mathbb{N}}\big),$$

where M_k are objects of C, and $\mu_k : M_k \to M_{k+1}$ are morphisms, called transitions. The *direct limit*

$$M = \lim_{k\to} M_k$$

need not exist in C; but if it does, then it is unique up to a unique isomorphism.

By an *inverse system* in the category C we mean data

$$\big(\{M_k\}_{k\in\mathbb{N}}, \{\mu_k\}_{k\in\mathbb{N}}\big),$$

where $\{M_k\}_{k\in\mathbb{N}}$ is a collection of objects, and $\mu_k : M_{k+1} \to M_k$ are morphisms, also called transitions. The *inverse limit*

$$M = \lim_{\leftarrow k} M_k$$

need not exist in C; but if it does, then it is unique up to a unique isomorphism.

Convention 12.1.1.

★ public 34 | new ★

When considering an inverse or a direct system $\{M_k\}_{k\in\mathbb{N}}$ in $\mathbf{C}_{\mathrm{str}}(A,\mathsf{M})$, we usually refer to it as a system in $\mathbf{C}(A,\mathsf{M})$, understanding implicitly that the morphisms are all strict. Likewise for limits.

Similarly, when we say that N is a subobject (resp. quotient object) of M in $\mathbf{C}(A,\mathsf{M})$ we mean that there is a given monomorphism $\phi : N \hookrightarrow M$ (resp. epimorphism $\phi : M \twoheadrightarrow N$) in $\mathbf{C}_{\mathrm{str}}(A,\mathsf{M})$.

Proposition 12.1.2.

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1. *Let $\{M_k\}_{k\in\mathbb{N}}$ be a direct system in $\mathbf{C}(A,\mathsf{M})$. Assume that for every i the direct limit $\lim_{k\to} M_k^i$ exists in M. Then the direct limit $M = \lim_{k\to} M_k$ exists in $\mathbf{C}(A,\mathsf{M})$, and in degree i it is $M^i = \lim_{k\to} M_k^i$.*
2. *Let $\{M_k\}_{k\in\mathbb{N}}$ be an inverse system in $\mathbf{C}(A,\mathsf{M})$. Assume that for every i the inverse limit $\lim_{\leftarrow k} M_k^i$ exists in M. Then the inverse limit $M = \lim_{\leftarrow k} M_k$ exists in $\mathbf{C}(A,\mathsf{M})$, and in degree i it is $M^i = \lim_{\leftarrow k} M_k^i$.*

Exercise 12.1.3. Prove Proposition 12.1.2.

The category $\mathbf{M}(\mathbb{K})$ of $\mathbb{K}$-modules has all direct and inverse limits.

Proposition 12.1.4. *Let C be a $\mathbb{K}$-linear category.*

(1) *Let $\{M_k\}_{k\in\mathbb{N}}$ be a direct system in C, and assume the direct limit $M = \lim_{k\to} M_k$ exists. Then for any object $N \in \mathsf{C}$, the canonical homomorphism*
$$\operatorname{Hom}_{\mathsf{C}}(M, N) \to \lim_{\leftarrow k} \operatorname{Hom}_{\mathsf{C}}(M_k, N)$$
in $\mathbf{M}(\mathbb{K})$ is bijective.

(2) *Let $\{M_k\}_{k\in\mathbb{N}}$ be an inverse system in C, and assume the inverse limit $M = \lim_{\leftarrow k} M_k$ exists. Then for any object $N \in \mathsf{C}$, the canonical homomorphism*
$$\operatorname{Hom}_{\mathsf{C}}(N, M) \to \lim_{\leftarrow k} \operatorname{Hom}_{\mathsf{C}}(N, M_k)$$
in $\mathbf{M}(\mathbb{K})$ is bijective.

Exercise 12.1.5. Prove Proposition 12.1.4.

Proposition 12.1.6.

(1) *Let*
$$\big(\{M_k\}_{k\in\mathbb{N}}, \{\mu_k\}_{k\in\mathbb{N}}\big)$$
be a direct system in M, and assume the direct limit $M = \lim_{k\to} M_k$ exists. Define $N_k := \operatorname{Im}(\mu_{k-1}) \subseteq M_k$, and let $\nu_k : N_{k+1} \to N_k$ be the morphism induced by μ_k. Then the direct limit $= \lim_{k\to} N_k$ exists, and the canonical morphism $N \to M$ is an isomorphism.

(2) *Let*
$$\big(\{M_k\}_{k\in\mathbb{N}}, \{\mu_k\}_{k\in\mathbb{N}}\big)$$
be an inverse system in M, and assume the inverse limit $M = \lim_{\leftarrow k} M_k$ exists. Define $N_k := \operatorname{Im}(\mu_k) \subseteq M_k$, and let $\nu_k : N_{k+1} \to N_k$ be the morphism induced by μ_k. Then the inverse limit $N = \lim_{\leftarrow k} N_k$ exists, and the canonical morphism $N \to M$ is an isomorphism.

Proof. (1) The direct systems $\{M_k\}_{k\in\mathbb{N}}$ and $\{N_k\}_{k\in\mathbb{N}}$ are sandwiched:
$$\begin{tikzcd}
\cdots \arrow[r] & N_k \arrow[r, "\alpha_k"] \arrow[rr, bend left, "\nu_k"] & M_k \arrow[r, "\beta_k"] \arrow[rr, bend right, "\mu_k"'] & N_{k+1} \arrow[r, "\alpha_{k+1}"] & M_{k+1} \arrow[r] & \cdots
\end{tikzcd}$$
The α_k are monomorphisms, and the β_k are epimorphisms. There is a system of morphisms $\phi_k : N_k \to M$. We will show that $\big(M, \{\phi_k\}_{k\in\mathbb{N}}\big)$ is a direct limit of $\{N_k\}_{k\in\mathbb{N}}$. Suppose L is some object, and $\psi_k : N_k \to L$ is a system of morphisms. Then $\beta_k \circ \psi_{k+1} : M_k \to L$ is a system of morphisms. Since $M = \lim_{\leftarrow k} M_k$, there is a unique morphism $\psi : M \to L$ inducing $\{\beta_k \circ \psi_{k+1}\}_{k\in\mathbb{N}}$. But this implies that
$$\phi_k \circ \psi = \psi_k : N_k \to L$$
for all k. If $\psi' : M \to L$ is some other morphism such that $\phi_k \circ \psi' = \psi_k$ for all k, then
$$\beta_{k-1} \circ \phi_k \circ \psi' = \beta_{k-1} \circ \psi_k : M_{k-1} \to L,$$
forcing $\psi' = \psi$.

(2) Similar. $\square$

We now show that in the category of complexes $\mathbf{C}(\mathbb{K})$ all direct limits exist, and that this operation is exact.

Proposition 12.1.7. *Let $\{M_k\}_{k\in\mathbb{N}}$ be a direct system in $\mathbf{C}(\mathbb{K})$.*

(1) *For any i let $M^i := \lim_{k\to} M^i_k \in \mathsf{M}(\mathbb{K})$. Let $M^i \to M^{i+1}$ be the homomorphisms induced by the differentials $M^i_k \to M^{i+1}_k$. Then M is a complex, and it is the direct limit of $\{M_k\}_{k\in\mathbb{N}}$.*
(2) *The canonical homomorphism*

$$\lim_{k\to} \mathrm{H}(M_k) \to \mathrm{H}(M)$$

is bijective.

In particular, if $\{M_k\}_{k\in\mathbb{N}}$ is a direct system of acyclic complexes, then $\lim_{k\to} M_k$ is acyclic.

Exercise 12.1.8. Prove Proposition 12.1.7.

It turns out that direct limits exist and are exact also in $\mathsf{C}(A)$.

Proposition 12.1.9. *Let $\{M_k\}_{k\in\mathbb{N}}$ be a direct system in $\mathsf{C}(A)$. Then the direct limit $M = \lim_{k\to} M_k$ exists in $\mathsf{C}(A)$, and the canonical homomorphism*

$$\lim_{k\to} \mathrm{H}(M_k) \to \mathrm{H}(M)$$

is bijective.

Proof. Let $M = \lim_{k\to} M_k$ be the direct limit in $\mathsf{C}(\mathbb{K})$, that Proposition 12.1.7 produces. By the universal property of limits, any element $a \in A^j$ induces morphisms $a : M^i \to M^{i+j}$, and in this way M becomes an object of $\mathsf{C}(A)$. It is now easy to see that M is a direct limit of $\{M_k\}_{k\in\mathbb{N}}$ in $\mathsf{C}(A)$ – this is almost the same as Proposition 12.1.7(1). The assertion about the cohomology follows from Proposition 12.1.7(2). □

We say that a direct system $\{M_k\}_{k\in\mathbb{N}}$ in M is *eventually constant* if $\mu_k : M_k \to M_{k+1}$ are isomorphisms for large k. The direct limit of an eventually constant direct system always exists: it is any M_k for large enough k.

Proposition 12.1.10. *Let $\{M_k\}_{k\in\mathbb{N}}$ be a direct system in $\mathsf{C}(A, \mathsf{M})$. Assume that for each i the direct system $\{M^i_k\}_{k\in\mathbb{N}}$ in M is eventually constant. Then the direct limit $M = \lim_{k\to} M_k$ exists in $\mathsf{C}(A, \mathsf{M})$, the direct limit $\lim_{k\to} \mathrm{H}(M_k)$ exists in $\mathsf{G}(\mathsf{M})$, and the canonical morphism*

$$\lim_{k\to} \mathrm{H}(M_k) \to \mathrm{H}(M)$$

is an isomorphism.

Proof. As mentioned above, for each i the limit $M^i = \lim_{k\to} M^i_k$ exists in M. By the universal property of limits, there are induced differentials $M^i \to M^{i+1}$, so the complex $M \in \mathsf{C}(\mathsf{M})$ exists. Again by the universal property, any element $a \in A^j$ induces morphisms $a : M^i \to M^{i+j}$, and in this way M becomes an object of $\mathsf{C}(A, \mathsf{M})$. It is now easy to see that M is a direct limit of $\{M_k\}_{k\in\mathbb{N}}$.

Regarding the cohomology: fix an integer i. Take k large enough such that $M^{i'}_k \to M^{i'}_{k'}$ are isomorphisms for all $k \le k'$ and $i-1 \le i' \le i+1$. Then $M^{i'}_{k'} \to M^{i'}$ are isomorphisms in this range, and therefore $\mathrm{H}^i(M_{k'}) \to \mathrm{H}^i(M)$ are isomorphisms for all $k \le k'$. We see that the direct system $\{\mathrm{H}^i(M_k)\}_{k\in\mathbb{N}}$ is eventually constant, and its direct limit is $\mathrm{H}^i(M)$. □

Exactness of inverse limits tends to be much more complicated that that of direct limits, even for $\mathbb{K}$-modules. We always have to make some condition on the inverse system (either it is eventually constant, or the weaker condition in Definition 12.1.13).

Proposition 12.1.11. *Let $\{M_k\}_{k\in\mathbb{N}}$ be an inverse system in $\mathsf{C}(A,\mathsf{M})$. Assume that for each i the inverse system $\{M_k^i\}_{k\in\mathbb{N}}$ in M is eventually constant. Then the inverse limit $M = \lim_{\leftarrow k} M_k$ exists in $\mathsf{C}(A,\mathsf{M})$, the inverse limit $\lim_{\leftarrow k} \mathrm{H}(M_k)$ exists in $\mathsf{G}(\mathsf{M})$, and the canonical morphism*

$$\mathrm{H}(M) \to \lim_{\leftarrow k} \mathrm{H}(M_k)$$

is an isomorphism.

Proof. The same as that of Proposition 12.1.10, but with arrows reversed. □

Proposition 12.1.12. *Let $\{M_k\}_{k\in\mathbb{N}}$ be an inverse system in $\mathsf{C}(A)$. Then the inverse limit $M = \lim_{\leftarrow k} M_k$ exists in $\mathsf{C}(A)$.*

Proof. It is like the beginning of the proof of Proposition 12.1.9: in each degree i we take $M^i := \lim_{\leftarrow k} M_k^i$, the inverse limit in $\mathsf{M}(\mathbb{K})$. There is an induced differential and induced action of A. We get a DG A-module M, and an easy verification shows that it is the inverse limit. □

Definition 12.1.13. Let

$$\big(\{M_k\}_{k\in\mathbb{N}}, \{\mu_k\}_{k\in\mathbb{N}}\big)$$

be an inverse system in $\mathsf{M}(\mathbb{K})$. For any $l \geq k$ let us write

$$\mu_{l,k} := \mathrm{id} \circ \mu_k \circ \cdots \circ \mu_{l-1} : M_l \to M_k.$$

We say that this inverse system has the *Mittag-Leffler* property if for any $k \in \mathbb{N}$ there exists some $l \geq k$ such that for every $l' \geq l$ there is equality

$$\mathrm{Im}(\mu_{l',k}) = \mathrm{Im}(\mu_{l,k})$$

of submodules of M_k.

Example 12.1.14. If the system

$$\big(\{M_k\}_{k\in\mathbb{N}}, \{\mu_k\}_{k\in\mathbb{N}}\big)$$

satisfies one of the following conditions, then in has the Mittag-Leffler property:

(a) The system has surjective transitions.
(b) The system is eventually constant.
(c) For any $k \in \mathbb{N}$ there exists some $l \geq k$ such that $\mu_{l,k} = 0$. This is called the *trivial Mittag-Leffler property*, or one says that the system is *pro-zero*.

Theorem 12.1.15 (Mittag-Leffler Argument)**.** *Let $\{M_k\}_{k\in\mathbb{N}}$ be an inverse system in $\mathsf{C}(A)$, with inverse limit $M = \lim_{\leftarrow k} M_k$. Assume the system satisfies these two conditions*:

(a) *For every $i \in \mathbb{Z}$ the inverse system $\{M_k^i\}_{k\in\mathbb{N}}$ in $\mathsf{M}(\mathbb{K})$ has the Mittag-Leffler property.*
(b) *For every $i \in \mathbb{Z}$ the inverse system $\{\mathrm{H}^i(M_k)\}_{k\in\mathbb{N}}$ in $\mathsf{M}(\mathbb{K})$ has the Mittag-Leffler property.*

Then the canonical homomorphisms

$$\mathrm{H}^i(M) \to \lim_{\leftarrow k} \mathrm{H}^i(M_k)$$

are bijective.

Proof. We can forget all about the graded A-module structure, and just view this as an inverse system in $\mathbf{C}(\mathbb{Z})$, i.e. and inverse system of complexes of abelian groups. Now this is a special case of [KS1, Proposition 1.12.4] or [EGA III, Ch. 0_{III}, Proposition 13.2.3] [EGA III]. $\square$

Corollary 12.1.16. *Let $\{M_k\}_{k\in\mathbb{N}}$ be an inverse system in $\mathbf{C}(A)$, with inverse limit $M = \lim_{\leftarrow k} M_k$. Assume the system satisfies these two conditions*:

(a) *For every $i \in \mathbb{Z}$ the inverse system $\{M_k^i\}_{k\in\mathbb{N}}$ has surjective transitions.*
(b) *For every k the DG module M_k is acyclic.*

Then M is acyclic.

Proof. Conditions (a) and (b) here imply conditions (a) and (b) of Theorem 12.1.15, respectively. $\square$

Remark 12.1.17. We will not attempt discussing direct or inverse limits in abstract abelian categories. Such definitions do exists (e.g. that of a *Grothendieck abelian category*), but this sort of thing is a source of anxiety (and often of errors).

Before going on, it is good to remind the roles of the cocycles and the coboundaries. Let $M \in \mathbf{C}(A, \mathsf{M})$. The object of coboundaries $\mathrm{Z}(M) \subseteq M$ is defined by

$$\mathrm{Z}^i(M) := \mathrm{Ker}\big(\mathrm{d} : M^i \to M^{i+1}\big).$$

The object of cocycles $\mathrm{B}(M) \subseteq M$ is defined by

$$\mathrm{B}^i(M) := \mathrm{Im}\big(\mathrm{d} : M^{i-1} \to M^i\big).$$

Note that $\mathrm{Z}(M)$ and $\mathrm{B}(M)$ live in $\mathbf{G}(\mathrm{Z}(A), \mathsf{M})$, and they have trivial differentials. There are exact sequences

$$0 \to \mathrm{Z}(M) \to M \to \mathrm{T}(\mathrm{B}(M)) \to 0 \tag{12.1.18}$$

and

$$0 \to \mathrm{B}(M) \to \mathrm{Z}(M) \to \mathrm{H}(M) \to 0 \tag{12.1.19}$$

in $\mathbf{G}_{\mathrm{str}}(\mathrm{Z}(A), \mathsf{M})$.

12.2. K-Projective Resolutions in $\mathbf{C}^-(\mathsf{M})$.

★ public 33| changed to “Resolutions in $\mathbf{C}(\mathsf{M})$” ★

Recall that M is some abelian category, and $\mathbf{C}(\mathsf{M})$ is the category of complexes in M. The strict category $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$ is abelian.

A subobject of a complex $M \in \mathbf{C}(\mathsf{M})$ is an object $N \in \mathbf{C}(\mathsf{M})$, together with a strict monomorphism $\phi : N \to M$. We shall use the abbreviation $N \subseteq M$ of $N \hookrightarrow M$. If $N, N' \subseteq M$, we say that $N' \subseteq N$ if the strict monomorphism $\phi' : N' \to M$ factors through $\phi : N \to M$; namely there is a morphism $\psi : N' \to N$ such that $\phi' = \phi \circ \psi$. Note there is at most one such ψ (this is what being a monomorphism means for ϕ). The morphisms ψ is itself monomorphism, and we call it a morphism of subobjects of M.

An *filtration* of a complex $M \in \mathbf{C}(\mathsf{M})$ is a collection $\{F_j(M)\}_{j \geq -1}$ of subobjects of M, such that $F_j(M) \subseteq F_{j+1}(M)$. This is a particular kind of direct system in $\mathbf{C}(\mathsf{M})$ (see Convention 12.1.1 for this slight inaccuracy). We say that $M = \lim_{j\to} F_j(M)$ if this limit exists in $\mathbf{C}(\mathsf{M})$, and the canonical morphism $\lim_{j\to} F_j(M) \to M$ is an isomorphism. There are also the subquotients

$$\mathrm{gr}^F_j(M) := F_j(M)/F_{j-1}(M) \in \mathbf{C}(\mathsf{M}).$$

The next definition is inspired by the work of Keller; see [Ke].

Definition 12.2.1. Let P be a complex in $\mathbf{C}(\mathsf{M})$.

(1) A *semi-projective filtration* on P is a filtration $F = \{F_j(P)\}_{j \geq -1}$ of P as an object of $\mathbf{C}(\mathsf{M})$, such that:
  - $F_{-1}(P) = 0$.
  - Each $\mathrm{gr}^F_j(P)$ is a complex of projective objects of M with zero differential.
  - $P = \lim_{j\to} F_j(P)$ in $\mathbf{C}(\mathsf{M})$.

(2) The complex P is called a *semi-projective complex* if it admits some semi-projective filtration.

Theorem 12.2.2. *Let M be an abelian category, and let P be a semi-projective complex in $\mathbf{C}(\mathsf{M})$. Then P is K-projective.*

Proof. Step 1. We start by proving that if $P = \mathrm{T}^k(Q)$, the shift of a projective object $Q \in \mathsf{M}$, then P is K-projective. This is easy: given an acyclic complex $N \in \mathbf{C}(\mathsf{M})$, we have

$$\mathrm{Hom}_{\mathsf{M}}(P, N) = \mathrm{Hom}_{\mathsf{M}}\big(\mathrm{T}^k(Q), N\big) \cong \mathrm{T}^{-k}\big(\mathrm{Hom}_{\mathsf{M}}(Q, N)\big)$$

in $\mathbf{C}(\mathbb{K})$. But $\mathrm{Hom}_{\mathsf{M}}(Q, -)$ is an exact functor $\mathsf{M} \to \mathbf{M}(\mathbb{K})$, so $\mathrm{Hom}_{\mathsf{M}}(Q, N)$ is an acyclic complex.

Step 2. Now P is a complex of projective objects of M with zero differential. This means that

$$P \cong \bigoplus_{k \in \mathbb{Z}} \mathrm{T}^k(Q_k)$$

in $\mathbf{C}(\mathsf{M})$, where each Q_k is a projective object in M. But then

$$\mathrm{Hom}_{\mathsf{M}}(P, N) \cong \prod_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{M}}\big(\mathrm{T}^k(Q_k), N\big).$$

This is an easy case of Proposition 12.1.4. By step 1 and the fact that a product of acyclic complexes in $\mathbf{C}(\mathbb{K})$ is acyclic (itself an easy case of the Mittag-Leffler argument), we conclude that $\mathrm{Hom}_{\mathsf{M}}(P, N)$ is acyclic.

Step 3. Fix a semi-projective filtration $F = \{F_j(P)\}_{j \in \mathbb{Z}}$ of P. Here we prove that for every j the complex $F_j(P)$ is K-projective. This is done by induction on $j \geq -1$. For $j = -1$ it is trivial. For $j \geq 0$ there is an exact sequence of complexes

$$0 \to F_{j-1}(P) \to F_j(P) \to \mathrm{gr}^F_j(P) \to 0 \tag{12.2.3}$$

in $\mathbf{C}(\mathsf{M})$. In each degree $i \in \mathbb{Z}$ the exact sequence

$$0 \to F_{j-1}(P)^i \to F_j(P)^i \to \mathrm{gr}^F_j(P)^i \to 0$$

in M splits, because $\mathrm{gr}^F_j(P)^i$ is a projective object. Thus the exact sequence (12.2.3) is split in the category $\mathbf{G}(\mathsf{M})$ of graded objects in M.

Let $N \in \mathbf{C}(\mathsf{M})$ be an acyclic complex. Applying the functor $\mathrm{Hom}_{\mathsf{M}}(-, N)$ to the sequence of complexes (12.2.3) we obtain a sequence

(12.2.4)
$$0 \to \mathrm{Hom}_{\mathsf{M}}\big(\mathrm{gr}_j^F(P), N\big) \to \mathrm{Hom}_{\mathsf{M}}\big(F_j(P), N\big) \to \mathrm{Hom}_{\mathsf{M}}\big(F_{j-1}(P), N\big) \to 0$$

in $\mathbf{C}(\mathbb{K})$. Because (12.2.3) is split in $\mathbf{G}(\mathsf{M})$, the sequence (12.2.4) is split in $\mathbf{G}(\mathbb{K})$. Therefore (12.2.4) is exact in $\mathbf{C}(\mathbb{K})$.

By the induction hypothesis the complex $\mathrm{Hom}_{\mathsf{M}}\big(F_{j-1}(P), N\big)$ is acyclic. By step 1 the complex $\mathrm{Hom}_{\mathsf{M}}\big(\mathrm{gr}_j^F(P), N\big)$ is acyclic. The long exact cohomology sequence associated to (12.2.4) shows that the complex $\mathrm{Hom}_{\mathsf{M}}\big(F_j(P), N\big)$ is acyclic too.

Step 4. We keep the semi-projective filtration $F = \{F_j(P)\}_{j\in\mathbb{Z}}$ from step 3. Take any acyclic complex $N \in \mathbf{C}(\mathsf{M})$. By Proposition 12.1.4 we know that

$$\mathrm{Hom}_{\mathsf{M}}(P, N) \cong \lim_{\leftarrow j} \mathrm{Hom}_{\mathsf{M}}\big(F_j(P), N\big)$$

in $\mathbf{C}(\mathbb{K})$. According to step 3 the complexes $\mathrm{Hom}_{\mathsf{M}}\big(F_j(P), N\big)$ are all acyclic. The exactness of the sequences (12.2.4) implies that the inverse system

$$\big\{\mathrm{Hom}_{\mathsf{M}}\big(F_j(P), N\big)\big\}_{j \geq -1}$$

in $\mathbf{C}(\mathbb{K})$ has surjective transitions. Now the Mittag-Leffler argument says that the inverse limit complex $\mathrm{Hom}_{\mathsf{M}}(P, N)$ is acyclic. □

Proposition 12.2.5. *Let* M *be an abelian category. If P is a bounded above complex of projectives, then P is a semi-projective complex.*

Proof.

★ $\boxed{\text{copy prf of prop 12.4.6}}$ ★

□

The next theorem is dual to [RD, Lemma 4.6(1)], in the sense of changing injectives to projectives. (See Theorem 12.4.7 for the injective case.) We give a much more detailed proof.

Theorem 12.2.6. *Let* M *be an abelian category with enough projectives. Any complex $M \in \mathbf{C}^-(\mathsf{M})$ admits a quasi-isomorphism $\rho : P \to M$, where P is a bounded above complex of projectives.*

Proof.

★ $\boxed{\text{copy and modify pf of Theorem 12.4.7}}$ ★

□

★ $\boxed{\text{old proof – to delete}}$ ★

Proof. Take any complex $M \in \mathbf{C}^-(\mathsf{M})$. Let i_1 be an integer such that $\mathrm{H}(M)$ is concentrated in degrees $\leq i_1$. Our procedure to produce a resolution will go like this: we start with the zero complex $F_{-1}(P)$, and enlarge it inductively to an ascending sequence of complexes

$$F_{-1}(P) \subseteq F_0(P) \subseteq F_1(P) \subseteq \cdots .$$

At the same time we construct a compatible sequence of morphisms of complexes $F_j(\rho) : F_j(P) \to M$. These properties will hold:

(a) The subquotient $F_j(P)/F_{j-1}(P)$ is concentrated in degrees $\leq i_1 - j$.
(b) The subquotient $F_j(P)/F_{j-1}(P)$ is a complex of projectives with zero differentials.
(c) Let $j \geq 0$. The morphism

$$\mathrm{H}^i(F_j(\rho)) : \mathrm{H}^i(F_j(P)) \to \mathrm{H}^i(M)$$

is an epimorphism for all i, and it is an isomorphism if $i \geq i_0 - j + 1$.

Property (a) guarantees that the direct limit $P := \lim_{j\to} F_j(P)$ exists in $\mathbf{C}(\mathsf{M})$. Indeed, the limit in $\mathbf{C}(\mathsf{M})$ is determined in each degree separately; and here in each degree i the ascending system $\{F_j(P)^i\}_{j\geq 0}$ is constant for $j \geq i_1 - i + 1$.

Now property (b) says that $F = \{F_j(P)\}_{j\in\mathbb{Z}}$ is a semi-projective complex. According to Theorem 12.2.2 the complex P is K-projective.

Since $F_j(P)^i \to P^i$ is an isomorphism for $j \geq i_1 - i + 1$, it follows that $\mathrm{H}^i(\rho) = \mathrm{H}^i(F_j(\rho))$ for $j \geq i_1 - i + 2$. By property (c) we conclude that $\mathrm{H}^i(\rho)$ is an isomorphism. So $\rho : P \to M$ is a quasi-isomorphism.

The construction is done in several steps.

Step 1. Here $j = 0$. Recall that $F_{-1}(P)$ is the zero complex. For any $i \in \mathbb{Z}$ we choose an epimorphism $\bar{\rho}_0^i : Q_0^i \to \mathrm{H}^i(M)$ from a projective object Q_0^i in M. If $\mathrm{H}^i(M) = 0$ then we take $Q_0^i = 0$. Since Q_0^i is projective, and there is an epimorphism $\mathrm{Z}^i(M) \to \mathrm{H}^i(M)$, the morphism $\bar{\rho}_0^i$ can be lifted to a morphism $\rho_0^i : Q_0^i \to \mathrm{Z}^i(M)$.

Define the complex $F_0(P) := \bigoplus_{i\in\mathbb{Z}} Q_0^i$, with Q_0^i in degree i, and with trivial differential. There is a morphism of complexes

$$F_0(\rho) : F_0(P) \to M$$

whose components are the ρ_0^i. Properties (a,b,c) hold for $j = 0$.

Step 2. Here $j \geq 1$, and

$$F_{j-1}(\rho) : F_{j-1}(P) \to M$$

has already been defined, and it satisfies properties (a,b,c).

Let

$$K^i := \operatorname{Ker}\Big(\mathrm{H}^i(F_{j-1}(\rho)) : \mathrm{H}^i(F_{j-1}(P)) \to \mathrm{H}^i(M)\Big).$$

Choose an epimorphism $\bar{\delta}_j^{i-1} : Q_j^{i-1} \to K^i$ from a projective object Q_j^{i-1} in M. If $K^i = 0$ then we take $Q_j^{i-1} = 0$. Now $K^i \subseteq \mathrm{H}^i(F_{j-1}(P))$, and there is an epimorphism $\mathrm{Z}^i(F_{j-1}(P)) \to \mathrm{H}^i(F_{j-1}(P))$; so we can lift $\bar{\delta}_j^{i-1}$ to a morphism

$$\delta_j^{i-1} : Q_j^{i-1} \to \mathrm{Z}^i(F_{j-1}(P)). \tag{12.2.7}$$

Define the complex

$$Q_j := \bigoplus_{i\in\mathbb{Z}} Q_j^i,$$

with Q_j^i in degree i, and with trivial differential. Next define the complex

$$F_j(P) := F_{j-1}(P) \oplus Q_j,$$

where the differential $\mathrm{d}_{F_j(P)}$ is the sum of the differential $\mathrm{d}_{F_{j-1}(P)}$ of $F_{j-1}(P)$, with the additional components δ_j^{i-1} from (12.2.7). This is is a differential because the δ_j^{i-1} go into the kernel of $\mathrm{d}_{F_{j-1}(P)}$.

By construction we have

$$F_j(P)/F_{j-1}(P) = Q_j$$

as complexes, and properties (a,b) hold.

Step 3. We now define $F_j(\rho)$. Consider the solid commutative diagram

(12.2.8)
$$\begin{tikzcd}
 & & M^{i-1} \arrow[d, "\mathrm{d}"] \\
Q_j^{i-1} \arrow[urr, dashed, bend left, "\rho_j^{i-1}"] \arrow[r, "\delta_j^{i-1}"] \arrow[dr, "\bar{\delta}_j^{i-1}"'] & \mathrm{Z}^i(F_{j-1}(P)) \arrow[r, "\mathrm{Z}^i(F_{j-1}(\rho))"] \arrow[d, two heads] & \mathrm{Z}^i(M) \arrow[d, two heads] \\
 & \mathrm{H}^i(F_{j-1}(P)) \arrow[r, "\mathrm{H}^i(F_{j-1}(\rho))"] & \mathrm{H}^i(M)
\end{tikzcd}$$

By definition of $\bar{\delta}_j^{i-1}$, the morphism from Q_j^{i-1} to $\mathrm{H}^i(M)$ is zero. This mean that the image of Q_j^{i-1} in $\mathrm{Z}^i(M)$ is actually inside $\mathrm{B}^i(M)$. Since Q_j^{i-1} is projective and $\mathrm{d} : M^{i-1} \to \mathrm{B}^i(M)$ is an epimorphism, we obtain a morphism

$$\rho_j^{i-1} : Q_j^{i-1} \to M^{i-1}.$$

This is the dashed arrow in (12.2.8).

Now we extend the morphism $F_{j-1}(\rho)$ to

$$F_j(\rho) : F_j(P) \to M$$

with the extra components ρ_j^i. The commutativity of the top portion in the diagram (12.2.8) says that $F_j(\rho)$ commutes with the differentials.

Step 4. It remains to verify that property (c) holds. Let us look at this commutative diagram:

(12.2.9)
$$\begin{tikzcd}
K^i \arrow[r] \arrow[dr, "\beta_i"'] & \mathrm{H}^i(F_{j-1}(P)) \arrow[r, two heads, "\mathrm{H}^i(F_{j-1}(\rho))"] \arrow[d, "\alpha_i"'] & \mathrm{H}^i(M) \\
 & \mathrm{H}^i(F_j(P)) \arrow[ur, "\mathrm{H}^i(F_j(\rho))"'] &
\end{tikzcd}$$

We know that $\mathrm{H}^i(F_{j-1}(\rho))$ is an epimorphism. This implies that $\mathrm{H}^i(F_j(\rho))$ is an epimorphism.

For $i > i_0 - j + 1$ we did not change anything when passing from F_{j-1} to F_j, so we still have an isomorphism in $\mathrm{H}^i(F_j(\rho))$. For $i < i_0 - j + 1$ we don't care.

As for the critical value $i = i_0 - j + 1$, notice that we did not increase the object of cocycles: $\mathrm{Z}^i(F_{j-1}(P) = \mathrm{Z}^i(F_j(P)$. This implies that α_i is an epimorphism. On the other hand, $\beta_i(K^i) = \delta_j^{i-1}(Q_j^{i-1}) = 0$. Therefore $\mathrm{H}^i(F_j(\rho))$ is an isomorphism. □

Corollary 12.2.10. *If* M *is an abelian category with enough projectives, then* $\mathbf{C}^-(\mathsf{M})$ *has enough K-projectives.*

Proof. Combine Theorems 12.2.2, Proposition 12.2.5 and 12.2.6. □

Remark 12.2.11. The proof of Theorem 12.2.6 gives a bit more information. for a graded object $N \in \mathbf{G}(\mathsf{M})$ we define

$$\sup(N) := \sup\, \{i \in \mathbb{Z} \mid N^i \neq 0\} \subseteq \mathbb{Z} \cup \{\pm\infty\}.$$

Take $M \in \mathbf{C}^{-}(\mathsf{M})$ with nonzero cohomology. Then $i_1 := \sup(\mathrm{H}(M))$ is an integer. The proof shows that we can find a semi-projective resolution $P \to M$ such that $\sup(P) = \sup(\mathrm{H}(M))$.

★ $\boxed{\text{Public 33 | to here 15/06}}$ ★

12.3. **K-Projective Resolutions in $\mathbf{C}(A)$.** In this subsection A is a DG ring (without any boundedness assumption).

Recall that the shift $\mathrm{T}^{-i}(A)$ is a DG A-module in which the element $\mathrm{t}^{-i}(1)$ is in degree i. This element is a cocycle, and when we forget the differentials, the graded module $\mathrm{T}^{-i}(A)^{\natural}$ is free over the graded ring $A^{\natural}$, with basis $\mathrm{t}^{-i}(1)$. Therefore, for any DG A-module M there is a canonical isomorphism

$$\mathrm{Hom}_A\big(\mathrm{T}^{-i}(A), M\big) \cong \mathrm{T}^{i}(M)$$

in $\mathbf{C}_{\mathrm{str}}(A)$, and canonical isomorphisms

$$\mathrm{Hom}_{\mathbf{C}_{\mathrm{str}}(A)}\big(\mathrm{T}^{-i}(A), M\big) \cong \mathrm{Z}^0\big(\mathrm{Hom}_A(\mathrm{T}^{-i}(A), M)\big) \cong \mathrm{Z}^i(M) \tag{12.3.1}$$

in $\mathbf{M}(\mathbb{K})$.

We begin with a definition that is very similar to Definition 12.2.1.

Definition 12.3.2. Let P be an object of $\mathbf{C}(\mathsf{A})$.

(1) We say that P is a *free DG A-module* if

$$P \cong \bigoplus_{s \in S} \mathrm{T}^{i_s}(A)$$

for some indexing set S and some integers i_s.

(2) A *semi-free filtration* on P is an ascending filtration $F = \{F_j(P)\}_{j \geq -1}$ of P in $\mathbf{C}(A)$, such that:
- $F_{-1}(P) = 0$.
- Each $\mathrm{gr}^F_j(P)$ is a free DG A-module.
- $P = \bigcup_j F_j(P)$.

(3) The DG module P is called a *semi-free* of it admits some semi-free filtration.

Example 12.3.3. If A is a ring, then a free DG A-module is a complex of free A-modules with zero differential. A semi-free DG A-module is also a complex of free A-modules, but there is a differential on it, and a subtle condition imposed by the existence of a filtration.

Exercise 12.3.4. Find a ring A, and a complex P of free A-modules, that is not semi-free. (Hint: Take the ring $A = \mathbb{K}[\epsilon]$. Find a complex P that is acyclic, but that $\mathbb{K} \otimes_A P$ is not acyclic. Use Theorem 12.3.5 to get a contradiction.)

Theorem 12.3.5. *Let P be an object of $\mathbf{C}(A)$. If P is semi-free, then it is K-projective.*

Proof. It is similar to the proof of Theorem 12.2.2.

Step 1. We start by proving that if $P = \mathrm{T}^i(A)$, a shift of A, then P is K-projective. This is easy: given an acyclic $N \in \mathbf{C}(A)$, we have

$$\mathrm{Hom}_A(P, N) = \mathrm{Hom}_A\big(\mathrm{T}^i(A), N\big) \cong \mathrm{T}^{-i}\big(\mathrm{Hom}_A(A, N)\big) \cong \mathrm{T}^{-i}(N)$$

in $\mathbf{C}(\mathbb{K})$, and this is acyclic.

Step 2. Now

$$P \cong \bigoplus_{s \in S} \mathrm{T}^{i_s}(A).$$

Then

$$\mathrm{Hom}_A(P, N) \cong \prod_{s \in S} \mathrm{Hom}_A\big(\mathrm{T}^{i_s}(A), N\big).$$

By step 1 and the fact that a product of acyclic complexes in $\mathbf{C}(\mathbb{K})$ is acyclic, we conclude that $\mathrm{Hom}_{\mathsf{M}}(P, N)$ is acyclic.

Step 3. Fix a semi-free filtration $F = \{F_j(P)\}_{j \geq -1}$ of P. Here we prove that for every j the DG module $F_j(P)$ is K-projective. This is done by induction on $j \geq -1$. For $j = -1$ it is trivial. For $j \geq 0$ there is an exact sequence

$$0 \to F_{j-1}(P) \to F_j(P) \to \mathrm{gr}_j^F(P) \to 0 \tag{12.3.6}$$

in the abelian category $\mathbf{C}_{\mathrm{str}}(A)$. Because $\mathrm{gr}_j^F(P)$ is a free DG module, it is a projective object in the abelian category $\mathbf{G}(A^\natural)$ of graded modules over the graded ring $A^\natural$, gotten by fogetting the differential of A. Therefore the sequence (12.3.6) is split exact in $\mathbf{G}(A^\natural)$.

Let $N \in \mathbf{C}(A)$ be an acyclic DG module. Applying the functor $\mathrm{Hom}_A(-, N)$ to the sequence (12.3.6) we obtain a sequence

$$0 \to \mathrm{Hom}_A\big(\mathrm{gr}_j^F(P), N\big) \to \mathrm{Hom}_A\big(F_j(P), N\big) \to \mathrm{Hom}_A\big(F_{j-1}(P), N\big) \to 0 \tag{12.3.7}$$

in $\mathbf{C}(\mathbb{K})$. If we forget differentials this is a sequence in $\mathbf{G}(\mathbb{K})$. Because (12.3.6) is split exact in $\mathbf{G}(A^\natural)$, it follows that (12.3.7) is split exact in $\mathbf{G}(\mathbb{K})$. Therefore (12.3.7) is exact in $\mathbf{C}(\mathbb{K})$.

By the induction hypothesis the DG $\mathbb{K}$-module $\mathrm{Hom}_A\big(F_{j-1}(P), N\big)$ is acyclic. By step 1 the DG module $\mathrm{Hom}_A\big(\mathrm{gr}_j^F(P), N\big)$ is acyclic. The long exact cohomology sequence associated to (12.3.7) shows that the DG module $\mathrm{Hom}_A\big(F_j(P), N\big)$ is acyclic too.

Step 4. We keep the semi-free filtration $F = \{F_j(P)\}_{j \geq -1}$ from step 3. Take any acyclic $N \in \mathbf{C}(\mathsf{M})$. By Proposition 12.1.4 we know that

$$\mathrm{Hom}_A(P, N) \cong \lim_{\leftarrow j} \mathrm{Hom}_A\big(F_j(P), N\big)$$

in $\mathbf{C}(\mathbb{K})$. According to step 3 the complexes $\mathrm{Hom}_A\big(F_j(P), N\big)$ are all acyclic. The exactness of the sequences (12.3.7) implies that the inverse system

$$\big\{\mathrm{Hom}_A\big(F_j(P), N\big)\big\}_{j \geq -1}$$

in $\mathbf{C}(\mathbb{K})$ has surjective transitions. Now the Mittag-Leffler argument says that the inverse limit complex $\mathrm{Hom}_A(P, N)$ is acyclic. □

Here is a result similar to Theorem 12.2.6. The proof is similar to one we found in the unpublished book [AFH].

Theorem 12.3.8. *Let A be any DG ring. Any $M \in \mathbf{C}(A)$ admits a quasi-isomorphism $\rho : P \to M$ from a semi-free DG module P.*

Proof. Step 1. In this step we construct $F_0(P)$. For any $i \in \mathbb{Z}$ the cohomology $\mathrm{H}^i(M)$ is an $\mathrm{H}^0(A)$-module. Choose a collection of $\mathrm{H}^0(A)$-module generators of $\mathrm{H}^i(M)$, indexed by a set S_0^i. Lift these elements to a collection $\{m_s\}_{s \in S_0^i}$ of elements of the module of cocycles $\mathrm{Z}^i(M)$. The collection $\{m_s\}_{s \in S_0^i}$ induces a homomorphism

$$\bigoplus_{s \in S_0^i} \mathrm{T}^{-i}(A) \to M \tag{12.3.9}$$

in $\mathbf{C}_{\mathrm{str}}(A)$. Cf. formula (12.3.1). Define the free DG module

$$F_0(P) := \bigoplus_{i \in \mathbb{Z}} \bigoplus_{s \in S_0^i} \mathrm{T}^{-i}(A), \tag{12.3.10}$$

and let

$$F_0(\rho) : F_0(P) \to M \tag{12.3.11}$$

be the homomorphism in $\mathbf{C}_{\mathrm{str}}(A)$ gotten by summing the homomorphisms (12.3.9). By constuction we see that

$$\mathrm{H}^i(F_0(\rho)) : \mathrm{H}^i(F_0(P)) \to \mathrm{H}^i(M) \tag{12.3.12}$$

is surjective for all i.

(2) In this step $j \geq 0$, and we have the following: a DG A-module $F_j(P)$, a homomorphism $F_j(\rho) : F_j(P) \to M$ in $\mathbf{C}_{\mathrm{str}}(A)$, and a collection of DG submodules $F_{j'}(P) \subseteq F_j(P)$ for $j' \leq j$. These satisfy: for all i the homomorphisms

$$\mathrm{H}^i(F_j(\rho)) : \mathrm{H}^i(F_j(P)) \to \mathrm{H}^i(M)$$

are surjective; $F_{-1}(P) = 0$; and the $F_{j'}(P)/F_{j'-1}(P)$ are free DG A-modules for all $j' \leq j$.

For any $i \in \mathbb{Z}$ let K_j^i be the kernel of $\mathrm{H}^i(F_j(\rho))$. So there is a short exact sequence

$$0 \to K_j^i \to \mathrm{H}^i(F_j(P)) \xrightarrow{\mathrm{H}^i(F_j(\rho))} \mathrm{H}^i(M) \to 0 \tag{12.3.13}$$

in $\mathbf{M}(\mathrm{H}^0(A))$. Choose a collection of $\mathrm{H}^0(A)$-module generators of K_j^i, indexed by a set S_{j+1}^i. Using the surjection $\mathrm{Z}^i(F_j(P)) \to \mathrm{H}^i(F_j(P))$, lift these elements to a collection $\{p_s\}_{s \in S_{j+1}^i}$ of elements of the module of cocycles $\mathrm{Z}^i(F_j(P))$. The collection $\{p_s\}_{s \in S_{j+1}^i}$ induces a homomorphism

$$\phi_{j+1}^i : \bigoplus_{s \in S_{j+1}^i} \mathrm{T}^{-i}(A) \to F_j(P) \tag{12.3.14}$$

in $\mathbf{C}_{\mathrm{str}}(A)$. Define the free DG module

$$Q_{j+1} := \bigoplus_{i \in \mathbb{Z}} \bigoplus_{s \in S_{j+1}^i} \mathrm{T}^{-i}(A) \tag{12.3.15}$$

and the homomorphism

$$\phi_{j+1} := \sum_i \phi_{j+1}^i : Q_{j+1} \to F_j(P) \tag{12.3.16}$$

in $\mathbf{C}_{\mathrm{str}}(A)$. Next define the DG module

$$F_{j+1}(P) := F_j(P) \oplus \mathrm{T}(Q_{j+1}), \tag{12.3.17}$$

in which the differential is

$$\mathrm{d}_{F_{j+1}(P)} := \mathrm{d}_{F_j(P)} + \mathrm{d}_{\mathrm{T}(Q_{j+1})} + \phi_{j+1} \circ \mathrm{t}^{-1} .$$

In other words, $F_{j+1}(P)$ is the standard cone on the strict homomorphism ϕ_{j+1}.

Note that $F_j(P)$ is a DG submodule of $F_{j+1}(P)$. Because the cocycles in $F_j(P)$ representing K_j^i become coboundaries in $F_{j+1}(P)$, it follows that for any i we have

$$K_j^i \subseteq \mathrm{Ker}\big(\mathrm{H}^i(F_j(P)) \to \mathrm{H}^i(F_{j+1}(P))\big). \tag{12.3.18}$$

(3) Now we extend the homomorphism $F_j(\rho) : F_j(P) \to M$ to a homomorphism $F_{j+1}(\rho) : F_{j+1}(P) \to M$. Consider the element $p_s \in \mathrm{Z}^i(F_j(P))$ for some index $s \in S_{j+1}^i$. Because the cohomology class of p_s is in K_j^i, the element $F_j(\rho)(p_s) \in M^i$ is a coboundary. Therefore we can find an element $m_s \in M^{i-1}$ such that $F_j(\rho)(p_s) = \mathrm{d}(m_s)$. From (12.3.15) we see that the collection of elements $\{m_s\}_{s \in \bigcup_i S_{j+1}^i}$ induces a strict homomorphism of DG modules

$$\rho'_{j+1} : \mathrm{T}(Q_{j+1}) \to M.$$

★ public 34 | need to fix rest of proof... ★

Define

$$F_{j+1}(\rho) := F_j(\rho) + \rho'_{j+1}$$

using (12.3.17). It is easy to see that this is a homomorphism of DG modules $F_{j+1}(P) \to M$.

(4) Let $P := \lim_{j \to} F_j(P)$ and

$$\rho := \lim_{j \to} F_j(\rho) : P \to M$$

in $\mathbf{C}_{\mathrm{str}}(A)$. The filtration $\{F_j(P)\}$ on P is clearly semi-free. It remains to prove that ρ is a quasi-isomorphism. In step 1 we saw that $\mathrm{H}^i(F_0(\rho))$ is surjective for all i. This implies that

$$\mathrm{H}^i(F_j(\rho)) : \mathrm{H}^i(F_j(P)) \to \mathrm{H}^i(M)$$

is surjective for all i and j. Define

$$L_j^i := \mathrm{Im}\big(\mathrm{H}^i(F_j(P)) \to \mathrm{H}^i(F_{j+1}(P))\big).$$

Because of formula (12.3.18) we have an exact sequence

$$K_j^i \to \mathrm{H}^i(F_j(P)) \to L_j^i \to 0.$$

By the definition of K_j^i it follows that

$$\mathrm{H}^i(F_j(\rho)) : L_j^i \to \mathrm{H}^i(M)$$

is bijective. Hence

$$\lim_{j \to} \mathrm{H}^i(F_j(\rho)) : \lim_{j \to} L_j^i \to \mathrm{H}^i(M)$$

is bijective. And by Proposition 12.1.6(1) we know that

$$\lim_{j \to} L_j^i = \lim_{j \to} \mathrm{H}^i(F_j(P)).$$

Finally, according to Proposition 12.1.10 we know that

$$\lim_{j \to} \mathrm{H}^i(F_j(P)) = \mathrm{H}^i(P).$$

So

$$\mathrm{H}^i(\rho) : \mathrm{H}^i(P) \to \mathrm{H}^i(M)$$

is bijective. □

Corollary 12.3.19. *Let A be any DG ring. The category $\mathsf{C}(A)$ has enough K-projectives.*

Proof. Combine Theorems 12.3.5 and 12.3.8. □

Remark 12.3.20. If A is a nonpositive DG ring (namely $A^i = 0$ for all $i > 0$), and if $M \in \mathsf{C}(A)$ has nonzero bounded above cohomology, then the proof of Theorem 12.3.8 gives a bit more information: we can find a semi-free resolution $P \to M$ such that $\sup(P) = \sup(\mathrm{H}(M))$.

★ public 34 | to here 22/06 ★

12.4. **K-Injective Resolutions in $\mathsf{K}^+(\mathsf{M})$.** In this subsection M is an abelian category, and $\mathsf{C}(\mathsf{M})$ is the category of complexes in M.

Let M be an object of $\mathsf{C}(\mathsf{M})$. A *quotient* of M is an object $N \in \mathsf{C}(\mathsf{M})$, together with an epimorphism $\phi : N \to M$ in $\mathsf{C}_{\mathrm{str}}(\mathsf{M})$. We shall use the abbreviation $M \twoheadrightarrow N$. If N, N' are both quotients of M, i.e. they are equipped with strict epimorphisms $\phi : M \twoheadrightarrow N$ and $\phi' : M \twoheadrightarrow N'$, then there can be at most one morphism $\psi : N \to N'$ in $\mathsf{C}_{\mathrm{str}}(\mathsf{M})$ such that $\phi' = \psi \circ \phi$; and this ψ must be an epimorphism. We call this ψ a morphism of quotients of M. See Convention 12.1.1 regarding strict morphisms.

A *cofiltration* of a complex $I \in \mathsf{C}(\mathsf{M})$ is an inverse system $\{G_q(I)\}_{q \geq -1}$ of quotients of I in $\mathsf{C}(\mathsf{M})$. We say that $I = \lim_{\leftarrow q} G_q(I)$ if this inverse limit exists in $\mathsf{C}(\mathsf{M})$, and the canonical morphism $I \to \lim_{\leftarrow q} G_q(I)$ is an isomorphism. The cofiltration gives rise to the subquotients

$$\mathrm{gr}^G_q(I) := \mathrm{Ker}\big(G_q(I) \to G_{q-1}(I)\big) \in \mathsf{C}(\mathsf{M}). \tag{12.4.1}$$

★ the name "cofiltration" is experimental ★

Definition 12.4.2. Let I be a complex in $\mathsf{C}(\mathsf{M})$.

(1) A *semi-injective cofiltration* on I is an cofiltration $G = \{G_q(I)\}_{q \geq -1}$ of I as an object of $\mathsf{C}(\mathsf{M})$, such that:
   - $G_{-1}(I) = 0$.
   - Each $\mathrm{gr}^G_q(I)$ is a complex of injective objects of M with zero differential.
   - $I = \lim_{\leftarrow q} G_q(I)$.

(2) The complex I is called a *semi-injective complex* if it admits some semi-injective cofiltration.

Theorem 12.4.3. *Let M be an abelian category, and let I be a semi-injective complex in $\mathsf{C}(\mathsf{M})$. Then I is K-injective.*

Proof. The proof is very similar to that of Theorem 12.2.2.

Step 1. We start by proving that if $I = \mathrm{T}^p(J)$, the shift of an injective object $J \in \mathsf{M}$, then I is K-projective. This is easy: given an acyclic complex $N \in \mathbf{C}(\mathsf{M})$, we have

$$\mathrm{Hom}_{\mathsf{M}}(N, I) = \mathrm{Hom}_{\mathsf{M}}\big(N, \mathrm{T}^p(J)\big) \cong \mathrm{T}^p\big(\mathrm{Hom}_{\mathsf{M}}(N, J)\big)$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. But $\mathrm{Hom}_{\mathsf{M}}(-, J)$ is an exact functor $\mathsf{M} \to \mathbf{M}(\mathbb{K})$, so $\mathrm{Hom}_{\mathsf{M}}(N, J)$ is an acyclic complex.

Step 2. Now I is a complex of injective objects of M with zero differential. This means that

$$I \cong \prod_{p \in \mathbb{Z}} \mathrm{T}^p(J_p)$$

in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, where each J_p is an injective object in M. But then

$$\mathrm{Hom}_{\mathsf{M}}(N, I) \cong \prod_{p \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{M}}\big(N, \mathrm{T}^p(J_p)\big).$$

This is an easy case of Proposition 12.1.4(2). By step 1 and the fact that a product of acyclic complexes in $\mathbf{C}(\mathbb{K})$ is acyclic (itself an easy case of the Mittag-Leffler argument), we conclude that $\mathrm{Hom}_{\mathsf{M}}(N, I)$ is acyclic.

Step 3. Fix a semi-injective cofiltration $G = \{G_q(I)\}_{q \geq -1}$ of I. Here we prove that for every q the complex $G_q(I)$ is K-injective. This is done by induction on q. For $q = -1$ it is trivial. For $q \geq 0$ there is an exact sequence of complexes

$$0 \to \mathrm{gr}_q^G(I) \to G_q(I) \to G_{q-1}(I) \to 0 \tag{12.4.4}$$

in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$. In each degree $p \in \mathbb{Z}$ the exact sequence

$$0 \to \mathrm{gr}_q^G(I)^p \to G_q(I)^p \to G_{q-1}(I)^p \to 0$$

in M splits, because $\mathrm{gr}_q^G(I)^p$ is an injective object. Thus the exact sequence (12.4.4) is split in the category $\mathbf{G}_{\mathrm{str}}(\mathsf{M})$ of graded objects in M.

Let $N \in \mathbf{C}(\mathsf{M})$ be an acyclic complex. Applying the functor $\mathrm{Hom}_{\mathsf{M}}(N, -)$ to the sequence of complexes (12.4.4) we obtain a sequence

$$0 \to \mathrm{Hom}_{\mathsf{M}}\big(N, \mathrm{gr}_q^G(I)\big) \to \mathrm{Hom}_{\mathsf{M}}\big(N, G^q(I)\big) \to \mathrm{Hom}_{\mathsf{M}}\big(N, G_{q-1}(I)\big) \to 0 \tag{12.4.5}$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. Because (12.4.4) is split in $\mathbf{G}_{\mathrm{str}}(\mathsf{M})$, the sequence (12.4.5) is split in $\mathbf{G}_{\mathrm{str}}(\mathbb{K})$. Therefore (12.4.5) is exact in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$.

By the induction hypothesis the complex $\mathrm{Hom}_{\mathsf{M}}\big(N, G_{q-1}(I)\big)$ is acyclic. By step 2 the complex $\mathrm{Hom}_{\mathsf{M}}\big(N, \mathrm{gr}_q^G(I)\big)$ is acyclic. The long exact cohomology sequence associated to (12.4.5) shows that the complex $\mathrm{Hom}_{\mathsf{M}}\big(N, G_q(I)\big)$ is acyclic too.

Step 4. We keep the semi-injective cofiltration $G = \{G_q(I)\}_{p \geq -1}$ from step 3. Take any acyclic complex $N \in \mathbf{C}(\mathsf{M})$. By Proposition 12.1.4 we know that

$$\mathrm{Hom}_{\mathsf{M}}(N, I) \cong \lim_{\leftarrow q} \mathrm{Hom}_{\mathsf{M}}\big(N, G_q(I)\big)$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. According to step 3 the complexes $\mathrm{Hom}_{\mathsf{M}}\big(N, G_q(I)\big)$ are all acyclic. The exactness of the sequences (12.4.5) implies that the inverse system

$$\big\{\mathrm{Hom}_{\mathsf{M}}\big(N, G_q(I)\big)\big\}_{q \geq -1}$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$ has surjective transitions. Now the Mittag-Leffler argument says that the inverse limit complex $\mathrm{Hom}_{\mathsf{M}}(N, I)$ is acyclic. $\square$

Proposition 12.4.6. *Let* M *be an abelian category. If* I *is a bounded below complex of injectives, then* I *is a semi-injective complex.*

Proof. We can assume that $I \neq 0$. Let p_0 be an integer such that $I^p = 0$ for all $p < p_0$. For $q \geq -1$ let $F_q(I)$ be the subcomplex of I defined by $F_q(I)^p := I^p$ if $p \geq p_0 + q + 1$, and $F_q(I)^p := 0$ otherwise. Then let $G_q(I) := I / F_q(I)$. The cofiltration $G = \{G_q(I)\}_{q \geq -1}$ is semi-injective. □

The next theorem is [RD, Lemma 4.6(1)]. See also [KS1, Proposition 1.7.7(i)]. We give a much more detailed proof.

Theorem 12.4.7. *Let* M *be an abelian category with enough injectives. Any complex* $M \in \mathbf{C}^+(\mathsf{M})$ *admits a quasi-isomorphism* $\rho : M \to I$ *into a bounded below complex of injectives* I.

Proof. Without loss of generality we can assume that $M^p = 0$ for all $p < 0$. The differential of the complex M is $\mathrm{d}_M^p : M^p \to M^{p+1}$. Choose a monomorphism $\rho^0 : M^0 \hookrightarrow I^0$ to some injective object I^0. We have a morphism

$$\delta^0 : M^0 \to (M^1 \oplus I^0)$$

whose components are d_M^0 and ρ^0. Next we choose a monomorphism

$$\psi^1 : \operatorname{Coker}(\delta^0) \hookrightarrow I^1$$

to some injective object I^1. So there is an exact sequence

$$0 \to M^0 \xrightarrow{\delta^0} (M^1 \oplus I^0) \xrightarrow{\psi^1} I^1 .$$

The components of ψ^1 are denoted by $\rho^1 : M^1 \to I^1$ and $\mathrm{d}_I^0 : I^0 \to I^1$.

Now $p \geq 1$, and we are given objects $I^0, \ldots, I^p$, and morphisms $\rho^0, \ldots, \rho^p$ and $d_I^0, \ldots, d_I^{p-1}$, that fit into this diagram:

(12.4.8)
$$\begin{tikzcd}
0 \arrow[r] & M^0 \arrow[r, "\mathrm{d}_M^0"] \arrow[d, "\rho^0"'] & \cdots \arrow[r] & M^{p-1} \arrow[r, "\mathrm{d}_M^{p-1}"] \arrow[d, "\rho^{p-1}"'] & M^p \arrow[r, "\mathrm{d}_M^p"] \arrow[d, "\rho^p"'] & M^{p+1} \\
0 \arrow[r] & I^0 \arrow[r, "\mathrm{d}_I^0"] & \cdots \arrow[r] & I^{p-1} \arrow[r, "\mathrm{d}_I^{p-1}"] & I^p &
\end{tikzcd}$$

Define the morphism

$$\delta^p : (M^p \oplus I^{p-1}) \to (M^{p+1} \oplus I^p)$$

whose components are $-\mathrm{d}_M^p$, ρ^p and d_I^{p-1}. Let us choose a monomorphism

$$\psi^{p+1} : \operatorname{Coker}(\delta^p) \hookrightarrow I^{p+1}$$

to an injective object I^{p+1}. We get an exact sequence

(12.4.9)
$$(M^p \oplus I^{p-1}) \xrightarrow{\delta^p} (M^{p+1} \oplus I^p) \xrightarrow{\psi^{p+1}} I^{p+1} .$$

The components of ψ^{p+1} are denoted by $\rho^{p+1} : M^{p+1} \to I^{p+1}$ and $\mathrm{d}_I^p : I^p \to I^{p+1}$. In this way we obtain the bigger diagram

(12.4.10)
$$\begin{tikzcd}
0 \arrow[r] & M^0 \arrow[r, "\mathrm{d}_M^0"] \arrow[d, "\rho^0"'] & \cdots \arrow[r] & M^{p-1} \arrow[r, "\mathrm{d}_M^{p-1}"] \arrow[d, "\rho^{p-1}"'] & M^p \arrow[r, "\mathrm{d}_M^p"] \arrow[d, "\rho^p"'] & M^{p+1} \arrow[d, "\rho^{p+1}"'] \\
0 \arrow[r] & I^0 \arrow[r, "\mathrm{d}_I^0"] & \cdots \arrow[r] & I^{p-1} \arrow[r, "\mathrm{d}_I^{p-1}"] & I^p \arrow[r, "\mathrm{d}_I^p"] & I^{p+1}
\end{tikzcd}$$

We do not know (yet) if this diagram is commutative.

We carry out this construction inductively for all $p \geq 1$, thus obtaining a diagram like (12.4.10) that goes infinitely to the right.

Because $\psi^{p+1} \circ \delta^p = 0$ in (12.4.9), it follows that $\mathrm{d}_I^{p+1} \circ \mathrm{d}_I^p = 0$. Letting $I^p := 0$ for negative p, the collection $I := \{I^p\}_{p \in \mathbb{Z}}$ becomes a complex, with differential $\{\mathrm{d}_I^p\}_{p \in \mathbb{Z}}$. The equality $\psi^{p+1} \circ \delta^p = 0$ also implies that

$$\rho^{p+1} \circ \mathrm{d}_M^p = \mathrm{d}_I^p \circ \rho^p, \tag{12.4.11}$$

so the collection $\rho := \{\rho^p\}_{p \in \mathbb{Z}}$ is a strict morphism of complexes $\rho : M \to I$.

Let us examine this commutative diagram with obvious vertical morphisms:

$$\begin{tikzcd}
M^p \oplus I^{p-1} \arrow[r, "\delta^p"] \arrow[d] & M^{p+1} \oplus I^p \arrow[r, "\delta^{p+1}"] \arrow[d] & M^{p+2} \oplus I^{p+1} \arrow[d] \\
M^p \oplus I^{p-1} \arrow[r, "\delta^p"] & M^{p+1} \oplus I^p \arrow[r, "\psi^{p+1}"] & I^{p+1}
\end{tikzcd} \tag{12.4.12}$$

The bottom row is exact – it is just (12.4.9). An easy calculation using (12.4.11) shows that $\delta^{p+1} \circ \delta^p = 0$. These two facts combine prove that the top row is also exact.

Let $J = \{J^p\}_{p \in \mathbb{Z}}$ be the complex with components $J^p := M^p \oplus I^{p-1}$ for $p \geq 1$, $J^0 := M^0$ and $J^p := 0$ for $p < 0$. The differential of J is $\{\delta^p\}_{p \in \mathbb{Z}}$. As we saw in the paragraph above, the complex J is acyclic. On the other hand, by the definition of the morphisms δ^p, we see that J is just the standard cone of the strict morphism of complexes $\mathrm{T}^{-1}(\rho) : \mathrm{T}^{-1}(M) \to \mathrm{T}^{-1}(I)$. Therefore ρ is a quasi-isomorphism. □

Corollary 12.4.13. *If* M *is an abelian category with enough injectives, then* $\mathbf{C}^+(\mathsf{M})$ *has enough K-injectives.*

Proof. Combine Theorems 12.4.7, Proposition 12.4.6 and Theorem 12.4.3. □

12.5. **K-Injective Resolutions in $\mathbf{K}(A)$.** Recall that we are working over a nonzero commutative base ring $\mathbb{K}$.

An *injective cogenerator* over $\mathbb{K}$ is an injective $\mathbb{K}$-module $\mathbb{K}^*$ with this property: if M is a nonzero $\mathbb{K}$-module, then $\mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K}^*)$ is nonzero. These always exist.

Example 12.5.1. If $\mathbb{K}$ is a field, we would usually prefer to take $\mathbb{K}^* = \mathbb{K}$.

For any nonzero ring $\mathbb{K}$ there is a canonical choice:

$$\mathbb{K}^* := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{K}, \mathbb{Q}/\mathbb{Z}).$$

It is usually a very big module...

In this subsection we fix an injective cogenerator $\mathbb{K}^*$.

Recall that A is a central DG $\mathbb{K}$-ring. The next definition is dual to Definition 12.3.2. We rely on Convention 12.1.1 regarding strict morphisms.

Definition 12.5.2. Let I be an object of $\mathbf{C}(\mathsf{A})$.

(1) We say that P is a *cofree DG A-module* if

$$I \cong \prod_{s \in S} \mathrm{T}^{i_s}\big(\mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K}^*)\big)$$

in $\mathbf{C}(A)$ for some indexing set S and some integers i_s.

(2) A *semi-cofree cofiltration* on I is a cofiltration $G = \{G_q(I)\}_{q \geq -1}$ of I as an object of $\mathbf{C}(A)$, such that:
   - $G_{-1}(I) = 0$.
   - Each $\operatorname{gr}^G_q(I)$ is a cofree DG A-module.
   - $I = \lim_{\leftarrow q} G_q(I)$.

(3) The complex I is called a *semi-cofree complex* if it admits a semi-cofree cofiltration.

Theorem 12.5.3. *Let I be an object of $\mathbf{C}(A)$. If I is semi-cofree, then it is K-injective.*

Proof. ???? □

Theorem 12.5.4. *Let A be any DG ring. Any $M \in \mathbf{C}(A)$ admits a quasi-isomorphism $\rho : M \to I$ to a semi-cofree DG module I.*

Proof. ???? □

Corollary 12.5.5. *Let A be any DG ring. The category $\mathbf{C}(A)$ has enough K-injectives.*

Proof. Combine Theorems 12.5.3 and 12.5.4. □

!!!

12.6. K-Flat Resolutions in $\mathsf{C}(\mathcal{A})$.

★ $\boxed{\text{The material involving geometry will be done later.}}$ ★

12.7. K-Injective Resolutions in $\mathsf{K}(\mathcal{A})$.

13. Derived Bifunctors

★ $\boxed{\text{Here we do } \mathrm{RHom}_A(-,-) \text{ and } - \otimes^{\mathrm{L}}_A -}$ ★

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