DERIVED CATEGORIES

BGU, 2016-17

course der cats IV | public 49 | date: 30 March 2017

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Contents

First Part [in book]

0. Introduction

This book develops the theory of *derived categories*, starting from the foundations, and going all the way to applications in algebra and geometry. The emphasis is on explicit constructions (with examples), as opposed to axiomatics. The most abstract concept we use is probably that of abelian category (which seems indispensable).

A special feature of this book is that most of the theory deals with $\mathbf{D}(A, \mathsf{M})$, the *derived category of DG A-modules in* M, where A is a DG (differential graded) ring and M is an abelian category. This covers most important examples that arise in algebra and geometry:

- The derived category $\mathbf{D}(A)$ of DG A-modules, for any DG ring A. This includes ordinary rings.
- The derived category $\mathbf{D}(\mathsf{M})$ for any abelian category M. This includes $\mathsf{M} = \mathsf{Mod}\,\mathcal{A}$, the category of sheaves of $\mathcal{A}$-modules on a ringed space $(X, \mathcal{A})$.

Furthermore, we work with *unbounded* derived categories. We prove existence of resolutions (bounded or unbounded) in several contexts.

The first half of the book (Sections 1-10) covers the general theory. This is done in an unorthodox manner, using DG categories as the source of derived categories and triangulated functors. Another departure from the tradition is that we only consider *pretriangulated categories*, thus sparing ourselves the burden of the octahedral axiom. In this part of the book we provide detailed proofs of all statements (except the routine ones, that are left as exercises). A more detailed description of the contents of the first half is in the Synopsis (subsection 0.2 of the Introduction).

The second half of the book (that is not yet written) shall start off with more of the general theory: derived bifunctor, and derived categories in geometry. This is in Sections 12-15).

After that we shall deal with a few specialized topics:

▷ Derived Categories in Commutative Algebra.
▷ Residues and Duality in Algebraic Geometry.
▷ Derived Categories in Noncommutative Algebra.

In this last portion of the book we shall leave out some of the proofs (but there are precise external references). Much of the material here is the state of the art, and is not included in any prior textbook.

The book is based on notes for advanced courses given at Ben Gurion University, in the academic years 2011-12 and 2015-16. The main sources for the first part of the book are [RD] and [KaSc1]; but the DG theory component is absent from those earlier texts, and is pretty much our own interpretation of folklore results.

0.1. **A Motivating Discussion: Duality.**

By way of introduction to the subject of derived categories, let us consider *duality*.

We begin with something elementary: linear algebra. Take a field $\mathbb{K}$. Given a $\mathbb{K}$-module M (i.e. a vector space), let

$$D(M) := \operatorname{Hom}_{\mathbb{K}}(M, \mathbb{K}),$$

be the dual module. There is a canonical homomorphism

$$\theta_M : M \to D(D(M)),$$

namely $\theta_M(m)(\phi) := \phi(m)$ for $m \in M$ and $\phi \in D(M)$. If M is finitely generated then θ_M is an isomorphism (actually this is "if and only if").

To formalize this situation, let $\mathsf{Mod}\,\mathbb{K}$ denote the category of $\mathbb{K}$-modules. Then

$$D : \mathsf{Mod}\,\mathbb{K} \to \mathsf{Mod}\,\mathbb{K}$$

is a contravariant functor, and

$$\theta : \mathrm{Id} \to D \circ D$$

is a natural transformation. Here Id is the identity functor of $\mathsf{Mod}\,\mathbb{K}$.

Now let us replace $\mathbb{K}$ by any nonzero commutative ring A. Again we can define a contravariant functor

$$D : \mathsf{Mod}\,A \to \mathsf{Mod}\,A, \quad D(M) := \mathrm{Hom}_A(M, A),$$

and a natural transformation $\theta : \mathrm{Id} \to D \circ D$. It is easy to see that $\theta_M : M \to D(D(M))$ is an isomorphism if M is a finitely generated free module. Of course we can't expect reflexivity (i.e. θ_M being an isomorphism) if M is not finitely generated; but what about a finitely generated module that is not free?

In order to understand this better, let us concentrate on the ring $A = \mathbb{Z}$. Since $\mathbb{Z}$-modules are just abelian groups, the category $\mathsf{Mod}\,\mathbb{Z}$ is often denoted by Ab. Let Ab_{f} be the full subcategory of finitely generated abelian groups. Any finitely generated abelian group is of the form $M \cong T \oplus F$, with F free and T finite. (The letters "T" and "F" stand for "torsion" and "free" respectively.) It is important to note that this is *not a canonical isomorphism*. There is a canonical short exact sequence

$$0 \to T \xrightarrow{\phi} M \xrightarrow{\psi} F \to 0, \tag{0.1.1}$$

but the decomposition $M \cong T \oplus F$ comes from *choosing a splitting* $\sigma : F \to M$ of this sequence.

Exercise 0.1.2. Prove that the exact sequence (0.1.1) is functorial (i.e. natural); namely there are functors $T, F : \mathsf{Ab}_{\mathrm{f}} \to \mathsf{Ab}_{\mathrm{f}}$, and natural transformations $\phi : T \to \mathrm{Id}$ and $\psi : \mathrm{Id} \to F$, such that for any $M \in \mathsf{Ab}_{\mathrm{f}}$, the group $T(M)$ is finite; the group $F(M)$ is free; and the sequence of homomorphisms

$$0 \to T(M) \xrightarrow{\phi_M} M \xrightarrow{\psi_M} F(M) \to 0 \tag{0.1.3}$$

is exact.

Next, prove that there does not exist a *functorial decomposition* of a finitely generated abelian group into a free part and a finite part. Namely, there is no natural transformation $\sigma : F \to \mathrm{Id}$, such that for every M, the homomorphism $\sigma_M : F(M) \to M$ splits the sequence (0.1.3). (Hint: find a counterexample.)

We know that for a free finitely generated abelian group F there is reflexivity, i.e. $\theta_F : F \to D(D(F))$ is an isomorphism. But for a finite abelian group T we have

$$D(T) = \mathrm{Hom}_{\mathbb{Z}}(T, \mathbb{Z}) = 0.$$

Thus, for a $M \in \mathsf{Ab}_{\mathrm{f}}$ with nonzero torsion subgroup T, reflexivity fails: $\theta_M : M \to D(D(M))$ is not an isomorphism.

On the other hand, for an abelian group M we can define another sort of dual:

$$D'(M) := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}).$$

There is a natural transformation $\theta' : \mathrm{Id} \to D' \circ D'$. For a finite abelian group T the homomorphism $\theta'_T : T \to D'(D'(T))$ is an isomorphism; this can be seen by decomposing T into cyclic groups, and for a finite cyclic group it is clear. So D' is a duality for finite abelian groups. (We may view the abelian group $\mathbb{Q}/\mathbb{Z}$ as the group of roots of 1 in $\mathbb{C}$, via the exponential function; and then D' becomes *Pontryagin Duality.*)

But for a finitely generated free abelian group F we get $D'(D'(F)) = \widehat{F}$, the profinite completion of F. So once more this is not a good duality for all finitely generated abelian groups.

We could try to be more clever and "patch" the two dualities D and D', into something that we will call $D \oplus D'$. This looks pleasing at first – but then we recall that the decomposition $M \cong T \oplus F$ of a finitely generated group is not functorial, so that $D \oplus D'$ can't be a functor.

This is where the *derived category* enters. For any commutative ring A there is the derived category $\mathbf{D}(\mathsf{Mod}\, A)$. Here is a very quick explanation of it.

Recall that a *complex* of A-modules is a diagram

$$M = \big(\cdots \to M^{-1} \xrightarrow{\mathrm{d}_M^{-1}} M^0 \xrightarrow{\mathrm{d}_M^0} M^1 \to \cdots \big) \tag{0.1.4}$$

in the category $\mathsf{Mod}\, A$. Namely the M^i are A-modules, and the d_M^i are homomorphisms. The condition is that $\mathrm{d}_M^{i+1} \circ \mathrm{d}_M^i = 0$. We sometimes write $M = \{M^i\}_{i \in \mathbb{Z}}$. The collection $\mathrm{d}_M = \{\mathrm{d}_M^i\}_{i \in \mathbb{Z}}$ is called the *differential* (or the coboundary operator) of M.

Given a second complex

$$N = \big(\cdots \to N^{-1} \xrightarrow{\mathrm{d}_N^{-1}} N^0 \xrightarrow{\mathrm{d}_N^0} N^1 \to \cdots \big),$$

a *homomorphism of complexes* $\phi : M \to N$ is a collection $\phi = \{\phi^i\}_{i \in \mathbb{Z}}$ of homomorphisms $\phi^i : M^i \to N^i$ in $\mathsf{Mod}\, A$ satisfying

$$\phi^{i+1} \circ \mathrm{d}_M^i = \mathrm{d}_N^i \circ \phi^i.$$

The resulting category is denoted by $\mathbf{C}(\mathsf{Mod}\, A)$.

The i-th *cohomology* of the complex M is

$$\mathrm{H}^i(M) := \frac{\mathrm{Ker}(\mathrm{d}_M^i)}{\mathrm{Im}(\mathrm{d}_M^{i-1})} \in \mathsf{Mod}\, A.$$

A homomorphism $\phi : M \to N$ in $\mathbf{C}(\mathsf{Mod}\, A)$ induces homomorphisms

$$\mathrm{H}^i(\phi) : \mathrm{H}^i(M) \to \mathrm{H}^i(N)$$

in $\mathsf{Mod}\, A$. We call ϕ a *quasi-isomorphism* if all the homomorphisms $\mathrm{H}^i(\phi)$ are isomorphisms.

The derived category $\mathbf{D}(\mathsf{Mod}\, A)$ is the localization of $\mathbf{C}(\mathsf{Mod}\, A)$ with respect to the quasi-isomorphisms. This means that $\mathbf{D}(\mathsf{Mod}\, A)$ has the same objects as $\mathbf{C}(\mathsf{Mod}\, A)$. There is a functor

$$\mathrm{Q} : \mathbf{C}(\mathsf{Mod}\, A) \to \mathbf{D}(\mathsf{Mod}\, A)$$

that is the identity of objects, and it sends quasi-isomorphisms to isomorphisms. Furthermore, any morphism in $\mathbf{D}(\mathsf{Mod}\, A)$ can be written as a fraction:

$$\mathrm{Q}(\phi) \circ \mathrm{Q}(\psi)^{-1},$$

where ϕ is a morphism in $\mathbf{C}(\mathsf{Mod}\, A)$, and ψ is a quasi-morphism in $\mathbf{C}(\mathsf{Mod}\, A)$. This is studied in Section 7 of the book.

A single A-module M^0 can be viewed as a complex M concentrated in degree 0:

$$M = \big(\cdots \to 0 \xrightarrow{0} M^0 \xrightarrow{0} 0 \to \cdots \big). \tag{0.1.5}$$

This turns out to be a fully faithful embedding

$$\mathsf{Mod}\, A \to \mathbf{D}(\mathsf{Mod}\, A). \tag{0.1.6}$$

The essential image of this embedding is the full subcategory of $\mathbf{D}(\mathsf{Mod}\, A)$ on the complexes M whose cohomology is concentrated in degree 0 (i.e. $\mathrm{H}^i(M) = 0$ for all $i \neq 0$). In this way we have *enlarged* the category of A-modules.

Here is a very important kind of quasi-isomorphism. Suppose M is a module and

$$\cdots \to P^{-2} \xrightarrow{\mathrm{d}_P^{-2}} P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \xrightarrow{\epsilon} M \to 0 \tag{0.1.7}$$

is a free resolution of it. We can view M as a complex concentrated in degree 0, by the embedding (0.1.6). Let P be the complex

$$P^{\cdot} = \big(\cdots \to P^{-2} \xrightarrow{\mathrm{d}_P^{-2}} P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \to 0 \to \cdots \big),$$

concentrated in nonpositive degrees. Then ϵ becomes a morphism of complexes

$$\epsilon : P \to M$$

with trivial components in nonzero degrees, and the exactness of the sequence (0.1.7) says that ϵ is actually a quasi-isomorphism. Thus

$$\mathrm{Q}(\epsilon) : P \to M$$

is an isomorphism in $\mathbf{D}(\mathsf{Mod}\, A)$.

Let us now return to $A = \mathbb{Z}$. The functor $D = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ from $\mathsf{Mod}\, \mathbb{Z}$ to itself has a *right derived functor*

$$\mathrm{R}D = \mathrm{RHom}_{\mathbb{Z}}(-, \mathbb{Z}),$$

which is a contravariant *triangulated functor*

$$\mathrm{R}D : \mathbf{D}(\mathsf{Mod}\, \mathbb{Z}) \to \mathbf{D}(\mathsf{Mod}\, \mathbb{Z}).$$

And there is a natural transformation of triangulated functors

$$\theta : \mathrm{Id} \to \mathrm{R}D \circ \mathrm{R}D.$$

Here is the way to calculate the value of the functor $\mathrm{R}D$ on a finitely generated abelian group M. Let us choose a free resolution of M like in (0.1.7). To be easy on ourselves, we can take it to be of this form:

$$P = \big(\cdots \to 0 \to P^{-1} \xrightarrow{\mathrm{d}_P^{-1}} P^0 \to 0 \to \cdots \big) = \big(\cdots \to 0 \to \mathbb{Z}^{r_1} \xrightarrow{\mathrm{d}} \mathbb{Z}^{r_0} \to 0 \cdots \big),$$

where $r_0, r_1 \in \mathbb{N}$ and d is a matrix of integers. Because $\mathrm{Q}(\epsilon) : P \to M$ is an isomorphism in $\mathbf{D}(\mathsf{Mod}\, \mathbb{Z})$, it suffices to calculate $\mathrm{R}D(P)$.

It is known that $\mathrm{R}D(P) = D(P)$ for bounded complexes of free modules, where $D(P)$ is calculated term by term. Thus

$$\mathrm{R}D(P) = D(P) = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) = \big(\cdots \to 0 \to \mathbb{Z}^{r_0} \xrightarrow{\mathrm{d}^*} \mathbb{Z}^{r_1} \to 0 \cdots\big),$$

a complex concentrated in degrees 0 and 1, with the transpose matrix d^* as its differential.

Because $\mathrm{R}D(P) = D(P)$ is itself a bounded complex of free modules, its derived dual is

$$\mathrm{R}D(\mathrm{R}D(P)) = D(D(P)) = \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}), \mathbb{Z}\big).$$

The canonical morphism

$$\theta_P : P \to D(D(P))$$

in $\mathbf{C}(\mathsf{Mod}\,\mathbb{Z})$ is an isomorphism in this case, because P^0 and P^{-1} are finite rank free modules. Therefore

$$\theta_M : M \to \mathrm{R}D(\mathrm{R}D(M))$$

is an isomorphism in $\mathbf{D}(\mathsf{Mod}\,\mathbb{Z})$. (For a more general statement see Subsection 13.2.) We see that $\mathrm{R}D$ *is a duality that holds for all finitely generated* $\mathbb{Z}$*-modules* !

Here is the connection between the derived duality $\mathrm{R}D$ and the "classical" dualities D and D'. Take a finitely generated abelian group M, with short exact sequence (0.1.1). There are functorial isomorphisms

$$\mathrm{H}^0(\mathrm{R}D(M)) \cong \mathrm{Ext}^0_{\mathbb{Z}}(M, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \cong D(M)$$

and

$$\mathrm{H}^1(\mathrm{R}D(M)) \cong \mathrm{Ext}^1_{\mathbb{Z}}(M, \mathbb{Z}) \cong D'(M).$$

The cohomologies $\mathrm{H}^i(\mathrm{R}D(M))$ vanish for $i \neq 0, 1$.

Note that $D(M) \cong D(F)$ and $D'(M) \cong D'(T)$. We see that if M is neither free nor finite, then $\mathrm{H}^0(\mathrm{R}D(M))$ and $\mathrm{H}^1(\mathrm{R}D(M))$ are both nonzero; so that the complex $D(M)$ is not isomorphic to an object of $\mathsf{Mod}\,\mathbb{Z}$, under the embedding (0.1.6).

This sort of duality holds for *many noetherian commutative rings* A. But the formula for the duality functor

$$\mathrm{R}D : \mathbf{D}(\mathsf{Mod}\,A) \to \mathbf{D}(\mathsf{Mod}\,A)$$

is somewhat different – it is

$$\mathrm{R}D(M) := \mathrm{RHom}_A(M, R),$$

where $R \in \mathbf{D}(\mathsf{Mod}\,A)$ is a *dualizing complex*. Such a dualizing complex is unique (up to a degree translation and tensoring with an invertible module).

Interestingly, the structure of the dualizing complex R depends on the geometry of the ring A (i.e. of the affine scheme $\mathrm{Spec}\,A$). If A is a regular ring (like $\mathbb{Z}$) then $R = A$ is dualizing. If A is Cohen-Macaulay (and $\mathrm{Spec}\,A$ is connected) then R is a single A-module. But if A is a more complicated ring, then R must live in several degrees.

Example 0.1.8. Consider the affine algebraic variety $X \subseteq \mathbf{A}^3_{\mathbb{R}}$ which is the union of a plane and a line, with coordinate ring

$$A = \mathbb{R}[t_1, t_2, t_3]/(t_3 \cdot t_1,\ t_3 \cdot t_2).$$

See figure 1. The dualizing complex R must live in two adjacent degrees; namely there is some i s.t. $\mathrm{H}^i(R)$ and $\mathrm{H}^{i+1}(R)$ are nonzero.

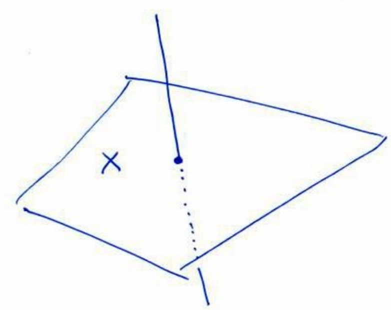

Figure 1. An algebraic variety that is connected but not equidimensional, and hence not Cohen-Macaulay.

One can also talk about dualizing complexes over *noncommutative rings*. (This is a favorite topic of mine!)

0.2. **Synopsis of the Book.** Here is a section-by-section description of the material in the book (the first half only).

Sections 1-2. These sections are pretty much a review of the standard material on categories and functors (especially abelian categories and additive functors) that is needed for the book. A reader who is familiar with this material can skip these sections. We do recommend looking at our notational convention, that are spelled out in Convention 1.2.1.

Section 3. A good understanding of *DG algebra* ("DG" is short for "differential graded") is essential in our approach to derived categories. We aim to study both the derived category $\mathbf{D}(\mathsf{M})$ of an abelian category M, and the derived category $\mathbf{D}(A)$ of DG modules over a DG ring A. In order to accomplish this, we introduce a new concept, that combines both these setups: the category $\mathbf{C}(A, \mathsf{M})$ of *DG A-modules in* M. See Subsection 3.7.

Actually, our methods can be expanded to handle the DG category $\mathbf{C}(\mathsf{A}, \mathsf{M})$ of DG A-modules in M, where A is a DG category (rather than a DG ring as above). This includes as a special case ($\mathsf{M} = \mathsf{Ab}$) the category $\mathbf{C}(\mathsf{A})$ of DG A-modules, in the sense of Keller; see Remark 3.7.7. We have decided to stick to the less general setup $\mathbf{C}(A, \mathsf{M})$ for these reasons:

(1) The treatment is much more streamlined and intuitive.
(2) Virtually all DG categories that occur in practice (in algebra and algebraic geometry) are full subcategories of $\mathbf{C}(A, \mathsf{M})$, for suitable A and M. A noteworthy instance is derived Morita theory for schemes (see Section 17.4), that fits nicely within our framework.

There do not exist (to our knowledge) detailed textbook references for DG algebra (by which we mean DG rings, DG modules, DG categories, DG functors and related constructions). Therefore we have included a lot of basic material in this section. Moreover, we present a new treatment of translations and cones, using the "little t operator", following our paper [Ye11]. Among other things, we prove (in Theorem 3.8.7) that the translation functor T of $\mathbf{C}(A, \mathsf{M})$ is a DG functor, and $\mathrm{t} : \mathrm{Id} \to \mathrm{T}$ is a degree -1 morphisms of DG functors from $\mathbf{C}(A, \mathsf{M})$ to itself.

Section 4. This section consists mostly of new material, some of it implicit in the paper [BoKa] on *pretriangulated DG categories.*

Inside the DG category $\mathbf{C}(A, \mathsf{M})$ there is the *strict category* $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$, that has all the objects, but its morphisms are the degree 0 cocycles. Any morphism $\phi : M \to N$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ gives rise to a *standard triangle*

$$M \xrightarrow{\phi} N \xrightarrow{e_\phi} \mathrm{Cone}(\phi) \xrightarrow{p_\phi} \mathrm{T}(M)$$

in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$.

Consider a DG functor

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N}), \tag{0.2.1}$$

where A and B are DG rings, and M and N are abelian categories. In Theorem 4.2.3 we show that there is a canonical isomorphism of DG functors

$$\tau_F : F \circ \mathrm{T} \xrightarrow{\simeq} \mathrm{T} \circ F \tag{0.2.2}$$

called the *translation isomorphism.* Then, in Theorem 4.3.7, we prove that F sends standard triangles in the $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ to standard triangles in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$.

We end this section with several examples of DG functors. These examples are prototypes – they can be easily extended to other setups.

Section 5. We start with the theory of *pretriangulated categories* and *triangulated functors*, following mainly [RD]. Because the *octahedral axiom* plays no role in our approach, we exclude it from the discussion, and this is the reason we do not talk about triangulated categories. In Subsection 5.4 we prove that the homotopy category $\mathbf{K}(A, \mathsf{M})$ is pretriangulated.

We conclude this section with Theorem 5.4.13. It says that given a DG functor F as in (0.2.1), with translation isomorphism τ_F from (0.2.2), the T-additive functor

$$(F, \tau_F) : \mathbf{K}(A, \mathsf{M}) \to \mathbf{K}(B, \mathsf{N})$$

is triangulated. This is possibly a new result (unifying well-known yet disparate examples).

Section 6. In this section we take a close look at *localization of categories.* We give a detailed proof of the theorem on Ore localization (also known as noncommutative localization). We then prove that the localization K_{S} of a pretriangulated category K at a multiplicatively closed set of cohomological origin S is a left and right Ore localization, the category K_{S} is pretriangulated, and the localization functor $\mathrm{Q} : \mathsf{K} \to \mathsf{K}_{\mathsf{S}}$ is triangulated.

Section 7. In the case of the pretriangulated category $\mathbf{K}(A, \mathsf{M})$, and the quasi-isomorphisms $\mathbf{S}(A, \mathsf{M})$ in it, we get the *derived category*

$$\mathbf{D}(A, \mathsf{M}) := \mathbf{K}(A, \mathsf{M})_{\mathbf{S}(A, \mathsf{M})},$$

and the triangulated localization functor

$$\mathrm{Q} : \mathbf{K}(A, \mathsf{M}) \to \mathbf{D}(A, \mathsf{M}).$$

We look at the full subcategories $\mathsf{K}^{\star}(A, \mathsf{M})$ of $\mathsf{K}(A, \mathsf{M})$ corresponding to boundedness conditions $\star$, and prove that their localizations with respect to quasi-isomorphisms embed fully faithfully in $\mathbf{D}(A, \mathsf{M})$. We also prove that the obvious functor $\mathsf{M} \to \mathbf{D}(\mathsf{M})$ is fully faithful.

Section 8. In this section we talk about *derived functors*. To make the definitions of the derived functors precise, we introduce some 2-*categorical notation* here.

The setting is general: we start from a triangulated functor $F : \mathsf{K} \to \mathsf{E}$ between pretriangulated categories, and a denominator set of cohomological origin $\mathsf{S} \subseteq \mathsf{K}$. A *right derived functor* of F is a pair $(\mathrm{R}F, \eta)$, where $\mathrm{R}F : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$ is a triangulated functor, and $\eta : F \to \mathrm{R}F \circ \mathrm{Q}$ is a morphism of triangulated functors. The pair $(\mathrm{R}F, \eta)$ has a universal property, making it unique up to a unique isomorphism. The *left derived functor* $(\mathrm{L}F, \eta)$ is defined similarly.

We provide a general existence theorem for derived functors. For the right derived functor we assume the existence of a pretriangulated category $\mathsf{J} \subseteq \mathsf{K}$ that is "right F-acyclic". Likewise for the left derived functor. This is the original result from [RD], but our proof is much more detailed.

Section 9. Here we specialize the general existence theorem from Section 8 to the case of the pretriangulated categories $\mathsf{K}^{\star}(A, \mathsf{M})$, for a DG ring A, and abelian category M and a boundedness condition $\star$. We define *K-injective DG modules*, and show they can be used to present any right derived functor (if there are enough of them). We also define *K-projective* and *K-flat DG modules*, and explain how they are used.

Section 10. In this section we prove existence of K-injective, K-projective and K-flat resolutions in several important cases of $\mathbf{C}^{\star}(A, \mathsf{M})$:

- K-projective resolutions in $\mathbf{C}^{-}(\mathsf{M})$, where M is an abelian category with enough projectives. This is classical (i.e. it is already in [RD]).
- K-projective resolutions in $\mathbf{C}(A)$, where A is any DG ring. This includes $\mathbf{C}(\mathsf{Mod}\, A)$, the category of unbounded complexes of modules over a ring A.
- K-injective resolutions in $\mathbf{C}^{+}(\mathsf{M})$, where M is an abelian category with enough injectives. This is classical too.
- K-injective resolutions in $\mathbf{C}(A)$, where A is any DG ring. This includes $\mathbf{C}(\mathsf{Mod}\, A)$ for any ring A.

Our proofs are explicit, and we use limits of complexes cautiously (since this is known to be a pitfall).

This ends the first half of the book. As mentioned before, the second half is yet to be written.

0.3. **Recommended Bibliography.** For further discussion of categories (and the related set theory), functors, and classical homological algebra, see the books [Mac2], [HiSt], [Rot], [GeMa], [KaSc1], [KaSc2], [Ne1], and [We].

Derived categories are treated in [RD] (the original reference), and in the last five books in the previous list. None of these references has emphasis on DG categories as the background out of which derived categories arise; indeed, most of these books do not even mention DG algebra.

Sources for algebraic geometry and modern differential geometry are [Har] and [KaSc1]. For commutative ring theory see the books [Eis], [Mats] and [AlKl]. For noncommutative ring theory see [Row] and [Rot].

Almost everything can be found in the evolving online reference [SP].

0.4. **Acknowledgments.** I want to thank the participants of the course on derived categories held at Ben Gurion University in Spring 2012, for correcting many of my mistakes (both in real time during the lectures, and afterwards when writing the notes [Ye7]). Thanks also to Joseph Lipman, Pierre Schapira, Amnon Neeman and Charles Weibel for helpful discussions during the preparation of that course. Vincent Beck, Yang Han and Lucas Simon sent me corrections and useful comments on the material in [Ye7].

As already mentioned, this book is being written while I am teaching a multi-semester course on the subject at Ben Gurion University, spanning the academic years 2015-16 and 2016-17. I wish to thank the participants of this course, and especially Rishi Vyas, Stephan Snigerov and Asaf Yekutieli, who contributed material and corrected numerous mistakes. Stephan also helped me prepare the manuscript for publication. Ben Gurion University is generous enough to permit this ongoing project. Some of this work was supported by the Israel Science Foundation grant no. 253/13.

1. Basic Facts on Categories

1.1. **Set Theory.** In this book we will not try to be precise about issues of set theory. The blanket assumption is that we are given a *Grothendieck universe* U. This is a "large" infinite set. A *small set*, or a U-small set, is a set S that is an element of U. We want all the products $\prod_{i \in I} S_i$ and disjoint unions $\coprod_{i \in I} S_i$, with I and S_i small sets, to be small sets too. (This requirement is not crucial for us, and it is more a matter of convenience. When dealing with higher categories, one usually needs a hierarchy of universes anyhow.) We assume that the axiom of choice holds in U.

A U-*category* is a category C whose set of objects $\mathrm{Ob}(\mathsf{C})$ is a subset of U, and for every $C, D \in \mathrm{Ob}(\mathsf{C})$ the set of morphisms $\mathrm{Hom}_{\mathsf{C}}(C, D)$ is small. If $\mathrm{Ob}(\mathsf{C})$ is also small, then C is called a *small category*. See [SGA 4] or [KaSc2, Section 1.1]. Another approach, involving "sets" vs "classes", can be found in [Ne1].

We denote by Set the category of all small sets. So $\mathrm{Ob}(\mathsf{Set}) = \mathsf{U}$, and Set is a U-category. A group (or a ring, etc.) is called small if its underlying set is small. We denote by Grp, Ab, Ring and $\mathsf{Ring}_{\mathsf{c}}$ the categories of small groups, small abelian groups, small rings and small commutative rings respectively. For a small ring A we denote by $\mathsf{Mod}\, A$ the category of all small left A-modules.

By default we work with U-categories, and from now on U will remain implicit. The one exception is when we deal with localization of categories, where we shall briefly encounter a set theoretical issue; but for most interesting cases this issue has an easy solution.

1.2. **Notation.** Let C be a category. We often write $C \in \mathsf{C}$ as an abbreviation for $C \in \mathrm{Ob}(\mathsf{C})$. For an object C, its identity automorphism is denoted by id_C. The identity functor of C is denoted by Id_{C}.

The opposite category of C is C^{op}. It has the same objects as C, but the morphism sets are

$$\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(C_0, C_1) := \mathrm{Hom}_{\mathsf{C}}(C_1, C_0),$$

and composition is reversed. The identity functor of C can be viewed as a contravariant functor $\mathrm{Id}_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}^{\mathrm{op}}$. A contravariant functor $F : \mathsf{C} \to \mathsf{D}$ is the same as a covariant functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$. Since we prefer dealing only with covariant functors, we make the following convention:

Convention 1.2.1. By default all functors will be covariant, unless explicitly mentioned otherwise.

We will try to keep the following font and letter conventions:

- $f : C \to D$ is a morphism between objects in a category.
- $F : \mathsf{C} \to \mathsf{D}$ is a functor between categories.
- $\eta : F \to G$ is morphism of functors (i.e. a natural transformation) between functors $F, G : \mathsf{C} \to \mathsf{D}$.
- $f, \phi, \alpha : M \to N$ are morphisms between objects in an abelian category M.
- $F : \mathsf{M} \to \mathsf{N}$ is an additive functor between abelian categories.
- The derived category of an abelian category M is $\mathbf{D}(\mathsf{M})$.
- If M is a module category, and $M \in \mathrm{Ob}(\mathsf{M})$, then elements of M will be denoted by $m, n, m_i, \ldots$.

1.3. **Epimorphisms and Monomorphisms.** Let C be a category. Recall that a morphism $f : C \to D$ in C is called an *isomorphism* if there is a morphism $g : D \to C$ such that $f \circ g = \mathrm{id}_D$ and $g \circ f = \mathrm{id}_C$. The morphism g is called the *inverse* of f, it is unique (if it exists), and it is denoted by f^{-1}. An isomorphism is often denoted by this shape of arrow: $f : C \xrightarrow{\simeq} D$.

A morphism $f : C \to D$ in C is called an *epimorphism* if it has the right cancellation property: for any $g, g' : D \to E$, $g \circ f = g' \circ f$ implies $g = g'$. An epimorphism is often denoted by this shape of arrow: $f : C \twoheadrightarrow D$.

A morphism $f : C \to D$ is called a *monomorphism* if it has the left cancellation property: for any $g, g' : E \to C$, $f \circ g = f \circ g'$ implies $g = g'$. A monomorphism is often denoted by this shape of arrow: $f : C \rightarrowtail D$.

Example 1.3.1. In Set the monomorphisms are the injections, and the epimorphisms are the surjections. A morphism $f : C \to D$ in Set that is both a monomorphism and an epimorphism is an isomorphism. The same holds in the category $\mathsf{Mod}\, A$ of left modules over a ring A.

This example could be misleading, because the property of being an epimorphism is often not preserved by forgetful functors, as the next exercise shows.

Exercise 1.3.2. Consider the category of rings Ring. (All rings have units, and ring homomorphisms are unital.) Show that the forgetful functor $\mathsf{Ring} \to \mathsf{Set}$ respects monomorphisms, but it does not respect epimorphisms. (Hint: show that the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in Ring.)

By a *subobject* of an object $C \in \mathsf{C}$ we mean a monomorphism $f : C' \rightarrowtail C$ in C. We sometimes write $C' \subseteq C$ in this situation, but this is only notational (and does not mean inclusion of sets). We say that two subobjects $f_0 : C'_0 \rightarrowtail C$ and $f_1 : C'_1 \rightarrowtail C$ of C are *isomorphic* if there is an isomorphism $g : C'_0 \xrightarrow{\simeq} C'_1$ such that $f_1 \circ g = f_0$.

Likewise, by a *quotient* of C we mean an epimorphism $g : C \twoheadrightarrow C''$ in C. There is an analogous notion of isomorphic quotients.

Exercise 1.3.3. Let C be a category, and let C be an object of C.

(1) Suppose $f_0 : C'_0 \rightarrowtail C$ and $f_1 : C'_1 \rightarrowtail C$ are subobjects of C. Show that there is at most one morphism $g : C'_0 \to C'_1$ such that $f_1 \circ g = f_0$; and if g exists, then it is a monomorphism.
(2) Show that isomorphism is an equivalence relation on the set of subobjects of C. Show that the set of equivalence classes of subobjects of C is partially ordered by "inclusion". (Ignore set-theoretical issues.)
(3) Formulate and prove the analogous statements for quotient objects.

An *initial object* in a category C is an object $C_0 \in \mathsf{C}$, such that for every object $C \in \mathsf{C}$ there is exactly one morphism $C_0 \to C$. Thus the set $\mathrm{Hom}_{\mathsf{C}}(C_0, C)$ is a singleton. A *terminal object* in C is an object $C_\infty \in \mathsf{C}$, such that for every object $C \in \mathsf{C}$ there is exactly one morphism $C \to C_\infty$.

Definition 1.3.4. A *zero object* in a category C is an object which is both initial and terminal.

Initial, terminal and zero objects are unique up to unique isomorphisms (but they need not exist).

Example 1.3.5. In Set, $\varnothing$ is an initial object, and any singleton is a terminal object. There is no zero object.

Example 1.3.6. In $\mathsf{Mod}\, A$, any trivial module (with only the zero element) is a zero object, and we denote this module by 0. This is allowed, since any other zero module is uniquely isomorphic to it.

1.4. **Products and Coproducts.** Let C be a category. By a *collection of objects of* C indexed by a (small) set I, we mean a function $I \to \mathrm{Ob}(\mathsf{C})$, $i \mapsto C_i$. We usually denote this collection like this: $\{C_i\}_{i \in I}$.

Given a a collection $\{C_i\}_{i \in I}$ of objects of C, its *product* is a pair $(C, \{p_i\}_{i \in I})$ consisting of an object $C \in \mathsf{C}$, and a collection $\{p_i\}_{i \in I}$ of morphisms $p_i : C \to C_i$, called *projections*. The pair $(C, \{p_i\}_{i \in I})$ must have this universal property: given any object D and morphisms $f_i : D \to C_i$, there is a unique morphism $f : D \to C$ s.t. $f_i = p_i \circ f$. Of course if a product $(C, \{p_i\}_{i \in I})$ exists, then it is unique up to a unique isomorphism; and we usually write $\prod_{i \in I} C_i := C$, leaving the projection morphisms implicit.

Example 1.4.1. In Set and $\mathsf{Mod}\, A$ all products exist, and they are the usual cartesian products.

For a collection $\{C_i\}_{i \in I}$ of objects of C, their *coproduct* is a pair $(C, \{e_i\}_{i \in I})$, consisting of an object C and a collection $\{e_i\}_{i \in I}$ of morphisms $e_i : C_i \to C$, called *embeddings*. The pair $(C, \{e_i\}_{i \in I})$ must have this universal property: given any object D and morphisms $f_i : C_i \to D$, there is a unique morphism $f : C \to D$ s.t. $f_i = f \circ e_i$. If a coproduct $(C, \{e_i\}_{i \in I})$ exists, then it is unique up to a unique isomorphism; and we write $\coprod_{i \in I} C_i := C$, leaving the embeddings implicit.

Example 1.4.2. In Set the coproduct is the disjoint union. In $\mathsf{Mod}\, A$ the coproduct is the direct sum.

1.5. **Equivalence of Categories.** Recall that a functor $F : \mathsf{C} \to \mathsf{D}$ is an *equivalence* if there exists a functor $G : \mathsf{D} \to \mathsf{C}$, and isomorphisms of functors (i.e. natural isomorphisms) $G \circ F \xrightarrow{\simeq} \mathrm{Id}_{\mathsf{C}}$ and $F \circ G \xrightarrow{\simeq} \mathrm{Id}_{\mathsf{D}}$. Such a functor G is called a *quasi-inverse* of F, and it is unique up to isomorphism (if it exists), and it is denoted by F^{-1}.

The functor $F : \mathsf{C} \to \mathsf{D}$ is *full* (resp. *faithful*) if every $C_0, C_1 \in \mathsf{C}$ the function

$$F : \mathrm{Hom}_{\mathsf{C}}(C_0, C_1) \to \mathrm{Hom}_{\mathsf{D}}\big(F(C_0), F(C_1)\big)$$

is surjective (resp. injective).

We know that $F : \mathsf{C} \to \mathsf{D}$ is an equivalence iff these two conditions hold:

(i) F is essentially surjective on objects. This means that for every $D \in \mathsf{D}$ there is some $C \in \mathsf{C}$ and an isomorphism $F(C) \xrightarrow{\simeq} D$.

(ii) F is fully faithful (i.e. full and faithful).

Exercise 1.5.1. If you are not sure about the last claim (characterization of equivalences), then prove it. (Hint: use the axiom of choice to construct a quasi-inverse of F.)

Example 1.5.2. Let C and D be categories. A functor $F : \mathsf{C} \to \mathsf{D}$ is called an *isomorphism of categories* if it is bijective on sets of objects and on sets of morphisms. It is clear that an isomorphism of categories is an equivalence. If F is an isomorphism of categories, then it has an inverse isomorphism $F^{-1} : \mathsf{D} \to \mathsf{C}$, which is unique. In practice, it is quite rare to find an isomorphism of categories.

1.6. **Bifunctors.** Let C and D be categories. Their product is the category $\mathsf{C} \times \mathsf{D}$ defined as follows: the set of objects is

$$\mathrm{Ob}(\mathsf{C} \times \mathsf{D}) := \mathrm{Ob}(\mathsf{C}) \times \mathrm{Ob}(\mathsf{D}).$$

The sets of morphisms are

$$\mathrm{Hom}_{\mathsf{C} \times \mathsf{D}}\big((C_0, D_0), (C_1, D_1)\big) := \mathrm{Hom}_{\mathsf{C}}(C_0, C_1) \times \mathrm{Hom}_{\mathsf{D}}(D_0, D_1).$$

The composition is

$$(f_1, g_1) \circ (f_0, g_0) := (f_1 \circ f_0, g_1 \circ g_0),$$

and the identity morphisms are $(\mathrm{id}_C, \mathrm{id}_D)$.

A *bifunctor*

$$F : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$$

is by definition a functor from the product category $\mathsf{C} \times \mathsf{D}$ to E. We say "bifunctor" because it is a functor of two arguments: $F(C, D) \in \mathsf{E}$. This will be especially useful when considering additive categories, because then we can talk about "bi-additive bifunctors".

1.7. **Representable Functors.** Let C be a category and $C \in \mathsf{C}$ an object. We get a functor

$$Y_C : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}, \quad Y_C := \mathrm{Hom}_{\mathsf{C}}(-, C),$$

called the *Yoneda functor*. This functor sends an object C' to the set $\mathrm{Hom}_{\mathsf{C}}(C', C)$, and a morphism $\psi : C' \to C''$ in C to the function

$$Y_C(\psi) := \mathrm{Hom}(\psi, \mathrm{id}_C) : \mathrm{Hom}_{\mathsf{C}}(C'', C) \to \mathrm{Hom}_{\mathsf{C}}(C', C).$$

Now suppose we are given a morphism $\phi : C_0 \to C_1$ in C. There is a morphism of functors (a natural transformation)

$$Y_\phi := \mathrm{Hom}_{\mathsf{C}}(-, \phi) : Y_{C_0} \to Y_{C_1}.$$

Here is the first formulation of the *Yoneda Lemma*.

Proposition 1.7.1 (Yoneda Lemma v1)**.** *Let C be a category, let $C_0, C_1 \in \mathsf{C}$ be objects, and let $\eta : Y_{C_0} \to Y_{C_1}$ be a morphism of functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$.*

(1) *There exists a unique morphism $\phi : C_0 \to C_1$ in C such that $Y_\phi = \eta$.*
(2) *If $\eta : Y_{C_0} \to Y_{C_1}$ is an isomorphism of functors, then $\phi : C_0 \to C_1$ is an isomorphism in C.*

See [KaSc2, Section 1.4] for a proof. The proof is not hard, but it is very confusing.

A functor $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is called *representable* if there is an isomorphism of functors $f : F \xrightarrow{\simeq} Y_C$ for some object $C \in \mathsf{C}$. By Proposition 1.7.1 the pair (C, f) is unique up to a unique isomorphism (if it exists). Note that the isomorphism of sets $f_C : F(C) \xrightarrow{\simeq} Y_C(C)$ gives a special element $\tilde{f} \in F(C)$ such that $f_C(\tilde{f}) = \mathrm{id}_C$.

Here is a fancier way to state this result. Consider the category $\mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$, whose objects are the functors $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$, and whose morphisms are the morphisms of functors. There is a set-theoretic difficulty here: the sets of objects and morphisms of $\mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ are too big (unless C is a small category); so this is not a U-category, and we must enlarge the universe.

Proposition 1.7.2 (Yoneda Lemma v2)**.** *The Yoneda functor*

$$Y : \mathsf{C} \to \mathsf{Fun}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set}), \quad C \mapsto Y_C, \quad \phi \mapsto Y_\phi$$

is fully faithful.

In other words, the Yoneda Lemma says that the functor Y is an equivalence from C to the category of representable functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$.

Dually, any $C \in \mathsf{C}$ gives rise to a functor

$$Y'_C : \mathsf{C} \to \mathsf{Set}, \quad Y'_C := \mathrm{Hom}_{\mathsf{C}}(C, -).$$

The identity automorphism id_C is a special element of the set $Y'_C(C)$.

A functor $F : \mathsf{C} \to \mathsf{Set}$ is called *corepresentable* if $F \cong Y'_C$ for some object $C \in \mathsf{C}$. The object C is said to corepresent the functor F. The dual Yoneda Lemma (v2) says that the functor Y' is an equivalence from C^{op} to the category of corepresentable functors $\mathsf{C} \to \mathsf{Set}$.

2. Abelian Categories and Additive Functors

The concept of *abelian category* is an extremely useful abstraction of module categories, introduced by Grothendieck in 1957. Before defining it (in Definition 2.3.8), we need some preparation.

2.1. Linear Categories.

Definition 2.1.1. Let $\mathbb{K}$ be a commutative ring. A $\mathbb{K}$-*linear category* is a category M, endowed with a $\mathbb{K}$-module structure on each of the sets of morphisms $\operatorname{Hom}_{\mathsf{M}}(M_0, M_1)$. The condition is this:

- For all $M_0, M_1, M_2 \in \mathsf{M}$ the composition function
$$\begin{aligned} &\operatorname{Hom}_{\mathsf{M}}(M_1, M_2) \times \operatorname{Hom}_{\mathsf{M}}(M_0, M_1) \to \operatorname{Hom}_{\mathsf{M}}(M_0, M_2) \\ &(\phi_2, \phi_1) \mapsto \phi_2 \circ \phi_1 \end{aligned}$$
is $\mathbb{K}$-bilinear.

If $\mathbb{K} = \mathbb{Z}$, we say that M is a *linear category*.

Let $\mathbb{K}$ be a commutative ring. By *central* $\mathbb{K}$-*ring* we mean a ring A, with a ring homomorphism $\mathbb{K} \to A$, such that the image of $\mathbb{K}$ is inside the center of A. (Many texts would call such A a "unital associative $\mathbb{K}$-algebra".)

Example 2.1.2. Let $\mathbb{K}$ be any nonzero commutative ring, and let n be a positive integer. Then the ring of matrices $A := \operatorname{Mat}_n(\mathbb{K})$ is a central $\mathbb{K}$-ring.

Proposition 2.1.3. *Let* M *be a* $\mathbb{K}$*-linear category.*

(1) *For any object* $M \in \mathsf{M}$*, the set*
$$\operatorname{End}_{\mathsf{M}}(M) := \operatorname{Hom}_{\mathsf{M}}(M, M),$$
with its given addition operation, and with the operation of composition, is a central $\mathbb{K}$*-ring.*

(2) *For any two objects* $M_0, M_1 \in \mathsf{M}$*, the set* $\operatorname{Hom}_{\mathsf{M}}(M_0, M_1)$*, with its given addition operation, and with the operations of composition, is a left module over the ring* $\operatorname{End}_{\mathsf{M}}(M_1)$*, and a right module over the ring* $\operatorname{End}_{\mathsf{M}}(M_0)$*. Furthermore, these left and right actions commute with each other.*

Proof. Exercise. □

This result can be reversed:

Example 2.1.4. Let A be a central $\mathbb{K}$-ring. Define a category M like this: there is a single object M, and its set of morphisms is $\operatorname{Hom}_{\mathsf{M}}(M, M) := A$. Composition in M is the multiplication of A. Then M is a $\mathbb{K}$-linear category.

Because of the above, in a linear category M, we often denote the identity automorphism of an object M by $1_M := \mathrm{id}_M \in \operatorname{End}_{\mathsf{M}}(M)$.

For a central $\mathbb{K}$-ring A, the opposite ring A^{op} has the same $\mathbb{K}$-module structure as A, but the multiplication is reversed.

Exercise 2.1.5. Let A be a nonzero ring. Let $P, Q \in \mathsf{Mod}\, A$ be distinct free A-modules of rank 1.

(1) Prove that there is a ring isomorphism $\operatorname{End}_{\mathsf{Mod}\, A}(P) \cong A^{\mathrm{op}}$. Is this ring isomorphism canonical?

(2) Let M be the full subcategory of $\mathsf{Mod}\, A$ on the set of objects $\{P, Q\}$. Compare the linear category M to the ring of matrices $\operatorname{Mats}_2(A^{\mathrm{op}})$.

2.2. **Additive Categories.**

Definition 2.2.1. An *additive category* is a linear category M satisfying these conditions:

(i) M has a zero object 0.
(ii) M has finite coproducts.

Observe that $\mathrm{Hom}_{\mathsf{M}}(M, N) \neq \varnothing$ for any $M, N \in \mathsf{M}$, since this is an abelian group. Also

$$\mathrm{Hom}_{\mathsf{M}}(M, 0) = \mathrm{Hom}_{\mathsf{M}}(0, M) = 0,$$

the zero abelian group. We denote the unique arrows $0 \to M$ and $M \to 0$ also by 0. So the numeral 0 has a lot of meanings; but they are (hopefully) clear from the contexts. The coproduct in a linear category M is usually denoted by $\bigoplus$; cf. Example 1.4.2.

Example 2.2.2. Let A be a $\mathbb{K}$-central ring. The category $\mathsf{Mod}\, A$ is a $\mathbb{K}$-linear additive category. The full subcategory $\mathsf{F} \subseteq \mathsf{Mod}\, A$ on the free modules is also additive.

Proposition 2.2.3. *Let M be a linear category. Let $\{M_i\}_{i \in I}$ be a finite collection of objects of M, and assume the coproduct $M = \bigoplus_{i \in I} M_i$ exists, with embeddings $e_i : M_i \to M$.*

(1) *For any i let $p_i : M \to M_i$ be the unique morphism s.t. $p_i \circ e_i = 1_{M_i}$, and $p_i \circ e_j = 0$ for $j \neq i$. Then $(M, \{p_i\}_{i \in I})$ is a product of the collection $\{M_i\}_{i \in I}$.*
(2) $\sum_{i \in I} e_i \circ p_i = 1_M$.

Proof. Exercise. □

Part (1) directly implies:

Corollary 2.2.4. *An additive category has finite products.*

Definition 2.2.5. Let M be an additive category, and let N be a full subcategory of M. We say that N is a *full additive subcategory* of M if N contains the zero object, and is closed under finite direct sums.

Exercise 2.2.6. In the situation of Definition 2.2.5, show that the category N is itself additive.

Example 2.2.7. Consider the linear category M from Example 2.1.4, built from a ring A. It does not have a zero object (unless the ring A is the zero ring), so it is not additive.

A more puzzling question is this: Does M have finite direct sums? This turns out to be equivalent to whether or not $A \cong A \oplus A$ as right A-modules. To see why, choose a fully faithful additive functor $F : \mathsf{M} \to \mathsf{Mod}\, A^{\mathrm{op}}$, that sends the unique object $M \in \mathsf{M}$ to a rank 1 free right A-module P. (We identify right A-modules with left A^{op}-modules.) Compare to Exercise 2.1.5.

Let $I := \{1, 2\}$, and let $\{M_i\}_{i \in I}$ be the only possible collection in M indexed by I (i.e. $M_i = M$). If there is a coproduct in M, then it must be $M_1 \oplus M_2 \cong M$. According to Proposition 2.4.2, we get

$$P \oplus P \cong F(M_1) \oplus F(M_2) \cong F(M) \cong P$$

in $\mathsf{Mod}\, A^{\mathrm{op}}$.

One can show that when A is nonzero and commutative, or nonzero and noetherian, then $A \ncong A \oplus A$ in $\mathsf{Mod}\, A^{\mathrm{op}}$. On the other hand, if we take a field $\mathbb{K}$, and a countable rank $\mathbb{K}$-module N, then $A := \mathrm{End}_{\mathbb{K}}(N)$ will satisfy $A \cong A \oplus A$.

Proposition 2.2.8. *Let M be a linear category, and $N \in \mathsf{M}$. The following conditions are equivalent:*

(i) *The ring $\mathrm{End}_{\mathsf{M}}(N)$ is trivial.*
(ii) *N is a zero object of M.*

Proof. (ii) $\Rightarrow$ (i): Since the set $\mathrm{End}_{\mathsf{M}}(N)$ is a singleton, it must be the trivial ring $(1 = 0)$.

(i) $\Rightarrow$ (ii): If the ring $\mathrm{End}_{\mathsf{M}}(N)$ is trivial, then all left and right modules over it must be trivial. Now use Proposition 2.1.3(2). □

2.3. **Abelian Categories.**

Definition 2.3.1. Let M be an additive category, and let $f : M \to N$ be a morphism in M. A *kernel* of f is a pair (K, k), consisting of an object $K \in \mathsf{M}$ and a morphism $k : K \to M$, with these properties:

(i) $f \circ k = 0$.
(ii) If $k' : K' \to M$ is a morphism in M such that $f \circ k' = 0$, then there is a unique morphism $g : K' \to K$ such that $k' = k \circ g$.

In other words, the object K represents the functor $\mathsf{M}^{\mathrm{op}} \to \mathsf{Ab}$,

$$K' \mapsto \{k' \in \mathrm{Hom}_{\mathsf{M}}(K', M) \mid f \circ k' = 0\}.$$

The kernel of f is of course unique up to a unique isomorphism (if it exists), and we denote if by $\mathrm{Ker}(f)$. Sometimes $\mathrm{Ker}(f)$ refers only to the object K, and other times it refers only to the morphism k; as usual, this should be clear from the context.

Definition 2.3.2. Let M be an additive category, and let $f : M \to N$ be a morphism in M. A *cokernel* of f is a pair (C, c), consisting of an object $C \in \mathsf{M}$ and a morphism $c : N \to C$, with these properties:

(i) $c \circ f = 0$.
(ii) If $c' : N \to C'$ is a morphism in M such that $c' \circ f = 0$, then there is a unique morphism $g : C \to C'$ such that $c' = g \circ c$.

In other words, the object C corepresents the functor $\mathsf{M} \to \mathsf{Ab}$,

$$C' \mapsto \{c' \in \mathrm{Hom}_{\mathsf{M}}(N, C') \mid c' \circ f = 0\}.$$

The cokernel of f is of course unique up to a unique isomorphism (if it exists), and we denote if by $\mathrm{Coker}(f)$. Sometimes $\mathrm{Coker}(f)$ refers only to the object C, and other times it refers only to the morphism c; as usual, this should be clear from the context.

Example 2.3.3. In $\mathsf{Mod}\, A$ all kernels and cokernels exist. Given $f : M \to N$, the kernel is $k : K \to M$, where

$$K := \{m \in M \mid f(m) = 0\},$$

and the k is the inclusion. The cokernel is $c : N \to C$, where $C := N/f(M)$, and c is the canonical projection.

Proposition 2.3.4. *Let M be an additive category, and let $f : M \to N$ be a morphism in M.*

(1) *If $k : K \to M$ is a kernel of f, then k is a monomorphism.*
(2) *If $c : N \to C$ is a cokernel of f, then c is an epimorphism.*

Proof. Exercise. □

Definition 2.3.5. Assume the additive category M has kernels and cokernels. Let $f : M \to N$ be a morphism in M.

(1) Define the *image* of f to be
$$\operatorname{Im}(f) := \operatorname{Ker}(\operatorname{Coker}(f)).$$
(2) Define the *coimage* of f to be
$$\operatorname{Coim}(f) := \operatorname{Coker}(\operatorname{Ker}(f)).$$

The image is familiar, but the coimage is not. The next diagram should help. We start with a morphism $f : M \to N$ in M. The kernel and cokernel of f fit into this diagram:
$$K \xrightarrow{k} M \xrightarrow{f} N \xrightarrow{c} C.$$
Inserting $\alpha := \operatorname{Coker}(k) = \operatorname{Coim}(f)$ and $\beta := \operatorname{Ker}(c) = \operatorname{Im}(f)$ we get the following commutative diagram (solid arrows):

(2.3.6)
$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[dr, "0"'] & M \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, dashed, "\gamma"] & N \arrow[r, "c"] & C \\
& M' \arrow[r, dashed, "f'"'] & N' \arrow[u, "\beta"'] \arrow[ur, "0"'] &
\end{tikzcd}$$

Since $c \circ f = 0$ there is a unique morphism γ making the diagram commutative. Now $\beta \circ \gamma \circ k = f \circ k = 0$; and β is a monomorphism; so $\gamma \circ k = 0$. Hence there is a unique morphism $f' : M' \to N'$ making the diagram commutative. We conclude that $f : M \to N$ induces a morphism

(2.3.7) $$f' : \operatorname{Coim}(f) \to \operatorname{Im}(f).$$

Definition 2.3.8. An *abelian category* is an additive category M with these extra properties:

(i) All morphisms in M admit kernels and cokernels.
(ii) For any morphism $f : M \to N$ in M, the induced morphism f' in equation (2.3.7) is an isomorphism.

Here is a less precise but (maybe) easier to remember way to state property (ii). Because $M' = \operatorname{Coker}(\operatorname{Ker}(f))$ and $N' = \operatorname{Ker}(\operatorname{Coker}(f))$, we see that

(2.3.9) $$\operatorname{Coker}(\operatorname{Ker}(f)) = \operatorname{Ker}(\operatorname{Coker}(f)).$$

From now on we forget all about the coimage.

Exercise 2.3.10. For any ring A, prove that the category $\mathsf{Mod}\, A$ is abelian.

This includes the category $\mathsf{Ab} = \mathsf{Mod}\, \mathbb{Z}$, from which the name derives.

Definition 2.3.11. Let M be an abelian category, and let N be a full subcategory of M. We say that N is a *full abelian subcategory* of M if N is closed under finite direct sums, kernels and cokernels.

Exercise 2.3.12. In the situation of Definition 2.3.11, the category N is itself abelian.

Example 2.3.13. Let M_1 be the category of finitely generated abelian groups, and let M_0 be the category of finite abelian groups. Then M_1 is a full abelian subcategory of Ab, and M_0 is a full abelian subcategory of M_1.

Exercise 2.3.14. Let N be the full subcategory of Ab whose objects are the finitely generated free abelian groups. It is an additive subcategory of Ab (since it is closed under direct sums).

(1) Show that N is closed under kernels in Ab.
(2) Show that N is not closed under cokernels in Ab, so it is not a full abelian subcategory of Ab.
(3) Show that N has cokernels (not the same as those of Ab). Still, it fails to be an abelian category.

Exercise 2.3.15. The category Grp is not linear of course. Still, it does have a zero object (the trivial group). Show that Grp has kernels and cokernels, but condition (ii) of Definition 2.3.8 fails.

Exercise 2.3.16. Let Hilb be the category of Hilbert spaces over $\mathbb{C}$. The morphisms are the continuous $\mathbb{C}$-linear homomorphisms. Show that Hilb is a $\mathbb{C}$-linear additive category with kernels and cokernels, but it is not an abelian category.

Exercise 2.3.17. Let A be a ring. Show that A is *left noetherian* iff the category $\mathsf{Mod}_{\mathrm{f}}\, A$ of finitely generated left modules is a full abelian subcategory of $\mathsf{Mod}\, A$.

Example 2.3.18. Let $(X, \mathcal{A})$ be a ringed space; namely X is a topological space and $\mathcal{A}$ is a sheaf of rings on X (see [Har, Sections II.1-2]). We denote by $\mathsf{PMod}\,\mathcal{A}$ the category of presheaves of left $\mathcal{A}$-modules on X. This is an abelian category. Given a morphism $f : \mathcal{M} \to \mathcal{N}$ in $\mathsf{PMod}\,\mathcal{A}$, its kernel is the presheaf $\mathcal{K}$ defined by

$$\Gamma(U, \mathcal{K}) := \operatorname{Ker}\big(f : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})\big)$$

on every open set $U \subseteq X$. The cokernel is the presheaf $\mathcal{C}$ defined by

$$\Gamma(U, \mathcal{C}) := \operatorname{Coker}\big(f : \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{N})\big).$$

Now let $\mathsf{Mod}\,\mathcal{A}$ be the full subcategory of $\mathsf{PMod}\,\mathcal{A}$ consisting of sheaves. It is a full additive subcategory of $\mathsf{PMod}\,\mathcal{A}$, closed under kernels. We know that $\mathsf{Mod}\,\mathcal{A}$ is not closed under cokernels inside $\mathsf{PMod}\,\mathcal{A}$, and hence it is not a full abelian subcategory.

However $\mathsf{Mod}\,\mathcal{A}$ is itself an abelian category, but with different cokernels. Indeed, for a morphism $f : \mathcal{M} \to \mathcal{N}$ in $\mathsf{Mod}\,\mathcal{A}$, its cokernel $\operatorname{Coker}_{\mathsf{Mod}\,\mathcal{A}}(f)$ is the sheafification of the presheaf $\operatorname{Coker}_{\mathsf{PMod}\,\mathcal{A}}(f)$.

Here is a general result about abelian categories.

Theorem 2.3.19 (Freyd & Mitchell)**.** *Let M be a small abelian category. Then M is equivalent to a full abelian subcategory of $\mathsf{Mod}\, A$, for a suitable ring A.*

This means that most of the time we can pretend that $\mathsf{M} \subseteq \mathsf{Mod}\, A$. This is a helpful heuristic; although in practice it is not a very useful fact.

Proposition 2.3.20. (1) *Let M be an additive category. Then the opposite category M^{op} is also additive.*
(2) *Let M be an abelian category. Then the opposite category M^{op} is also abelian.*

Proof. (1) First note that

$$\mathrm{Hom}_{\mathsf{M}^{\mathrm{op}}}(M,N) = \mathrm{Hom}_{\mathsf{M}}(N,M),$$

so this is an abelian group. The bilinearity of the composition in M^{op} is clear, and the zero objects are the same. Existence of finite coproducts in M^{op} is because of existence of finite products in M; see Proposition 2.2.3(1).

(2) M^{op} has kernels and cokernels, since $\mathrm{Ker}_{\mathsf{M}^{\mathrm{op}}}(f) = \mathrm{Coker}_{\mathsf{M}}(f)$ and vice versa. Also the symmetric condition (ii) of Definition 2.3.8 holds. □

Proposition 2.3.21. *Let $f : M \to N$ be a morphism in an abelian category M.*

1. *f is a monomorphism iff* $\mathrm{Ker}(f) = 0$.
2. *f is an epimorphism iff* $\mathrm{Coker}(f) = 0$.
3. *f is an isomorphism iff it is both a monomorphism and an epimorphism.*

Proof. Exercise. □

2.4. **Additive Functors.**

Definition 2.4.1. Let M and N be $\mathbb{K}$-linear categories. A functor $F : \mathsf{M} \to \mathsf{N}$ is called a $\mathbb{K}$-*linear functor* if for every $M_0, M_1 \in \mathsf{M}$ the function

$$F : \mathrm{Hom}_{\mathsf{M}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{N}}(F(M_0), F(M_1))$$

is a $\mathbb{K}$-linear homomorphism.

A $\mathbb{Z}$-linear functor is also called an *additive functor*.

Additive functors commute with finite direct sums. More precisely:

Proposition 2.4.2. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between linear categories, let $\{M_i\}_{i \in I}$ be a finite collection of objects of M, and assume that the direct sum $(M, \{e_i\}_{i \in I})$ of the collection $\{M_i\}_{i \in I}$ exists in M. Then $\big(F(M), \{F(e_i)\}_{i \in I}\big)$ is a direct sum of the collection $\{F(M_i)\}_{i \in I}$ in N.*

Exercise 2.4.3. Prove Proposition 2.4.2. (Hint: use Proposition 2.2.3.)

Note that the proposition above also talks about finite products, because of Proposition 2.2.3.

Example 2.4.4. Let $A \to B$ be a ring homomorphism. The corresponding forgetful functor

$$F : \mathsf{Mod}\, B \to \mathsf{Mod}\, A$$

(also called restriction of scalars) is additive. The functor

$$G : \mathsf{Mod}\, A \to \mathsf{Mod}\, B$$

defined by $G(M) := B \otimes_A M$, called extension of scalars, is also additive.

Proposition 2.4.5. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between linear categories. Then*:

1. *For any $M \in \mathsf{M}$ the function*

   $$F : \mathrm{End}_{\mathsf{M}}(M) \to \mathrm{End}_{\mathsf{N}}\big(F(M)\big)$$

   is a ring homomorphism.

(2) *For any* $M_0, M_1 \in \mathsf{M}$ *the function*

$$F : \mathrm{Hom}_{\mathsf{M}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{N}}\big(F(M_0), F(M_1)\big)$$

is a homomorphism of left $\mathrm{End}_{\mathsf{M}}(M_1)$*-modules, and of right* $\mathrm{End}_{\mathsf{M}}(M_0)$*-modules.*

(3) *If* M *is a zero object of* M*, then* $F(M)$ *is a zero object of* N*.*

Proof. (1) By Definition 2.4.1 the function F respects addition. By the definition of a functor, it respects multiplication and units.

(2) Immediate from the definitions, like (1).

(3) Combine part (1) with Proposition 2.2.8. ∎

Definition 2.4.6. Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between abelian categories.

(1) F is called *left exact* if it commutes with kernels. Namely for any morphism $\phi : M_0 \to M_1$ in M, with kernel $k : K \to M_0$, the morphism $F(k) : F(K) \to F(M_0)$ is a kernel of $F(\phi) : F(M_0) \to F(M_1)$.
(2) F is called *right exact* if it commutes with cokernels. Namely for any morphism $\phi : M_0 \to M_1$ in M, with cokernel $c : M_1 \to C$, the morphism $F(c) : F(M_1) \to F(C)$ is a cokernel of $F(\phi) : F(M_0) \to F(M_1)$.
(3) F is called *exact* if it is both left exact and right exact.

This is illustrated in the following diagrams. Suppose $\phi : M_0 \to M_1$ is a morphism in M, with kernel K and cokernel C. Applying F to the diagram

$$K \xrightarrow{k} M_0 \xrightarrow{\phi} M_1 \xrightarrow{c} C$$

we get the solid arrows in

$$\begin{tikzcd}
F(K) \arrow[r, "F(k)"] \arrow[dr, dashed, "\psi"'] & F(M_0) \arrow[r, "F(\phi)"] & F(M_1) \arrow[r, "F(c)"] \arrow[d, dashed] & F(C) \\
& \mathrm{Ker}_{\mathsf{N}}(F(\phi)) \arrow[u, dashed] & \mathrm{Coker}_{\mathsf{N}}(F(\phi)) \arrow[ur, dashed, "\chi"'] &
\end{tikzcd}$$

Because N is abelian, we get the vertical dashed arrows: the kernel and cokernel of $F(\phi)$. The slanted dashed arrows exist and are unique because $F(\phi) \circ F(k) = 0$ and $F(c) \circ F(\phi) = 0$. Left exactness requires ψ to be an isomorphism, and right exactness requires χ to be an isomorphism.

Definition 2.4.7. Let M be an abelian category. An *exact sequence* in M is a diagram

$$\cdots \to M_0 \xrightarrow{\phi_0} M_1 \xrightarrow{\phi_1} M_2 \to \cdots$$

(finite or infinite on either side), such that for every index i for which ϕ_{i-1} and ϕ_i are both defined, the composition $\phi_i \circ \phi_{i-1}$ is zero, and the induced morphism $\mathrm{Im}(\phi_{i-1}) \to \mathrm{Ker}(\phi_i)$ is an isomorphism.

As usual, a *short exact sequence* is one of the form

$$0 \to M_0 \to M_1 \to M_2 \to 0. \tag{2.4.8}$$

Proposition 2.4.9. *Let* $F : \mathsf{M} \to \mathsf{N}$ *be an additive functor between abelian categories.*

(1) *The functor F is left exact iff for every short exact sequence* (2.4.8) *in* M*, the sequence*
$$0 \to F(M_0) \to F(M_1) \to F(M_2)$$
is exact in N*.*

(2) *The functor F is right exact iff for every short exact sequence* (2.4.8) *in* M*, the sequence*
$$F(M_0) \to F(M_1) \to F(M_2) \to 0$$
is exact in N*.*

Exercise 2.4.10. Prove Proposition 2.4.9. (Hint: $M_0 \cong \operatorname{Ker}(M_1 \to M_2)$ etc.)

Example 2.4.11. Let A be a commutative ring, and let M be a fixed A-module. Define functors $F, G : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$ and $H : (\mathsf{Mod}\, A)^{\mathrm{op}} \to \mathsf{Mod}\, A$ like this: $F(N) := M \otimes_A N$, $G(N) := \operatorname{Hom}_A(M, N)$ and $H(N) := \operatorname{Hom}_A(N, M)$. Then F is right exact, and G and H are left exact.

Proposition 2.4.12. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between abelian categories. If F is an equivalence then it is exact.*

Proof. We will prove that F respects kernels; the proof for cokernels is similar. Take a morphism $\phi : M_0 \to M_1$ in M, with kernel K. We have this diagram (solid arrows):

$$\begin{tikzcd}
M \arrow[d, dashed, "\psi"'] \arrow[dr, dashed, "\theta"] & & \\
K \arrow[r, "k"] & M_0 \arrow[r, "\phi"] & M_1
\end{tikzcd}$$

Applying F we obtain this diagram (solid arrows):

$$\begin{tikzcd}
N = F(M) \arrow[d, dashed, "F(\psi)"'] \arrow[dr, dashed, "\bar{\theta}"] & & \\
F(K) \arrow[r, "F(k)"] & F(M_0) \arrow[r, "F(\phi)"] & F(M_1)
\end{tikzcd}$$

in N. Suppose $\bar{\theta} : N \to F(M_0)$ is a morphism in N s.t. $F(\phi) \circ \bar{\theta} = 0$. Since F is essentially surjective on objects, there is some $M \in \mathsf{M}$ with an isomorphism $\alpha : F(M) \xrightarrow{\simeq} N$. After replacing N with $F(M)$ and $\bar{\theta}$ with $\bar{\theta} \circ \alpha$, we can assume that $N = F(M)$.

Now since F is fully faithful, there is a unique $\theta : M \to M_0$ s.t. $F(\theta) = \bar{\theta}$; and $\phi \circ \theta = 0$. So there is a unique $\psi : M \to K$ s.t. $\theta = k \circ \psi$. It follows that $F(\psi) : F(M) \to F(K)$ is the unique morphism s.t. $\bar{\theta} = F(k) \circ F(\psi)$. $\square$

Here is a result that could afford another proof of the previous proposition.

Proposition 2.4.13. *Let $F : \mathsf{M} \to \mathsf{N}$ be an additive functor between linear categories. Assume F is an equivalence, with quasi-inverse G. Then $G : \mathsf{N} \to \mathsf{M}$ is an additive functor.*

Exercise 2.4.14. Prove Proposition 2.4.13.

2.5. **Projective Objects.** In this subsection M is an abelian category.

A *splitting* of an epimorphism $\psi : M \to M''$ in M is a morphism $\alpha : M'' \to M$ s.t. $\psi \circ \alpha = 1_{M''}$. A splitting of a monomorphism $\phi : M' \to M$ is a morphism $\beta : M \to M'$ s.t. $\beta \circ \phi = 1_{M'}$. A splitting of a short exact sequence

$$0 \to M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \to 0 \tag{2.5.1}$$

is a splitting of the epimorphism ψ, or equivalently a splitting of the monomorphism ϕ. The short exact sequence is said to be *split* if it has some splitting.

Exercise 2.5.2. Show how to get from a splitting of ϕ to a splitting of ψ, and vice versa. Show how any of those gives rise to an isomorphism $M \cong M' \oplus M''$.

Definition 2.5.3. An object $P \in \mathsf{M}$ is called a *projective object* if for any morphism $\gamma : P \to N$ and any *epimorphism* $\psi : M \twoheadrightarrow N$, there exists a morphism $\tilde{\gamma} : P \to N$ such that $\psi \circ \tilde{\gamma} = \gamma$.

This is described in the following commutative diagram in M :

$$\begin{tikzcd} & P \arrow[dl, dashed, "\tilde{\gamma}"'] \arrow[d, "\gamma"] \\ M \arrow[r, two heads, "\psi"'] & N \end{tikzcd}$$

Proposition 2.5.4. *The following conditions are equivalent for $P \in \mathsf{M}$:*

(i) *P is projective.*

(ii) *The additive functor*

$$\mathrm{Hom}_{\mathsf{M}}(P, -) : \mathsf{M} \to \mathsf{Ab}$$

is exact.

(iii) *Any short exact sequence* (2.5.1) *with $M'' = P$ is split.*

Proof. Exercise. □

Definition 2.5.5. We say M *has enough projectives* if every $M \in \mathsf{M}$ admits an epimorphism $P \to M$ from a projective object P.

Exercise 2.5.6. Let A be a ring.

(1) Prove that an A-module P is projective iff it is a direct summand of a free module; i.e. $P \oplus P' \cong Q$ for some module P' and free module Q.

(2) Prove that the category $\mathsf{Mod}\, A$ has enough projectives.

Exercise 2.5.7. Let M be the category of finite abelian groups. Prove that the only projective object in M is 0. So M does not have enough projectives. (Hint: use Proposition 2.5.4.)

Example 2.5.8. Consider the scheme $X := \mathbf{P}^1_{\mathbb{K}}$, the projective line over a field $\mathbb{K}$. (If the reader prefers, he/she can assume $\mathbb{K}$ is algebraically closed, so X is a classical algebraic variety.) The structure sheaf (sheaf of functions) is $\mathcal{O}_X$. The category $\mathsf{Coh}\, \mathcal{O}_X$ of coherent $\mathcal{O}_X$-modules is abelian (it is a full abelian subcategory of $\mathsf{Mod}\, \mathcal{O}_X$, cf. Example 2.3.18). One can show that the only projective object of $\mathsf{Coh}\, \mathcal{O}_X$ is 0, but this is quite involved.

Let us only indicate why $\mathcal{O}_X$ is not projective. Denote by t_0, t_1 the homogeneous coordinates of X. These belong to $\Gamma(X, \mathcal{O}_X(1))$, so each determines a homomorphism of sheaves $t_j : \mathcal{O}_X(i) \to \mathcal{O}_X(i+1)$. We get a sequence

$$0 \to \mathcal{O}_X(-2) \xrightarrow{\left[\begin{smallmatrix} t_0 & t_1 \end{smallmatrix}\right]} \mathcal{O}_X(-1)^2 \xrightarrow{\left[\begin{smallmatrix} -t_1 \\ t_0 \end{smallmatrix}\right]} \mathcal{O}_X \to 0$$

in $\mathsf{Coh}\,\mathcal{O}_X$, which is known to be exact. Because $\Gamma(X, \mathcal{O}_X) = \mathbb{K}$, and $\Gamma(X, \mathcal{O}_X(-1)) = 0$, this sequence is not split.

2.6. **Injective Objects.** In this subsection M is an abelian category.

Definition 2.6.1. An object $I \in \mathsf{M}$ is called an *injective object* if for any morphism $\gamma : M \to I$ and any *monomorphism* $\psi : M \rightarrowtail N$, there exists a morphism $\tilde{\gamma} : N \to I$ such that $\tilde{\gamma} \circ \psi = \gamma$.

This is depicted in the following commutative diagram in M :

$$\begin{tikzcd}
I & \\
M \arrow[u, "\gamma"] \arrow[r, tail, "\psi"'] & N \arrow[ul, dashed, "\tilde{\gamma}"']
\end{tikzcd}$$

Proposition 2.6.2. *The following conditions are equivalent for $I \in \mathsf{M}$:*

(i) *I is injective.*

(ii) *The additive functor*

$$\mathrm{Hom}_{\mathsf{M}}(-, I) : \mathsf{M}^{\mathrm{op}} \to \mathsf{Ab}$$

is exact.

(iii) *Any short exact sequence* (2.5.1) *with $M' = I$ is split.*

Proof. Exercise. □

Example 2.6.3. Let A be a ring. Unlike projectives, the structure of injective objects in $\mathsf{Mod}\,A$ is very complicated, and not much is known (except that they exist). However if A is a commutative noetherian ring then we know this: every injective module I is a direct sum of indecomposable injective modules. And these indecomposables are parametrized by $\mathrm{Spec}\,A$, the set of prime ideals of A. These facts are due to Matlis; see [RD, pages 120-122] for details.

Definition 2.6.4. We say M *has enough injectives* if every $M \in \mathsf{M}$ admits a monomorphism $M \to I$ to an injective object I.

Here are a few results about injective objects. Recall that modules over a ring are always left modules by default.

Proposition 2.6.5. *Let $f : A \to B$ be a ring homomorphism, and let I be an injective A-module. Then $J := \mathrm{Hom}_A(B, I)$ is an injective B-module.*

Proof. Note that B is a left A-module via f, and a right B-module. This makes J into a left B-module. In a formula: for $\phi \in J$ and $b, b' \in B$ we have $(b \cdot \phi)(b') = \phi(b' \cdot b)$.

Now given any $N \in \mathsf{Mod}\,B$ there is an isomorphism

$$\mathrm{Hom}_B(N, J) = \mathrm{Hom}_B(N, \mathrm{Hom}_A(B, I)) \cong \mathrm{Hom}_A(N, I). \tag{2.6.6}$$

This is a natural isomorphism (of functors in N). So the functor $\operatorname{Hom}_B(-, J)$ is exact, and hence J is injective. □

Theorem 2.6.7 (Baer Criterion). *Let A be a ring and I an A-module. Assume that every A-module homomorphism $\mathfrak{a} \to I$ from a left ideal $\mathfrak{a} \subseteq A$ extends to a homomorphism $A \to I$. Then the module I is injective.*

Proof. Consider an A-module M, a submodule $N \subseteq M$, and a homomorphism $\gamma : N \to I$. We have to prove that γ extends to a homomorphism $M \to I$. Look at the pairs (N', γ') consisting of a submodule $N' \subseteq M$ that contains N, and a homomorphism $\gamma' : N' \to I$ that extends γ. The set of all such pairs is ordered by inclusion, and it satisfies the conditions of Zorn's Lemma. Therefore there exists a maximal pair (N', γ'). We claim that $N' = M$.

Otherwise, there is an element $m \in M$ that does not belong to N'. Define $N'' := N' + A \cdot m$, so $N' \subsetneq N'' \subseteq M$. Let

$$\mathfrak{a} := \{a \in A \mid a \cdot m \in N'\},$$

which is a left ideal of A. There is a short exact sequence

$$0 \to \mathfrak{a} \xrightarrow{\alpha} N' \oplus A \to N'' \to 0$$

of A-modules, where $\alpha(a) := (a \cdot m, -a)$. Let $\phi : \mathfrak{a} \to I$ be the homomorphism $\phi(a) := \gamma'(a \cdot m)$. By assumption, it extends to a homomorphism $\tilde{\phi} : A \to I$. We get a homomorphism

$$\gamma' + \tilde{\phi} : N' \oplus A \to I$$

that vanishes on the image of α. Thus there is an induced homomorphism $\gamma'' : N'' \to I$. This contradicts the maximality of (N', γ'). □

Lemma 2.6.8. *The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is injective.*

Proof. By the Baer criterion, it is enough to consider a homomorphism $\gamma : \mathfrak{a} \to \mathbb{Q}/\mathbb{Z}$ for an ideal $\mathfrak{a} = n \cdot \mathbb{Z} \subseteq \mathbb{Z}$. We may assume that $n \neq 0$. Say $\gamma(n) = r + \mathbb{Z}$ with $r \in \mathbb{Q}$. Then we can extend γ to $\tilde{\gamma} : \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ with $\tilde{\gamma}(1) := r/n + \mathbb{Z}$. □

Lemma 2.6.9. *Let $\{I_x\}_{x \in X}$ be a collection of injective objects of M. If the product $\prod_{x \in X} I_x$ exists in M, then it is an injective object.*

Proof. Exercise. □

Theorem 2.6.10. *Let A be any ring. The category $\mathsf{Mod}\, A$ has enough injectives.*

Proof. Step 1. Here $A = \mathbb{Z}$. Take any nonzero $\mathbb{Z}$-module M and any nonzero $m \in M$. Consider the cyclic submodule $M' := \mathbb{Z} \cdot m \subseteq M$. There is a homomorphism $\gamma' : M' \to \mathbb{Q}/\mathbb{Z}$ s.t. $\gamma'(m) \neq 0$. Indeed, if $M' \cong \mathbb{Z}$, then we take any $r \in \mathbb{Q} - \mathbb{Z}$; and if $M' \cong \mathbb{Z}/(n)$ for some $n > 0$, then we take $r := 1/n$. In either case, we define $\gamma'(m) := r + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$. Since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, γ' extends to a homomorphism $\gamma : M \to \mathbb{Q}/\mathbb{Z}$. By construction we have $\gamma(m) \neq 0$.

Step 2. Now A is any ring, M is any nonzero A-module, and $m \in M$ a nonzero element. Define the A-module $I := \operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$, which, according to Lemma 2.6.8 and Proposition 2.6.5, is an injective A-module. Let $\gamma : M \to \mathbb{Q}/\mathbb{Z}$ be a $\mathbb{Z}$-linear homomorphism such that $\gamma(m) \neq 0$. Such γ exists by step 1. Let $\theta : I \to \mathbb{Q}/\mathbb{Z}$ be the $\mathbb{Z}$-linear homomorphism that sends an element $\chi \in I$ to $\chi(1) \in \mathbb{Q}/\mathbb{Z}$. The adjunction formula (2.6.6) gives an A-module homomorphism $\psi : M \to I$ s.t. $\theta \circ \psi = \gamma$. We note that $(\theta \circ \psi)(m) = \gamma(m) \neq 0$, and hence $\psi(m) \neq 0$.

Step 3. Here A and M are arbitrary. Let I be as in step 2. For any nonzero $m \in M$ there is an A-linear homomorphism $\psi_m : M \to I$ such that $\psi_m(m) \neq 0$. For $m = 0$ let $\psi_0 : M \to I$ be an arbitrary homomorphism (e.g. $\psi_0 = 0$). Define the A-module $J := \prod_{m \in M} I$. There is a homomorphism $\psi := \prod_{m \in M} \psi_m : M \to J$, and it is easy to check that ψ is a monomorphism. By Lemma 2.6.9, J is an injective A-module. □

Exercise 2.6.11. At the price of getting a bigger injective module, we can make the construction of injective resolutions functorial. Let $I := \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ as above. Given an A-module M, consider the set

$$X(M) := \mathrm{Hom}_A(M, I) \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}).$$

Let $J(M) := \prod_{\psi \in X(M)} I$. There is a "tautological" homomorphism $\phi_M : M \to J(M)$. Show that ϕ_M is a monomorphism, $J : M \mapsto J(M)$ is a functor, and $\phi : \mathrm{Id} \to J$ is a natural transformation.

Is the functor $J : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$ additive?

Example 2.6.12. Let N be the category of torsion abelian groups, and M the category of finite abelian groups. Then $\mathsf{N} \subseteq \mathsf{Ab}$ and $\mathsf{M} \subseteq \mathsf{N}$ are full abelian subcategories. M has no projectives nor injectives except 0 (see Exercise 2.5.7 regarding projectives). The only projective in N is 0. However, it can be shown that N has enough injectives; see [Har, Lemma III.3.2] or [Ye1, Proposition 4.6].

Proposition 2.6.13. *If A is a left noetherian ring, then any direct sum of injective A-modules is an injective module.*

Exercise 2.6.14. Prove Proposition 2.6.13. (Hint: use the Baer criterion.)

Remark 2.6.15. Actually, the converse of Proposition 2.6.13 is also true: if every direct sum of injective A-modules is injective, then A is left noetherian. But experience tells us that this fact is not very important...

Exercise 2.6.16. Here we study injectives in the category $\mathsf{Ab} = \mathsf{Mod}\, \mathbb{Z}$. By Lemma 2.6.8, the module $I := \mathbb{Q}/\mathbb{Z}$ is injective. For a (positive) prime number p, we denote by $\widehat{\mathbb{Z}}_p$ the ring of p-adic integers, and by $\widehat{\mathbb{Q}}_p$ its field of fractions (namely the p-adic completions of $\mathbb{Z}$ and $\mathbb{Q}$ respectively). Define the abelian group $I_p := \widehat{\mathbb{Q}}_p/\widehat{\mathbb{Z}}_p$.

(1) Show that I_p is an injective object of Ab.
(2) Show that I_p is indecomposable (i.e. it is not the direct sum of two nonzero objects).
(3) Show that $I \cong \bigoplus_p I_p$.
(4) The theory tells us that there is another indecomposable injective object in Ab, besides the I_p. Try to identify it.

Remark 2.6.17. Let $\mathbb{K}$ be a field and $A := \mathbb{K}[t]$, the polynomial ring in one variable. As we very well know, the categories $\mathsf{Mod}\, A$ and $\mathsf{Mod}\, \mathbb{Z}$ share many properties. Let $A^* := \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$, which is of course an injective A-module (because $\mathbb{K}$ is an injective $\mathbb{K}$-module). The structure of A^*, as a direct sum of indecomposable injectives, was used to cook up a counterexample in [Ye5, Section 6].

The abelian category $\mathsf{Mod}\, \mathcal{A}$ associated to a ringed space $(X, \mathcal{A})$ was introduced in Example 2.3.18.

Proposition 2.6.18. *Let $(X, \mathcal{A})$ be a ringed space. The category $\mathsf{Mod}\,\mathcal{A}$ has enough injectives.*

Proof. Let $\mathcal{M}$ be an $\mathcal{A}$-module. Take a point $x \in X$. The stalk $\mathcal{M}_x$ is a module over the ring $\mathcal{A}_x$, and by Theorem 2.6.10 we can find an embedding $\phi_x : \mathcal{M}_x \to I_x$ into an injective $\mathcal{A}_x$-module. Let $g_x : \{x\} \to X$ be the inclusion, which we may view as a map of ringed spaces from $(\{x\}, \mathcal{A}_x)$ to $(X, \mathcal{A})$. Define $\mathcal{I}_x := g_{x*}(I_x)$, which is an $\mathcal{A}$-module (in fact it is a constant sheaf supported on the closed set $\overline{\{x\}} \subseteq X$). The adjunction formula gives rise to a sheaf homomorphism $\psi_x : \mathcal{M} \to \mathcal{I}_x$. Since the functor $g_x^* : \mathsf{Mod}\,\mathcal{A} \to \mathsf{Mod}\,\mathcal{A}_x$ is exact, the adjunction formula shows that $\mathcal{I}_x$ is an injective object of $\mathsf{Mod}\,\mathcal{A}$.

Finally let $\mathcal{J} := \prod_{x \in X} \mathcal{I}_x$. This is an injective $\mathcal{A}$-module. There is a homomorphism $\psi := \prod_{x \in X} \psi_x : \mathcal{M} \to \mathcal{J}$ in $\mathsf{Mod}\,\mathcal{A}$. This is a monomorphism, since for every point x, letting $\mathcal{J}_x$ be the stalk of the sheaf $\mathcal{J}$ at x, the composition $\mathcal{M}_x \xrightarrow{\psi_x} \mathcal{J}_x \xrightarrow{p_x} \mathcal{I}_x$ is the embedding $\phi_x : \mathcal{M}_x \to I_x$. $\square$

3. Differential Graded Algebra

In this section we fix a nonzero commutative base ring $\mathbb{K}$. (It seems more relaxing to have a base ring $\mathbb{K}$ around, rather than working with the universal base $\mathbb{K} = \mathbb{Z}$.) Throughout, "DG" stands for "differential graded".

3.1. **Graded Algebra.**
Before entering the DG world, it is good to understand the graded world.

A *graded* $\mathbb{K}$*-module* is a $\mathbb{K}$-module M equipped with a decomposition $M = \bigoplus_{i\in\mathbb{Z}} M^i$ into $\mathbb{K}$-submodules. The $\mathbb{K}$-module M^i is called the degree i component of M. The elements of M^i are called homogeneous elements of degree i.

Suppose M and N are graded $\mathbb{K}$-modules. For any integer i let

$$(M \otimes_{\mathbb{K}} N)^i := \bigoplus_{j\in\mathbb{Z}} (M^j \otimes_{\mathbb{K}} N^{i-j}).$$

Then

$$M \otimes_{\mathbb{K}} N = \bigoplus_{i\in\mathbb{Z}} (M \otimes_{\mathbb{K}} N)^i, \tag{3.1.1}$$

is a graded $\mathbb{K}$-module.

A $\mathbb{K}$-linear homomorphism $\phi : M \to N$ is said to be of degree i if $\phi(M^j) \subseteq N^{j+i}$ for all j. We denote by $\mathrm{Hom}_{\mathbb{K}}(M,N)^i$ the $\mathbb{K}$-module of degree i homomorphisms $M \to N$. In other words

$$\mathrm{Hom}_{\mathbb{K}}(M,N)^i = \prod_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{K}}(M^j, N^{j+i}).$$

Then

$$\mathrm{Hom}_{\mathbb{K}}(M,N) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{K}}(M,N)^i \tag{3.1.2}$$

is a graded $\mathbb{K}$-module. A degree 0 homomorphism $\phi : M \to N$ is sometimes called a *strict homomorphism of graded* $\mathbb{K}$*-modules*.

If M_0, M_1, M_2 are graded $\mathbb{K}$-modules, and $\phi_k : M_{k-1} \to M_k$ are $\mathbb{K}$-linear homomorphisms of degrees i_k, then $\phi_2 \circ \phi_1 : M_0 \to M_2$ is a $\mathbb{K}$-linear homomorphism of degree $i_1 + i_2$. The identity automorphism $1_M : M \to M$ has degree 0.

A *graded ring* is a ring A, equipped with decomposition as an abelian group $A = \bigoplus_{i\in\mathbb{Z}} A^i$, such that the unit element $1 \in A^0$, and $A^i \cdot A^j \subseteq A^{i+j}$. A *central graded* $\mathbb{K}$*-ring* is a graded ring A, together with a ring homomorphism $\mathbb{K} \to A^0$, such that the image of $\mathbb{K}$ is central in A (i.e. $\lambda \cdot a = a \cdot \lambda$ for all $\lambda \in \mathbb{K}$ and $a \in A$). A *homomorphism of central graded* $\mathbb{K}$*-rings* $f : A \to B$ is a ring homomorphism that respects the gradings and the homomorphisms from $\mathbb{K}$. As always for ring homomorphisms, f must preserve units, i.e. $f(1_A) = 1_B$.

Example 3.1.3. Let M be a graded $\mathbb{K}$-module. Then the graded module

$$\mathrm{End}_{\mathbb{K}}(M) := \mathrm{Hom}_{\mathbb{K}}(M,M),$$

with the operation of composition, is a central graded $\mathbb{K}$-ring.

The manipulation of graded modules and homomorphisms involves the *Koszul sign rule*, that says this (in deliberately vague terms): suppose a and b are homogeneous elements of degrees i and j. Let $f(\dots, a, b, \dots)$ be a multilinear graded expression, such that the expression $f(\dots,b,a,\dots)$ is also defined (i.e. it makes sense to transpose a and b), and for the ungraded expression $f^{\natural}$ there is equality

$$f^{\natural}(\dots,b,a,\dots) = f^{\natural}(\dots,a,b,\dots).$$

Then

$$f(\dots,b,a,\dots) = (-1)^{ij} \cdot f(\dots,a,b,\dots).$$

The next examples ought to clarify how the Koszul sign rule is used.

Example 3.1.4. Suppose that for $k = 0, 1$ we are given graded $\mathbb{K}$-module homomorphisms $\phi_k : M_k \to N_k$ of degrees i_k. Then the homomorphism

$$\phi_0 \otimes \phi_1 \in \operatorname{Hom}_{\mathbb{K}}(M_0 \otimes_{\mathbb{K}} M_1, N_0 \otimes_{\mathbb{K}} N_1)^{i_0+i_1}$$

acts on a tensor $m_0 \otimes m_1 \in M_0 \otimes_{\mathbb{K}} M_1$, with $m_k \in M_k^{j_k}$, like this:

$$(\phi_0 \otimes \phi_1)(m_0 \otimes m_1) := (-1)^{i_1 \cdot j_0} \cdot \phi_0(m_0) \otimes \phi_1(m_1) \in N_0 \otimes_{\mathbb{K}} N_1.$$

The sign is because ϕ_1 and m_0 were transposed.

Example 3.1.5. Suppose we are given graded $\mathbb{K}$-module homomorphisms $\phi_0 : N_0 \to M_0$ and $\phi_1 : M_1 \to N_1$ of degrees i_0 and i_1. Then the homomorphism

$$\operatorname{Hom}(\phi_0,\phi_1) \in \operatorname{Hom}_{\mathbb{K}}\big(\operatorname{Hom}_{\mathbb{K}}(M_0,M_1), \operatorname{Hom}_{\mathbb{K}}(N_0,N_1)\big)^{i_0+i_1}$$

acts on $\gamma \in \operatorname{Hom}_{\mathbb{K}}(M_0,M_1)^j$ as follows: for an element $n_0 \in N_0^k$ we have

$$\operatorname{Hom}(\phi_0,\phi_1)(\gamma)(n_0) := (-1)^{i_0 \cdot (i_1+j)}(\phi_1 \circ \gamma \circ \phi_0)(n_0) \in N_1^{k+i_0+i_1+j}.$$

The sign is because ϕ_0 jumped across ϕ_1 and γ.

Example 3.1.6. Let A and B be central graded $\mathbb{K}$-rings. Then $A \otimes_{\mathbb{K}} B$ is a central graded $\mathbb{K}$-ring, with multiplication

$$(a_0 \otimes b_0) \cdot (a_1 \otimes b_1) := (-1)^{i_1 \cdot j_0} \cdot (a_0 \cdot a_1) \otimes (b_0 \cdot b_1)$$

for elements $a_k \in A^{i_k}$ and $b_k \in B^{j_k}$.

Example 3.1.7. The Koszul sign rule influences the meaning of commutativity for graded rings. A graded ring A is called *weakly commutative* if $b \cdot a = (-1)^{ij} \cdot a \cdot b$ for all $a \in A^i$ and $b \in A^j$.

Remark 3.1.8. Following [Ye11], there are two variants of commutativity for DG rings. Besides the weakly commutative graded rings from the example above, there are also *strongly commutative* graded rings; the extra condition is that $a^2 = 0$ if the degree i is odd. If $\mathbb{K}$ has characteristic 0 (i.e. it contains $\mathbb{Q}$), then the two variants of commutativity coincide.

Exercise 3.1.9. Let A be a central graded $\mathbb{K}$-ring. Elements $a \in A^i$ and $b \in A^j$ are said to graded-commute if $b \cdot a = (-1)^{ij} \cdot a \cdot b$. A homogeneous element $a \in A$ is called graded-central if it graded-commutes with all other homogeneous elements. The *graded center* of A is the $\mathbb{K}$-linear span of all graded-central homogeneous elements in A. Show that the graded center of A is a graded subring of A.

Let A be a central graded $\mathbb{K}$-ring. A *graded left A-module* is a left A-module M, equipped with a $\mathbb{K}$-module decomposition $M = \bigoplus_{i \in \mathbb{Z}} M^i$, such that $A^i \cdot M^j \subseteq M^{i+j}$. We can also talk about graded right A-modules, and graded bimodules. But our default option is that modules are left modules.

If M is a graded $\mathbb{K}$-module, A is a central graded $\mathbb{K}$-ring, and $f : A \to \mathrm{End}_{\mathbb{K}}(M)$ is a graded $\mathbb{K}$-ring homomorphism, then M becomes a graded A-module, with action $a \cdot m := f(a)(m)$. Any graded A-module structure on M arises this way.

Lemma 3.1.10. *Let A be a central graded $\mathbb{K}$-ring, let M be a right graded A-module, and let N be a left graded A-module. Then the $\mathbb{K}$-module $M \otimes_A N$ has a direct sum decomposition*

$$M \otimes_A N = \bigoplus_{i \in \mathbb{Z}} (M \otimes_A N)^i,$$

where $(M \otimes_A N)^i$ is the $\mathbb{K}$-linear span of the tensors $m \otimes n$ with $m \in M^j$ and $n \in N^{i-j}$.

Proof. There is a canonical surjection of $\mathbb{K}$-modules

$$M \otimes_{\mathbb{K}} N \to M \otimes_A N.$$

Its kernel is the $\mathbb{K}$-submodule $L \subseteq M \otimes_{\mathbb{K}} N$ generated by the elements

$$(m \cdot a) \otimes n - m \otimes (a \cdot n),$$

for $m \in M^j$, $n \in N^k$ and $a \in A^l$. So L is a graded submodule of $M \otimes_{\mathbb{K}} N$, and therefore so is the quotient. Finally, by formula (3.1.1) the i-th homogeneous component of $M \otimes_A N$ is precisely $(M \otimes_A N)^i$. □

Definition 3.1.11. Let A be a central graded $\mathbb{K}$-ring, and let M, N be graded A-modules. For any $i \in \mathbb{Z}$ define $\mathrm{Hom}_A(M, N)^i$ to be the subset of $\mathrm{Hom}_{\mathbb{K}}(M, N)^i$ consisting of the homomorphisms $\phi : M \to N$ such that

$$\phi(a \cdot m) = (-1)^{ik} \cdot a \cdot \phi(m)$$

for all $a \in A^k$. Next let

$$\mathrm{Hom}_A(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_A(M, N)^i.$$

Suppose C is a $\mathbb{K}$-linear category (Definition 2.1.1). Since the composition of morphisms is $\mathbb{K}$-bilinear, for any triple of objects $M_0, M_1, M_2 \in \mathsf{C}$, composition can be expressed as a $\mathbb{K}$-linear homomorphism

$$\begin{aligned} &\mathrm{Hom}_{\mathsf{C}}(M_1, M_2) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathsf{C}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{C}}(M_0, M_2) \\ &\phi_2 \otimes \phi_1 \mapsto \phi_2 \circ \phi_1. \end{aligned}$$

We refer to it as the composition homomorphism. It will be used in the following definition.

Definition 3.1.12. A *graded $\mathbb{K}$-linear category* is a $\mathbb{K}$-linear category C, endowed with a grading on each of the $\mathbb{K}$-modules $\mathrm{Hom}_{\mathsf{C}}(M_0, M_1)$. The conditions are these:

(a) For any object M, the identity automorphism 1_M has degree 0.
(b) For any triple of objects $M_0, M_1, M_2 \in \mathsf{C}$, the composition homomorphism

$$\mathrm{Hom}_{\mathsf{C}}(M_1, M_2) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathsf{C}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{C}}(M_0, M_2)$$

is a strict homomorphism of graded $\mathbb{K}$-modules.

In item (b) we use the graded module structure on a tensor product from equation (3.1.1). A morphism $\phi \in \mathrm{Hom}_{\mathsf{C}}(M_0, M_1)^i$ is called a morphism of degree i.

Definition 3.1.13. Let C be a graded $\mathbb{K}$-linear category. We define C^0 to be the subcategory of C on all objects, but the morphisms are only the degree 0 morphisms.

Example 3.1.14. Let A be a central graded $\mathbb{K}$-ring. Define $\mathbf{G}(A)$ to be the category whose objects are the graded A-modules. For $M, N \in \mathbf{G}(A)$, the set of morphisms is the graded module

$$\operatorname{Hom}_{\mathbf{G}(A)}(M, N) := \operatorname{Hom}_A(M, N)$$

from Definition 3.1.11. Then $\mathbf{G}(A)$ is a graded $\mathbb{K}$-linear category. The morphisms in the subcategory $\mathbf{G}^0(A)$ are the strict homomorphisms of graded A-modules.

Definition 3.1.15. Let C and D be graded $\mathbb{K}$-linear categories. A functor $F : \mathsf{C} \to \mathsf{D}$ is called a *graded $\mathbb{K}$-linear functor* if it satisfies this condition:

▷ For any pair of objects $M_0, M_1 \in \mathsf{C}$, the function

$$F : \operatorname{Hom}_{\mathsf{C}}(M_0, M_1) \to \operatorname{Hom}_{\mathsf{D}}\big(F(M_0), F(M_1)\big)$$

is a strict homomorphism of graded $\mathbb{K}$-modules.

Example 3.1.16. Let A be a central graded $\mathbb{K}$-ring. We can view A as a category A with a single object, and it is a $\mathbb{K}$-linear graded category. If $f : A \to B$ is a homomorphism of central graded $\mathbb{K}$-rings, then passing to single-object categories we get a $\mathbb{K}$-linear graded functor $F : \mathsf{A} \to \mathsf{B}$.

Recall that "morphism of functors" is synonymous with "natural transformation".

Definition 3.1.17. Let $F, G : \mathsf{C} \to \mathsf{D}$ be $\mathbb{K}$-linear graded functors between $\mathbb{K}$-linear graded categories, and let $i \in \mathbb{Z}$. A *degree i morphism of graded functors* $\eta : F \to G$ is a collection $\eta = \{\eta_M\}_{M \in \mathsf{C}}$ of morphisms

$$\eta_M \in \operatorname{Hom}_{\mathsf{D}}\big(F(M), G(M)\big)^i,$$

such that for any morphism $\phi \in \operatorname{Hom}_{\mathsf{C}}(M_0, M_1)^j$, there is equality

$$G(\phi) \circ \eta_{M_0} = (-1)^{ij} \cdot \eta_{M_1} \circ F(\phi)$$

inside

$$\operatorname{Hom}_{\mathsf{D}}\big(F(M_0), G(M_1)\big)^{i+j}.$$

Definition 3.1.18. Let M be a $\mathbb{K}$-linear category. A *graded object in* M is a collection $\{M^i\}_{i \in \mathbb{Z}}$ of objects $M^i \in \mathsf{M}$.

Because we did not assume that M has countable direct sums, the graded objects are "external" to M; cf. Example 3.1.21.

Suppose $M = \{M^i\}_{i \in \mathbb{Z}}$ and $N = \{N^i\}_{i \in \mathbb{Z}}$ are graded objects in M. For any integer i we define the $\mathbb{K}$-module

$$\operatorname{Hom}_{\mathsf{M}}(M, N)^i := \prod_{j \in \mathbb{Z}} \operatorname{Hom}_{\mathsf{M}}(M^j, N^{j+i}).$$

We get a graded $\mathbb{K}$-module

$$\operatorname{Hom}_{\mathsf{M}}(M, N) := \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathsf{M}}(M, N)^i. \tag{3.1.19}$$

Definition 3.1.20. Let M be a $\mathbb{K}$-linear category. The *category of graded objects in* M is the $\mathbb{K}$-linear graded category $\mathbf{G}(\mathsf{M})$, whose objects are the graded objects in M, and the morphism sets are the graded modules

$$\operatorname{Hom}_{\mathbf{G}(\mathsf{M})}(M, N) := \operatorname{Hom}_{\mathsf{M}}(M, N)$$

from equation (3.1.19). The composition operation is the obvious one.

Example 3.1.21. Suppose $\mathsf{M} = \mathsf{Mod}\, A$, the category of modules over a central $\mathbb{K}$-ring A. For any $M = \{M^i\}_{i \in \mathbb{Z}} \in \mathbf{G}(\mathsf{M})$ let $F(M) := \bigoplus_{i \in \mathbb{Z}} M^i$. Then $F(M)$ is a graded A-module, as discussed earlier, so $F(M)$ is an object of the category $\mathbf{G}(A)$ from Example 3.1.14. It is not hard to verify that

$$F : \mathbf{G}(\mathsf{M}) \to \mathbf{G}(A)$$

is an equivalence of $\mathbb{K}$-linear graded categories.

In the next definition we combine graded rings and linear categories, to concoct a new hybrid.

Definition 3.1.22. Let M be a $\mathbb{K}$-linear category, and let A be a central graded $\mathbb{K}$-ring. A *graded A-module in* M is an object $M \in \mathbf{G}(\mathsf{M})$, together with graded $\mathbb{K}$-ring homomorphism $f : A \to \operatorname{End}_{\mathsf{M}}(M)$.

What the definition says is that any element $a \in A^i$ gives rise to a degree i endomorphism $f(a)$ of the graded object $M = \{M^i\}_{i \in \mathbb{Z}}$. In turn, this means that for every j, $f(a) : M^j \to M^{j+i}$ is a morphism in M. The operation f satisfies $f(1_A) = 1_M$ and $f(a_1 \cdot a_2) = f(a_1) \circ f(a_2)$

Example 3.1.23. If $A = \mathbb{K}$, then $\mathbf{G}(A, \mathsf{M}) = \mathbf{G}(\mathsf{M})$; and if $\mathsf{M} = \mathsf{Mod}\, \mathbb{K}$, then $\mathbf{G}(A, \mathsf{M}) = \mathbf{G}(A)$.

The next definition is a variant of Definition 3.1.11.

Definition 3.1.24. Let M be a $\mathbb{K}$-linear category, and let A be a central graded $\mathbb{K}$-ring. For $M, N \in \mathbf{G}(A, \mathsf{M})$ and $i \in \mathbb{Z}$ we define $\operatorname{Hom}_{A,\mathsf{M}}(M, N)^i$ to be the subset of $\operatorname{Hom}_{\mathsf{M}}(M, N)^i$ consisting of the morphisms $\phi : M \to N$ such that

$$\phi \circ f_M(a) = (-1)^{ik} \cdot f_N(a) \circ \phi$$

for all $a \in A^k$. Next let

$$\operatorname{Hom}_{A,\mathsf{M}}(M, N) := \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{A,\mathsf{M}}(M, N)^i.$$

This is a graded $\mathbb{K}$-module.

Definition 3.1.25. Let M be a $\mathbb{K}$-linear category, and let A be a central graded $\mathbb{K}$-ring. The *category of graded A-modules in* M is the $\mathbb{K}$-linear graded category $\mathbf{G}(A, \mathsf{M})$ whose objects are the graded A-modules in M, and the morphism sets are the graded $\mathbb{K}$-modules

$$\operatorname{Hom}_{\mathbf{G}(A,\mathsf{M})}(M_0, M_1) := \operatorname{Hom}_{A,\mathsf{M}}(M_0, M_1)$$

from Definition 3.1.24.

Notice that forgetting the action of A is a faithful $\mathbb{K}$-linear graded functor $\mathbf{G}(A, \mathsf{M}) \to \mathbf{G}(\mathsf{M})$. As in any graded category, there is the subcategory $\mathbf{G}^0(A, \mathsf{M}) \subseteq \mathbf{G}(A, \mathsf{M})$ of strict morphisms.

Exercise 3.1.26. Show that if M is an abelian $\mathbb{K}$-linear category, then $\mathbf{G}^0(A, \mathsf{M})$ is an abelian category.

3.2. DG $\mathbb{K}$-modules.

Definition 3.2.1. A *DG $\mathbb{K}$-module* is a graded $\mathbb{K}$-module $M = \bigoplus_{i \in \mathbb{Z}} M^i$, together with a $\mathbb{K}$-linear operator $\mathrm{d}_M : M \to M$ of degree 1, called the differential, satisfying $\mathrm{d}_M \circ \mathrm{d}_M = 0$.

When there is no danger of confusion, we may write d instead of d_M.

Definition 3.2.2. Let M and N be DG $\mathbb{K}$-modules. A *strict homomorphism of DG $\mathbb{K}$-modules* is a $\mathbb{K}$-linear homomorphism $\phi : M \to N$ that commutes with the differentials and respects the gradings. The resulting category is denoted by $\mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K}$.

Remark 3.2.3. The name "strict morphism of DG modules", and the corresponding notation $\mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K}$, are new. We introduced them to distinguish $\mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K}$ from the DG category $\mathsf{DGMod}\, \mathbb{K}$ that contains it; cf. Definitions 3.4.1 and 3.4.4.

Suppose M and N are DG $\mathbb{K}$-modules. Their tensor product $M \otimes_{\mathbb{K}} N$ was defined, as a graded module, in equation (3.1.1). We put on it the differential

$$\mathrm{d}(m \otimes n) := \mathrm{d}_M(m) \otimes n + (-1)^i \cdot m \otimes \mathrm{d}_N(n) \tag{3.2.4}$$

for $m \in M^i$ and $n \in N^j$. In this way $M \otimes_{\mathbb{K}} N$ becomes a DG $\mathbb{K}$-module. We sometimes write $\mathrm{d}_{M \otimes_{\mathbb{K}} N}$ for the differential.

The Hom graded module $\mathrm{Hom}_{\mathbb{K}}(M, N)$ was introduced in equation(3.1.2). There is a differential on it:

$$\mathrm{d}(\phi) := \mathrm{d}_N \circ \phi - (-1)^i \cdot \phi \circ \mathrm{d}_M \tag{3.2.5}$$

for $\phi \in \mathrm{Hom}_{\mathbb{K}}(M, N)^i$. When we need to emphasize where d acts, we sometimes denote it by $\mathrm{d}_{\mathrm{Hom}_{\mathbb{K}}(M,N)}$.

Let M be a DG $\mathbb{K}$-module. The module of degree i cocycles of M is

$$\mathrm{Z}^i(M) := \mathrm{Ker}(\mathrm{d}|_{M^i}) \subseteq M^i, \tag{3.2.6}$$

and the module of degree i coboundaries is

$$\mathrm{B}^i(M) := \mathrm{Im}(\mathrm{d}|_{M^{i-1}}) \subseteq M^i. \tag{3.2.7}$$

Since $\mathrm{d} \circ \mathrm{d} = 0$ we have $\mathrm{B}^i(N) \subseteq \mathrm{Z}^i(N)$. The i-th cohomology is

$$\mathrm{H}^i(M) := \mathrm{Z}^i(M) / \mathrm{B}^i(M). \tag{3.2.8}$$

These are all $\mathbb{K}$-modules, and in fact they are functors

$$\mathrm{Z}^i, \mathrm{B}^i, \mathrm{H}^i : \mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K} \to \mathsf{Mod}\, \mathbb{K}.$$

Let us record the following result, whose easy proof we leave out.

Proposition 3.2.9. *For DG $\mathbb{K}$-modules M and N, there is equality*

$$\mathrm{Hom}_{\mathsf{DGMod}_{\mathrm{str}}\, \mathbb{K}}(M, N) = \mathrm{Z}^0\big(\mathrm{Hom}_{\mathbb{K}}(M, N)\big)$$

of these submodules of $\mathrm{Hom}_{\mathbb{K}}(M, N)$.

3.3. **DG Rings and Modules.**

Definition 3.3.1. A *DG ring* is a graded ring $A = \bigoplus_{i \in \mathbb{Z}} A^i$, together with an operator $\mathrm{d}_A : A \to A$ of degree 1 called the differential, satisfying the equation $\mathrm{d}_A \circ \mathrm{d}_A = 0$, and the graded Leibniz rule

$$\mathrm{d}_A(a \cdot b) = \mathrm{d}_A(a) \cdot b + (-1)^i \cdot a \cdot \mathrm{d}_A(b)$$

for all $a \in A^i$ and $b \in A^j$.

We sometimes write d instead of d_A.

Definition 3.3.2. Let A and B be DG rings. A *homomorphism of DG rings* $f : A \to B$ is a ring homomorphism that commutes with the differentials and respects the gradings. The resulting category is denoted by DGRing.

Rings are viewed as DG rings concentrated in degree 0 (and with trivial differentials). Thus the category of rings Ring is a full subcategory of DGRing.

Definition 3.3.3. We say that A is a *central DG* $\mathbb{K}$*-ring* if there is a given DG ring homomorphism $\mathbb{K} \to A$, whose image is central in A.

We denote by $\mathsf{DGRing} /_{\mathrm{ce}} \mathbb{K}$ the category of central DG $\mathbb{K}$-rings, in which the morphisms $f : A \to B$ are the homomorphisms in DGRing that respect the given structural homomorphisms from $\mathbb{K}$.

Proposition 3.3.4. *Let A be a central DG $\mathbb{K}$-ring. Then the differential d_A is $\mathbb{K}$-linear. In particular, the image of $\mathbb{K}$ is contained in $\mathrm{Z}^0(A)$.*

Proof. Exercise. $\square$

Of course when $\mathbb{K} = \mathbb{Z}$ we have $\mathsf{DGRing} /_{\mathrm{ce}} \mathbb{K} = \mathsf{DGRing}$.

Here are few examples of DG rings. First a silly example.

Example 3.3.5. Let A be a central graded $\mathbb{K}$-ring. Then A, with the trivial differential, is a central DG $\mathbb{K}$-ring.

Example 3.3.6. Let X be a differentiable (i.e. of type C^∞) manifold over $\mathbb{R}$. The de Rham complex A of X is a central DG $\mathbb{R}$-ring, with the wedge product and the exterior differential. See [KaSc1, Section 2.9.7] for details. In fact A is a commutative DG ring, in the sense of Remark 3.1.8.

The next example is the algebraic analogue of the previous one.

Example 3.3.7. Let C be a commutative $\mathbb{K}$-ring. Then the algebraic de Rham complex $A := \Omega_{C/\mathbb{K}} = \bigoplus_{p \geq 0} \Omega^p_{C/\mathbb{K}}$ is a strongly commutative DG $\mathbb{K}$-ring. See [Eis, Exercise 16.15] or [Mats, Section 25].

Example 3.3.8. Let M be a DG $\mathbb{K}$-module. Consider the DG $\mathbb{K}$-module

$$\mathrm{End}_{\mathbb{K}}(M) := \mathrm{Hom}_{\mathbb{K}}(M, M).$$

Composition of endomorphisms is an associative multiplication on $\mathrm{End}_{\mathbb{K}}(M)$ that respects the grading, and the graded Leibniz rule holds. We see that $\mathrm{End}_{\mathbb{K}}(M)$ is a central DG $\mathbb{K}$-ring.

Example 3.3.9. Let C be a commutative ring and let $c \in C$ be an element. The *Koszul complex* of c is the DG C-module $\mathrm{K}(C; c)$ defined as follows. In degree 0 we let $\mathrm{K}^0(C; c) := C$. In degree -1, $\mathrm{K}^{-1}(C; c)$ is a free C-module of rank 1, with basis element x. All other homogeneous components are trivial. The differential d is determined by what it does to the basis element $x \in \mathrm{K}^{-1}(C;c)$, and we let $\mathrm{d}(x) := c \in \mathrm{K}^0(C;c)$.

To make $\mathrm{K}(C;c)$ into a DG ring, we treat x as an odd variable (in the sense of strongly commutative DG rings – see Remark 3.1.8). This dictates $x^2 = 0$. It is easy to verify that $\mathrm{K}(C;c)$ is a central DG C-ring.

Example 3.3.10. Let A and B be central DG $\mathbb{K}$-rings. The graded ring $A \otimes_{\mathbb{K}} B$ from Example 3.1.6, with the differential (3.2.4), is a central DG $\mathbb{K}$-ring.

Example 3.3.11. Let C be a commutative ring and let $\boldsymbol{c} = (c_1, \ldots, c_n)$ be a sequence of elements in C. By combining Examples 3.3.9 and 3.3.10 we obtain the Koszul complex

$$\mathrm{K}(C;\boldsymbol{c}) := \mathrm{K}(C;c_1) \otimes_C \cdots \otimes_C \mathrm{K}(C;c_n).$$

This is a strongly commutative DG C-ring. In the classical literature the multiplicative structure of $\mathrm{K}(C;\boldsymbol{c})$ has been usually ignored; see [Eis] and [Mats].

Definition 3.3.12. Let A be a central DG $\mathbb{K}$-ring. The *opposite DG ring* A^{op} is the same DG $\mathbb{K}$-module as A, but the multiplication $\cdot^{\mathrm{op}}$ is reversed and twisted by signs:

$$a \cdot^{\mathrm{op}} b := (-1)^{ij} \cdot b \cdot a$$

for $a \in A^i$ and $b \in A^j$.

Definition 3.3.13. Let A be a central DG $\mathbb{K}$-ring. A *left DG A-module* is a graded left A-module $M = \bigoplus_{i \in \mathbb{Z}} M^i$, with an operator $\mathrm{d}_M : M \to M$ of degree 1 called the differential, satisfying $\mathrm{d}_M \circ \mathrm{d}_M = 0$ and

$$\mathrm{d}_M(a \cdot m) = \mathrm{d}_A(a) \cdot m + (-1)^i \cdot a \cdot \mathrm{d}_M(m)$$

for $a \in A^i$ and $m \in M^j$.

Right DG A-modules are defined likewise, but we won't deal with them much. This is because right DG A-modules are left DG modules over the opposite DG ring A^{op}. More precisely, if M is a right DG A-module, then the formula

$$a \cdot m := (-1)^{ij} \cdot m \cdot a, \tag{3.3.14}$$

for $a \in A^i$ and $m \in M^j$, makes M in to a left DG A^{op}-module.

So we make this convention for the rest of the book (analogous to Convention 1.2.1):

Convention 3.3.15. By default, DG modules are *left DG modules.* In particular, a module over a ring is by default a left module.

Proposition 3.3.16. *Let A be a central DG $\mathbb{K}$-ring, and let M be a DG $\mathbb{K}$-module.*

1. *Suppose $f : A \to \mathrm{End}_{\mathbb{K}}(M)$ is a DG $\mathbb{K}$-ring homomorphism. Then the formula $a \cdot m := f(a)(m)$, for $a \in A^i$ and $m \in M^j$, makes M into a DG A-module.*
2. *Conversely, any DG A-module structure on M, that's compatible with the DG $\mathbb{K}$-module structure, arises in this way from a DG $\mathbb{K}$-ring homomorphism $f : A \to \mathrm{End}_{\mathbb{K}}(M)$.*

Proof. Exercise. □

Definition 3.3.17. Let M and N be DG A-modules. A *strict homomorphism of DG A-modules* is a $\mathbb{K}$-linear homomorphism $\phi : M \to N$ that respects the differentials, the gradings and the action of A. The resulting category is denoted by $\mathsf{DGMod}_{\mathrm{str}}\, A$.

Exercise 3.3.18. Let A be a DG ring. Show that the cocycles $\mathrm{Z}(A) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Z}^i(A)$ are a graded subring of A, and the coboundaries $\mathrm{B}(A) := \bigoplus_{i \in \mathbb{Z}} \mathrm{B}^i(A)$ are a two-sided ideal of $\mathrm{Z}(A)$. Thus the cohomology $\mathrm{H}(A) := \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^i(A)$ is a graded ring. (Compare to Definition 3.4.4.)

Next show that given a DG A-module M, its cohomology $\mathrm{H}(M)$ is a graded $\mathrm{H}(A)$-module.

Definition 3.3.19. Let A be a central DG $\mathbb{K}$-ring, let M be a right DG A-module, and let N be a left DG A-module. By Lemma 3.1.10, $M \otimes_A N$ is a graded $\mathbb{K}$-module. We make it into a DG $\mathbb{K}$-module with the differential from formula (3.2.4).

Definition 3.3.20. Let A be a central DG $\mathbb{K}$-ring, and let M, N be left DG A-modules. The graded $\mathbb{K}$-module $\mathrm{Hom}_A(M, N)$ from Definition 3.1.11 is made into a DG $\mathbb{K}$-module with the differential from (3.2.5).

Generalizing Proposition 3.2.9, for DG A-modules M and N there is equality

$$\mathrm{Hom}_{\mathsf{DGMod}_{\mathrm{str}}\, A}(M, N) = \mathrm{Z}^0\big(\mathrm{Hom}_A(M, N)\big).$$

3.4. **DG Categories.** In Definition 3.1.12 we saw graded categories. Here is the DG version.

Definition 3.4.1. A $\mathbb{K}$*-linear DG category* is a $\mathbb{K}$-linear category C, endowed with a DG $\mathbb{K}$-module structure on each of the morphism $\mathbb{K}$-modules $\mathrm{Hom}_{\mathsf{C}}(M_0, M_1)$. The conditions are these:

(a) For any object M, the identity automorphism 1_M is a degree 0 cocycle in $\mathrm{Hom}_{\mathsf{C}}(M, M)$.

(b) For any triple of objects $M_0, M_1, M_2 \in \mathsf{C}$, the composition homomorphism

$$\mathrm{Hom}_{\mathsf{C}}(M_1, M_2) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathsf{C}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{C}}(M_0, M_2)$$

is a strict homomorphism of DG $\mathbb{K}$-modules.

Definition 3.4.2. Let C be a $\mathbb{K}$-linear DG category.

(1) A morphism $\phi \in \mathrm{Hom}_{\mathsf{C}}(M, N)^i$ is called a *degree i morphism*.
(2) A morphism $\phi \in \mathrm{Hom}_{\mathsf{C}}(M, N)$ is called a *cocycle* if $\mathrm{d}(\phi) = 0$.
(3) A morphism $\phi : M \to N$ in C is called a *strict morphism* if it is a degree 0 cocycle.

Lemma 3.4.3. *Let C be a $\mathbb{K}$-linear DG category, and for $i = 1, 2, 3$ let $\phi_i : M_{i-1} \to M_i$ be a morphism in C of degree k_i.*

(1) *The morphism $\phi_2 \circ \phi_1$ has degree $k_1 + k_2$, and*

$$\mathrm{d}(\phi_2 \circ \phi_1) = \mathrm{d}(\phi_2) \circ \phi_1 + (-1)^{k_2} \cdot \phi_2 \circ \mathrm{d}(\phi_1).$$

(2) *If ϕ_1 and ϕ_2 are cocycles, then so is $\phi_2 \circ \phi_1$.*
(3) *If ϕ_2 is a coboundary, and ϕ_1 and ϕ_3 are cocycles, then $\phi_3 \circ \phi_2 \circ \phi_1$ is a coboundary.*

Proof. (1) This is just a rephrasing of item (b) in Definition 3.4.1.

(2) This is immediate from (1).

(3) Say $\phi_2 = \mathrm{d}(\psi_2)$ for some degree $k_2 - 1$ morphism $\psi_2 : M_1 \to M_2$. Then

$$\phi_3 \circ \phi_2 \circ \phi_1 = \mathrm{d}((-1)^{k_3} \cdot \phi_3 \circ \psi_2 \circ \phi_1).$$

□

The previous lemma makes the next definition possible.

Definition 3.4.4. Let C be a $\mathbb{K}$-linear DG category.

(1) The *strict category* of C is the category $\mathrm{Str}(\mathsf{C}) = \mathsf{C}_{\mathrm{str}}$, with the same objects as C, but with strict morphisms only. Thus
$$\mathrm{Hom}_{\mathrm{Str}(\mathsf{C})}(M, N) = \mathrm{Z}^0\big(\mathrm{Hom}_{\mathsf{C}}(M, N)\big).$$

(2) The *homotopy category* of C is the category $\mathrm{Ho}(\mathsf{C})$, with the same objects as C, and with morphism sets
$$\mathrm{Hom}_{\mathrm{Ho}(\mathsf{C})}(M, N) := \mathrm{H}^0\big(\mathrm{Hom}_{\mathsf{C}}(M, N)\big).$$

(3) We denote by
$$\mathrm{P} : \mathrm{Str}(\mathsf{C}) \to \mathrm{Ho}(\mathsf{C})$$
the functor which is the identity on objects, and sends a strict morphism to its homotopy class.

The categories $\mathrm{Str}(\mathsf{C})$ and $\mathrm{Ho}(\mathsf{C})$ are $\mathbb{K}$-linear. The inclusion $\mathrm{Str}(\mathsf{C}) \to \mathsf{C}$ and $\mathrm{P} : \mathrm{Str}(\mathsf{C}) \to \mathrm{Ho}(\mathsf{C})$ are $\mathbb{K}$-linear functors. The first is faithful (injective on morphisms), and the second is full (surjective on morphisms).

Example 3.4.5. If A is a $\mathbb{K}$-linear DG category, then for every object $x \in \mathsf{A}$, its set of endomorphisms $A := \mathrm{End}_{\mathsf{A}}(x)$ is a central DG $\mathbb{K}$-ring. Conversely, any central DG $\mathbb{K}$-ring A can be viewed as a $\mathbb{K}$-linear DG category with a single object.

Example 3.4.6. Let A be a central DG $\mathbb{K}$-ring. The set of DG A-modules forms a $\mathbb{K}$-linear DG category $\mathsf{DGMod}\,A$, in which the morphism DG modules are

$$\mathrm{Hom}_{\mathsf{DGMod}\,A}(M, N) := \mathrm{Hom}_A(M, N)$$

from Definition 3.3.20. The strict category here is

$$\mathrm{Str}(\mathsf{DGMod}\,A) = \mathsf{DGMod}_{\mathrm{str}}\,A;$$

cf. Definition 3.3.17.

Remark 3.4.7. The fact that the concept of "DG category" includes both DG rings (Example 3.4.5) and DG modules over them (Example 3.4.6) is a source of frequent confusion. See Remarks 3.4.9 and 3.7.7.

Here is the categorical version of Definition 3.3.12.

Definition 3.4.8. Let C be a $\mathbb{K}$-linear DG category. The *opposite DG category* C^{op} has the same set of objects. The morphism DG modules are

$$\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(M, N) := \mathrm{Hom}_{\mathsf{C}}(N, M).$$

The composition is reversed with signs:

$$\phi \circ^{\mathrm{op}} \psi := (-1)^{ij} \cdot \psi \circ \phi$$

for composable homogeneous morphisms ϕ and ψ in C of degrees i and j.

Remark 3.4.9. There is a vast theory on DG categories. See the relatively old references [Kel], [BoKa]; for more modern accounts try to search the internet (there are numerous accounts, of many flavors). In this book we shall be exclusively concerned with the categories $\mathbf{C}(A, \mathsf{M})$, to be introduced in Subsection 3.7, that have a lot more structure than other DG categories. See Remark 3.7.7 regarding the DG category $\mathbf{C}(\mathsf{A}) = \mathbf{C}(\mathsf{A}, \mathsf{Mod}\,\mathbb{K})$ of left DG modules over a $\mathbb{K}$-linear DG category A, in the sense of [Kel].

Here is a useful result.

Proposition 3.4.10. *Let $\phi : M \to N$ be a degree i isomorphism in the $\mathbb{K}$-linear DG category C. Assume ϕ is a cocycle, namely $\mathrm{d}(\phi) = 0$. Then its inverse $\phi^{-1} : N \to M$ is also a cocycle.*

Proof. According the Leibniz rule (Lemma 3.4.3(1)), and the fact that 1_M is a cocycle, we have

$$0 = \mathrm{d}(1_M) = \mathrm{d}(\phi^{-1} \circ \phi) = \mathrm{d}(\phi^{-1}) \circ \phi + (-1)^{-i} \cdot \phi^{-1} \circ \mathrm{d}(\phi) = \mathrm{d}(\phi^{-1}) \circ \phi.$$

Because ϕ is an isomorphism, we conclude that $\mathrm{d}(\phi^{-1}) = 0$. □

3.5. **DG Functors.** Here C and D are $\mathbb{K}$-linear DG categories (see Definition 3.4.1). When we forget differentials, C and D become $\mathbb{K}$-linear graded categories. So we can talk about graded functors $\mathsf{C} \to \mathsf{D}$, as in Definition 3.1.15.

The differential of the DG $\mathbb{K}$-module $\mathrm{Hom}_{\mathsf{C}}(M_0, M_1)$, for objects $M_0, M_1 \in \mathsf{C}$, will be denoted by d_{C}. Likewise in D.

Recall the meaning of a strict homomorphism of DG $\mathbb{K}$-modules: it has degree 0 and commutes with the differentials.

Definition 3.5.1. Let C and D be $\mathbb{K}$-linear DG categories. A functor $F : \mathsf{C} \to \mathsf{D}$ is called a $\mathbb{K}$-*linear DG functor* if it satisfies this condition:

▷ For any pair of objects $M_0, M_1 \in \mathsf{C}$, the function

$$F : \mathrm{Hom}_{\mathsf{C}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{D}}\big(F(M_0), F(M_1)\big)$$

is a strict homomorphism of DG $\mathbb{K}$-modules.

In other words, F is a DG functor if it is a graded functor, and

$$\mathrm{d}_{\mathsf{D}} \circ F = F \circ \mathrm{d}_{\mathsf{C}} \tag{3.5.2}$$

as degree 1 homomorphisms

$$\mathrm{Hom}_{\mathsf{C}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{D}}\big(F(M_0), F(M_1)\big).$$

Example 3.5.3. Let $f : A \to B$ be a homomorphism of central DG $\mathbb{K}$-rings. Define the DG categories C and D as follows: $\mathrm{Ob}(\mathsf{C}) := \{x\}$, $\mathrm{End}_{\mathsf{C}}(x) := A$, $\mathrm{Ob}(\mathsf{D}) := \{y\}$ and $\mathrm{End}_{\mathsf{D}}(y) := B$. Then f becomes a $\mathbb{K}$-linear DG functor $F : \mathsf{C} \to \mathsf{D}$.

Other examples of DG functors, more relevant to our study, will be given in Subsection 4.4.

Definition 3.5.4. Let $F, G : \mathsf{C} \to \mathsf{D}$ be $\mathbb{K}$-linear DG functors.

(1) A *degree i morphism of DG functors* $\eta : F \to G$ is a degree i morphism of graded functors, as in Definition 3.1.17.

(2) Let $\eta : F \to G$ be a degree i morphism of DG functors. For any object $M \in \mathsf{C}$ there is a degree $i+1$ morphism

$$\mathrm{d}_{\mathsf{D}}(\eta_M) : F(M) \to G(M)$$

in D. We let

$$\mathrm{d}_{\mathsf{D}}(\eta) := \big\{ \mathrm{d}_{\mathsf{D}}(\eta_M) \big\}_{M \in \mathsf{C}}.$$

(3) A *strict morphism of DG functors* is a degree 0 morphism of graded functors $\eta : F \to G$ such that $\mathrm{d}_{\mathsf{D}}(\eta) = 0$.

Proposition 3.5.5. *The collection of morphisms* $\mathrm{d}_{\mathsf{D}}(\eta)$ *defined above is a degree* $i+1$ *morphism of DG functors* $F \to G$.

Proof. Exercise. □

The categories $\mathrm{Str}(\mathsf{C}) = \mathsf{C}_{\mathrm{str}}$ and $\mathrm{Ho}(\mathsf{C})$ were introduced in Definition 3.4.4.

Proposition 3.5.6. *Let* $F : \mathsf{C} \to \mathsf{D}$ *be a* $\mathbb{K}$*-linear DG functor. Then* F *induces* $\mathbb{K}$*-linear functors*

$$\mathrm{Str}(F) : \mathrm{Str}(\mathsf{C}) \to \mathrm{Str}(\mathsf{D})$$

and

$$\mathrm{Ho}(F) : \mathrm{Ho}(\mathsf{C}) \to \mathrm{Ho}(\mathsf{D}).$$

Proof. Because F is a DG functor, it sends 0-cocycles in $\mathrm{Hom}_{\mathsf{C}}(M_0, M_1)$ to 0-cocycles in $\mathrm{Hom}_{\mathsf{D}}(F(M_0), F(M_1))$. The same for 0-coboundaries. □

By abuse of notation, and when there is no danger for confusion, we will sometimes write F instead of $\mathrm{Str}(F)$ or $\mathrm{Ho}(F)$.

Exercise 3.5.7. Let A and C be $\mathbb{K}$-linear DG categories, and assume A is small. Define $\mathsf{DGFun}(\mathsf{A}, \mathsf{C})$ to be the set of $\mathbb{K}$-linear DG functors $F : \mathsf{A} \to \mathsf{C}$. Show that $\mathsf{DGFun}(\mathsf{A}, \mathsf{C})$ is a $\mathbb{K}$-linear DG category, where the morphisms are from Definition 3.5.4(1), and their differentials are from Definition 3.5.4(2).

3.6. **Complexes in Abelian Categories.** Here we recall facts about complexes from the classical homological theory, and place them within our context. In this subsection M is a $\mathbb{K}$-linear abelian category.

Remark 3.6.1. Actually, almost everything we do here makes sense when M is just a $\mathbb{K}$-linear category (not necessarily abelian). However, requiring it to be abelian eliminates the confusion between "ring-like" and "module-like" categories (only the latter are abelian). Cf. Remark 3.4.7.

A *complex* of objects of M, or a complex in M, is a diagram

$$\big(\cdots \to M^{-1} \xrightarrow{\mathrm{d}_M^{-1}} M^0 \xrightarrow{\mathrm{d}_M^{0}} M^1 \xrightarrow{\mathrm{d}_M^{1}} M^2 \to \cdots \big) \tag{3.6.2}$$

of objects and morphisms in M, such that $\mathrm{d}_M^{i+1} \circ \mathrm{d}_M^i = 0$. The collection of objects $M := \{M^i\}_{i \in \mathbb{Z}}$ is nothing but a graded object of M, as defined in Subsection 3.1. The collection of morphisms $\mathrm{d}_M := \{\mathrm{d}_M^i\}_{i \in \mathbb{Z}}$ is called a *differential*, or a *coboundary operator*. Thus a complex is a pair (M, d_M) made up of a graded object M and a differential d_M on it. We sometimes write d instead of d_M or d_M^i. At other times we leave the differential implicit, and just refer to the complex as M.

Let N be another complex in M. A *strict morphism of complexes* $\phi : M \to N$ is a collection $\phi = \{\phi^i\}_{i \in \mathbb{Z}}$ of morphisms $\phi^i : M^i \to N^i$ in M, such that

$$\mathrm{d}_N^i \circ \phi^i = \phi^{i+1} \circ \mathrm{d}_M^i. \tag{3.6.3}$$

Note that a strict morphism $\phi : M \to N$ can be viewed as a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & M^i \arrow[r, "\mathrm{d}_M^i"] \arrow[d, "\phi^i"'] & M^{i+1} \arrow[r] \arrow[d, "\phi^{i+1}"'] & \cdots \\
\cdots \arrow[r] & N^i \arrow[r, "\mathrm{d}_N^i"] & N^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

in M. The identity automorphism 1_M of the complex M is a strict morphism.

Remark 3.6.4. In most textbooks, what we call "strict morphism of complexes" is simply called a "morphism of complexes". See Remark 3.2.3 for an explanation.

Let us denote by $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$ the category of complexes in M, with strict morphisms. This is a $\mathbb{K}$-linear abelian category. Indeed, the direct sum of complexes is the degree-wise direct sum, i.e. $(M \oplus N)^i = M^i \oplus N^i$. The same for kernels and cokernels. If N is a full abelian subcategory of M, then $\mathbf{C}_{\mathrm{str}}(\mathsf{N})$ is a full abelian subcategory of $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$.

Any single object $M \in \mathsf{M}$ can be viewed as a complex

$$M' := \big(\cdots \to 0 \to M \to 0 \to \cdots\big),$$

where M is in degree 0; the differential of this complex is of course zero. The assignment $M \mapsto M'$ is a fully faithful $\mathbb{K}$-linear functor $\mathsf{M} \to \mathbf{C}_{\mathrm{str}}(\mathsf{M})$.

Let M, N be complexes in M. As in (3.1.19) there is a graded $\mathbb{K}$-module $\mathrm{Hom}_{\mathsf{M}}(M, N)$. It is a DG $\mathbb{K}$-module with differential d given by the formula

$$\mathrm{d}(\phi) := \mathrm{d}_N \circ \phi - (-1)^i \cdot \phi \circ \mathrm{d}_M \tag{3.6.5}$$

for $\phi \in \mathrm{Hom}_{\mathsf{M}}(M, N)^i$. It is easy to check that $\mathrm{d} \circ \mathrm{d} = 0$. We sometimes denote this differential by $\mathrm{d}_{\mathrm{Hom}_{\mathsf{M}}(M,N)}$.

Thus, an element $\phi \in \mathrm{Hom}_{\mathsf{M}}(M, N)^i$ is a collection $\phi = \{\phi^j\}_{j \in \mathbb{Z}}$ of morphisms $\phi^j : M^j \to N^{j+i}$. In a diagram, for $i = 2$, it looks like this:

$$\begin{tikzcd}
\cdots \arrow[r] & M^j \arrow[r, "\mathrm{d}"] \arrow[drr, "\phi^j"] & M^{j+1} \arrow[r, "\mathrm{d}"] \arrow[drr, "\phi^{j+1}"] & M^{j+2} \arrow[r, "\mathrm{d}"] & M^{j+3} \arrow[r] & \cdots \\
\cdots \arrow[r] & N^j \arrow[r, "\mathrm{d}"] & N^{j+1} \arrow[r, "\mathrm{d}"] & N^{j+2} \arrow[r, "\mathrm{d}"] & N^{j+3} \arrow[r] & {}
\end{tikzcd}$$

Since ϕ does not have to commute with the differentials, this is usually not a commutative diagram!

For a triple of complexes M_0, M_1, M_2, there are $\mathbb{K}$-linear homomorphisms

$$\mathrm{Hom}_{\mathsf{M}}(M_1, M_2)^{i_2} \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathsf{M}}(M_0, M_1)^{i_1} \to \mathrm{Hom}_{\mathsf{M}}(M_0, M_2)^{i_1 + i_2},$$

$\phi_2 \otimes \phi_1 \mapsto \phi_2 \circ \phi_1$.

Lemma 3.6.6. *The composition homomorphism*

$$\mathrm{Hom}_{\mathsf{M}}(M_1, M_2) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathsf{M}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{M}}(M_0, M_2)$$

is a strict homomorphism of DG $\mathbb{K}$*-modules.*

Proof. This is a good exercise. □

The lemma justifies the next definition.

Definition 3.6.7. Let $\mathbf{C}(\mathsf{M})$ be the $\mathbb{K}$-linear DG category whose objects are the complexes in M, and the morphism DG $\mathbb{K}$-modules are $\mathrm{Hom}_{\mathsf{M}}(M, N)$ from formulas (3.1.19) and (3.6.5).

It is clear, from comparing formulas (3.6.5) and (3.6.3), that the strict morphisms of complexes defined at the top of this subsection are the same as those from Definition 3.4.4(1). In other words, $\mathrm{Str}(\mathbf{C}(\mathsf{M})) = \mathbf{C}_{\mathrm{str}}(\mathsf{M})$.

Remark 3.6.8. A possible ambiguity could arise in the meaning of $\mathrm{Hom}_{\mathsf{M}}(M, N)$ if $M, N \in \mathsf{M}$: does it mean the $\mathbb{K}$-module of morphisms in the category M ? Or, if we view M and N as complexes by the canonical embedding $\mathsf{M} \subseteq \mathbf{C}(\mathsf{M})$, does $\mathrm{Hom}_{\mathsf{M}}(M, N)$ mean the complex of $\mathbb{K}$-modules defined for complexes? It turns out that there is no actual difficulty: since the complex of $\mathbb{K}$-modules $\mathrm{Hom}_{\mathsf{M}}(M, N)$ is concentrated in degree 0, we may view it as a single $\mathbb{K}$-module, and this is precisely the $\mathbb{K}$-module of morphisms in the category M.

When $\mathsf{M} = \mathsf{Mod}\, A$ for a ring A, there is no essential distinction between complexes and DG modules:

Proposition 3.6.9. *Let A be a central $\mathbb{K}$-ring. Given a complex $M \in \mathbf{C}(\mathsf{Mod}\, A)$, with notation as in* (3.6.2)*, define the DG A-module*

$$F(M) := \bigoplus_{i \in \mathbb{Z}} M^i,$$

with differential $\mathrm{d} := \sum_{i \in \mathbb{Z}} \mathrm{d}_M^i$*. Then the functor*

$$F : \mathbf{C}(\mathsf{Mod}\, A) \to \mathsf{DGMod}\, A$$

is an equivalence of $\mathbb{K}$-linear DG categories.

The proof is an exercise. The only hard part in it is to choose good notation.

3.7. **The DG Category $\mathbf{C}(A, \mathsf{M})$.** We now combine material from previous subsections. The concept introduced in the definition below is new. It is the DG version of Definition 3.1.22.

Definition 3.7.1. Let M be a $\mathbb{K}$-linear abelian category, and let A be a central DG $\mathbb{K}$-ring. A *DG A-module in* M is an object $M \in \mathbf{C}(\mathsf{M})$, together with a DG $\mathbb{K}$-ring homomorphism $f : A \to \mathrm{End}_{\mathsf{M}}(M)$.

If M is a DG A-module in M, then after forgetting the differentials, M becomes a graded A-module in M; see Definition 3.1.22.

Definition 3.7.2. Let M be a $\mathbb{K}$-linear abelian category, let A be a central DG $\mathbb{K}$-ring, and let M, N be DG A-modules in M. In Definition 3.1.24 we introduced the graded $\mathbb{K}$-module $\mathrm{Hom}_{A,\mathsf{M}}(M, N)$. This is made into a DG $\mathbb{K}$-module with differential

$$\mathrm{d}(\phi) := \mathrm{d}_N \circ \phi - (-1)^i \cdot \phi \circ \mathrm{d}_M$$

for $\phi \in \mathrm{Hom}_{A,\mathsf{M}}(M, N)^i$.

When we have to be specific, we denote the differential of $\mathrm{Hom}_{A,\mathsf{M}}(M,N)$ by $\mathrm{d}_{\mathrm{Hom}}$, $\mathrm{d}_{A,\mathsf{M}}$, or $\mathrm{d}_{\mathrm{Hom}_{A,\mathsf{M}}(M,N)}$.

As we have seen before (in Lemma 3.6.6, and Conversely in Lemma 3.4.3), given $\phi_k \in \mathrm{Hom}_{A,\mathsf{M}}(M_{k-1}, M_k)^{i_k}$ for $k = 1, 2$, we have

$$\phi_2 \circ \phi_1 \in \mathrm{Hom}_{A,\mathsf{M}}(M_0, M_2)^{i_1+i_2},$$

and

$$\mathrm{d}(\phi_2 \circ \phi_1) = \mathrm{d}(\phi_2) \circ \phi_1 + (-1)^{i_2} \cdot \phi_2 \circ \mathrm{d}(\phi_1).$$

Also the identity automorphism $1_M = \mathrm{id}_M$ belongs to $\mathrm{Hom}_{A,\mathsf{M}}(M,M)^0$, and $\mathrm{d}(1_M) = 0$. Therefore the next definition is legitimate.

Definition 3.7.3. Let M be a $\mathbb{K}$-linear abelian category, and let A be a central DG $\mathbb{K}$-ring. The $\mathbb{K}$-linear DG category of DG A-modules in M is denoted by $\mathbf{C}(A, \mathsf{M})$. The morphism DG module are

$$\mathrm{Hom}_{\mathbf{C}(A,\mathsf{M})}(M_0, M_1) := \mathrm{Hom}_{A,\mathsf{M}}(M_0, M_1)$$

from Definition 3.7.2. The composition is that of $\mathbf{C}(\mathsf{M})$.

Notice that forgetting the action of A is a faithful $\mathbb{K}$-linear DG functor $\mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(\mathsf{M})$. On the other hand, forgetting the differentials is a fully faithful $\mathbb{K}$-linear graded functor $\mathbf{C}(A, \mathsf{M}) \to \mathbf{G}(A, \mathsf{M})$.

Example 3.7.4. If $A = \mathbb{K}$, then $\mathbf{C}(A, \mathsf{M}) = \mathbf{C}(\mathsf{M})$; and if $\mathsf{M} = \mathsf{Mod}\,\mathbb{K}$, then $\mathbf{C}(A, \mathsf{M}) = \mathsf{DGMod}\,A$. Because of this, we sometimes write $\mathbf{C}(A) := \mathsf{DGMod}\,A$.

Definition 3.7.5. In the situation of Definition 3.7.3:

(1) The strict category of $\mathbf{C}(A, \mathsf{M})$ (see Definition 3.4.4(1)) is denoted by $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$.
(2) The homotopy category of $\mathbf{C}(A, \mathsf{M})$ (see Definition 3.4.4(2)) is denoted by $\mathbf{K}(A, \mathsf{M})$.

The next proposition is merely an interpretation of the definitions; but it is worth mentioning.

Proposition 3.7.6. *Let $\phi : M \to N$ be a degree 0 morphism in $\mathbf{C}(A, \mathsf{M})$. The next two conditions are equivalent*:

(i) *ϕ is strict.*
(ii) *$\phi \circ \mathrm{d}_M = \mathrm{d}_N \circ \phi$.*

Remark 3.7.7. Here is a generalization of Definition 3.7.3. Instead of a central DG $\mathbb{K}$-ring A we can take a small $\mathbb{K}$-linear DG category A. We then define the $\mathbb{K}$-linear DG category

$$\mathbf{C}(\mathsf{A}, \mathsf{M}) := \mathsf{DGFun}(\mathsf{A}, \mathbf{C}(\mathsf{M}))$$

as in Exercise 3.5.7.

This is indeed a generalization of Definition 3.7.3: when A has a single object x, and we write $A := \mathrm{End}_{\mathsf{A}}(x)$, then the functor $M \mapsto M(x)$ is an isomorphism of DG categories $\mathbf{C}(\mathsf{A}, \mathsf{M}) \xrightarrow{\simeq} \mathbf{C}(A, \mathsf{M})$.

In the special case of $\mathsf{M} = \mathsf{Mod}\,\mathbb{K}$, the DG category $\mathbf{C}(\mathsf{A}, \mathsf{M})$ is what Keller [Kel] calls the DG category of *left DG* A*-modules.*

Practically everything we do in this book for $\mathbf{C}(A, \mathsf{M})$ holds in the more general context of $\mathbf{C}(\mathsf{A}, \mathsf{M})$. However, in the more general context a lot of the intuition is lost, and some aspects become pretty cumbersome. This is the reason we decided to stick with the less general context.

Definition 3.7.8. Let M be a $\mathbb{K}$-linear abelian category, and let A be a central DG $\mathbb{K}$-ring. For any integer i let

$$\mathrm{H}^i : \mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \to \mathsf{M}$$

be the $\mathbb{K}$-linear functor, that sends a complex M to its i-th cohomology $\mathrm{H}^i(M) \in \mathsf{M}$ as in (3.2.8), and that sends a strict morphism $\phi : M_0 \to M_1$ to the morphism

$$\mathrm{H}^i(\phi) : \mathrm{H}^i(M_0) \to \mathrm{H}^i(M_1).$$

3.8. **The Translation Functor.** As before, we fix a $\mathbb{K}$-linear abelian category M, and a central DG $\mathbb{K}$-ring A.

The translation functor goes back to the beginnings of derived categories – see Remark 3.8.11. The treatment in this subsection (with the operator t) is taken from [Ye11, Section 1].

Definition 3.8.1. Let $M = \{M^i\}_{i \in \mathbb{Z}}$ be a graded module in M, i.e. an object of $\mathbf{G}(\mathsf{M})$. The *translation* of M is the object

$$\mathrm{T}(M) = \{\mathrm{T}(M)^i\}_{i \in \mathbb{Z}} \in \mathbf{G}(\mathsf{M})$$

defined as follows: the graded component of degree i of $\mathrm{T}(M)$ is $\mathrm{T}(M)^i := M^{i+1}$.

Definition 3.8.2 (The little t operator)**.** Let $M = \{M^i\}_{i \in \mathbb{Z}}$ be a graded module in M, i.e. an object of $\mathbf{G}(\mathsf{M})$. We define

$$\mathrm{t}_M : M \to \mathrm{T}(M)$$

to be the degree -1 morphism of graded objects of M, that in every degree $i+1$ is identity morphism

$$\mathrm{t}_M := 1_{M^{i+1}} : M^{i+1} \xrightarrow{\simeq} M^{i+1} = \mathrm{T}(M)^i$$

of the object M^{i+1} in M.

Note that the morphism

$$\mathrm{t}_M \in \mathrm{Hom}_{\mathbf{G}(\mathsf{M})}\big(M, \mathrm{T}(M)\big)^{-1}$$

is invertible, with inverse

$$\mathrm{t}_M^{-1} \in \mathrm{Hom}_{\mathbf{G}(\mathsf{M})}\big(\mathrm{T}(M), M\big)^{1}.$$

Definition 3.8.3. Let $M = \{M^i\}_{i \in \mathbb{Z}}$ be a DG A-module in M, i.e. an object of $\mathbf{C}(A, \mathsf{M})$. The *translation* of M is the object

$$\mathrm{T}(M) \in \mathbf{C}(A, \mathsf{M})$$

defined as follows.

(1) As graded object of M, it is as specified in Definition 3.8.1.
(2) The differential $\mathrm{d}_{\mathrm{T}(M)}$ is defined by the formula

$$\mathrm{d}_{\mathrm{T}(M)} := -\,\mathrm{t}_M \circ \mathrm{d}_M \circ \mathrm{t}_M^{-1}\,.$$

(3) Let $f_M : A \to \mathrm{End}_{\mathsf{M}}(M)$ be the DG ring homomorphism that determines the action of A on M. Then

$$f_{\mathrm{T}(M)} : A \to \mathrm{End}_{\mathsf{M}}(\mathrm{T}(M))$$

is defined by

$$f_{\mathrm{T}(M)}(a) := (-1)^j \cdot \mathrm{t}_M \circ f_M(a) \circ \mathrm{t}_M^{-1}$$

for $a \in A^j$.

Thus, the differential $\mathrm{d}_{\mathrm{T}(M)} = \{\mathrm{d}^i_{\mathrm{T}(M)}\}_{i\in\mathbb{Z}}$ makes this diagram in M commutative for every i :

$$\begin{tikzcd} \mathrm{T}(M)^i \arrow[r, "\mathrm{d}^i_{\mathrm{T}(M)}"] & \mathrm{T}(M)^{i+1} \\ M^{i+1} \arrow[u, "\mathrm{t}_M"] \arrow[r, "-\mathrm{d}^{i+1}_M"] & M^{i+2} \arrow[u, "\mathrm{t}_M"'] \end{tikzcd}$$

And the left A-module structure makes this diagram in M commutative for every i and every $a \in A^j$:

$$\begin{tikzcd} \mathrm{T}(M)^i \arrow[r, "f_{\mathrm{T}(M)}(a)"] & \mathrm{T}(M)^{i+j} \\ M^{i+1} \arrow[u, "\mathrm{t}_M"] \arrow[r, "(-1)^j \cdot f_M(a)"] & M^{i+j+1} \arrow[u, "\mathrm{t}_M"'] \end{tikzcd}$$

Warning: t_M is not a morphism in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$, because it has degree -1.

Proposition 3.8.4. *The morphisms* t_M *and* t_M^{-1} *are cocycles, in the DG* $\mathbb{K}$*-modules* $\mathrm{Hom}_{A,\mathsf{M}}\big(M, \mathrm{T}(M)\big)$ *and* $\mathrm{Hom}_{A,\mathsf{M}}\big(\mathrm{T}(M), M\big)$ *respectively.*

Proof. We use the notation $\mathrm{d}_{\mathrm{Hom}}$ for the differential in the DG module $\mathrm{Hom}_{A,\mathsf{M}}\big(M, \mathrm{T}(M)\big)$. Let us calculate. Because t_M has degree -1, we have

$$\begin{aligned} \mathrm{d}_{\mathrm{Hom}}(\mathrm{t}_M) &= \mathrm{d}_{\mathrm{T}(M)} \circ \mathrm{t}_M + \mathrm{t}_M \circ \mathrm{d}_M \\ &= (-\,\mathrm{t}_M \circ \mathrm{d}_M \circ \mathrm{t}_M^{-1}) \circ \mathrm{t}_M + \mathrm{t}_M \circ \mathrm{d}_M = 0. \end{aligned}$$

As for t_M^{-1} : this is done using the graded Leibniz rule, just like in the proof Proposition 3.4.10. □

Definition 3.8.5. Given a morphism

$$\phi \in \mathrm{Hom}_{A,\mathsf{M}}(M, N)^i,$$

we define the morphism

$$\mathrm{T}(\phi) \in \mathrm{Hom}_{A,\mathsf{M}}\big(\mathrm{T}(M), \mathrm{T}(N)\big)^i$$

to be

$$\mathrm{T}(\phi) := (-1)^i \cdot \mathrm{t}_N \circ \phi \circ \mathrm{t}_M^{-1} .$$

To clarify this definition, let us write $\phi = \{\phi^j\}_{j\in\mathbb{Z}}$, so that $\phi^j : M^j \to N^{j+i}$ is a morphism in M. Then

$$\mathrm{T}(\phi)^j : \mathrm{T}(M)^j \to \mathrm{T}(N)^{j+i}$$

is

$$\mathrm{T}(\phi)^j = (-1)^i \cdot \mathrm{t}_N \circ \phi^{j+1} \circ \mathrm{t}_M^{-1} .$$

The corresponding commutative diagram in M, for each i, j, is:

(3.8.6)
$$\begin{tikzcd} \mathrm{T}(M)^j \arrow[r, "\mathrm{T}(\phi)^j"] & \mathrm{T}(N)^{j+i} \\ M^{j+1} \arrow[u, "\mathrm{t}_M"] \arrow[r, "(-1)^i \cdot \phi^{j+1}"] & N^{j+i} \arrow[u, "\mathrm{t}_N"'] \end{tikzcd}$$

Theorem 3.8.7. *Let* M *be* $\mathbb{K}$*-linear abelian category and let* A *be a central DG* $\mathbb{K}$*-ring.*

1. *The assignments $M \mapsto \mathrm{T}(M)$ and $\phi \mapsto \mathrm{T}(\phi)$ are a $\mathbb{K}$-linear DG functor*
$$\mathrm{T} : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(A, \mathsf{M}).$$
2. *The collection* $\mathrm{t} := \{\mathrm{t}_M\}_{M \in \mathbf{C}(A,\mathsf{M})}$ *is a degree -1 isomorphism*
$$\mathrm{t} : \mathrm{Id} \to \mathrm{T}$$
of DG functors from $\mathbf{C}(A, \mathsf{M})$ to itself.

Proof. (1) Take morphisms $\phi_1 : M_0 \to M_1$ and $\phi_2 : M_1 \to M_2$, of degrees i_1 and i_2 respectively. Then
$$\begin{aligned}\mathrm{T}(\phi_2 \circ \phi_1) &= (-1)^{i_1+i_2} \cdot \mathrm{t}_{M_2} \circ (\phi_2 \circ \phi_1) \circ \mathrm{t}_{M_0}^{-1} \\ &= (-1)^{i_1+i_2} \cdot \mathrm{t}_{M_2} \circ \phi_2 \circ (\mathrm{t}_{M_1}^{-1} \circ \mathrm{t}_{M_1}) \circ\ \phi_1 \circ \mathrm{t}_{M_0}^{-1} \\ &= \big((-1)^{i_2} \cdot \mathrm{t}_{M_2} \circ \phi_2 \circ \mathrm{t}_{M_1}^{-1}\big) \circ \big((-1)^{i_1} \cdot \mathrm{t}_{M_1} \circ \phi_1 \circ \mathrm{t}_{M_0}^{-1}\big) \\ &= \mathrm{T}(\phi_2) \circ \mathrm{T}(\phi_1).\end{aligned}$$
Clearly $\mathrm{T}(1_M) = 1_M$, and
$$\mathrm{T}(\lambda \cdot \phi + \psi) = \lambda \cdot \mathrm{T}(\phi) + \mathrm{T}(\psi)$$
for all $\lambda \in \mathbb{K}$ and $\phi, \psi \in \mathrm{Hom}_{A,\mathsf{M}}(M_0, M_1)^i$. So T is a $\mathbb{K}$-linear graded functor.

By Proposition 3.8.4 we know that $\mathrm{d} \circ \mathrm{t} = -\,\mathrm{t} \circ \mathrm{d}$ and $\mathrm{d} \circ \mathrm{t}^{-1} = -\,\mathrm{t}^{-1} \circ \mathrm{d}$, This implies that for any morphism ϕ in $\mathbf{C}(A, \mathsf{M})$, we have $\mathrm{T}(\mathrm{d}(\phi)) = \mathrm{d}(\mathrm{T}(\phi))$. So T is a DG functor.

(2) Take any $\phi \in \mathrm{Hom}_{A,\mathsf{M}}(M_0, M_1)^i$. We have to prove that
$$\mathrm{t}_{M_1} \circ \phi = (-1)^i \cdot \mathrm{T}(\phi) \circ \mathrm{t}_{M_0}$$
as elements of $\mathrm{Hom}_{A,\mathsf{M}}(M_0, \mathrm{T}(M_1))^{i+1}$. But by Definition 3.8.5 we have
$$\mathrm{T}(\phi) \circ \mathrm{t}_{M_0} = \big((-1)^i \cdot \mathrm{t}_{M_1} \circ \phi \circ \mathrm{t}_{M_0}^{-1}\big) \circ \mathrm{t}_{M_0} = (-1)^i \cdot \mathrm{t}_{M_1} \circ \phi.$$
□

Definition 3.8.8. We call T the *translation functor* of the DG category $\mathbf{C}(A, \mathsf{M})$.

Corollary 3.8.9.

1. *The functor T is an automorphism of the category $\mathbf{C}(A, \mathsf{M})$.*
2. *For any $k, l \in \mathbb{Z}$ there is an equality of functors $\mathrm{T}^l \circ \mathrm{T}^k = \mathrm{T}^{l+k}$.*
3. *For any k the functor*
$$\mathrm{T}^k : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(A, \mathsf{M})$$
is an auto-equivalence of DG categories.

Proof. (1) This is because the functor T is bijective on the set of objects of $\mathbf{C}(A, \mathsf{M})$ and on the sets of morphisms.

(2) By part (1) of this corollary, the inverse T^{-1} is a uniquely defined functor (not just up to an isomorphism of functors).

(3) By part (1) of the theorem above. □

Proposition 3.8.10. *Consider any $M \in \mathbf{C}(A, \mathsf{M})$.*

1. *There is equality*
$$\mathrm{t}_{\mathrm{T}(M)} = -\,\mathrm{T}(\mathrm{t}_M)$$
of degree -1 morphisms $\mathrm{T}(M) \to \mathrm{T}^2(M)$ in $\mathbf{C}(A, \mathsf{M})$.

(2) *There is equality*

$$\mathrm{t}_{\mathrm{T}^{-1}(M)} = -\,\mathrm{T}^{-1}(\mathrm{t}_M)$$

of degree -1 *morphisms*

$$\mathrm{T}^{-1}(M) \to \mathrm{T}(\mathrm{T}^{-1}(M)) = M = \mathrm{T}^{-1}(\mathrm{T}(M))$$

in $\mathbf{C}(A, \mathsf{M})$.

Proof. (1) This is an easy calculation, using Definition 3.8.5:

$$\mathrm{T}(\mathrm{t}_M) = -\,\mathrm{t}_{\mathrm{T}(M)} \circ \mathrm{t}_M \circ \mathrm{t}_M^{-1} = -\,\mathrm{t}_{\mathrm{T}(M)}\,.$$

(2) A similar calculation. □

Remark 3.8.11. There are several names in the literature for the translation functor T : *twist*, *shift* and *suspension*. There are also several notations: $\mathrm{T}(M) = M[1] = \Sigma M$. In the later part of this book we shall use the notation $M[k] := \mathrm{T}^k(M)$ for the k-th translation.

3.9. **The Standard Cone of a Strict Morphism.** As before, we fix a $\mathbb{K}$-linear abelian category M, and a central DG $\mathbb{K}$-ring A. Here is the cone construction in $\mathbf{C}(A, \mathsf{M})$, as it looks using the operator t.

Definition 3.9.1. Let $\phi : M \to N$ be a strict morphism in $\mathbf{C}(A, \mathsf{M})$. The *standard cone of* ϕ is the object $\mathrm{Cone}(\phi) \in \mathbf{C}(A, \mathsf{M})$ defined as follows. As a graded A-module in M we let

$$\mathrm{Cone}(\phi) := N \oplus \mathrm{T}(M).$$

The differential $\mathrm{d}_{\mathrm{Cone}}$ is this: if we express the graded module as a column

$$\mathrm{Cone}(\phi) = \begin{bmatrix} N \\ \mathrm{T}(M) \end{bmatrix},$$

then $\mathrm{d}_{\mathrm{Cone}}$ is left multiplication by the matrix

$$\mathrm{d}_{\mathrm{Cone}} := \begin{bmatrix} \mathrm{d}_N & \phi \circ \mathrm{t}_M^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}(M)} \end{bmatrix}$$

of degree 1 morphisms of graded A-module in M.

In other words,

$$\mathrm{d}^i_{\mathrm{Cone}} : \mathrm{Cone}(\phi)^i \to \mathrm{Cone}(\phi)^{i+1}$$

is

$$\mathrm{d}^i_{\mathrm{Cone}} = \mathrm{d}^i_N + \mathrm{d}^i_{\mathrm{T}(M)} + \phi^{i+1} \circ \mathrm{t}_M^{-1},$$

where $\phi^{i+1} \circ \mathrm{t}_M^{-1}$ is the composed morphism

$$\mathrm{T}(M)^i \xrightarrow{\mathrm{t}_M^{-1}} M^{i+1} \xrightarrow{\phi^{i+i}} N^{i+1}.$$

Let us denote by

$$e_\phi : N \to N \oplus \mathrm{T}(M) \tag{3.9.2}$$

the embedding, and by

$$p_\phi : N \oplus \mathrm{T}(M) \to \mathrm{T}(M) \tag{3.9.3}$$

the projection. Thus, as matrices we have

$$e_\phi = \begin{bmatrix} 1_N \\ 0 \end{bmatrix} \quad \text{and} \quad p_\phi = \begin{bmatrix} 0 & 1_{\mathrm{T}(M)} \end{bmatrix}.$$

The standard cone of ϕ sits in the exact sequence

$$0 \to N \xrightarrow{e_\phi} \operatorname{Cone}(\phi) \xrightarrow{p_\phi} \mathrm{T}(M) \to 0 \tag{3.9.4}$$

in the abelian category $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$.

Definition 3.9.5. Let $\phi : M \to N$ be a morphism in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$. The diagram

$$M \xrightarrow{\phi} N \xrightarrow{e_\phi} \operatorname{Cone}(\phi) \xrightarrow{p_\phi} \mathrm{T}(M)$$

in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$ is called the *standard triangle* associated to ϕ.

The cone construction is functorial, in the following sense.

Proposition 3.9.6. *Let*

$$\begin{tikzcd} M_0 \arrow[r, "\phi_0"] \arrow[d, "\psi"'] & N_0 \arrow[d, "\chi"] \\ M_1 \arrow[r, "\phi_1"] & N_1 \end{tikzcd}$$

be a commutative diagram in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$. *Then*

$$(\chi, \mathrm{T}(\psi)) : \operatorname{Cone}(\phi_0) \to \operatorname{Cone}(\phi_1) \tag{3.9.7}$$

is a morphism in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$, *and the diagram*

$$\begin{tikzcd} M_0 \arrow[r, "\phi_0"] \arrow[d, "\psi"'] & N_0 \arrow[r, "e_{\phi_0}"] \arrow[d, "\chi"'] & \operatorname{Cone}(\phi_0) \arrow[r, "p_{\phi_0}"] \arrow[d, "{(\chi, \mathrm{T}(\psi))}"'] & \mathrm{T}(M_0) \arrow[d, "\mathrm{T}(\psi)"'] \\ M_1 \arrow[r, "\phi_1"] & N_1 \arrow[r, "e_{\phi_1}"] & \operatorname{Cone}(\phi_1) \arrow[r, "p_{\phi_1}"] & \mathrm{T}(M_1) \end{tikzcd}$$

in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$ *is commutative.*

Proof. This is a simple consequence of the definitions. □

4. Properties of DG Functors

In this section we fix central DG $\mathbb{K}$-rings A and B, and $\mathbb{K}$-linear abelian categories M and N. The $\mathbb{K}$-linear DG categories $\mathbf{C}(A, \mathsf{M})$ and $\mathbf{C}(B, \mathsf{N})$ were introduced in Subsection 3.7. Graded functors, DG functors, and morphisms between them, were introduced in Subsection 3.5.

Some of the material in this section is new – such as the gauge in Definition 4.1.1, and its role in the the characterization of DG functors in Theorem 4.1.2. We think that Theorem 4.3.7, which says that a DG functor commutes with cones, is also a new result.

4.1. **The Gauge of a Graded Functor.** The next definition is new.

Definition 4.1.1. Let

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$$

be a $\mathbb{K}$-linear graded functor. For any object $M \in \mathbf{C}(A, \mathsf{M})$ let

$$\gamma_{F,M} := \mathrm{d}_{F(M)} - F(\mathrm{d}_M) \in \mathrm{Hom}_{B,\mathsf{N}}\big(F(M), F(M)\big)^1 .$$

The collection of morphisms

$$\gamma_F := \{\gamma_{F,M}\}_{M \in \mathbf{C}(A,\mathsf{M})}$$

is called the *gauge of F*.

Theorem 4.1.2. [1] *The following two conditions are equivalent for a $\mathbb{K}$-linear graded functor*

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N}).$$

(i) *F is a DG functor.*
(ii) *The gauge γ_F is a degree 1 morphism of graded functors $\gamma_F : F \to F$.*

Proof. Recall that F is a DG functor (condition (i)) iff

$$(F \circ \mathrm{d}_{A,\mathsf{M}})(\phi) = (\mathrm{d}_{B,\mathsf{N}} \circ F)(\phi) \tag{4.1.3}$$

for every $\phi \in \mathrm{Hom}_{A,\mathsf{M}}(M_0, M_1)^i$. And γ_F is a degree 1 morphism of graded functors (condition (ii)) iff

$$\gamma_{F,M_1} \circ F(\phi) = (-1)^i \cdot F(\phi) \circ \gamma_{F,M_0} \tag{4.1.4}$$

for every such ϕ.

Here is the calculation. Because F is a graded functor, we get

$$\begin{aligned} F\big(\mathrm{d}_{A,\mathsf{M}}(\phi)\big) &= F\big(\mathrm{d}_{M_1} \circ \phi - (-1)^i \cdot \phi \circ \mathrm{d}_{M_0}\big) \\ &= F(\mathrm{d}_{M_1}) \circ F(\phi) - (-1)^i \cdot F(\phi) \circ F(\mathrm{d}_{M_0}) \end{aligned} \tag{4.1.5}$$

and

$$\mathrm{d}_{B,\mathsf{N}}\big(F(\phi)\big) = \mathrm{d}_{F(M_1)} \circ F(\phi) - (-1)^i \cdot F(\phi) \circ \mathrm{d}_{F(M_0)} . \tag{4.1.6}$$

Using equations (4.1.5) and (4.1.6), and the definition of γ_F, we obtain

$$\begin{aligned} &(F \circ \mathrm{d}_{A,\mathsf{M}} - \mathrm{d}_{B,\mathsf{N}} \circ F)(\phi) = F\big(\mathrm{d}_{A,\mathsf{M}}(\phi)\big) - \mathrm{d}_{B,\mathsf{N}}\big(F(\phi)\big) \\ &\quad = \big(F(\mathrm{d}_{M_1}) - \mathrm{d}_{F(M_1)}\big) \circ F(\phi) - (-1)^i \cdot F(\phi) \circ \big(F(\mathrm{d}_{M_0}) - \mathrm{d}_{F(M_0)}\big) \\ &\quad = -\gamma_{F,M_1} \circ F(\phi) + (-1)^i \cdot F(\phi) \circ \gamma_{F,M_0} . \end{aligned} \tag{4.1.7}$$

[1] This theorem is due to Rishi Vyas.

Finally, the vanishing of the first expression in (4.1.7) is the same as equality in (4.1.3); whereas the vanishing of the last expression in (4.1.7) is the same as equality in (4.1.4). □

4.2. **The Translation Isomorphism of a DG Functor.** The translation functor of $\mathbf{C}(A, \mathsf{M})$ will be denoted here by $\mathrm{T}_{A,\mathsf{M}}$. Recall that for an object $M \in \mathbf{C}(A, \mathsf{M})$, we have the little t operator

$$\mathrm{t}_M \in \mathrm{Hom}_{A,\mathsf{M}}\big(M, \mathrm{T}_{A,\mathsf{M}}(M)\big)^{-1}.$$

This is an isomorphism in $\mathbf{C}(A, \mathsf{M})$. Likewise for the DG category $\mathbf{C}(B, \mathsf{N})$.

Definition 4.2.1. Let

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$$

be a $\mathbb{K}$-linear DG functor. For an object $M \in \mathbf{C}(A, \mathsf{M})$, let

$$\tau_{F,M} : F(\mathrm{T}_{A,\mathsf{M}}(M)) \to \mathrm{T}_{B,\mathsf{N}}(F(M))$$

be the isomorphism

$$\tau_{F,M} := \mathrm{t}_{F(M)} \circ F(\mathrm{t}_M)^{-1}$$

in $\mathbf{C}(B, \mathsf{N})$, called the *translation isomorphism* of the functor F at the object M.

The isomorphism $\tau_{F,M}$ sits in the following commutative diagram

$$\begin{tikzcd}
F(\mathrm{T}_{A,\mathsf{M}}(M)) \arrow[r, "\tau_{F,M}"] & \mathrm{T}_{B,\mathsf{N}}(F(M)) \\
F(M) \arrow[u, "F(\mathrm{t}_M)"] \arrow[ur, "\mathrm{t}_{F(M)}"'] &
\end{tikzcd}$$

of isomorphisms in the category $\mathbf{C}(B, \mathsf{N})$.

Proposition 4.2.2. *$\tau_{F,M}$ is an isomorphism in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$.*

Proof. We know that $\tau_{F,M}$ is an isomorphism in $\mathbf{C}(B, \mathsf{N})$. It suffices to prove that both $\tau_{F,M}$ and its inverse $\tau_{F,M}^{-1}$ are strict morphisms. Now by Proposition 3.8.4, t_M and t_M^{-1} are cocycles. Therefore, $F(\mathrm{t}_M)$ and $F(\mathrm{t}_M)^{-1} = F(\mathrm{t}_M^{-1})$ are cocycles. For the same reason, $\mathrm{t}_{F(M)}$ and $\mathrm{t}_{F(M)}^{-1}$ are cocycles. But $\tau_{F,M} = \mathrm{t}_{F(M)} \circ F(\mathrm{t}_M)^{-1}$, and $\tau_{F,M}^{-1} = F(\mathrm{t}_M) \circ \mathrm{t}_{F(M)}^{-1}$. □

Theorem 4.2.3. *Let*

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$$

be a $\mathbb{K}$-linear DG functor. Then the collection $\tau_F := \{\tau_{F,M}\}_{M \in \mathbf{C}(A,\mathsf{M})}$ is an isomorphism

$$\tau_F : F \circ \mathrm{T}_{A,\mathsf{M}} \xrightarrow{\simeq} \mathrm{T}_{B,\mathsf{N}} \circ F$$

of functors

$$\mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \to \mathbf{C}_{\mathrm{str}}(B, \mathsf{N}).$$

The slogan summarizing this theorem is “A DG functor commutes with translations”.

Proof. In view of Proposition 4.2.2, all we need to prove is that τ_F is a morphism of functors (i.e. it is a natural transformation).

Let $\phi : M_0 \to M_1$ be a morphism in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. We must prove that the diagram

$$\begin{tikzcd}
(F \circ \mathrm{T}_{A,\mathsf{M}})(M_0) \arrow[r, "\tau_{F,M_0}"] \arrow[d, "(F\circ\mathrm{T}_{A,\mathsf{M}})(\phi)"'] & (\mathrm{T}_{B,\mathsf{N}} \circ F)(M_0) \arrow[d, "(\mathrm{T}_{B,\mathsf{N}} \circ F)(\phi)"] \\
(F \circ \mathrm{T}_{A,\mathsf{M}})(M_1) \arrow[r, "\tau_{F,M_1}"] & (\mathrm{T}_{B,\mathsf{N}} \circ F)(M_1)
\end{tikzcd}$$

in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$ is commutative. This will be true if the next diagram

$$\begin{tikzcd}
(F \circ \mathrm{T}_{A,\mathsf{M}})(M_0) \arrow[d, "(F\circ\mathrm{T}_{A,\mathsf{M}})(\phi)"'] & F(M_0) \arrow[l, "F(\mathrm{t}_{M_0})"'] \arrow[r, "\mathrm{t}_{F(M_0)}"] \arrow[d, "F(\phi)"'] & (\mathrm{T}_{B,\mathsf{N}} \circ F)(M_0) \arrow[d, "(\mathrm{T}_{B,\mathsf{N}} \circ F)(\phi)"] \\
(F \circ \mathrm{T}_{A,\mathsf{M}})(M_1) & F(M_1) \arrow[l, "F(\mathrm{t}_{M_1})"'] \arrow[r, "\mathrm{t}_{F(M_1)}"] & (\mathrm{T}_{B,\mathsf{N}} \circ F)(M_1)
\end{tikzcd}$$

in $\mathbf{C}(B, \mathsf{N})$, whose horizontal arrows are isomorphisms, is commutative. For this to be true, it is enough to prove that both squares in this diagram are commutative. This is true by Theorem 3.8.7(2) □

Recall that the translation T and all its powers are DG functors. To finish this subsection, we calculate their translation isomorphisms.

Proposition 4.2.4. *For any integer k, the translation isomorphism of the DG functor T^k is*

$$\tau_{\mathrm{T}^k} = (-1)^k \cdot \mathrm{id}_{\mathrm{T}^{k+1}},$$

where $\mathrm{id}_{\mathrm{T}^{k+1}}$ is the identity automorphism of the functor T^{k+1}.

Proof. By Definition 4.2.1 and Proposition 3.8.10(1), for $k = 1$ the formula is

$$\tau_{\mathrm{T},M} = \mathrm{t}_{\mathrm{T}(M)} \circ \mathrm{T}(\mathrm{t}_M)^{-1} = -\,\mathrm{id}_{\mathrm{T}^2(M)},$$

where $\mathrm{id}_{\mathrm{T}^2(M)}$ is the identity automorphism of the DG module $\mathrm{T}^2(M)$. Hence $\tau_{\mathrm{T}} = -\,\mathrm{id}_{\mathrm{T}^2}$. For other integers k the calculation is similar. □

4.3. **Cones and DG Functors.**

Definition 4.3.1. The subcategory $\mathbf{C}^0(A, \mathsf{M})$ of $\mathbf{C}(A, \mathsf{M})$ is defined to be the subcategory on all objects, but with degree 0 morphisms only.

There are inclusions of categories (faithful functors, identities on objects)

$$\mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \xrightarrow{\subseteq} \mathbf{C}^0(A, \mathsf{M}) \xrightarrow{\subseteq} \mathbf{C}(A, \mathsf{M}).$$

Forgetting the differentials is a fully faithful functor

$$\mathbf{C}^0(A, \mathsf{M}) \to \mathbf{G}^0(A, \mathsf{M}); \tag{4.3.2}$$

see Definition 3.1.13.

Let

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$$

be a $\mathbb{K}$-linear DG functor. Given a morphism $\phi : M_0 \to M_1$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$, we have a morphism

$$F(\phi) : F(M_0) \to F(M_1)$$

in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$, and objects $F(\mathrm{Cone}_{A,\mathsf{M}}(\phi))$ and $\mathrm{Cone}_{B,\mathsf{N}}(F(\phi))$ in $\mathbf{C}(B, \mathsf{N})$. By definition (and the fully faithful functor (4.3.2)) there is a canonical isomorphism

$$\mathrm{Cone}_{A,\mathsf{M}}(\phi) \cong M_1 \oplus \mathrm{T}_{A,\mathsf{M}}(M_0) \tag{4.3.3}$$

in $\mathbf{C}^0(A, \mathsf{M})$. Since F is an additive functor, it commutes with finite direct sums, and therefore there is a canonical isomorphism

$$F(\mathrm{Cone}_{A,\mathsf{M}}(\phi)) \cong F(M_1) \oplus F(\mathrm{T}_{A,\mathsf{M}}(M_0)) \tag{4.3.4}$$

in $\mathbf{C}^0(B, \mathsf{N})$. And by definition there is a canonical isomorphism

$$\mathrm{Cone}_{B,\mathsf{N}}(F(\phi)) \cong F(M_1) \oplus \mathrm{T}_{B,\mathsf{N}}(F(M_0)) \tag{4.3.5}$$

in $\mathbf{C}^0(B, \mathsf{N})$. Warning: the isomorphisms (4.3.3), (4.3.4) and (4.3.5) are usually not strict! They are degree 0 isomorphisms of graded modules, but they might not commute with the differentials; see Proposition 3.7.6. The differentials on the right sides are diagonal matrices, but on the left sides they are upper-triangular matrices (see Definition 3.9.1).

Lemma 4.3.6. *Let*

$$F, G : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$$

be $\mathbb{K}$-linear graded functors, and let $\eta : F \to G$ be a degree j morphism of graded functors. Suppose $M \cong M_0 \oplus M_1$ in $\mathbf{C}^0(A, \mathsf{M})$, with embeddings $e_i : M_i \to M$ and projections $p_i : M \to M_i$. Then

$$\eta_M = \big(G(e_0), G(e_1)\big) \circ (\eta_{M_0}, \eta_{M_1}) \circ \big(F(p_0), F(p_1)\big),$$

as degree j morphisms $F(M) \to G(M)$ in $\mathbf{C}(B, \mathsf{N})$.

The lemma says that the diagram

$$\begin{tikzcd}
F(M) \arrow[r, "\big(F(p_0){,}F(p_1)\big)"] \arrow[d, "\eta_M"'] & F(M_0) \oplus F(M_1) \arrow[d, "(\eta_{M_0}{,}\eta_{M_1})"] \\
G(M) & G(M_0) \oplus G(M_1) \arrow[l, "\big(G(e_0){,}G(e_1)\big)"']
\end{tikzcd}$$

in $\mathbf{C}(B, \mathsf{N})$ is commutative.

Proof. It suffices to prove that the diagram below is commutative for $i = 0, 1$:

$$\begin{tikzcd}
F(M_i) \arrow[r, "F(e_i)"] \arrow[d, "\eta_{M_i}"'] \arrow[rr, bend left, "\mathrm{id}"] & F(M) \arrow[r, "F(p_i)"] \arrow[d, "\eta_M"'] & F(M_i) \arrow[d, "\eta_{M_i}"'] \\
G(M_i) \arrow[r, "G(e_i)"] \arrow[rr, bend right, "\mathrm{id}"'] & G(M) \arrow[r, "G(p_i)"] & G(M_i)
\end{tikzcd}$$

This is true because η is a morphism of functors (a natural transformation). $\square$

Theorem 4.3.7. *Let*
$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$$
be a $\mathbb{K}$-linear DG functor, and let $\phi : M_0 \to M_1$ be a morphism in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. Define the isomorphism
$$\mathrm{cone}(F, \phi) : F(\mathrm{Cone}_{A,\mathsf{M}}(\phi)) \to \mathrm{Cone}_{B,\mathsf{N}}(F(\phi))$$
in $\mathbf{C}^0(B, \mathsf{N})$ to be
$$\mathrm{cone}(F, \phi) := (\mathrm{id}_{F(M_1)}, \tau_{F,M_0}).$$
Then:

(1) *The isomorphism* $\mathrm{cone}(F, \phi)$ *is strict*; *namely it commutes with the differentials.*

(2) *The diagram*

$$\begin{tikzcd}
F(M_0) \arrow[r, "F(\phi)"] \arrow[d, "="'] & F(M_1) \arrow[r, "F(e_\phi)"] \arrow[d, "="'] & F(\mathrm{Cone}_{A,\mathsf{M}}(\phi)) \arrow[r, "F(p_\phi)"] \arrow[d, "\mathrm{cone}(F,\phi)"'] & F(\mathrm{T}_{A,\mathsf{M}}(M_0)) \arrow[d, "\tau_{F,M_0}"'] \\
F(M_0) \arrow[r, "F(\phi)"] & F(M_1) \arrow[r, "e_{F(\phi)}"] & \mathrm{Cone}_{B,\mathsf{N}}(F(\phi)) \arrow[r, "p_{F(\phi)}"] & \mathrm{T}_{B,\mathsf{N}}(F(M_0))
\end{tikzcd}$$

in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$ is commutative.

When defining $\mathrm{cone}(F, \phi)$ above, we are using the decompositions (4.3.4) and (4.3.5) in the category $\mathbf{C}^0(B, \mathsf{N})$, and the isomorphism τ_{F,M_0} from Definition 4.2.1.

The slogan summarizing this theorem is "A DG functor sends standard triangles to standard triangles".

Proof. (1) To save space let us write $\theta := \mathrm{cone}(F, \phi)$. We have to prove that $\mathrm{d}_{B,\mathsf{N}}(\theta) = 0$. Let's write $P := \mathrm{Cone}_{A,\mathsf{M}}(\phi)$ and $Q := \mathrm{Cone}_{B,\mathsf{N}}(F(\phi))$. Recall that
$$\mathrm{d}_{B,\mathsf{N}}(\theta) = \mathrm{d}_Q \circ \theta - \theta \circ \mathrm{d}_{F(P)}.$$
We have to prove that this is the zero element in $\mathrm{Hom}_{B,\mathsf{N}}\big(F(P), Q\big)^1$.

Writing the cones as column modules:
$$P = \begin{bmatrix} M_1 \\ \mathrm{T}_{A,\mathsf{M}}(M_0) \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} F(M_1) \\ \mathrm{T}_{B,\mathsf{N}}(F(M_0)) \end{bmatrix},$$
the matrices representing the morphisms in question are
$$\theta = \begin{bmatrix} \mathrm{id}_{F(M_1)} & 0 \\ 0 & \tau_{F,M_0} \end{bmatrix}, \quad \mathrm{d}_P = \begin{bmatrix} \mathrm{d}_{M_1} & \phi \circ \mathrm{t}_{M_0}^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}_{A,\mathsf{M}}(M_0)} \end{bmatrix}$$
and
$$\mathrm{d}_Q = \begin{bmatrix} \mathrm{d}_{F(M_1)} & F(\phi) \circ \mathrm{t}_{F(M_0)}^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}_{B,\mathsf{N}}(F(M_0))} \end{bmatrix}.$$

Let us write $\gamma := \gamma_F$ for simplicity. According to Theorem 4.1.2, the gauge $\gamma : F \to F$ is a degree 1 morphism of functors $\mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$. Because the decomposition (4.3.3) is in the category $\mathbf{C}^0(A, \mathsf{M})$, Lemma 4.3.6 tells us that γ_P decomposes too, i.e.
$$\gamma_P = \begin{bmatrix} \gamma_{M_1} & 0 \\ 0 & \gamma_{\mathrm{T}_{A,\mathsf{M}}(M_0)} \end{bmatrix}.$$

By definition of γ_P we have

$$\mathrm{d}_{F(P)} = F(\mathrm{d}_P) + \gamma_P \in \mathrm{Hom}_{B,\mathsf{N}}\big(F(P), F(P)\big)^1.$$

It follows that

$$\begin{aligned}
\mathrm{d}_{F(P)} &= F(\mathrm{d}_P) + \gamma_P \\
&= \begin{bmatrix} F(\mathrm{d}_{M_1}) & F(\phi \circ \mathrm{t}_{M_0}^{-1}) \\ 0 & F(\mathrm{d}_{\mathrm{T}_{A,\mathsf{M}}(M_0)}) \end{bmatrix} + \begin{bmatrix} \gamma_{M_1} & 0 \\ 0 & \gamma_{\mathrm{T}_{A,\mathsf{M}}(M_0)} \end{bmatrix} \\
&= \begin{bmatrix} F(\mathrm{d}_{M_1}) + \gamma_{M_1} & F(\phi \circ \mathrm{t}_{M_0}^{-1}) \\ 0 & F(\mathrm{d}_{\mathrm{T}_{A,\mathsf{M}}(M_0)}) + \gamma_{\mathrm{T}_{A,\mathsf{M}}(M_0)} \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{d}_{F(M_1)} & F(\phi \circ \mathrm{t}_{M_0}^{-1}) \\ 0 & \mathrm{d}_{F(\mathrm{T}_{A,\mathsf{M}}(M_0))} \end{bmatrix}.
\end{aligned}$$

Finally we will check that $\theta \circ \mathrm{d}_{F(P)}$ and $\mathrm{d}_Q \circ \theta$ are equal as matrices of morphisms. We do that in each matrix position separately. The two left positions in the matrices $\theta \circ \mathrm{d}_{F(P)}$ and $\mathrm{d}_Q \circ \theta$ agree trivially. The bottom right positions in these matrices are $\tau_{F,M_0} \circ \mathrm{d}_{F(\mathrm{T}_{A,\mathsf{M}}(M_0))}$ and $\mathrm{d}_{\mathrm{T}_{B,\mathsf{N}}(F(M_0))} \circ \tau_{F,M_0}$ respectively; they are equal by Proposition 4.2.2. And in the top right positions we have $F(\phi \circ \mathrm{t}_{M_0}^{-1})$ and $F(\phi) \circ \mathrm{t}_{F(M_0)}^{-1} \circ \tau_{F,M_0}$ respectively. Now $F(\phi \circ \mathrm{t}_{M_0}^{-1}) = F(\phi) \circ F(\mathrm{t}_{M_0}^{-1})$; so it suffices to prove that $F(\mathrm{t}_{M_0}^{-1}) = \mathrm{t}_{F(M_0)}^{-1} \circ \tau_{F,M_0}$. This is immediate from the definition of τ_{F,M_0}.

(2) By definition of $\theta = \mathrm{cone}(F, \phi)$, the diagram is commutative in $\mathbf{C}^0(B, \mathsf{N})$. But by part (1) we know that all morphisms in it lie in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$. ☐

Corollary 4.3.8. *In the situation of Theorem* 4.3.7*, the diagram*

$$\begin{tikzcd}
F(M_0) \arrow[r, "F(\phi)"] \arrow[d, "="'] & F(M_1) \arrow[r, "F(e_\phi)"] \arrow[d, "="'] & F(\mathrm{Cone}_{A,\mathsf{M}}(\phi)) \arrow[r, "\tau_{F,M_0} \circ F(p_\phi)"] \arrow[d, "\mathrm{cone}(F,\phi)"'] & \mathrm{T}_{B,\mathsf{N}}(F(M_0)) \arrow[d, "="'] \\
F(M_0) \arrow[r, "F(\phi)"] & F(M_1) \arrow[r, "e_{F(\phi)}"] & \mathrm{Cone}_{B,\mathsf{N}}(F(\phi)) \arrow[r, "p_{F(\phi)}"] & \mathrm{T}_{B,\mathsf{N}}(F(M_0))
\end{tikzcd}$$

is an isomorphism of triangles in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$.

Proof. Just rearrange the diagram in item (2) of the theorem. ☐

4.4. **Examples of DG Functors.** Recall that M and N are $\mathbb{K}$-linear categories, and A and B are central DG $\mathbb{K}$-rings. Here are three examples of DG functors, of various types. These examples should serve as templates for constructing other DG functors.

Example 4.4.1. Here $A = B = \mathbb{K}$, so $\mathbf{C}(A, \mathsf{M}) = \mathbf{C}(\mathsf{M})$ and $\mathbf{C}(B, \mathsf{N}) = \mathbf{C}(\mathsf{N})$. Let $F : \mathsf{M} \to \mathsf{N}$ be a $\mathbb{K}$-linear functor. It extends to a functor

$$\mathbf{C}(F) : \mathbf{C}(\mathsf{M}) \to \mathbf{C}(\mathsf{N})$$

as follows: on objects, a complex

$$M = \big(\{M^i\}_{i \in \mathbb{Z}}, \{\mathrm{d}_M^i\}_{i \in \mathbb{Z}}\big) \in \mathbf{C}(\mathsf{M})$$

goes to the complex

$$\mathbf{C}(F)(M) := \big(\{F(M^i)\}, \{F(\mathrm{d}_M^i)\}\big) \in \mathbf{C}(\mathsf{N}).$$

A morphism $\phi = \{\phi^j\}$ in $\mathbf{C}(\mathsf{M})$ goes to the morphism $\mathbf{C}(\phi) := \{F(\phi^j)\}$ in $\mathbf{C}(\mathsf{N})$. A slightly tedious calculation shows that $\mathbf{C}(F)$ is a $\mathbb{K}$-linear DG functor.

Given a complex $M \in \mathbf{C}(\mathsf{M})$, let $N := \mathbf{C}(F)(M) \in \mathbf{C}(\mathsf{N})$. Then the translations are

$$\mathrm{T}_{\mathsf{N}}(N) = \mathbf{C}(F)\big(\mathrm{T}_{\mathsf{M}}(M)\big);$$

and $\mathbf{C}(F)(\mathrm{t}_M) = \mathrm{t}_N$. So the translation isomorphism

$$\tau_{\mathbf{C}(F)} : \mathbf{C}(F) \circ \mathrm{T}_{\mathsf{M}} \xrightarrow{\simeq} \mathrm{T}_{\mathsf{N}} \circ \mathbf{C}(F)$$

of functors $\mathbf{C}_{\mathrm{str}}(\mathsf{M}) \to \mathbf{C}_{\mathrm{str}}(\mathsf{N})$ is equality.

Let $\phi : M_0 \to M_1$ be a morphism in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, whose image under $\mathbf{C}(F)$ is the morphism $\psi : N_0 \to N_1$ in $\mathbf{C}_{\mathrm{str}}(\mathsf{N})$. Then

$$\mathrm{Cone}(\psi) = N_1 \oplus \mathrm{T}_{\mathsf{N}}(N_0) = \mathbf{C}(F)\big(\mathrm{Cone}(\phi)\big)$$

as graded objects of N, with differential

$$\mathrm{d}_{\mathrm{Cone}(\psi)} = \begin{bmatrix} \mathrm{d}_{N_1} & \psi \circ \mathrm{t}_{N_0}^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}(N_0)} \end{bmatrix} = \mathbf{C}(F)\Bigg(\begin{bmatrix} \mathrm{d}_{M_1} & \phi \circ \mathrm{t}_{M_0}^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}(M_0)} \end{bmatrix}\Bigg) = \mathbf{C}(F)\big(\mathrm{d}_{\mathrm{Cone}(\phi)}\big).$$

We see that the cone isomorphism θ is equality, and the gauge $\gamma_{\mathbf{C}(F)}$ is zero.

The next example is much more complicated, and we work out the full details (only once – later on, such details will be left to the reader).

Example 4.4.2. Let A and B be central DG $\mathbb{K}$-rings, and fix some

$$N \in \mathsf{DGMod}(B \otimes_{\mathbb{K}} A^{\mathrm{op}}).$$

In other words, N is a DG B-A-bimodule. For any $M \in \mathsf{DGMod}\, A$ we have a DG $\mathbb{K}$-module

$$F(M) := N \otimes_A M,$$

as in Definition 3.3.19. The differential of $F(M)$ is

$$\mathrm{d}_{F(M)} = \mathrm{d}_N \otimes \mathrm{id}_M + \mathrm{id}_N \otimes \mathrm{d}_M. \tag{4.4.3}$$

See Example 3.1.4 regarding the Koszul sign rule that's involved. But $F(M)$ has a structure of a DG B-module: for any $b \in B$, $n \in N$ and $m \in M$, the action is

$$b \cdot (n \otimes m) := (b \cdot n) \otimes m.$$

Clearly

$$F : \mathbf{C}(A) = \mathsf{DGMod}\, A \to \mathbf{C}(B) = \mathsf{DGMod}\, B$$

is a $\mathbb{K}$-linear functor. We will show that it is actually a DG functor.

Let $M_0, M_1 \in \mathbf{C}(A)$, and consider the $\mathbb{K}$-linear homomorphism

$$F : \mathrm{Hom}_A(M_0, M_1) \to \mathrm{Hom}_B(N \otimes_A M_0, N \otimes_A M_1). \tag{4.4.4}$$

Take any $\phi \in \mathrm{Hom}_A(M_0, M_1)^i$. Then

$$F(\phi) \in \mathrm{Hom}_B(N \otimes_A M_0, N \otimes_A M_1)$$

is the homomorphism that on a homogeneous tensor $n \otimes m \in (N \otimes_A M_0)^{k+j}$, with $n \in N^k$ and $m \in M_0^j$, has the value

$$F(\phi)(n \otimes m) = (-1)^{ik} \cdot n \otimes \phi(m) \in (N \otimes_A M_1)^{k+j+i}.$$

In other words,

$$F(\phi) = \mathrm{id}_N \otimes \phi. \tag{4.4.5}$$

We see that the homomorphism $F(\phi)$ has degree i. So F is a graded functor.

Let us calculate γ_F, the gauge of F. From (4.4.5) and (4.4.3) we get

$$\gamma_{F,M} = \mathrm{d}_N \otimes \mathrm{id}_M,$$

which is often a nonzero endomorphism of $F(M)$. Still, take any degree i morphism $\phi : M_0 \to M_1$ in $\mathbf{C}(A)$. Then

$$\begin{aligned} \gamma_{M_1} \circ F(\phi) &= (\mathrm{d}_N \otimes \mathrm{id}_{M_1}) \circ (\mathrm{id}_N \otimes \phi) \\ &= \mathrm{d}_N \otimes \phi = (-1)^i \cdot (\mathrm{id}_N \otimes \phi) \circ (\mathrm{d}_N \otimes \mathrm{id}_{M_0}) = (-1)^i \cdot F(\phi) \circ \gamma_{M_0}. \end{aligned}$$

We see that γ_F satisfies the condition of Definition 3.5.4(1), which is really Definition 3.1.17. By Theorem 4.1.2, F is a DG functor. (It is possible to calculate directly that F is a DG functor, but this takes more work.)

Finally let us figure out what is the translation isomorphism τ_F of the functor F. Take $M \in \mathbf{C}(A)$. Then

$$\tau_{F,M} : F(\mathrm{T}_A(M)) \to \mathrm{T}_B(F(M))$$

is an isomorphism in $\mathbf{C}_{\mathrm{str}}(B)$. By Definition 4.2.1 we have $\tau_{F,M} := \mathrm{t}_{F(M)} \circ F(\mathrm{t}_M)^{-1}$. Take any $n \in N^k$ and $m \in M^{j+1}$, so that

$$n \otimes \mathrm{t}_M(m) \in (N \otimes_A \mathrm{T}_A(M))^{k+j} = F(\mathrm{T}_A(M))^{k+j},$$

a typical degree $k+j$ element of $F(\mathrm{T}_A(M))$. But

$$n \otimes \mathrm{t}_M(m) = (-1)^k \cdot (\mathrm{id}_N \otimes \mathrm{t}_M)(n \otimes m) = (-1)^k \cdot F(\mathrm{t}_M)(n \otimes m).$$

Therefore

$$\tau_{F,M}(n \otimes \mathrm{t}_M(m)) = (-1)^k \cdot \mathrm{t}_{F(M)}(n \otimes m) \in \mathrm{T}_B(F(M))^{k+j}.$$

Observe that when N is concentrated in degree 0, we are back in the situation of Example 4.4.1, in which there are no sign twists, and $\tau_{F,M}$ is "equality".

Example 4.4.6. Let A and B be central DG $\mathbb{K}$-rings, and fix some

$$N \in \mathsf{DGMod}(A \otimes_{\mathbb{K}} B^{\mathrm{op}}).$$

For any $M \in \mathsf{DGMod}\, A$ we define

$$F(M) := \mathrm{Hom}_A(N, M).$$

This is a DG B-module: for any $b \in B^i$ and $\phi \in \mathrm{Hom}_A(N, M)^j$, the homomorphism $b \cdot \phi \in \mathrm{Hom}_A(N, M)^{i+j}$ has value

$$(b \cdot \phi)(n) := (-1)^{i \cdot (j+k)} \cdot \phi(n \cdot b) \in M^{i+j+k}$$

on $n \in N^k$. As in the previous example,

$$F : \mathbf{C}(A) = \mathsf{DGMod}\, A \to \mathbf{C}(B) = \mathsf{DGMod}\, B$$

is a $\mathbb{K}$-linear graded functor.

The value of the gauge γ_F at $M \in \mathbf{C}(A)$ is

$$\gamma_{F,M} = \mathrm{Hom}(\mathrm{d}_N, \mathrm{id}_M).$$

See Example 3.1.5 regarding this notation. Namely for

$$\psi \in F(M)^j = \mathrm{Hom}_A(N, M)^j$$

we have

$$\gamma_{F,M}(\psi) = (-1)^j \cdot \psi \circ \mathrm{d}_N.$$

It is not too hard to check that γ_F is a degree 1 morphism of functors. Hence, by Theorem 4.1.2, F is a DG functor.

The formula for the translation isomorphism τ_F is as follows. Take $M \in \mathbf{C}(A)$. Then

$$\tau_{F,M} : F(\mathrm{T}_A(M)) = \mathrm{Hom}_A(N, \mathrm{T}_A(M)) \to \mathrm{T}_B(F(M)) = \mathrm{T}_B(\mathrm{Hom}_A(N, M))$$

is, by definition, $\tau_{F,M} = \mathrm{t}_{F(M)} \circ F(\mathrm{t}_M)^{-1}$. Now

$$F(\mathrm{t}_M)^{-1} = \mathrm{Hom}(\mathrm{id}_N, \mathrm{t}_M^{-1}).$$

So given any $\psi \in F(\mathrm{T}_A(M))^k$, we have

$$\tau_{F,M}(\psi) = \mathrm{t}_{F(M)}(\mathrm{t}_M^{-1} \circ \psi) \in \mathrm{T}_B(F(M))^k.$$

5. Pretriangulated Categories and Triangulated Functors

In this section we introduce pretriangulated categories and triangulated functors. "Pretriangulated" means that the octahedral axiom is not required to hold. There is one result here that seems to be new: Theorem 5.4.13, which asserts that a DG functor between DG module categories induces a triangulated functor between the associated homotopy categories.

As in previous sections, we fix a base commutative ring $\mathbb{K}$. All linear categories and linear functors here are implicitly assumed to be $\mathbb{K}$-linear. In particular, this assumption says that all DG rings are central $\mathbb{K}$-rings, and all DG ring homomorphisms are $\mathbb{K}$-linear.

5.1. **T-Additive Categories.** Recall that a functor is called an isomorphism of categories if it is bijective of sets of objects and on sets of morphisms; see Example 1.5.2.

Definition 5.1.1. Let K be an additive category. A *translation* on K is an additive automorphism T of K, called the *translation functor*. The pair (K, T) is called a *T-additive category.*

Remark 5.1.2. Some texts give a more relaxed definition: T is only required to be an additive auto-equivalence of K. The resulting theory is more complicated (it is 2-categorical, but most texts try to suppress this fact).

Later in the book we will write $M[k] := \mathrm{T}^k(M)$, the k-th translation of an object M.

Definition 5.1.3. Suppose $(\mathsf{K}, \mathrm{T}_{\mathsf{K}})$ and $(\mathsf{L}, \mathrm{T}_{\mathsf{L}})$ are T-additive categories. A *T-additive functor* between them is a pair (F, τ), consisting of an additive functor $F : \mathsf{K} \to \mathsf{L}$, together with an isomorphism

$$\tau : F \circ \mathrm{T}_{\mathsf{K}} \xrightarrow{\simeq} \mathrm{T}_{\mathsf{L}} \circ F$$

of functors $\mathsf{K} \to \mathsf{L}$, called a *translation isomorphism.*

Definition 5.1.4. Let $(\mathsf{K}_i, \mathrm{T}_i)$ be T-additive categories, for $i = 0, 1, 2$, and let

$$(F_i, \tau_i) : (\mathsf{K}_{i-1}, \mathrm{T}_{i-1}) \to (\mathsf{K}_i, \mathrm{T}_i)$$

be T-additive functors. The composition

$$(F, \tau) = (F_2, \tau_2) \circ (F_1, \tau_1)$$

is the T-additive functor $(\mathsf{K}_0, \mathrm{T}_0) \to (\mathsf{K}_2, \mathrm{T}_2)$ defined as follows: the functor is $F := F_2 \circ F_1$, and the translation isomorphism

$$\tau : F \circ T_0 \xrightarrow{\simeq} T_2 \circ F$$

is $\tau := \tau_2 \circ F_2(\tau_1)$.

Definition 5.1.5. Suppose $(\mathsf{K}, \mathrm{T}_{\mathsf{K}})$ and $(\mathsf{L}, \mathrm{T}_{\mathsf{L}})$ are T-additive categories, and

$$(F, \tau), (G, \nu) : (\mathsf{K}, \mathrm{T}_{\mathsf{K}}) \to (\mathsf{L}, \mathrm{T}_{\mathsf{L}})$$

are T-additive functors. A *morphism of T-additive functors*

$$\eta : (F, \tau) \to (G, \nu)$$

is a morphism of functors $\eta : F \to G$, such that for every object $M \in \mathsf{K}$ this diagram in L is commutative:

$$\begin{tikzcd} F(\mathrm{T}_{\mathsf{K}}(M)) \arrow[r, "\tau_M"] \arrow[d, "\eta_{\mathrm{T}_{\mathsf{K}}(M)}"'] & \mathrm{T}_{\mathsf{L}}(F(M)) \arrow[d, "\mathrm{T}_{\mathsf{L}}(\eta_M)"] \\ G(\mathrm{T}_{\mathsf{K}}(M)) \arrow[r, "\nu_M"] & \mathrm{T}_{\mathsf{L}}(G(M)) \end{tikzcd} .$$

Definition 5.1.6. Let (K, T) be an additive category with translation. A *triangle* in (K, T) is a diagram

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

in K.

Definition 5.1.7. Let (K, T) be a T-additive category. Suppose

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

and

$$L' \xrightarrow{\alpha'} M' \xrightarrow{\beta'} N' \xrightarrow{\gamma'} \mathrm{T}(L')$$

are triangles in (K, T). A *morphism of triangles* between them is a commutative diagram

$$\begin{tikzcd} L \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[r, "\beta"] \arrow[d, "\psi"'] & N \arrow[r, "\gamma"] \arrow[d, "\chi"'] & \mathrm{T}(L) \arrow[d, "\mathrm{T}(\phi)"'] \\ L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L') \end{tikzcd}$$

in K.

The morphism of triangles (ϕ, ψ, χ) is called an isomorphism if ϕ, ψ and χ are all isomorphisms.

Remark 5.1.8. Why "triangle"? This is because sometimes a triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

is written as a diagram

$$\begin{tikzcd} & N \arrow[dl, "\gamma"'] & \\ L \arrow[rr, "\alpha"] & & M \arrow[ul, "\beta"'] \end{tikzcd}$$

But here γ is a morphism "of degree 1".

5.2. **Pretriangulated Categories.**

Definition 5.2.1. A *pretriangulated category* is a T-additive category (K, T), equipped with a set of triangles called *distinguished triangles*. The following axioms have to be satisfied:

(TR1) (a) Any triangle that is isomorphic to a distinguished triangle is also a distinguished triangle.
(b) For every morphism $\alpha : L \to M$ in K there is a distinguished triangle

$$L \xrightarrow{\alpha} M \to N \to \mathrm{T}(L).$$

(c) For every object M the triangle

$$M \xrightarrow{1_M} M \to 0 \to \mathrm{T}(M)$$

is distinguished.

(TR2) A triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

is distinguished iff the triangle

$$M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L) \xrightarrow{-\mathrm{T}(\alpha)} \mathrm{T}(M)$$

is distinguished.

(TR3) Suppose

$$\begin{array}{ccccccc} L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \xrightarrow{\gamma} & \mathrm{T}(L) \\ \phi \downarrow & & \psi \downarrow & & & & \\ L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & \xrightarrow{\gamma'} & \mathrm{T}(L') \end{array}$$

is a commutative diagram in K in which the rows are distinguished triangles. Then there exists a morphism $\chi : N \to N'$ such that the diagram

$$\begin{array}{ccccccc} L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \xrightarrow{\gamma} & \mathrm{T}(L) \\ \phi \downarrow & & \psi \downarrow & & \chi \downarrow & & \downarrow \mathrm{T}(\phi) \\ L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' & \xrightarrow{\gamma'} & \mathrm{T}(L') \end{array} .$$

is a morphism of triangles.

Remark 5.2.2. The numbering of the axioms we use is taken from [RD]; the numbering in [Sc], [KaSc1] [KaSc2] and [Ne1] is different.

In the situation that we care about, namely $\mathsf{K} = \mathbf{K}(A, \mathsf{M})$, the distinguished triangles will be those triangles that are isomorphic, in $\mathbf{K}(A, \mathsf{M})$, to the standard triangles in $\mathbf{C}(A, \mathsf{M})$ from Definition 3.9.5. See Definition 5.4.3 below for the precise statement.

The object N in item (b) of axiom (TR1) is referred to as a *cone* on $\alpha : L \to M$. We should think of the cone as something combining "the cokernel" and "the kernel" of α.

Axiom (TR2) says that if we "turn" a distinguished triangle we remain with a distinguished triangle.

Axiom (TR3) says that a commutative square (ϕ, ψ) induces a morphism χ on the cones of the horizontal morphisms, that fits into a morphism of distinguished triangles (ϕ, ψ, χ). Note however that the new morphism χ is *not unique*; in other words, *cones are not functorial*. This fact has some deep consequences in many applications. However, in the situations that will interest us, namely when $\mathsf{K} = \mathbf{K}(A, \mathsf{M})$, the cones come from the standard cones in $\mathbf{C}(A, \mathsf{M})$; and the standard cones in $\mathbf{C}(A, \mathsf{M})$ are functorial (Definition 3.9.6).

Remark 5.2.3. There is a fourth axiom in the literature, called the *octahedral axiom*; it is axiom (TR4) in [RD]. Keeping with the traditional usage, the name *triangulated category* is reserved for a pretriangulated category that also satisfies this extra axiom.

Because the octahedral axiom is extremely cumbersome to state (and prove), and also it does not play any role in the study of derived categories, we have decided to ignore it completely in our book.

For the role of the octahedral axiom in the structure of abstract triangulated categories, see the book [Ne1]. It is not known whether the octahedral axiom is a consequence of the other axioms; there was a recent paper by Maccioca (arxiv:1506.00887) claiming that, but it had a fatal error in it.

The reader should not confuse the meaning of the name "pretriangulated category", as used here, with the "pretriangulated DG category" from [BoKa]. See Remark 5.4.15.

For a category K there is a canonical contravariant functor $\mathrm{op} : \mathsf{K} \to \mathsf{K}^{\mathrm{op}}$, that is the identity on objects, and reverses the arrows. Note that op is an anti-isomorphism of categories (i.e. a contravariant isomorphism), so its inverse op^{-1} is unique.

Proposition 5.2.4. *Let K be a pretriangulated category, and let $\mathrm{op} : \mathsf{K} \to \mathsf{K}^{\mathrm{op}}$ be the canonical contravariant functor from K to its opposite category. Define a translation T^{op} on K^{op} by the formula $\mathrm{T}^{\mathrm{op}} := \mathrm{op} \circ \mathrm{T}^{-1} \circ \mathrm{op}^{-1}$. The distinguished triangles in K^{op} are defined to be the triangles*

$$N \xrightarrow{\mathrm{op}(\beta)} M \xrightarrow{\mathrm{op}(\alpha)} L \xrightarrow{\mathrm{op}(-\mathrm{T}^{-1}(\gamma))} \mathrm{T}^{\mathrm{op}}(N),$$

where $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$ is any distinguished triangle in K. Then $(\mathsf{K}^{\mathrm{op}}, \mathrm{T}^{\mathrm{op}})$ is a pretriangulated category.

Proof. This is an exercise. (Hint: use the proof of Proposition 5.3.3 below.) □

Proposition 5.2.5. *Let K be a pretriangulated category. If*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

is a distinguished triangle in K, then $\beta \circ \alpha = 0$.

Proof. By axioms (TR1) and (TR3) we have a commutative diagram

$$\begin{tikzcd}
L \arrow[r, "1_L"] \arrow[d, "1_L"'] & L \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[r] \arrow[d] & \mathrm{T}(L) \arrow[d, "\mathrm{T}(1_L)"'] \\
L \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & N \arrow[r, "\gamma"] & \mathrm{T}(L) .
\end{tikzcd}$$

We see that $\beta \circ \alpha$ factors through 0. □

5.3. **Triangulated and Cohomological Functors.** Suppose K and L are T-additive categories, with translation functors T_{K} and T_{L} respectively. The notion of T-additive functor $F : \mathsf{K} \to \mathsf{L}$ was defined in Definition 5.1.3 In that definition we also introduced the notion of morphism $\eta : F \to G$ between T-additive functors.

Definition 5.3.1. Let K and L be pretriangulated categories.

(1) A *triangulated functor* from K to L is a T-additive functor

$$(F, \tau) : \mathsf{K} \to \mathsf{L}$$

that satisfies this condition: for any distinguished triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}_{\mathsf{K}}(L)$$

in K, the triangle

$$F(L) \xrightarrow{F(\alpha)} F(M) \xrightarrow{F(\beta)} F(N) \xrightarrow{\tau_L \circ F(\gamma)} \mathrm{T}_{\mathsf{L}}(F(L))$$

is a distinguished triangle in L.

(2) Suppose $(G, \nu) : \mathsf{K} \to \mathsf{L}$ is another triangulated functor. A *morphism of triangulated functors* $\eta : (F, \tau) \to (G, \nu)$ is a morphism of T-additive functors, as in Definition 5.1.5 .

Sometimes we keep the translation isomorphism τ implicit, and refer to F as a triangulated functor.

Definition 5.3.2. Let K be a pretriangulated category, and let M be an abelian category. A *cohomological functor* $F : \mathsf{K} \to \mathsf{M}$ is an additive functor, such that for every distinguished triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

in K, the sequence

$$F(L) \xrightarrow{F(\alpha)} F(M) \xrightarrow{F(\beta)} F(N)$$

is exact in M.

Proposition 5.3.3. *Let $F : \mathsf{K} \to \mathsf{M}$ be a cohomological functor, and let*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

be a distinguished triangle in K. Then the sequence

$$\cdots \to F(\mathrm{T}^i(L)) \xrightarrow{F(\mathrm{T}^i(\alpha))} F(\mathrm{T}^i(M)) \xrightarrow{F(\mathrm{T}^i(\beta))} F(\mathrm{T}^i(N)) \xrightarrow{F(\mathrm{T}^i(\gamma))} F(\mathrm{T}^{i+1}(L))$$
$$\xrightarrow{F(\mathrm{T}^{i+1}(\alpha))} F(\mathrm{T}^{i+1}(M)) \to \cdots$$

in M is exact.

Proof. By axiom (TR2) we have distinguished triangles

$$\mathrm{T}^i(L) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\alpha)} \mathrm{T}^i(M) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\beta)} \mathrm{T}^i(N) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\gamma)} \mathrm{T}^{i+1}(L),$$

$$\mathrm{T}^i(M) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\beta)} \mathrm{T}^i(N) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\gamma)} \mathrm{T}^{i+1}(L) \xrightarrow{(-1)^{i+1} \cdot \mathrm{T}^{i+1}(\alpha)} \mathrm{T}^{i+1}(M)$$

and

$$\mathrm{T}^i(N) \xrightarrow{(-1)^i \cdot \mathrm{T}^i(\gamma)} \mathrm{T}^{i+1}(L) \xrightarrow{(-1)^{i+1} \cdot \mathrm{T}^{i+1}(\alpha)} \mathrm{T}^{i+1}(M) \xrightarrow{(-1)^{i+1} \cdot \mathrm{T}^{i+1}(\beta)} \mathrm{T}^{i+1}(N).$$

Now use the definition, noting that multiplying morphisms in an exact sequence by -1 preserves exactness. □

Proposition 5.3.4. *Let K be a pretriangulated category. For any $P \in \mathsf{K}$ the functors*

$$\mathrm{Hom}_{\mathsf{K}}(-, P) : \mathsf{K}^{\mathrm{op}} \to \mathsf{Ab}$$

and

$$\mathrm{Hom}_{\mathsf{K}}(P, -) : \mathsf{K} \to \mathsf{Ab}$$

are cohomological functors.

Proof. We will prove the covariant statement; the contravariant statement is an immediate consequence, since

$$\mathrm{Hom}_{\mathsf{K}}(M, P) = \mathrm{Hom}_{\mathsf{K}^{\mathrm{op}}}(P, M),$$

and K^{op} is pretriangulated (with the correct pretriangulated structure to make this true).

Consider a distinguished triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

in K. We have to prove that the sequence

$$\mathrm{Hom}_{\mathsf{K}}(P, L) \xrightarrow{\mathrm{Hom}(1_P, \alpha)} \mathrm{Hom}_{\mathsf{K}}(P, M) \xrightarrow{\mathrm{Hom}(1_P, \beta)} \mathrm{Hom}_{\mathsf{K}}(P, N)$$

is exact. In view of Proposition 5.2.5, all we need to show is that for any $\psi : P \to M$ s.t. $\beta \circ \psi = 0$, there is some $\phi : P \to L$ s.t. $\psi = \alpha \circ \phi$. In a picture, we must show that the diagram below (solid arrows)

$$\begin{tikzcd}
P \arrow[r, "1"] \arrow[d, dashed, "\phi"'] & P \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & \mathrm{T}(P) \arrow[d, dashed, "\mathrm{T}(\phi)"'] \\
L \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & N \arrow[r, "\gamma"] & \mathrm{T}(L) .
\end{tikzcd}$$

can be completed (dashed arrow). This is true by (TR2) (= turning) and and (TR3) (= extending). $\square$

Proposition 5.3.5. *Let* K *be a pretriangulated category, and let*

$$\begin{tikzcd}
L \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[r, "\beta"] \arrow[d, "\psi"'] & N \arrow[r, "\gamma"] \arrow[d, "\chi"'] & \mathrm{T}(L) \arrow[d, "\mathrm{T}(\phi)"'] \\
L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L') .
\end{tikzcd}$$

be a morphism of distinguished triangles. If ϕ *and* ψ *are isomorphisms, then* χ *is also an isomorphism.*

Proof. Take an arbitrary $P \in \mathsf{K}$, and let $F := \mathrm{Hom}_{\mathsf{K}}(P, -)$. We get a commutative diagram

$$\begin{tikzcd}
F(L) \arrow[r, "F(\alpha)"] \arrow[d, "F(\phi)"'] & F(M) \arrow[r, "F(\beta)"] \arrow[d, "F(\psi)"'] & F(N) \arrow[r, "F(\gamma)"] \arrow[d, "F(\chi)"'] & F(\mathrm{T}(L)) \arrow[r, "F(\mathrm{T}(\alpha))"] \arrow[d, "F(\mathrm{T}(\phi))"'] & F(\mathrm{T}(M)) \arrow[d, "F(\mathrm{T}(\psi))"'] \\
F(L') \arrow[r, "F(\alpha')"] & F(M') \arrow[r, "F(\beta')"] & F(N') \arrow[r, "F(\gamma')"] & F(\mathrm{T}(L')) \arrow[r, "F(\mathrm{T}(\alpha'))"] & F(\mathrm{T}(M'))
\end{tikzcd}$$

in Ab. By Proposition 5.3.4(2) the rows in the diagram are exact sequences. Since the other vertical arrows are isomorphisms, it follows that

$$F(\chi) : \mathrm{Hom}_{\mathsf{K}}(P, N) \to \mathrm{Hom}_{\mathsf{K}}(P, N')$$

is an isomorphism of abelian groups. By forgetting structure, we see that $F(\chi)$ is an isomorphism of sets.

We now use the Yoneda Lemma. Let us write $Y_N := \mathrm{Hom}_{\mathsf{K}}(-, N)$ and $Y_{N'} := \mathrm{Hom}_{\mathsf{K}}(-, N')$, viewed as functors $\mathsf{K}^{\mathrm{op}} \to \mathsf{Set}$. For any object $P \in \mathsf{K}$ we have isomorphisms of sets $Y_N(P) \cong F(N)$ and $Y_{N'}(P) \cong F(N')$. The calculation above shows that the morphism of functors $Y(\chi) : Y_N \to Y_{N'}$ is an isomorphism. According to Proposition 1.7.1(2), the morphism $\chi : N \to N'$ in K is an isomorphism. □

Proposition 5.3.6. *Let* K *be a pretriangulated category, and let*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

be a distinguished triangle in it. The two conditions below are equivalent:

(i) $\alpha : L \to M$ *is an isomorphism.*
(ii) $N \cong 0$.

Proof. Exercise. (Hint: use Proposition 5.3.5.) □

Question 5.3.7. Let K and L be pretriangulated categories, and let $F : \mathsf{K} \to \mathsf{L}$ be an additive functor. Is it true that there is at most one isomorphism of functors $\tau : F \circ \mathrm{T}_{\mathsf{K}} \xrightarrow{\simeq} \mathrm{T}_{\mathsf{L}} \circ F$ such that the pair (F, τ) is a triangulated functor?

5.4. **The Homotopy Category is Pretriangulated.** In this subsection we consider an abelian category M and a DG ring A (everything over the base ring $\mathbb{K}$). These ingredients give rise to the $\mathbb{K}$-linear DG category $\mathbf{C}(A, \mathsf{M})$ of DG A-module in M, as in Subsection 3.7.

The strict category $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ and the homotopy category $\mathbf{K}(A, \mathsf{M})$ was introduced in Definition 3.7.5. Recall that these linear categories have the same objects as $\mathbf{C}(A, \mathsf{M})$. The morphisms groups are

$$\mathrm{Hom}_{\mathbf{C}_{\mathrm{str}}(A,\mathsf{M})}(M_0, M_1) = \mathrm{Z}^0\big(\mathrm{Hom}_{\mathbf{C}(A,\mathsf{M})}(M_0, M_1)\big)$$

and

$$\mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(M_0, M_1) = \mathrm{H}^0\big(\mathrm{Hom}_{\mathbf{C}(A,\mathsf{M})}(M_0, M_1)\big).$$

Thus the morphisms $M_0 \to M_1$ in $\mathbf{K}(A, \mathsf{M})$ are the homotopy classes $\bar{\phi} : M_0 \to M_1$ of the morphisms $\phi : M_0 \to M_1$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$.

Recall the full additive functor

$$\mathrm{P} : \mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \to \mathbf{K}(A, \mathsf{M}) \tag{5.4.1}$$

from Definition 3.4.4, that is the identity on objects, and on morphisms it is $\mathrm{P}(\phi) := \bar{\phi}$.

Consider the translation functor T from Definition 3.8.8. Since T is a DG functor from $\mathbf{C}(A, \mathsf{M})$ to itself (see Corollary 3.8.9), it restricts to a linear functor from $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ to itself, and it induces a linear functor $\bar{\mathrm{T}}$ from $\mathbf{K}(A, \mathsf{M})$ to itself, such that $P \circ \mathrm{T} = \bar{\mathrm{T}} \circ P$.

Proposition 5.4.2.

(1) *The category* $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$*, equipped with the translation functor* T*, is a T-additive category.*
(2) *The category* $\mathbf{K}(A, \mathsf{M})$*, equipped with the translation functor* $\bar{\mathrm{T}}$*, is a T-additive category.*
(3) *Let* $\tau : \mathrm{P} \circ \mathrm{T} \xrightarrow{\simeq} \bar{\mathrm{T}} \circ \mathrm{P}$ *be equality. Then the pair*

$$(\mathrm{P}, \tau) : \mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \to \mathbf{K}(A, \mathsf{M})$$

is a T-additive functor.

Proof. (1) We need to prove that $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$ is additive. Of course the zero complex is a zero object. Next we consider finite direct sums. Let $M_1, \ldots, M_r$ be a finite collection of objects in $\mathsf{C}(A, \mathsf{M})$. Each M_i is a DG A-module in M, and we write it as $M_i = \{M_i^j\}_{j \in \mathbb{Z}}$. In each degree j the direct sum $M^j := \bigoplus_{i=1}^r M_i^j$ exists in M. Let $M := \{M^j\}_{j \in \mathbb{Z}}$ be the resulting graded object in M. The differential $\mathrm{d}_M : M^j \to M^{j+1}$ exists by the universal property of direct sums; so we obtain a complex $M \in \mathsf{C}(\mathsf{M})$. The DG A-module structure on M is defined similarly: for $a \in A^k$, there is an induced degree k morphism $f(a) : M \to M$ in $\mathsf{C}(\mathsf{M})$. Thus M becomes an object of $\mathsf{C}(A, \mathsf{M})$. But the embeddings $e_i : M_i \to M$ are strict morphisms, so $(M, \{e_i\})$ is a coproduct of the collection $\{M_i\}$ in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$.

(2) Now consider the category $\mathsf{K}(A, \mathsf{M})$. Because the functor $\mathrm{P} : \mathsf{C}_{\mathrm{str}}(A, \mathsf{M}) \to \mathsf{K}(A, \mathsf{M})$ is additive, and is bijective on objects, part (1) above and Proposition 2.4.2 say that $\mathsf{K}(A, \mathsf{M})$ is an additive category.

(3) Clear. □

From now on we denote by T, instead of by $\bar{\mathrm{T}}$, the translation functor of $\mathsf{K}(A, \mathsf{M})$.

Definition 5.4.3. A triangle

$$L \xrightarrow{\bar{\alpha}} M \xrightarrow{\bar{\beta}} N \xrightarrow{\bar{\gamma}} \mathrm{T}(L)$$

in $\mathsf{K}(A, \mathsf{M})$ is said to be a *distinguished triangle* if there is a standard triangle

$$L' \xrightarrow{\alpha'} M' \xrightarrow{\beta'} N' \xrightarrow{\gamma'} \mathrm{T}(L')$$

in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$, as in Definition 3.9.5, and an isomorphism of triangles

$$\begin{tikzcd}
L' \arrow[r, "\mathrm{P}(\alpha')"] \arrow[d, "\bar{\phi}"'] & M' \arrow[r, "\mathrm{P}(\beta')"] \arrow[d, "\bar{\psi}"'] & N' \arrow[r, "\mathrm{P}(\gamma')"] \arrow[d, "\bar{\chi}"'] & \mathrm{T}(L') \arrow[d, "\mathrm{T}(\bar{\phi})"'] \\
L \arrow[r, "\bar{\alpha}"] & M \arrow[r, "\bar{\beta}"] & N \arrow[r, "\bar{\gamma}"] & \mathrm{T}(L)
\end{tikzcd} .$$

in $\mathsf{K}(A, \mathsf{M})$.

Theorem 5.4.4. *The T-additive category $\mathsf{K}(A, \mathsf{M})$, with the set of distinguished triangles defined above, is a pretriangulated category.*

The proof is after three lemmas.

Lemma 5.4.5. *Let $M \in \mathsf{C}(A, \mathsf{M})$, and consider the cone $N := \mathrm{Cone}(1_M)$. Then the DG module N is null-homotopic, i.e. $0 \to N$ is an isomorphism in $\mathsf{K}(A, \mathsf{M})$.*

Proof. We shall exhibit a homotopy θ from 0_N to 1_N. Recall from Subsection 3.9 that

$$N = \mathrm{Cone}(1_M) = M \oplus \mathrm{T}(M) = \begin{bmatrix} M \\ \mathrm{T}(M) \end{bmatrix}$$

as graded modules, with differential whose matrix presentation is

$$\mathrm{d}_N = \begin{bmatrix} \mathrm{d}_M & \mathrm{t}_M^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}(M)} \end{bmatrix} .$$

And by the definition in Subsection 3.8 we have

$$\mathrm{d}_{\mathrm{T}(M)} = -\,\mathrm{t}_M \circ \mathrm{d}_M \circ \mathrm{t}_M^{-1} .$$

Define $\theta : N \to N$ to be the degree -1 morphism with matrix presentation

$$\theta := \begin{bmatrix} 0 & 0 \\ \mathrm{t}_M & 0 \end{bmatrix}.$$

Then, using the formulas above for d_N and $\mathrm{d}_{\mathrm{T}(M)}$, we get

$$\mathrm{d}_N \circ \theta + \theta \circ \mathrm{d}_N = \begin{bmatrix} 1_M & 0 \\ 0 & 1_{\mathrm{T}(M)} \end{bmatrix} = 1_N.$$

□

Exercise 5.4.6. Here is a generalization of Lemma 5.4.5. Consider a morphism $\phi : M_0 \to M_1$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. Show that the three conditions below are equivalent:

(i) ϕ is a homotopy equivalence.
(ii) $\bar{\phi}$ is an isomorphism in $\mathbf{K}(A, \mathsf{M})$.
(iii) The DG module $\mathrm{Cone}(\phi)$ is null-homotopic.

Try to do this directly, not using Proposition 5.3.4(2) and Theorem 5.4.4.

The next lemma is based on [KaSc1, Lemma 1.4.2].

Lemma 5.4.7. *Consider a morphism $\alpha : L \to M$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$, the standard triangle*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

associated to α, and the standard triangle

$$M \xrightarrow{\beta} N \xrightarrow{\phi} P \xrightarrow{\psi} \mathrm{T}(M)$$

associated to β, all in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. So $N = \mathrm{Cone}(\alpha)$ and and $P = \mathrm{Cone}(\beta)$. There is a morphism $\rho : \mathrm{T}(L) \to P$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ s.t. $\bar{\rho}$ is an isomorphism in $\mathbf{K}(A, \mathsf{M})$, and the diagram

$$\begin{tikzcd} M \arrow[r, "\bar{\beta}"] \arrow[d, "\bar{1}_M"'] & N \arrow[r, "\bar{\gamma}"] \arrow[d, "\bar{1}_N"'] & \mathrm{T}(L) \arrow[r, "-\mathrm{T}(\bar{\alpha})"] \arrow[d, "\bar{\rho}"'] & \mathrm{T}(M) \arrow[d, "\bar{1}_{\mathrm{T}(M)}"'] \\ M \arrow[r, "\bar{\beta}"] & N \arrow[r, "\bar{\phi}"] & P \arrow[r, "\bar{\psi}"] & \mathrm{T}(M) \end{tikzcd}$$

commutes in $\mathbf{K}(A, \mathsf{M})$.

Proof. Note that $N = M \oplus \mathrm{T}(L)$ and $P = N \oplus \mathrm{T}(M) = M \oplus \mathrm{T}(L) \oplus \mathrm{T}(M)$ as graded module. Thus P and d_P have the following matrix presentations:

$$P = \begin{bmatrix} M \\ \mathrm{T}(L) \\ \mathrm{T}(M) \end{bmatrix}, \quad \mathrm{d}_P = \begin{bmatrix} \mathrm{d}_M & \alpha \circ \mathrm{t}_L^{-1} & \mathrm{t}_M^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}(L)} & 0 \\ 0 & 0 & \mathrm{d}_{\mathrm{T}(M)} \end{bmatrix}.$$

Define morphisms $\rho : \mathrm{T}(L) \to P$ and $\chi : P \to \mathrm{T}(L)$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ by the matrix presentations

$$\rho := \begin{bmatrix} 0 \\ 1_{\mathrm{T}(L)} \\ -\mathrm{T}(\alpha) \end{bmatrix}, \quad \chi := \begin{bmatrix} 0 & 1_{\mathrm{T}(L)} & 0 \end{bmatrix}.$$

Direct calculations show that:

- $\chi \circ \rho = 1_{\mathrm{T}(L)}$.
- $\rho \circ \gamma = \rho \circ \chi \circ \phi$.
- $\psi \circ \rho = -\mathrm{T}(\alpha)$.

It remains to prove that $\rho \circ \chi$ is homotopic to 1_P. Define a degree -1 morphism $\theta : P \to P$ by the matrix

$$\theta := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathrm{t}_M & 0 & 0 \end{bmatrix}.$$

Then a direct calculation, using the equalities

$$\mathrm{t}_M \circ \mathrm{d}_M + \mathrm{d}_{\mathrm{T}(M)} \circ \mathrm{t}_M = 0$$

and

$$\mathrm{T}(\alpha) = \mathrm{t}_M \circ \alpha \circ \mathrm{t}_L^{-1}$$

gives

$$\theta \circ \mathrm{d}_P + \mathrm{d}_P \circ \theta = 1_P - \rho \circ \chi.$$

□

Lemma 5.4.8. *Consider a standard triangle*

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. *For any integer* k, *the triangle*

$$\mathrm{T}^k(L) \xrightarrow{\mathrm{T}^k(\alpha)} \mathrm{T}^k(M) \xrightarrow{\mathrm{T}^k(\beta)} \mathrm{T}^k(N) \xrightarrow{(-1)^k \cdot \mathrm{T}^k(\gamma)} \mathrm{T}^{k+1}(L)$$

is isomorphic, in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$, *to a standard triangle.*

Proof. Combine Corollary 3.8.9, Corollary 4.3.8 with $F = \mathrm{T}$, and Proposition 4.2.4. □

Proof of Theorem 5.4.4. (TR1): By definition the set of distinguished triangles in $\mathbf{K}(A, \mathsf{M})$ is closed under isomorphisms. This establishes item (a).

As for item (b): consider any morphism $\bar{\alpha} : L \to M$ in $\mathbf{K}(A, \mathsf{M})$. It is represented by a morphism $\alpha : L \to M$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. Take the standard triangle on α in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. Its image in $\mathbf{K}(A, \mathsf{M})$ has the desired property.

Finally, Lemma 5.4.5 shows that the triangle

$$M \xrightarrow{\bar{1}_M} M \to 0 \to \mathrm{T}(M)$$

is isomorphic in $\mathbf{K}(A, \mathsf{M})$ to the triangle

$$M \xrightarrow{\bar{1}_M} M \xrightarrow{\bar{e}} \mathrm{Cone}(1_M) \xrightarrow{\bar{p}} \mathrm{T}(M).$$

The latter is the image of a standard triangle, and so it is distinguished.

(TR2): Consider the triangles

$$L \xrightarrow{\bar{\alpha}} M \xrightarrow{\bar{\beta}} N \xrightarrow{\bar{\gamma}} \mathrm{T}(L) \tag{5.4.9}$$

and

$$M \xrightarrow{\bar{\beta}} N \xrightarrow{\bar{\gamma}} \mathrm{T}(L) \xrightarrow{-\mathrm{T}(\bar{\alpha})} \mathrm{T}(M) \tag{5.4.10}$$

in $\mathbf{K}(A, \mathsf{M})$. If (5.4.9) is distinguished, then by Lemma 5.4.7 so is (5.4.10).

Conversely, if (5.4.10) is distinguished, then by turning it 5 times, and using the previous step (namely by Lemma 5.4.7), we see that the triangle

$$\mathrm{T}^2(L) \xrightarrow{\mathrm{T}^2(\bar{\alpha})} \mathrm{T}^2(M) \xrightarrow{\mathrm{T}^2(\bar{\beta})} \mathrm{T}^2(N) \xrightarrow{\mathrm{T}^2(\bar{\gamma})} \mathrm{T}^3(L)$$

is distinguished. According to Lemma 5.4.8 (with $k = -2$), the triangle gotten by applying T^{-2} to this is distinguished. But this is just the triangle (5.4.9).

(TR3): Consider a commutative diagram in $\mathbf{K}(A, \mathsf{M})$:

(5.4.11)
$$\begin{tikzcd} \bar{L} \arrow[r, "\bar{\alpha}"] \arrow[d, "\bar{\phi}"'] & \bar{M} \arrow[r, "\bar{\beta}"] \arrow[d, "\bar{\psi}"'] & \bar{N} \arrow[r, "\bar{\gamma}"] & \mathrm{T}(\bar{L}) \\ \bar{L}' \arrow[r, "\bar{\alpha}'"] & \bar{M}' \arrow[r, "\bar{\beta}'"] & \bar{N}' \arrow[r, "\bar{\gamma}'"] & \mathrm{T}(\bar{L}') \end{tikzcd}$$

where the horizontal triangles are distinguished. By definition the rows in (5.4.11) are isomorphic in $\mathbf{K}(A, \mathsf{M})$ to the images under the functor P of standard triangles in $\mathbf{C}(A, \mathsf{M})$. These are the rows in diagram (5.4.12) below. The vertical morphisms in (5.4.11) are also induced from morphisms in $\mathbf{C}(A, \mathsf{M})$, i.e. $\bar{\phi} = \mathrm{P}(\phi)$ and $\bar{\psi} = \mathrm{P}(\psi)$. Thus (5.4.11) is isomorphic to the image under P of the following diagram:

(5.4.12)
$$\begin{tikzcd} L \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[r, "\beta"] \arrow[d, "\psi"'] & N \arrow[r, "\gamma"] & \mathrm{T}(L) \\ L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L') \end{tikzcd}$$

Warning: the diagram (5.4.12) is only commutative up to homotopy in $\mathbf{C}(A, \mathsf{M})$.

Since the rows in (5.4.12) are standard triangles (see Definition 3.9.5), the objects N and N' are cones: $N = \mathrm{Cone}(\alpha)$ and $N' = \mathrm{Cone}(\alpha')$. The commutativity up to homotopy of this diagram means that there is a degree -1 morphism $\theta : L \to M'$ in $\mathbf{C}(A, \mathsf{M})$ such that

$$\alpha' \circ \phi = \psi \circ \alpha + \mathrm{d}(\theta).$$

Define the morphism

$$\chi : N = \begin{bmatrix} M \\ \mathrm{T}(L) \end{bmatrix} \to N' = \begin{bmatrix} M' \\ \mathrm{T}(L') \end{bmatrix}$$

by the matrix presentation

$$\chi := \begin{bmatrix} \psi & \theta \circ \mathrm{t}_L^{-1} \\ 0 & \mathrm{T}(\phi) \end{bmatrix}.$$

An easy calculation shows that χ is a morphism in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$, and that there are equalities $\mathrm{T}(\phi) \circ \gamma = \gamma' \circ \chi$ and $\chi \circ \beta = \beta' \circ \psi$. Therefore, when we apply the functor P, and conjugate by the original isomorphism between (5.4.11) and the image of (5.4.12), we obtain a commutative diagram

$$\begin{tikzcd} \bar{L} \arrow[r, "\bar{\alpha}"] \arrow[d, "\bar{\phi}"'] & \bar{M} \arrow[r, "\bar{\beta}"] \arrow[d, "\bar{\psi}"'] & \bar{N} \arrow[r, "\bar{\gamma}"] \arrow[d, "\bar{\chi}"'] & \mathrm{T}(\bar{L}) \arrow[d, "\mathrm{T}(\bar{\phi})"'] \\ \bar{L}' \arrow[r, "\bar{\alpha}'"] & \bar{M}' \arrow[r, "\bar{\beta}'"] & \bar{N}' \arrow[r, "\bar{\gamma}'"] & \mathrm{T}(\bar{L}') \end{tikzcd}$$

in $\mathbf{K}(A, \mathsf{M})$, where $\bar{\chi}$ is conjugate to $\mathrm{P}(\chi)$. □

We now add a second DG ring B, and a second additive category N. DG functors were introduced in Subsection 3.5.

Consider a DG functor

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N}).$$

From Theorem 4.2.3 we know that the translation isomorphism is an isomorphism of DG functors

$$\tau_F : F \circ \mathrm{T}_{A,\mathsf{M}} \xrightarrow{\simeq} \mathrm{T}_{B,\mathsf{N}} \circ F.$$

Therefore, when we pass to the homotopy categories, and writing $\bar{F} := \mathrm{Ho}(F)$, we get a T-additive functor

$$(\bar{F}, \bar{\tau}_F) : \mathbf{K}(A, \mathsf{M}) \to \mathbf{K}(B, \mathsf{N}).$$

Theorem 5.4.13. *Let*

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$$

be a DG functor, with translation isomorphism τ_F. Then the T-additive functor

$$(\bar{F}, \bar{\tau}_F) : \mathbf{K}(A, \mathsf{M}) \to \mathbf{K}(B, \mathsf{N})$$

is a triangulated functor.

Proof. Take a distinguished triangle

$$L \xrightarrow{\bar{\alpha}} M \xrightarrow{\bar{\beta}} N \xrightarrow{\bar{\gamma}} \mathrm{T}(L)$$

in $\mathbf{K}(A, \mathsf{M})$. Since we are only interested in triangles up to isomorphism, we can assume that this is the image under the functor P of a standard triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L)$$

in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. According to Theorem 4.3.7 and Corollary 4.3.8, there is a standard triangle

$$L' \xrightarrow{\alpha'} M' \xrightarrow{\beta'} N' \xrightarrow{\gamma'} \mathrm{T}(L')$$

in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$, and a commutative diagram

$$\begin{tikzcd}
F(L) \arrow[r, "F(\alpha)"] \arrow[d, "\phi"'] & F(M) \arrow[r, "F(\beta)"] \arrow[d, "\psi"'] & F(N) \arrow[r, "\tau_{F,L} \circ F(\gamma)"] \arrow[d, "\chi"'] & \mathrm{T}(F(L)) \arrow[d, "\mathrm{T}(\phi)"'] \\
L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L')
\end{tikzcd}$$

in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$, in which the vertical arrows are isomorphisms. (Actually, we can take $L' = F(L)$, $\phi = \mathrm{id}_{F(L)}$, etc.) After applying the functor P to this diagram, we see that the condition in Definition 5.3.1(1) is satisfied. □

Corollary 5.4.14. *For any integer k, the pair $\big(\mathrm{T}^k, (-1)^k \cdot \mathrm{id}_{\mathrm{T}^{k+1}}\big)$ is a triangulated functor from $\mathbf{K}(A, \mathsf{M})$ to itself.*

Proof. Combine Theorems 5.4.13 and Proposition 4.2.4. □

Remark 5.4.15. In [BoKa], Bondal and Kapranov introduce the concept of *pretriangulated DG category*. This is a DG category C for which the homotopy category $\mathrm{Ho}(\mathsf{C})$ is canonically triangulated (the details of the definition are too complicated to mention here). Our DG categories $\mathbf{C}(A, \mathsf{M})$ are pretriangulated in the sense of [BoKa]; but they have a lot more structure (e.g. the objects have cohomologies too).

Suppose C and C' are pretriangulated DG categories. In [BoKa] there is a (rather complicated) definition of *pre-exact DG functor* $F : \mathsf{C} \to \mathsf{C}'$. It is stated there that if F is a pre-exact DG functor, then $\mathrm{Ho}(F) : \mathrm{Ho}(\mathsf{C}) \to \mathrm{Ho}(\mathsf{C}')$ is a triangulated functor. This is analogous to our Theorem 5.4.13. Presumably, Theorems 4.2.3 and 4.3.7 imply that any DG functor $F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(A', \mathsf{M}')$ is pre-exact in the sense of [BoKa]; but we did not verify this.

6. Localization of Categories

Most of this section is devoted to the general theory of *Ore localization of categories*. In the last subsection we talk about localization of a pretriangulated category K with respect to a denominator set of cohomological origin $\mathsf{S} \subseteq \mathsf{K}$.

6.1. The Formalism of Localization.

We will start with a category A, without even assuming it is linear. Still we use the notation A, because it will be suggestive to think about a linear category A with a single object, which is just a ring A. The reason is that our localization procedure is the same as that in noncommutative ring theory (the only change being that we allow multiple objects).

The emphasis will be on morphisms rather than on objects. Thus it will be convenient to write

$$\mathsf{A}(M, N) := \mathrm{Hom}_{\mathsf{A}}(M, N)$$

for $M, N \in \mathrm{Ob}(\mathsf{A})$. We sometimes use the notation $a \in \mathsf{A}$ for a morphism $a \in \mathsf{A}(M, N)$, leaving the objects implicit. When we write $b \circ a$ for $a, b \in \mathsf{A}$, we implicitly mean that these morphisms are composable.

For heuristic purposes, we can think of A as a linear category (e.g. living inside some category of modules), with objects $M, N, \ldots$. For any given object M, we then have a genuine ring $\mathsf{A}(M) := \mathsf{A}(M, M)$.

Definition 6.1.1. Let A be a category. A *multiplicatively closed set of morphisms* in A is a subcategory $\mathsf{S} \subseteq \mathsf{A}$ such that $\mathrm{Ob}(\mathsf{S}) = \mathrm{Ob}(\mathsf{A})$.

In other words, for any pair of objects $M, N \in \mathsf{A}$ there is a subset $\mathsf{S}(M, N) \subseteq \mathsf{A}(M, N)$, such that $1_M \in \mathsf{S}(M, M)$, and such that for any $s \in \mathsf{S}(L, M)$ and $t \in \mathsf{S}(M, N)$, the composition $t \circ s \in \mathsf{S}(L, N)$.

Using our shorthand, we can write the definition like this: $1_M \in \mathsf{S}$, and $s, t \in \mathsf{S}$ implies $t \circ s \in \mathsf{S}$.

If $A = \mathsf{A}$ is a single object linear category, namely a ring, then $S = \mathsf{S}$ is a multiplicatively closed set in the sense of ring theory.

There are various notions of localization in the literature. We restrict attention to two of them.

Definition 6.1.2. Let S be a multiplicatively closed set of morphisms in a category A. A *localization* of A with respect to S is a pair $(\mathsf{A}_{\mathsf{S}}, \mathrm{Q})$, consisting of a category A_{S} and a functor $\mathrm{Q} : \mathsf{A} \to \mathsf{A}_{\mathsf{S}}$, called the localization functor, having the following properties:

- (L1) There is equality $\mathrm{Ob}(\mathsf{A}_{\mathsf{S}}) = \mathrm{Ob}(\mathsf{A})$, and Q is the identity on objects.
- (L2) For every $s \in \mathsf{S}$, the morphism $\mathrm{Q}(s) \in \mathsf{A}_{\mathsf{S}}$ is invertible (i.e. it is an isomorphism).
- (L3) Suppose B is a category, and $F : \mathsf{A} \to \mathsf{B}$ is a functor such that $F(s)$ is invertible for every $s \in \mathsf{S}$. Then there is a unique functor $F_{\mathsf{S}} : \mathsf{A}_{\mathsf{S}} \to \mathsf{B}$ such that $F_{\mathsf{S}} \circ \mathrm{Q} = F$ as functors $\mathsf{A} \to \mathsf{B}$.

In a commutative diagram:

$$\begin{tikzcd}
\mathsf{S} \arrow[r, "\mathrm{inc}"] & \mathsf{A} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] & \mathsf{B} \\
& \mathsf{A}_{\mathsf{S}} \arrow[ur, "F_{\mathsf{S}}"'] &
\end{tikzcd}$$

In the ring case, $F : A \to B$ is a ring homomorphism, etc.

Proposition 6.1.3. *A localization (in the sense of Definition 6.1.2) is unique up to a unique isomorphism. Namely if* $(\mathsf{A}'_{\mathsf{S}}, \mathrm{Q}')$ *is another localization, then there is a unique functor* $G : \mathsf{A}_{\mathsf{S}} \to \mathsf{A}'_{\mathsf{S}}$ *which is the identity on objects, bijective on morphisms, and* $G \circ \mathrm{Q} = \mathrm{Q}'$.

Proof. Exercise. □

A localization in this general sense always exists, but often it is of little value, because there is no practical way to describe the morphisms in it.

6.2. **Ore Localization.** There is a better notion of localization. The references here are [RD], [GaZi], [We], [KaSc1], [Ste] and [Row]. The first four references talk about localization of categories; and the last two talk about noncommutative rings. It seems that historically, this *noncommutative calculus of fractions* was discovered by Ore and Asano in ring theory, around 1930. There was progress in the categorical side, notably by Gabriel around 1960.

In this subsection we mostly follow the treatment of [Ste]; but we sometimes use diagrams instead of formulas involving letters – this is the only way the author was able to understand the proofs!

Definition 6.2.1. Let S be a multiplicatively closed set of morphisms in a category A. A *right Ore localization* of A with respect to S is a pair $(\mathsf{A}_{\mathsf{S}}, \mathrm{Q})$, consisting of a category A_{S} and a functor $\mathrm{Q} : \mathsf{A} \to \mathsf{A}_{\mathsf{S}}$, having the following properties:

(RO1) There is equality $\mathrm{Ob}(\mathsf{A}_{\mathsf{S}}) = \mathrm{Ob}(\mathsf{A})$, and Q is the identity on objects.
(RO2) For every $s \in \mathsf{S}$, the morphism $\mathrm{Q}(s) \in \mathsf{A}_{\mathsf{S}}$ is an isomorphism.
(RO3) Every morphism $q \in \mathsf{A}_{\mathsf{S}}$ can be written as $q = \mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1}$ for some $a \in \mathsf{A}$ and $s \in \mathsf{S}$.
(RO4) Suppose $a, b \in \mathsf{A}$ satisfy $\mathrm{Q}(a) = \mathrm{Q}(b)$. Then $a \circ s = b \circ s$ for some $s \in \mathsf{S}$.

The letters "RO" stand for "right Ore". We refer to the expression $q = \mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1}$ as a *right fraction representation of* q.

Remark 6.2.2. There is an obvious notion of *left Ore localization*, with properties (LO1)-(LO4) that are identical to (RO1)-(RO4) respectively, except that in the last two the compositions are reversed: $q = \mathrm{Q}(s)^{-1} \circ \mathrm{Q}(a)$ and $s \circ a = s \circ b$. The results to follow in this section all have "left" versions, with identical proofs (just a matter of reversing some arrows or compositions), and so they will be omitted.

To reinforce the last remark, we give:

Proposition 6.2.3. *Let* S *be a multiplicatively closed set in a category* A*, and let* $\mathrm{Q} : \mathsf{A} \to \mathsf{A}_{\mathsf{S}}$ *be a functor. Prove that* $\mathrm{Q} : \mathsf{A} \to \mathsf{A}_{\mathsf{S}}$ *is a right Ore localization of* A *with respect to* S *if and only if* $\mathrm{Q}^{\mathrm{op}} : \mathsf{A}^{\mathrm{op}} \to (\mathsf{A}^{\mathrm{op}})_{\mathsf{S}^{\mathrm{op}}}$ *is a left Ore localization of* A^{op} *with respect to* S^{op}.

Exercise 6.2.4. Prove Proposition 6.2.3.

Lemma 6.2.5. *Let* $(\mathsf{A}_{\mathsf{S}}, \mathrm{Q})$ *be a right Ore localization, let* $a_1, a_2 \in \mathsf{A}$ *and* $s_1, s_2 \in \mathsf{S}$. *The following conditions are equivalent*:

(i) $\mathrm{Q}(a_1) \circ \mathrm{Q}(s_1)^{-1} = \mathrm{Q}(a_2) \circ \mathrm{Q}(s_2)^{-1}$ *in* A_{S}.
(ii) *There are* $b_1, b_2 \in \mathsf{A}$ *s.t.* $a_1 \circ b_1 = a_2 \circ b_2$*, and* $s_1 \circ b_1 = s_2 \circ b_2 \in \mathsf{S}$.

Proof. (ii) $\Rightarrow$ (i): Since $\mathrm{Q}(s_i)$ and $\mathrm{Q}(s_i \circ b_i)$ are invertible, it follows that $\mathrm{Q}(b_i)$ are invertible. So

$$\begin{aligned}\mathrm{Q}(a_1) \circ \mathrm{Q}(s_1)^{-1} &= \mathrm{Q}(a_1) \circ \mathrm{Q}(b_1) \circ \mathrm{Q}(b_1)^{-1} \circ \mathrm{Q}(s_1)^{-1} \\ &= \mathrm{Q}(a_2) \circ \mathrm{Q}(b_2) \circ \mathrm{Q}(b_2)^{-1} \circ \mathrm{Q}(s_2)^{-1} = \mathrm{Q}(a_2) \circ \mathrm{Q}(s_2)^{-1}.\end{aligned}$$

(i) $\Rightarrow$ (ii): By property (RO3) there are $c \in \mathsf{A}$ and $u \in \mathsf{S}$ s.t.

$$\mathrm{Q}(s_2)^{-1} \circ \mathrm{Q}(s_1) = \mathrm{Q}(c) \circ \mathrm{Q}(u)^{-1}. \tag{6.2.6}$$

Rewriting this equation we get

$$\mathrm{Q}(s_1 \circ u) = \mathrm{Q}(s_2 \circ c). \tag{6.2.7}$$

It is given that

$$\mathrm{Q}(a_1) = \mathrm{Q}(a_2) \circ \mathrm{Q}(s_2)^{-1} \circ \mathrm{Q}(s_1).$$

Plugging (6.2.6) into it we obtain

$$\mathrm{Q}(a_1) = \mathrm{Q}(a_2) \circ \mathrm{Q}(c) \circ \mathrm{Q}(u)^{-1}.$$

Rearranging this equation we get

$$\mathrm{Q}(a_1 \circ u) = \mathrm{Q}(a_2 \circ c). \tag{6.2.8}$$

By property (RO4) there is $v \in \mathsf{S}$ s.t.

$$a_1 \circ u \circ v = a_2 \circ c \circ v.$$

Likewise, from equation (6.2.7) and property (RO4), there is $v' \in \mathsf{S}$ s.t.

$$s_1 \circ u \circ v' = s_2 \circ c \circ v'.$$

Again using property (RO3), there are $d \in \mathsf{A}$ and $w \in \mathsf{S}$ s.t.

$$\mathrm{Q}(v)^{-1} \circ \mathrm{Q}(v') = \mathrm{Q}(d) \circ \mathrm{Q}(w)^{-1}.$$

Rearranging we get

$$\mathrm{Q}(v' \circ w) = \mathrm{Q}(v \circ d).$$

By property (RO4) there is $w' \in \mathsf{S}$ s.t.

$$v' \circ w \circ w' = v \circ d \circ w'.$$

Define

$$b_1 := u \circ v \circ d \circ w' \ , \ b_2 := c \circ v \circ d \circ w' \ .$$

Then

$$\begin{aligned}s_1 \circ b_1 &= s_1 \circ u \circ v \circ d \circ w' = s_1 \circ u \circ v' \circ w \circ w' \\ &= s_2 \circ c \circ v' \circ w \circ w' = s_2 \circ b_2,\end{aligned}$$

and it is in S. Also

$$a_1 \circ b_1 = a_1 \circ u \circ v \circ d \circ w' = a_2 \circ c \circ v \circ d \circ w' = a_2 \circ b_2.$$

□

Proposition 6.2.9. *A right Ore localization* $(\mathsf{A}_\mathsf{S}, \mathrm{Q})$ *is a localization in the sense of Definition* 6.1.2.

Proof. Say B is a category, and $F : \mathsf{A} \to \mathsf{B}$ is a functor such that $F(s)$ is an isomorphism for every $s \in \mathsf{S}$.

The uniqueness of a functor $F_{\mathsf{S}} : \mathsf{A}_{\mathsf{S}} \to \mathsf{B}$ satisfying $F_{\mathsf{S}} \circ \mathrm{Q} = F$ is clear from property (RO3). We have to prove existence.

Define F_{S} to be F on objects, and

$$F_{\mathsf{S}}(q) := F(a_1) \circ F(s_1)^{-1},$$

where

$$q = \mathrm{Q}(a_1) \circ \mathrm{Q}(s_1)^{-1} \in \mathsf{A}_{\mathsf{S}},\ a_1 \in \mathsf{A},\ s_1 \in \mathsf{S}$$

is any presentation of q as a right fraction, that exists by (RO3). We have to prove that this is well defined. So suppose that $q = \mathrm{Q}(a_2) \circ \mathrm{Q}(s_2)^{-1}$ is another presentation of q. Let $b_1, b_2 \in \mathsf{A}$ be as in Lemma 6.2.5. Since $F(s_i)$ and $F(s_i \circ b_i)$ are invertible, then so is $F(b_i)$. We get

$$F(a_2) = F(a_1) \circ F(b_1) \circ F(b_2)^{-1}$$

and

$$F(s_2) = F(s_1) \circ F(b_1) \circ F(b_2)^{-1}.$$

Hence

$$F(a_2) \circ F(s_2)^{-1} = F(a_1) \circ F(s_1)^{-1}.$$

It remains to prove that F_{S} is a functor. Since the identity 1_M of the object M in A_{S} can be presented as $1_M = \mathrm{Q}(1_M) \circ \mathrm{Q}(1_M)^{-1}$, we see that $F_{\mathsf{S}}(1_M) = 1_{F(M)}$.

Next let q_1 and q_2 be morphisms in A_{S}, such that composition $q_2 \circ q_1$ exists (i.e. the target of q_1 is the source of q_2). We have to show that $F_{\mathsf{S}}(q_2 \circ q_1)$ equals $F_{\mathsf{S}}(q_2) \circ F_{\mathsf{S}}(q_1)$. Choose presentations $q_i = \mathrm{Q}(a_i) \circ \mathrm{Q}(s_i)^{-1}$, so that

$$F_{\mathsf{S}}(q_2) \circ F_{\mathsf{S}}(q_1) = F(a_2) \circ F(s_2)^{-1} \circ F(a_1) \circ F(s_1)^{-1}. \tag{6.2.10}$$

By property (R03) there is a right fraction presentation

$$\mathrm{Q}(s_2)^{-1} \circ \mathrm{Q}(a_1) = \mathrm{Q}(b) \circ \mathrm{Q}(t)^{-1} \tag{6.2.11}$$

for some $b \in \mathsf{A}$ and $t \in \mathsf{S}$. Because

$$\mathrm{Q}(a_1 \circ t) = \mathrm{Q}(s_2 \circ b),$$

by (RO4) there is some $r \in \mathsf{S}$ such that

$$a_1 \circ t \circ r = s_2 \circ b \circ r.$$

Therefore

$$F(a_1 \circ t \circ r) = F(s_2 \circ b \circ r),$$

which implies, by canceling the invertible morphism $F(r)$ and rearranging, that

$$F(s_2)^{-1} \circ F(a_1) = F(b) \circ F(t)^{-1} \tag{6.2.12}$$

in B.

Let us continue. Using equation (6.2.11) we have

$$\begin{aligned} q_2 \circ q_1 &= \mathrm{Q}(a_2) \circ \mathrm{Q}(s_2)^{-1} \circ \mathrm{Q}(a_1) \circ \mathrm{Q}(s_1)^{-1} \\ &= \mathrm{Q}(a_2) \circ \mathrm{Q}(b) \circ \mathrm{Q}(t)^{-1} \circ \mathrm{Q}(s_1)^{-1} = \mathrm{Q}(a_2 \circ b) \circ \mathrm{Q}(s_1 \circ t)^{-1}. \end{aligned}$$

Using this presentation of $q_2 \circ q_1$, and the equality (6.2.12), we obtain

$$\begin{aligned} F_{\mathsf{S}}(q_2 \circ q_1) &= F(a_2 \circ b) \circ F(s_1 \circ t)^{-1} = F(a_2) \circ F(b) \circ F(t)^{-1} \circ F(s_1)^{-1} \\ &= F(a_2) \circ F(s_2)^{-1} \circ F(a_1) \circ F(s_1)^{-1}. \end{aligned}$$

This is the same as (6.2.10). □

Corollary 6.2.13. *Let S be a multiplicatively closed set of morphisms in a category A. Assume that $(\mathsf{A}_\mathsf{S}, \mathrm{Q})$ and $(\mathsf{A}'_\mathsf{S}, \mathrm{Q}')$ are either right Ore localizations or left Ore localizations of A with respect to S. Then there is a unique isomorphism of localizations*

$$(\mathsf{A}_\mathsf{S}, \mathrm{Q}) \cong (\mathsf{A}'_\mathsf{S}, \mathrm{Q}'),$$

as in Proposition 6.1.3.

Proof. By Proposition 6.2.9 (in its right or left versions, as the case may be), both $(\mathsf{A}_\mathsf{S}, \mathrm{Q})$ and $(\mathsf{A}'_\mathsf{S}, \mathrm{Q}')$ are localizations in the sense of Definition 6.1.2. Hence, by Proposition 6.1.3, there is a unique isomorphism $(\mathsf{A}_\mathsf{S}, \mathrm{Q}) \cong (\mathsf{A}'_\mathsf{S}, \mathrm{Q}')$. □

Definition 6.2.14. Let S be multiplicatively closed set of morphisms in a category A. We say that S is a *right denominator set* if it satisfies these two conditions:

(RD1) (Right Ore condition) Given $a \in \mathsf{A}$ and $s \in \mathsf{S}$, there exist $b \in \mathsf{A}$ and $t \in \mathsf{S}$ such that $a \circ t = s \circ b$.

(RD2) (Right Cancellation condition) Given $a, b \in \mathsf{A}$ and $s \in \mathsf{S}$ such that $s \circ a = s \circ b$, there exists $t \in \mathsf{S}$ such that $a \circ t = b \circ t$.

In commutative diagrams:

$$\begin{tikzcd} & K \arrow[dl, dashed, "t"'] \arrow[dr, dashed, "b"] & \\ M \arrow[dr, "a"'] & & N \arrow[dl, "s"] \\ & L & \end{tikzcd} \qquad \begin{tikzcd} K \arrow[d, dashed, "t"'] \\ M \arrow[d, bend right, "a"'] \arrow[d, bend left, "b"] \\ N \arrow[d, "s"'] \\ L \end{tikzcd}$$

There is a similar left version of this definition, with conditions (LD1) and (LD2).
Here is the main theorem regarding Ore localization.

Theorem 6.2.15. *The following conditions are equivalent for a category A and a multiplicatively closed set of morphisms $\mathsf{S} \subseteq \mathsf{A}$.*

(i) *The right Ore localization $(\mathsf{A}_\mathsf{S}, \mathrm{Q})$ exists.*

(ii) S *is a right denominator set.*

The proof of Theorem 6.2.15 is after some preparation. The hard part is proving that (ii) ⇒ (i). The general idea is the same as in commutative localization: we consider the set of pairs of morphisms $\mathsf{A} \times \mathsf{S}$, and define a relation $\sim$ on it, with the hope that this is an equivalence relation, and that the quotient set A_S will be a category, and it will have the desired properties.

Let's assume that S is a right denominator set. For any $M, N \in \mathrm{Ob}(A)$ consider the set

$$(\mathsf{A} \times \mathsf{S})(M, N) := \coprod_{L \in \mathrm{Ob}(A)} \mathsf{A}(L, N) \times \mathsf{S}(L, M).$$

Remark 6.2.16. The set $(\mathsf{A} \times \mathsf{S})(M, N)$ could be big, namely not an element of the initial universe U. This would require the introduction of a larger universe, say V, in which U is an element. And the resulting category A_{S} will be a V-category.

We will ignore this issue. Moreover, in many cases of interest (derived categories where there are DG enhancements, such as the K-injective enhancement), there will be an alternative presentation of A_{S} as a U-category. We will refer to this when we get to it.

An element $(a, s) \in (\mathsf{A} \times \mathsf{S})(M, N)$ can be pictured as a diagram

(6.2.17)
$$\begin{tikzcd} & L \arrow[dl, "s"'] \arrow[dr, "a"] & \\ M & & N \end{tikzcd}$$

in A. This diagram will eventually represent the right fraction

$$\mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1} : M \to N.$$

Definition 6.2.18. We define a relation $\sim$ on the set $\mathsf{A} \times \mathsf{S}$ like this:

$$(a_1, s_1) \sim (a_2, s_2)$$

if there exist $b_1, b_2 \in \mathsf{A}$ s.t.

$$a_1 \circ b_1 = a_2 \circ b_2 \text{ and } s_1 \circ b_1 = s_2 \circ b_2 \in \mathsf{S}\,.$$

Note that the relation $\sim$ imposes condition (ii) of Lemma 6.2.5.

Here it is in a commutative diagram, in which we have made the objects explicit:

(6.2.19)
$$\begin{tikzcd} & K \arrow[dl, "b_1"'] \arrow[dr, "b_2"] & \\ L_1 \arrow[d, "s_1"'] \arrow[drr, "a_1"' near end] & & L_2 \arrow[d, "a_2"] \arrow[dll, "s_2" near end] \\ M & & N \end{tikzcd}$$

The arrows ending at M are in S.

Lemma 6.2.20. *If the right Ore condition holds then the relation $\sim$ is an equivalence.*

Proof. Reflexivity: take $K := L$ and $b_i := 1_L : L \to L$. Symmetry is trivial.

Now to prove transitivity. Suppose we are given $(a_1, s_1) \sim (a_2, s_2)$ and $(a_2, s_2) \sim (a_3, s_3)$. So we have the solid part of the first diagram in Figure 2, and it is commutative. The arrows ending at M are in S.

By condition (RD1) applied to $K \to M \leftarrow J$ there are $t \in \mathsf{S}$ and $d \in \mathsf{A}$ s.t.

$$(s_3 \circ c_3) \circ d = (s_1 \circ b_1) \circ t.$$

Comparing arrows $I \to M$ in this diagram, we see that

$$s_2 \circ (b_2 \circ t) = s_1 \circ b_1 \circ t = s_3 \circ c_3 \circ d = s_2 \circ (c_2 \circ d).$$

By (RD2) there is $u \in \mathsf{S}$ s.t.

$$(b_2 \circ t) \circ u = (c_2 \circ d) \circ u.$$

$$
\begin{tikzcd}
 & H \arrow[d, dashed, "u"] & \\
 & I \arrow[dl, dashed, "t"'] \arrow[dr, dashed, "d"] & \\
K \arrow[d, "b_1"'] \arrow[dr, "b_2"] & & J \arrow[dl, "c_2"'] \arrow[d, "c_3"] \\
L_1 \arrow[d, "s_1"'] \arrow[drr, "a_1"' near end] & L_2 \arrow[dl, "s_2"'] \arrow[dr, "a_2"] & L_3 \arrow[d, "a_3"] \arrow[dll, "s_3" near end] \\
M & & N
\end{tikzcd}
\qquad
\begin{tikzcd}
 & H \arrow[dl, "b_1 \circ t \circ u"'] \arrow[dr, "c_3 \circ d \circ u"] & \\
L_1 \arrow[d, "s_1"'] \arrow[drr, "a_1"' near end] & & L_3 \arrow[d, "a_3"] \arrow[dll, "s_3" near end] \\
M & & N
\end{tikzcd}
$$

Figure 2.

So all paths $H \to M$ are equal and belong to S, and all paths $H \to N$ are equal. Now delete the object L_2 and the arrows going through it. Then delete the objects I, J, K, but keep the paths going through them. We get the second diagram in Figure 2. It is commutative, and all arrows ending at M are in S. This is evidence for $(a_1, s_1) \sim (a_3, s_3)$. □

Proof of Theorem 6.2.15.

Step 1. In this step we prove (i) ⇒ (ii). Take $a \in \mathsf{A}$ and $s \in \mathsf{S}$. Consider $q := \mathrm{Q}(s)^{-1} \circ \mathrm{Q}(a)$. By (RO3) there are $b \in \mathsf{A}$ and $t \in \mathsf{S}$ s.t. $q = \mathrm{Q}(b) \circ \mathrm{Q}(t)^{-1}$. So

$$\mathrm{Q}(s \circ b) = \mathrm{Q}(a \circ t).$$

By (RO4) there is $u \in \mathsf{S}$ s.t.

$$(s \circ b) \circ u = (a \circ t) \circ u.$$

We read this as

$$s \circ (b \circ u) = a \circ (t \circ u),$$

and note that $t \circ u \in \mathsf{S}$. So (RD1) holds.

Next $a, b \in \mathsf{A}$ and $s \in \mathsf{S}$ s.t. $s \circ a = s \circ b$. Then $\mathrm{Q}(s \circ a) = \mathrm{Q}(s \circ b)$. But $\mathrm{Q}(s)$ is invertible, so $\mathrm{Q}(a) = \mathrm{Q}(b)$. By (RO4) there is $t \in \mathsf{S}$ s.t. $a \circ t = b \circ t$. We have proved (RD2).

Step 2. Now we assume that condition (ii) holds, and we define the morphism sets $\mathsf{A}_{\mathsf{S}}(M, N)$, composition between them, and the identity morphisms.

For any $M, N \in \mathrm{Ob}(\mathsf{A})$ let

$$\mathsf{A}_{\mathsf{S}}(M, N) := \frac{(\mathsf{A} \times \mathsf{S})(M, N)}{\sim},$$

$$\begin{tikzcd}
 & & K \arrow[dl, "u"'] \arrow[dr, "c"] & & \\
 & L_1 \arrow[dl, "s_1"'] \arrow[dr, "a_1"] & & L_2 \arrow[dl, "s_2"'] \arrow[dr, "a_2"] & \\
M_0 & & M_1 & & M_2
\end{tikzcd}$$

FIGURE 3.

$$\begin{tikzcd}
J_1 \arrow[d, dashed, "b_1"'] \arrow[dr, dashed, "b_1'"] & K \arrow[dl, "u"'] \arrow[drr, "c"] & & K' \arrow[dll, "u'"'] \arrow[dr, "c'"] & J_2 \arrow[dl, dashed, "b_2"'] \arrow[d, dashed, "b_2'"] \\
L_1 \arrow[d, "s_1"'] \arrow[drr, "a_1"'] & L_1' \arrow[dl, "s_1'"] \arrow[dr, "a_1'"] & & L_2 \arrow[dl, "s_2"'] \arrow[dr, "a_2"] & L_2' \arrow[dll, "s_2'"] \arrow[d, "a_2'"] \\
M_0 & & M_1 & & M_2
\end{tikzcd}$$

FIGURE 4.

where $\sim$ is the relation from Definition 6.2.18, which is an equivalence relation by Lemma 6.2.20.

We define composition like this. Given $q_1 \in \mathsf{A}_{\mathsf{S}}(M_0, M_1)$ and $q_2 \in \mathsf{A}_{\mathsf{S}}(M_1, M_2)$, choose representatives $(a_i, s_i) \in (\mathsf{A} \times \mathsf{S})(M_{i-1}, M_i))$. We use the notation $q_i = \overline{(a_i, s_i)}$ to indicate this. By (RD1) there are $c \in \mathsf{A}$ and $u \in \mathsf{S}$ s.t. $s_2 \circ c = a_1 \circ u$. The composition

$$q_2 \circ q_1 \in \mathsf{A}_{\mathsf{S}}(M_0, M_2)$$

is defined to be

$$q_2 \circ q_1 := \overline{(a_2 \circ c, s_1 \circ u)} \in (\mathsf{A} \times \mathsf{S})(M_0, M_2)).$$

The idea behind the formula can be seen in the diagram in Figure 3.

We have to verify that this definition is independent of the representatives. So suppose we take other representatives $q_i = \overline{(a_i', s_i')}$, and we choose morphisms u', c' to construct the composition. This is the solid part of the diagram in Figure 4, and it is a commutative diagram. We must prove that

$$\overline{(a_2 \circ c, s_1 \circ u)} = \overline{(a_2' \circ c', s_1' \circ u')}.$$

There are morphisms b_i, b_i' the are evidence for $(a_i, s_i) \sim (a_i', s_i')$. They are depicted as the dashed arrows in Figure 4. That whole diagram is commutative. The morphisms $J_1 \to M_0$, $K \to M_0$, $K' \to M_0$ and $J_2 \to M_1$ are all in S.

Choose $v_1 \in \mathsf{S}$ and $d_1 \in \mathsf{A}$ s.t. the first diagram in Figure 5 is commutative. This can be done by (RD1).

$$
\begin{tikzcd}
& I_1 \arrow[dl, "v_1"'] \arrow[dr, "d_1"] & \\
J_1 \arrow[dr, "b_1"'] & & K \arrow[dl, "u"] \\
& L_1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& I_2 \arrow[d, dashed, "\tilde{v}"] & \\
& \tilde{I}_2 \arrow[dl, dashed, "\tilde{v}_2"'] \arrow[dr, dashed, "\tilde{d}_2"] & \\
K' \arrow[dr, "c'"'] & & J_2 \arrow[dl, "b_2'"] \\
& L_2' \arrow[d, "s_2'"'] & \\
& M_1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& I_2 \arrow[dl, "v_2"'] \arrow[dr, "d_2"] & \\
K' \arrow[dr, "c'"'] & & J_2 \arrow[dl, "b_2'"] \\
& L_2' &
\end{tikzcd}
$$

Figure 5.

Consider the solid part of the middle diagram in Figure 5. Since $J_2 \to M_1$ is in S, by (RD1) there are $\tilde{v}_2 \in \mathsf{S}$ and $\tilde{d}_2 \in \mathsf{A}$ s.t. the two paths $\tilde{I}_2 \to M_1$ are equal. By (RD2) there is $\tilde{v} \in \mathsf{S}$ s.t. the two paths $I_2 \to L_2'$ are equal. We get the commutative diagram in the middle of Figure 5. Next, defining $d_2 := \tilde{d}_2 \circ \tilde{v}$ and $v_2 := \tilde{v}_2 \circ \tilde{v} \in \mathsf{S}$, we obtain the third commutative diagram in Figure 5.

We now embed the first and third diagrams from Figure 5 into the diagram in Figure 4. This gives us the solid diagram in Figure 6, and it is commutative. The morphisms $I_1 \to M_0$ belong to S.

Choose $w \in \mathsf{S}$ and $e \in \mathsf{A}$, starting at an object H, to fill the diagram $I_1 \to M_0 \leftarrow I_2$, using (RD1). The path $H \to I_1 \to M_0$ is in S, and all the paths $H \to M_0$ are equal. But we could have failure of commutativity in the paths $H \to L_1'$ and $H \to L_2$.

The two paths $H \to L_1'$ in Figure 6 satisfy

$$s_1' \circ (b_1' \circ v_1 \circ w) = s_1' \circ (u' \circ v_2 \circ e).$$

Therefore there is $w' \in \mathsf{S}$ s.t.

$$(b_1' \circ v_1 \circ w) \circ w' = (u' \circ v_2 \circ e) \circ w'.$$

Next, the two paths $H' \to L_2$ satisfy

$$s_2 \circ (c \circ d_1 \circ w \circ w') = s_2 \circ (b_2 \circ d_2 \circ e \circ w');$$

this is because we can take a detour through L_1'. Therefore there is $w'' \in \mathsf{S}$ s.t.

$$(c \circ d_1 \circ w \circ w') \circ w'' = (b_2 \circ d_2 \circ e \circ w') \circ w''.$$

Now all paths $H'' \to M_2$ in Figure 6 are equal. All paths $H'' \to M_0$ are equal and are in S.

Erase the objects M_1, J_1, J_2 and all arrows touching them from Figure 6. Then erase H, H', but keep the paths through them. We obtain the commutative diagram in Figure 7. This is evidence for

$$(a_2 \circ c, s_1 \circ u) \sim (a_2' \circ c', s_1' \circ u').$$

The proof that composition is well-defined is done.

$$
\begin{tikzcd}
H'' \arrow[r, dashed, "w''"] & H' \arrow[r, dashed, "w'"] & H \arrow[dl, dashed, "w"'] \arrow[dr, dashed, "e"] & & \\
& I_1 \arrow[dl, "v_1"'] \arrow[d, "d_1"] & & I_2 \arrow[d, "v_2"'] \arrow[dr, "d_2"] & \\
J_1 \arrow[d, "b_1"'] \arrow[dr, "b'_1"] & K \arrow[dl, "u"] \arrow[drr, "c"] & & K' \arrow[dll, "u'"'] \arrow[dr, "c'"'] & J_2 \arrow[dl, "b_2"'] \arrow[d, "b'_2"] \\
L_1 \arrow[d, "s_1"'] \arrow[drr, "a_1"] & L'_1 \arrow[dl, "s'_1"] \arrow[dr, "a'_1"] & & L_2 \arrow[dl, "s_2"'] \arrow[dr, "a_2"] & L'_2 \arrow[dll, "s'_2"] \arrow[d, "a'_2"] \\
M_0 & & M_1 & & M_2
\end{tikzcd}
$$

FIGURE 6.

$$
\begin{tikzcd}
& & H'' \arrow[dl, "w \circ w' \circ w''"'] \arrow[dr, "e \circ w' \circ w''"] & & \\
& I_1 \arrow[d, "d_1"] & & I_2 \arrow[d, "v_2"'] & \\
& K \arrow[dl, "u"] \arrow[drr, "c"] & & K' \arrow[dll, "u'"'] \arrow[dr, "c'"'] & \\
L_1 \arrow[d, "s_1"'] & L'_1 \arrow[dl, "s'_1"] & & L_2 \arrow[dr, "a_2"'] & L'_2 \arrow[d, "a'_2"] \\
M_0 & & & & M_2
\end{tikzcd}
$$

FIGURE 7.

The identity morphism 1_M of an object M is $\overline{(1_M, 1_M)}$.

Step 3. We have to verify the associativity and the identity properties of composition in A_S. Namely that A_S is a category. This seems to be not too hard, given Step 2, and we leave it as an exercise!

Step 4. The functor $\mathrm{Q} : \mathsf{A} \to \mathsf{A}_\mathsf{S}$ is defined to be $\mathrm{Q}(M) := M$ on objects, and $\mathrm{Q}(a) := \overline{(a, 1_M)}$ for $a : M \to N$ in A. We have to verify this is a functor... Again, an exercise.

Step 5. Finally we verify properties (RO1)-(RO4). (RO1) is clear. The inverse of $\mathrm{Q}(s)$ is $\overline{(1, s)}$, so (RO2) holds.

It is not hard to see that

$$\overline{(a, s)} = \overline{(a, 1)} \circ \overline{(1, s)};$$

this is (RO3).

If $\mathrm{Q}(a_1) = \mathrm{Q}(a_2)$, then $(a_1, 1_M) \sim (a_2, 1_M)$; so there are $b_1, b_2 \in \mathsf{A}$ s.t. $a_1 \circ b_1 = a_2 \circ b_2$ and $1 \circ b_1 = 1 \circ b_2 \in \mathsf{S}$. Writing $s := b_1 \in \mathsf{S}$, we get $a_1 \circ s = a_2 \circ s$. This proves (RO4). □

Proposition 6.2.21. *Let* A *be a category, let* S *be a right denominator set in* A*, and let* $(\mathsf{A}_\mathsf{S}, \mathrm{Q})$ *be the right Ore localization. For any two morphisms* $q_1, q_2 : M \to N$ *in* A_S *there is a common denominator. Namely we can write*

$$q_i = \mathrm{Q}(a_i) \circ \mathrm{Q}(s)^{-1}$$

for suitable $a_i \in \mathsf{A}$ *and* $s \in \mathsf{S}$.

Proof. Choose representatives $q_i = \mathrm{Q}(a'_i) \circ \mathrm{Q}(s'_i)^{-1}$. By (RD1) applied to $L_1 \to M \leftarrow L_2$, there are $b \in \mathsf{A}$ and $t \in \mathsf{S}$ s.t. the diagram above M commutes:

$$\begin{tikzcd}
& L \arrow[dl, "t"'] \arrow[dr, "b"] & \\
L_1 \arrow[d, "s'_1"'] \arrow[drr, "a'_1"' near end] & & L_2 \arrow[d, "a'_2"] \arrow[dll, "s'_2"' near start] \\
M & & N
\end{tikzcd}$$

Write $s := s'_1 \circ t = s'_2 \circ b$, $a_1 := a'_1 \circ t$ and $a_2 := a'_2 \circ b$. By Lemma 6.2.5 we get $q_i = \mathrm{Q}(a_i) \circ \mathrm{Q}(s)^{-1}$. □

Exercise 6.2.22. Let A be a category, let S be a right denominator set in A. Let Y be a subset of $\mathrm{Ob}(\mathsf{A})$, and let B and T be the full subcategories of A and S respectively on the set of objects Y.

(1) Is T a right denominator set in B ?
(2) Show that if T is a right denominator set in B, then the inclusion functor $F : \mathsf{B} \to \mathsf{A}$ extends uniquely to a functor $F_\mathsf{T} : \mathsf{B}_\mathsf{T} \to \mathsf{A}_\mathsf{S}$.
(3) Assume that T is a right denominator set in B. Is the functor F_T full or faithful?

We will return to these questions later.

6.3. **Localization of Linear Categories.** Until now in this section we dealt with arbitrary categories. In this and the subsequent subsection, our categories will be linear over some commutative base ring $\mathbb{K}$ (that will be implicit in everything). This includes the case $\mathbb{K} = \mathbb{Z}$ of course.

For convenience we only talk about right denominator sets here. All the statements hold equally for left denominator sets; cf. Remark 6.2.2 and Proposition 6.2.3.

Theorem 6.3.1. *Let A be a $\mathbb{K}$-linear category, let S be a right denominator set in A, and let $(\mathsf{A}_\mathsf{S}, \mathrm{Q})$ be the right Ore localization.*

(1) *The category A_S has a unique $\mathbb{K}$-linear structure, such that $\mathrm{Q} : \mathsf{A} \to \mathsf{A}_\mathsf{S}$ is a $\mathbb{K}$-linear functor.*
(2) *Suppose B is another $\mathbb{K}$-linear category, and $F : \mathsf{A} \to \mathsf{B}$ is a $\mathbb{K}$-linear functor s.t. $F(s)$ is invertible for every $s \in \mathsf{S}$. Let $F_\mathsf{S} : \mathsf{A}_\mathsf{S} \to \mathsf{B}$ be the localization of F. Then F_S is a $\mathbb{K}$-linear functor.*
(3) *If A is an additive category, then so is A_S.*

Proof. (1) Let $q_i : M \to N$ be morphisms in A_S. Choose common denominator presentations $q_i = \mathrm{Q}(a_i) \circ \mathrm{Q}(s)^{-1}$. Since Q must be an additive functor, we have to define

$$\mathrm{Q}(a_1) + \mathrm{Q}(a_2) := \mathrm{Q}(a_1 + a_2). \tag{6.3.2}$$

By the distributive law (bilinearity of composition) we must define

$$q_1 + q_2 := \mathrm{Q}(a_1 + a_2) \circ \mathrm{Q}(s)^{-1}.$$

For $\lambda \in \mathbb{K}$ we must define

$$\lambda \cdot q_i := \mathrm{Q}(\lambda \cdot a_i) \circ \mathrm{Q}(s)^{-1}.$$

The usual tricks are then used to prove independence of representatives. For instance, to prove that (6.3.2) is independent of choices, suppose that $\mathrm{Q}(a_1) = \mathrm{Q}(a_1')$ and $\mathrm{Q}(a_2) = \mathrm{Q}(a_2')$. Then, by (RO4), there are $t_1, t_2 \in \mathsf{S}$ such that $a_1 \circ t_1 = a_1' \circ t_1$ and $a_2 \circ t_2 = a_2' \circ t_2$. By (RD1) there exist $b \in \mathsf{A}$ and $v \in \mathsf{S}$ s.t. $t_1 \circ b = t_2 \circ v$. Let $t_3 := t_2 \circ v \in \mathsf{S}$. Then

$$(a_1 + a_2) \circ t_3 = (a_1' + a_2') \circ t_3,$$

and hence

$$\mathrm{Q}(a_1 + a_2) = \mathrm{Q}(a_1' + a_2').$$

In this way A_S is a $\mathbb{K}$-linear category, and Q is a $\mathbb{K}$-linear functor.

(2) The only option for F_S is $F_\mathsf{S}(q_i) := F(a_i) \circ F(s)^{-1}$. The usual tricks are used to prove independence of representatives.

(3) Clear from Propositions 2.4.2 and 2.4.5. □

Example 6.3.3. Let A be a ring, which we can think of as a one object linear category A. In this context, Theorem 6.3.1 is one of the most important results in ring theory. See [Row, Ste].

Example 6.3.4. Suppose A is a commutative ring, and S is a multiplicatively closed set in it. Because A is commutative, the denominator conditions hold automatically. The localized category A_S is the single object category, with endomorphism set A_S. This is simply the usual commutative localization.

Note that if S contains a nilpotent element, then the ring A_S is trivial.

The observation above should serve as a warning: localization can sometimes kill everything. This is the singularity effect: dividing by zero!

Fortunately, the localization procedure (7.0.1), that gives rise to the derived category, does not cause any catastrophe, as we shall see in Proposition 6.4.10.

Remark 6.3.5. Suppose A is a ring and S is a right denominator set in it. Then the right Ore localization A_S is *flat* as left A-module. See [Row, Theorem 3.1.20]. I have no idea if something like this is true for linear categories with more than one object.

Proposition 6.3.6. *Let* (K, T) *be a T-additive* $\mathbb{K}$*-linear category, let* S *be a right denominator set in* K *such that* $\mathrm{T}(\mathsf{S}) = \mathsf{S}$*, and let* $\mathrm{Q} : \mathsf{K} \to \mathsf{K}_\mathsf{S}$ *be the localization functor.*

(1) *There is a unique* $\mathbb{K}$*-linear automorphism* T_S *of the category* K_S*, such that*
$$\mathrm{T}_\mathsf{S} \circ \mathrm{Q} = \mathrm{Q} \circ \mathrm{T}$$
as functors $\mathsf{K} \to \mathsf{K}_\mathsf{S}$.

(2) *Let* τ *be the identity automorphism of the functor* $\mathrm{Q} \circ \mathrm{T}$*. Then*
$$(\mathrm{Q}, \tau) : (\mathsf{K}, \mathrm{T}) \to (\mathsf{K}_\mathsf{S}, \mathrm{T}_\mathsf{S})$$
is a T-additive functor.

Proof. (1) By the assumption the functor $\mathrm{Q} \circ \mathrm{T} : \mathsf{K} \to \mathsf{K}_\mathsf{S}$ sends the morphisms in S to isomorphism. By the property (L3) of localization in Definition 6.1.2, the functor $\mathrm{T}_\mathsf{S} : \mathsf{K}_\mathsf{S} \to \mathsf{K}_\mathsf{S}$ satisfying $\mathrm{T}_\mathsf{S} \circ \mathrm{Q} = \mathrm{Q} \circ \mathrm{T}$ exists and is unique. Similarly, there is a unique functor $\mathrm{T}_\mathsf{S}^{-1} : \mathsf{K}_\mathsf{S} \to \mathsf{K}_\mathsf{S}$ satisfying $\mathrm{T}_\mathsf{S}^{-1} \circ \mathrm{Q} = \mathrm{Q} \circ \mathrm{T}^{-1}$. An easy calculation shows that
$$\mathrm{T}_\mathsf{S}^{-1} \circ \mathrm{T}_\mathsf{S} = \mathrm{Id} = \mathrm{T}_\mathsf{S} \circ \mathrm{T}_\mathsf{S}^{-1} .$$
Hence T_S is an automorphism of K_S.

(2) This is clear. □

The composition of T-additive functors was defined in Definition 5.1.4.

Proposition 6.3.7. *In the situation of Proposition* 6.3.6*, suppose* $(\mathsf{K}', \mathrm{T}')$ *is another T-additive* $\mathbb{K}$*-linear category, and*
$$(F, \nu) : (\mathsf{K}, \mathrm{T}) \to (\mathsf{K}', \mathrm{T}')$$
is a T-additive $\mathbb{K}$*-linear functor, such that* $F(s)$ *is invertible for any* $s \in \mathsf{S}$*. Let* $F_\mathsf{S} : \mathsf{K}_\mathsf{S} \to \mathsf{K}'$ *be the localized functor. Then there is a unique isomorphism*
$$\nu_\mathsf{S} : F_\mathsf{S} \circ \mathrm{T}_\mathsf{S} \xrightarrow{\simeq} \mathrm{T}' \circ F_\mathsf{S}$$
of functors $\mathsf{K}_\mathsf{S} \to \mathsf{K}'$*, such that*
$$(F, \nu) = (F_\mathsf{S}, \nu_\mathsf{S}) \circ (\mathrm{Q}, \tau)$$
as T-additive functors $(\mathsf{K}, \mathrm{T}) \to (\mathsf{K}', \mathrm{T}')$.

Exercise 6.3.8. Prove Proposition 6.3.7.

6.4. **Localization of Pretriangulated Categories.** Let K be a pretriangulated category, with translation functor T.

Proposition 6.4.1. *Suppose $H : \mathsf{K} \to \mathsf{M}$ is a cohomological functor, where M is some abelian category. Let*

$$\mathsf{S} := \{s \in \mathsf{K} \mid H(\mathrm{T}^i(s)) \text{ is invertible for all } i \in \mathbb{Z}\}.$$

Then S is a left and right denominator set in K.

Proof. It is clear that S is closed under composition and contains the identity morphisms. So it is a multiplicatively closed set.

Let's prove that condition (RD1) of Definition 6.2.14 holds. Suppose we are given morphisms $L \xrightarrow{a} N \xleftarrow{s} M$ with $s \in \mathsf{S}$. We need to find morphisms $L \xleftarrow{t} K \xrightarrow{b} M$ with $t \in \mathsf{S}$ and such that $a \circ t = s \circ b$.

Consider the solid commutative diagram

$$\begin{tikzcd}
K \arrow[r, "t"] \arrow[d, dashed, "b"'] & L \arrow[r, "c\circ a"] \arrow[d, "a"] & P \arrow[r] \arrow[d, "="] & \mathrm{T}(K) \arrow[d, dashed, "\mathrm{T}(b)"] \\
M \arrow[r, "s"] & N \arrow[r, "c"] & P \arrow[r] & \mathrm{T}(M)
\end{tikzcd}$$

where the bottom row is a distinguished triangle built on $M \xrightarrow{s} N$, and and the top row is a distinguished triangle built on $L \xrightarrow{c\circ a} P$, then turned $120°$ to the right. By axiom (TR3) there is a morphism b making the diagram commutative. Thus $a \circ t = s \circ b$. Since $H(\mathrm{T}^i(s))$ are invertible for all $i \in \mathbb{Z}$, it follows that $H(\mathrm{T}^i(P)) = 0$. But then $H(\mathrm{T}^i(t))$ are invertible for all $i \in \mathbb{Z}$, so $t \in \mathsf{S}$.

Next we prove condition (RD2) of Definition 6.2.14. Because we are in an additive category, this condition is simplified: given $a \in \mathsf{K}$ and $s \in \mathsf{S}$ satisfying $s \circ a = 0$, we have to find $t \in \mathsf{S}$ satisfying $a \circ t = 0$.

Say the objects involved are $L \xrightarrow{a} M \xrightarrow{s} N$. Take a distinguished triangle built on s and then turned:

$$P \xrightarrow{b} M \xrightarrow{s} N \to \mathrm{T}(P).$$

We get an exact sequence

$$\mathrm{Hom}_{\mathsf{K}}(L, P) \xrightarrow{b\,\circ\,-} \mathrm{Hom}_{\mathsf{K}}(L, M) \xrightarrow{s\,\circ\,-} \mathrm{Hom}_{\mathsf{K}}(L, N).$$

Since $s \circ a = 0$, there is $c : L \to P$ s.t. $a = b \circ c$. Now look at a distinguished triangle built on c, and then turned:

$$K \xrightarrow{t} L \xrightarrow{c} P \to \mathrm{T}(K).$$

We know that $c \circ t = 0$; hence $a \circ t = b \circ c \circ t = 0$. But $(s \in \mathsf{S}) \Rightarrow (H(\mathrm{T}^i(P)) = 0$ for all $i) \Rightarrow (t \in \mathsf{S})$.

The left versions (LD1) and (LD2) are proved the same way. $\square$

Definition 6.4.2. A *denominator set of cohomological origin* in K is a denominator set $\mathsf{S} \subseteq \mathsf{K}$ that arises from a cohomological functor H, as in Proposition 6.4.1. The morphisms in S are called *quasi-isomorphisms relative to H*.

Theorem 6.4.3. *Let (K, T) be a pretriangulated category, let S be a denominator set of cohomological origin in K, and let*

$$(\mathrm{Q}, \tau) : (\mathsf{K}, \mathrm{T}) \to (\mathsf{K}_{\mathsf{S}}, \mathrm{T}_{\mathsf{S}})$$

be the T-additive functor from Proposition 6.3.6. *The T-additive category* $(\mathsf{K}_{\mathsf{S}}, \mathrm{T}_{\mathsf{S}})$ *has a unique pretriangulated structure such that these two properties hold*:

(i) *The pair* (Q, τ) *is a triangulated functor.*
(ii) *Suppose* $(\mathsf{K}', \mathrm{T}')$ *is another pretriangulated category, and*

$$(F, \nu) : (\mathsf{K}, \mathrm{T}) \to (\mathsf{K}', \mathrm{T}')$$

is a triangulated functor, such that $F(s)$ *is invertible for every* $s \in \mathsf{S}$. *Let*

$$(F_{\mathsf{S}}, \nu_{\mathsf{S}}) : (\mathsf{K}_{\mathsf{S}}, \mathrm{T}_{\mathsf{S}}) \to (\mathsf{K}', \mathrm{T}')$$

be the T-additive functor from Proposition 6.3.7. *Then* $(F_{\mathsf{S}}, \nu_{\mathsf{S}})$ *is a triangulated functor.*

Proof. Since S is of cohomological origin we have $\mathrm{T}(\mathsf{S}) = \mathsf{S}$. Recall that the translation isomorphism τ is the identity automorphism of the functor $\mathrm{Q} \circ \mathrm{T}$; see Proposition 6.3.6. So we will ignore it.

Step 1. The distinguished triangles in K_{S} are defined to be those triangles that are isomorphic to the images under Q of distinguished triangles in K. Let us verify the axioms of pretriangulated category.

(TR1). By definition every triangle that's isomorphic to a distinguished triangle is distinguished; and the triangle

$$M \xrightarrow{1_M} M \to 0 \to \mathrm{T}(M)$$

in K_{S} is clearly distinguished.

Suppose we are given a morphism $\alpha : L \to M$ in K_{S}. We have to build a distinguished triangle on it. Choose a fraction presentation $\alpha = \mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1}$. Using condition (LD1) we can find $b \in \mathsf{K}$ and $t \in \mathsf{S}$ such that $t \circ a = b \circ s$. These fit into the solid commutative diagram

$$\begin{tikzcd} L \arrow[rr, bend left, dashed, "\alpha"] \arrow[dr, "b"'] & K \arrow[l, "s"'] \arrow[r, "a"] & M \arrow[dl, "t"] \\ & \tilde{K} & \end{tikzcd}$$

in K. (The dashed arrow α is in K_{S}.)

Consider the solid commutative diagram below, where the rows are distinguished triangles built on a and b respectively.

(6.4.4)
$$\begin{tikzcd} K \arrow[r, "a"] \arrow[d, "s"'] & M \arrow[r, "e"] \arrow[d, "t"'] & N \arrow[r, "c"] \arrow[d, dashed, "u"'] & \mathrm{T}(K) \arrow[d, "\mathrm{T}(s)"'] \\ L \arrow[r, "b"] & \tilde{K} \arrow[r] & P \arrow[r, "d"] & \mathrm{T}(L) \end{tikzcd}$$

By (TR3) there is a morphism u that makes the whole diagram commutative. Since $s, t \in \mathsf{S}$ and H is a cohomological functor, it follows that $u \in \mathsf{S}$. Applying the functor Q to (6.4.4), and using the isomorphism $\mathrm{Q}(t) : M \to \tilde{K}$ to replace $\tilde{K}$ with

M, we get the commutative diagram

$$\begin{tikzcd}
K \arrow[r, "\mathrm{Q}(a)"] \arrow[d, "\mathrm{Q}(s)"'] & M \arrow[r, "\mathrm{Q}(e)"] \arrow[d, "\mathrm{Q}(1_M)"'] & N \arrow[r, "\mathrm{Q}(c)"] \arrow[d, "\mathrm{Q}(u)"'] & \mathrm{T}(K) \arrow[d, "\mathrm{T}(\mathrm{Q}(s))"'] \\
L \arrow[r, "\alpha"] & M \arrow[r, "\mathrm{Q}(u \circ e)"] & P \arrow[r, "\mathrm{Q}(d)"] & \mathrm{T}(L)
\end{tikzcd}$$

in K_{S}. The top row is a distinguished triangle, and the vertical arrows are isomorphisms. So the bottom row is a distinguished triangle. This is the triangle we were looking for.

(TR2). Turning: this is trivial.

(TR3). We are given the solid commutative diagram in K_{S}, where the rows are distinguished triangles:

(6.4.5)
$$\begin{tikzcd}
L \arrow[r, "\alpha"] \arrow[d, "\phi"'] & M \arrow[r, "\beta"] \arrow[d, "\psi"'] & N \arrow[r, "\gamma"] \arrow[d, dashed, "\chi"'] & \mathrm{T}(L) \arrow[d, "\mathrm{T}(\phi)"'] \\
L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L')
\end{tikzcd}$$

and we have to find χ to complete the diagram.

By replacing the rows with isomorphic triangles, we can assume they come from K. Thus we can replace (6.4.5) with this diagram:

(6.4.6)
$$\begin{tikzcd}
L \arrow[r, "\mathrm{Q}(\alpha)"] \arrow[d, "\phi"'] & M \arrow[r, "\mathrm{Q}(\beta)"] \arrow[d, "\psi"'] & N \arrow[r, "\mathrm{Q}(\gamma)"] \arrow[d, dashed, "\chi"'] & \mathrm{T}(L) \arrow[d, "\mathrm{T}(\phi)"'] \\
L' \arrow[r, "\mathrm{Q}(\alpha')"] & M' \arrow[r, "\mathrm{Q}(\beta')"] & N' \arrow[r, "\mathrm{Q}(\gamma')"] & \mathrm{T}(L')
\end{tikzcd}$$

in which $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ are morphisms in K. It is a commutative diagram. Let us choose fraction presentations $\phi = \mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1}$ and $\psi = \mathrm{Q}(b) \circ \mathrm{Q}(t)^{-1}$. Then the solid diagram (6.4.6) comes from applying Q to the diagram

(6.4.7)
$$\begin{tikzcd}
L \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & N \arrow[r, "\gamma"] & \mathrm{T}(L) \\
\tilde{L} \arrow[u, "s"] \arrow[d, "a"'] & \tilde{M} \arrow[u, "t"] \arrow[d, "b"'] & & \mathrm{T}(\tilde{L}) \arrow[u, "\mathrm{T}(s)"'] \arrow[d, "\mathrm{T}(a)"'] \\
L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L')
\end{tikzcd}$$

in K. Here the rows are distinguished triangles in K; but the diagram might fail to be commutative.

By axiom (RO3) we can find $c \in \mathsf{K}$ and $u \in \mathsf{S}$ s.t.

$$\mathrm{Q}(t)^{-1} \circ \mathrm{Q}(\alpha) \circ \mathrm{Q}(s) = \mathrm{Q}(c) \circ \mathrm{Q}(u)^{-1}.$$

This is the solid diagram:

$$\begin{tikzcd}
 & & L \arrow[r, "\alpha"] & M \\
\tilde{L}'' \arrow[r, dashed, "u'"] & \tilde{L}' \arrow[r, "u"] \arrow[rr, bend right, "c"'] & \tilde{L} \arrow[u, "s"] \arrow[d, "a"] & \tilde{M} \arrow[u, "t"'] \arrow[d, "b"] \\
 & & L' \arrow[r, "\alpha'"] & M'
\end{tikzcd}$$

Thus

$$\mathrm{Q}(\alpha \circ s \circ u) = \mathrm{Q}(t \circ c).$$

By (RO4) there is $u' \in \mathsf{S}$ s.t.

$$(\alpha \circ s \circ u) \circ u' = (t \circ c) \circ u'.$$

We get

$$\phi = \mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1} = \mathrm{Q}(a \circ u \circ u') \circ \mathrm{Q}(s \circ u \circ u')^{-1}$$

in K_{S}. Thus, after substituting $\tilde{L} := \tilde{L}''$, $s := s \circ u \circ u'$, $a := a \circ u \circ u'$ and $c := c \circ u'$, we get a new diagram

(6.4.8)
$$\begin{tikzcd}
L \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & N \arrow[r, "\gamma"] & \mathrm{T}(L) \\
\tilde{L} \arrow[u, "s"] \arrow[r, "c"] \arrow[d, "a"'] & \tilde{M} \arrow[u, "t"] \arrow[d, "b"] & & \mathrm{T}(\tilde{L}) \arrow[u, "\mathrm{T}(s)"'] \arrow[d, "\mathrm{T}(a)"'] \\
L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L')
\end{tikzcd}$$

in K instead of (6.4.7). In this new diagram the top left square is commutative; but maybe the bottom left square is not commutative.

When we apply Q to the diagram (6.4.8), the whole diagram, including the bottom left square, becomes commutative, since (6.4.6) is commutative. Again using condition (RO4), there is $v \in \mathsf{S}$ s.t.

$$(\alpha' \circ a) \circ v = (b \circ c) \circ v.$$

In a diagram:

$$\begin{tikzcd}
 & L \arrow[r, "\alpha"] & M \\
\tilde{L}' \arrow[r, "v"] & \tilde{L} \arrow[u, "s"] \arrow[r, "c"] \arrow[d, "a"'] & \tilde{M} \arrow[u, "t"'] \arrow[d, "b"'] \\
 & L' \arrow[r, "\alpha'"] & M'
\end{tikzcd}$$

Performing the replacements $\tilde{L} := \tilde{L}'$, $s := s \circ v$, $c := c \circ v$ and $a := a \circ v$ we now have a commutative square also at the bottom left of (6.4.8). Since $\gamma \circ \beta = 0$ and $\gamma' \circ \beta' = 0$, in fact the whole diagram (6.4.8) in K is now commutative.

Now by (TR1) we can embed the morphism c in a distinguished triangle. We get the solid diagram

$$\begin{tikzcd}
L \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & N \arrow[r, "\gamma"] & \mathrm{T}(L) \\
\tilde{L} \arrow[r, "c"] \arrow[u, "s"] \arrow[d, "a"'] & \tilde{M} \arrow[r, "\tilde{\beta}"] \arrow[u, "t"'] \arrow[d, "b"'] & \tilde{N} \arrow[r, "\tilde{\gamma}"] \arrow[u, dashed, "w"] \arrow[d, dashed, "d"] & \mathrm{T}(\tilde{L}) \arrow[u, "\mathrm{T}(s)"'] \arrow[d, "\mathrm{T}(a)"] \\
L' \arrow[r, "\alpha'"] & M' \arrow[r, "\beta'"] & N' \arrow[r, "\gamma'"] & \mathrm{T}(L')
\end{tikzcd} \tag{6.4.9}$$

in K. The rows are distinguished triangles. Since $\tilde{\gamma} \circ \tilde{\beta} = 0$, the solid diagram is commutative. By (TR3) there are morphisms w and d that make the whole diagram commutative. Now the morphism $w \in \mathsf{S}$ by the usual long exact sequence argument. The morphism

$$\chi := \mathrm{Q}(d) \circ \mathrm{Q}(w)^{-1} : N \to N'$$

solves the problem.

Step 2. Suppose (F, ν) is a triangulated functor as in condition (ii). By Proposition 6.3.7 this extends uniquely to a T-additive functor $(F_{\mathsf{S}}, \nu_{\mathsf{S}})$. The construction of the pretriangulated structure on $(\mathsf{K}_{\mathsf{S}}, \mathrm{T}_{\mathsf{S}})$ in the previous steps, and the defining property of the translation isomorphism ν_{S} in Proposition 6.3.7, show that $(F_{\mathsf{S}}, \nu_{\mathsf{S}})$ is a triangulated functor.

Step 3. At this point $(\mathsf{K}_{\mathsf{S}}, \mathrm{T}_{\mathsf{S}})$ is a pretriangulated category, and conditions (i)-(ii) of the theorem are satisfied. We need to prove the uniqueness of the pretriangulated structure on $(\mathsf{K}_{\mathsf{S}}, \mathrm{T}_{\mathsf{S}})$. Condition (i) says that we can't have less distinguished triangles than those we declared. We can't have more distinguished triangles, because of condition (ii). □

Proposition 6.4.10. *Consider the situation of Proposition* 6.4.1 *and Theorem* 6.4.3*.*

(1) *The cohomological functor* $H : \mathsf{K} \to \mathsf{M}$ *factors into* $H = H_{\mathsf{S}} \circ \mathrm{Q}$*, where* $H_{\mathsf{S}} : \mathsf{K}_{\mathsf{S}} \to \mathsf{M}$ *is a cohomological functor.*
(2) *Let* M *be an object of* K*. The object* $\mathrm{Q}(M)$ *is zero in* K_{S} *iff the objects* $H(\mathrm{T}^i((M))$ *are zero in* M *for all* i*.*

Proof. (1) The existence and uniqueness of the functor H_{S} are by the universal property (L3) in Definition 6.1.2. We leave it as an exercise to show that H_{S} is a cohomological functor.

(2) Since H_{S} is an additive functor, if $\mathrm{Q}(M) = 0$, then so is $H(M) = H_{\mathsf{S}}(\mathrm{Q}(M))$. And of course $\mathrm{Q}(M) = 0$ iff $\mathrm{Q}(\mathrm{T}^i(M)) = 0$ for all i.

For the converse, let $\phi : 0 \to M$ be the zero morphism in K. If $H(\mathrm{T}^i(M)) = 0$ for all i, then $H(\mathrm{T}^i(\phi)) : 0 \to H(\mathrm{T}^i(M))$ are isomorphisms for all i. Then $\phi \in \mathsf{S}$, and so $\mathrm{Q}(\phi) : 0 \to \mathrm{Q}(M)$ is an isomorphism in K_{S}. □

7. The Derived Category $\mathbf{D}(A, \mathsf{M})$

In this section there is a commutative base ring $\mathbb{K}$, that shall remain implicit most of the time. We fix a central DG $\mathbb{K}$-ring A, and a $\mathbb{K}$-linear abelian category M. The DG category $\mathbf{C}(A, \mathsf{M})$ was introduced in Subsection 3.7, and the pretriangulated category $\mathbf{K}(A, \mathsf{M})$ was introduced in Subsection 5.4.

The functor $\mathrm{H}^0 : \mathbf{K}(A, \mathsf{M}) \to \mathsf{M}$ is a cohomological functor, in the sense of Definition 5.3.2. The resulting denominator set is denoted by $\mathbf{S}(A, \mathsf{M})$, and its elements are called *quasi-isomorphisms*. The *derived category* of (A, M) is the pretriangulated category

$$\mathbf{D}(A, \mathsf{M}) := \mathbf{K}(A, \mathsf{M})_{\mathbf{S}(A,\mathsf{M})}. \tag{7.0.1}$$

7.1. Definition of the Derived Category.

Proposition 7.1.1. *Let* M *be an abelian category and let* A *be a DG ring. The functor*

$$\mathrm{H}^0 : \mathbf{K}(A, \mathsf{M}) \to \mathsf{M}$$

is cohomological.

Proof. Clearly H^0 is additive. Consider a distinguished triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L) \tag{7.1.2}$$

in $\mathbf{K}(A, \mathsf{M})$. We can assume that it is the image of a standard triangle in $\mathbf{C}(A, \mathsf{M})$, namely that N is the cone associated to α, as in Definition 3.9.5, $\beta = e_\alpha$ and $\gamma = p_\alpha$. By construction, the cone N sits in an exact sequence of complexes

$$0 \to M \xrightarrow{e_\alpha} N \xrightarrow{p_\alpha} \mathrm{T}(L) \to 0. \tag{7.1.3}$$

Consider the diagram

$$\begin{tikzcd}
\mathrm{H}^{-1}(\mathrm{T}(L)) \arrow[r, "\mathrm{conn}"] \arrow[d, "\mathrm{H}(\mathrm{t}_L^{-1})"'] & \mathrm{H}^0(M) \arrow[r, "\mathrm{H}^0(e_\alpha)"] \arrow[d, "="'] & \mathrm{H}^0(N) \arrow[d, "="'] \\
\mathrm{H}^0(L) \arrow[r, "\mathrm{H}^0(\alpha)"] & \mathrm{H}^0(M) \arrow[r, "\mathrm{H}^0(\beta)"] & \mathrm{H}^0(N)
\end{tikzcd}$$

in M, where the first row is part of the long exact cohomology sequence for (7.1.3), and the second row comes from (7.1.2). The first square is commutative because any lifting represents the connecting homomorphism (cf. [Rot, Theorem 6.2]). The second square is also commutative. It follows that the diagram is commutative, and that the bottom row is exact. $\square$

Definition 7.1.4. A morphism ϕ in $\mathbf{K}(A, \mathsf{M})$ is called a *quasi-isomorphism* if the morphisms $\mathrm{H}^i(\phi)$ in M are isomorphisms for all i.

The set of quasi-isomorphisms in $\mathbf{K}(A, \mathsf{M})$ is denoted by $\mathbf{S}(A, \mathsf{M})$.

Note that $\mathrm{H}^i = \mathrm{H}^0 \circ \mathrm{T}^i$. By Proposition 7.1.1 the functor H^0 is cohomological. Therefore $\mathbf{S}(A, \mathsf{M})$ is a denominator set of cohomological origin, Theorem 6.4.3 applies to it, and the next definition makes sense.

Definition 7.1.5. Let M be a $\mathbb{K}$-linear abelian category and A a central DG $\mathbb{K}$-ring. The *derived category* of (A, M) is the $\mathbb{K}$-linear pretriangulated category

$$\mathbf{D}(A, \mathsf{M}) := \mathbf{K}(A, \mathsf{M})_{\mathbf{S}(A,\mathsf{M})}.$$

In our situation we have additive functors

$$\mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \xrightarrow{\mathrm{P}} \mathbf{K}(A, \mathsf{M}) \xrightarrow{\mathrm{Q}} \mathbf{D}(A, \mathsf{M}),$$

that are the identity on objects. Recall that the functor P sends a strict morphism of DG modules to its homotopy class; and Q is the localization functor with respect to quasi-isomorphisms.

Definition 7.1.6. Let M be an abelian category and let A be a DG ring. Define the functor

$$\tilde{\mathrm{Q}} := \mathrm{Q} \circ \mathrm{P} : \mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \to \mathbf{D}(A, \mathsf{M}).$$

This definition will only be used in the present section.

It is sometimes convenient to describe morphisms in $\mathbf{D}(A, \mathsf{M})$ in terms of the functor $\tilde{\mathrm{Q}}$. A morphism $s \in \mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ is called a quasi-isomorphism if $\mathrm{P}(s)$ is a quasi-isomorphism in $\mathbf{K}(A, \mathsf{M})$; i.e. if all the $\mathrm{H}^i(s)$ are isomorphisms.

Proposition 7.1.7.

(1) *Any morphism ϕ in $\mathbf{D}(A, \mathsf{M})$ can be written as a right fraction*

$$\phi = \tilde{\mathrm{Q}}(a) \circ \tilde{\mathrm{Q}}(s)^{-1}$$

where $a, s \in \mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ and s is a quasi-isomorphism.

(2) *The kernel of $\tilde{\mathrm{Q}}$ is this: $\tilde{\mathrm{Q}}(a) = 0$ in $\mathbf{D}(A, \mathsf{M})$ iff there exists a quasi-isomorphism s in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ such that $a \circ s$ is a coboundary in $\mathbf{C}(A, \mathsf{M})$.*

Proof. (1) This is because of property (RO3) of Definition 6.2.1 and the fact that P is full.

(2) Property (RO4) of Definition 6.2.1 tells us what the kernel of Q is; and by definition the kernel of P is the 0-coboundaries. □

Of course there is a left version of this proposition.

Recall that $\mathbf{G}(\mathsf{M})$ is the category of graded objects of M. For any DG module $M \in \mathbf{D}(A, \mathsf{M})$, its cohomology $\mathrm{H}(M)$ is an object of $\mathbf{G}(\mathsf{M})$, and this is a functor.

Corollary 7.1.8. *The functor*

$$\mathrm{H} : \mathbf{D}(A, \mathsf{M}) \to \mathbf{G}(\mathsf{M})$$

is conservative. Namely a morphism $\phi : M \to N$ in $\mathbf{D}(A, \mathsf{M})$ is an isomorphism if and only if the morphism

$$\mathrm{H}(\phi) : \mathrm{H}(M) \to \mathrm{H}(N)$$

in $\mathbf{G}(\mathsf{M})$ is an isomorphism.

Proof. One implication is trivial. For the other direction, assume that $\mathrm{H}(\phi)$ is an isomorphism. We can write ϕ as a right fraction: $\phi = \mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1}$ where $a \in \mathbf{K}(A, \mathsf{M})$ and $s \in \mathbf{S}(A, \mathsf{M})$. Then

$$\mathrm{H}(\phi) = \mathrm{H}(\mathrm{Q}(a)) \circ \mathrm{H}(\mathrm{Q}(s))^{-1}.$$

By definition $\mathrm{H}(\mathrm{Q}(s))$ is an isomorphism. Hence $\mathrm{H}(\mathrm{Q}(a))$ is an isomorphism. But then $a \in \mathbf{S}(A, \mathsf{M})$ too, and therefore $\mathrm{Q}(a)$ is an isomorphism in $\mathbf{D}(A, \mathsf{M})$. It follows that ϕ is an isomorphism in $\mathbf{D}(A, \mathsf{M})$. □

Exercise 7.1.9. Here $\mathsf{M} = \mathsf{Mod}\,\mathbb{K}$, so $\mathbf{K}(A, \mathsf{M}) = \mathbf{K}(A)$. Show that the functor $\mathrm{H}^0 : \mathbf{K}(A) \to \mathsf{Mod}\,\mathbb{K}$ is corepresentable by the object $A \in \mathbf{K}(A)$ (see Subsection 1.7).

7.2. Localization of Subcategories of $\mathbf{K}(A, \mathsf{M})$.

Definition 7.2.1. Let K be a pretriangulated category. A *full pretriangulated subcategory* of K is a subcategory $\mathsf{L} \subseteq \mathsf{K}$ satisfying these conditions:

(a) L is a full additive subcategory (see Definition 2.2.5).
(b) L is closed under translations, i.e. $L \in \mathsf{L}$ iff $\mathrm{T}(L) \in \mathsf{L}$.
(c) L is closed under distinguished triangles, i.e. if

$$L' \to L \to L'' \to \mathrm{T}(L)$$

is a distinguished triangle in K s.t. $L', L \in \mathsf{L}$, then also $L'' \in \mathsf{L}$.

Observe that L itself is pretriangulated, and the inclusion $\mathsf{L} \to \mathsf{K}$ is a triangulated functor.

Denominator sets of cohomological origin were introduced in Definition 6.4.2. By Theorem 6.4.3, if $\mathsf{S} \subseteq \mathsf{K}$ is a denominator set of cohomological origin, then the localization K_{S} is a pretriangulated category.

Example 7.2.2. This is the most important example for us: $\mathsf{K} = \mathbf{K}(A, \mathsf{M})$, $H = \mathrm{H}^0 : \mathbf{K}(A, \mathsf{M}) \to \mathsf{M}$ and $\mathsf{S} = \mathbf{S}(A, \mathsf{M})$. Here $\mathsf{K}_{\mathsf{S}} = \mathbf{D}(A, \mathsf{M})$, the derived category.

Proposition 7.2.3. *Let K be a pretriangulated category, let S be a denominator set of cohomological origin in K, and let K' be a full pretriangulated subcategory of K. Then $\mathsf{S}' := \mathsf{K}' \cap \mathsf{S}$ is a denominator set of cohomological origin in K', the Ore localization $\mathsf{K}'_{\mathsf{S}'}$ exists, and $\mathsf{K}'_{\mathsf{S}'}$ is a pretriangulated category.*

Proof. Let $H : \mathsf{K} \to \mathsf{M}$ be a cohomological functor that determines S. The functor $H|_{\mathsf{K}'} : \mathsf{K}' \to \mathsf{M}$ is also cohomological, and the set of morphisms S' satisfies

$$\mathsf{S}' = \{s \in \mathsf{K}' \mid H|_{\mathsf{K}'}(\mathrm{T}^i(s)) \text{ is an isomorphism for all } i\}.$$

Hence Proposition 6.4.1 and Theorem 6.4.3 apply. □

In the situation of the proposition, the localization functor is denoted by $\mathrm{Q}' : \mathsf{K}' \to \mathsf{K}'_{\mathsf{S}'}$.

Proposition 7.2.4. *In the situation of Proposition 7.2.3, let $F : \mathsf{K}' \to \mathsf{E}$ be a triangulated functor into some pretriangulated category E. Assume that for every $s \in \mathsf{S}'$, the morphism $F(s)$ is an isomorphism in E. Then there is a unique triangulated functor $F_{\mathsf{S}'} : \mathsf{K}'_{\mathsf{S}'} \to \mathsf{E}$ that extends F; Namely $F_{\mathsf{S}'} \circ \mathrm{Q}' = F$ as functors $\mathsf{K}' \to \mathsf{E}$.*

Proof. This is part of Theorem 6.4.3. □

In particular we can look at the functor $F : \mathsf{K}' \xrightarrow{\mathrm{inc}} \mathsf{K} \xrightarrow{\mathrm{Q}} \mathsf{K}_{\mathsf{S}}$, and its extension $F_{\mathsf{S}'} : \mathsf{K}'_{\mathsf{S}'} \to \mathsf{K}_{\mathsf{S}}$. We are interested in sufficient conditions for the functor $F_{\mathsf{S}'}$ to be fully faithful.

Proposition 7.2.5. *Let K be a pretriangulated category, let S be a denominator set of cohomological origin in K, and let $\mathsf{K}' \subseteq \mathsf{K}$ be a full pretriangulated subcategory. Define $\mathsf{S}' := \mathsf{K}' \cap \mathsf{S}$. Assume either of these conditions holds:*

(r) *Let $M \in \mathrm{Ob}(\mathsf{K})$. If there exists a morphism $s : M \to L$ in S with $L \in \mathrm{Ob}(\mathsf{K}')$, there exists a morphism $t : K \to M$ in S with $K \in \mathrm{Ob}(\mathsf{K}')$.*
(l) *The same, but with arrows reversed.*

Then the functor $F_{\mathsf{S}'} : \mathsf{K}'_{\mathsf{S}'} \to \mathsf{K}_{\mathsf{S}}$ is fully faithful.

Proof. We will prove the proposition under condition (r); the other condition is done the same way.

Let $L_1, L_2 \in \mathrm{Ob}(\mathsf{K}')$, and let $q : L_1 \to L_2$ be a morphism in K_{S}. Choose a presentation $q = \mathrm{Q}(a) \circ \mathrm{Q}(s)^{-1}$ with $s : M \to L_1$ a morphism in S and $a : M \to L_2$ a morphism in K. By condition (r) we can find a morphism $t : K \to M$ in S with $K \in \mathrm{Ob}(\mathsf{K}')$.

$$\begin{tikzcd} & K \arrow[d, "t"'] & \\ & M \arrow[dl, "s"'] \arrow[dr, "a"] & \\ L_1 \arrow[rr, dashed, "q"] & & L_2 \end{tikzcd}$$

Then $q = \mathrm{Q}(a \circ t) \circ \mathrm{Q}(s \circ t)^{-1}$. But $s \circ t \in \mathsf{S}'$ and $a \circ t \in \mathsf{K}'$, so q is in the image of the functor $F_{\mathsf{S}'}$. We see that $F_{\mathsf{S}'}$ is full.

Now let $q' : L_1 \to L_2$ be a morphism in $\mathsf{K}'_{\mathsf{S}'}$ such that $F_{\mathsf{S}'}(q') = 0$. Let us denote the localization functor $\mathsf{K}' \to \mathsf{K}'_{\mathsf{S}'}$ by Q'. Choose a presentation $q' = \mathrm{Q}'(a) \circ \mathrm{Q}'(s)^{-1}$, with $s : N \to L_1$ a morphism in S' and $a : N \to L_2$ a morphism in K'. Because $F_{\mathsf{S}'}(q') = 0$, and using Lemma 6.2.5, there is a morphism $u : M \to N$ in K such that $a \circ u = 0$ and $s \circ u \in \mathsf{S}$. Note that $u \in \mathsf{S}$. By condition (r), applied to $u : M \to N$, there is a morphism $t : K \to M$ in S such that $K \in \mathrm{Ob}(\mathsf{K}')$.

$$\begin{tikzcd} & K \arrow[d, "t"'] & \\ & M \arrow[d, "u"'] & \\ & N \arrow[dl, "s"'] \arrow[dr, "a"] & \\ L_1 \arrow[rr, dashed, "q'"] & & L_2 \end{tikzcd}$$

Then we have

$$q' = \mathrm{Q}'(a \circ u \circ t) \circ \mathrm{Q}'(s \circ u \circ t)^{-1} = 0.$$

This proves that $F_{\mathsf{S}'}$ is faithful. $\square$

7.3. **Boundedness Conditions.** A graded object $M = \{M^i\}_{i \in \mathbb{Z}}$ of M is said to be *bounded above* if the set $\{i \mid M^i \neq 0\}$ is bounded above. Likewise we define *bounded below* and *bounded* graded objects.

Definition 7.3.1. We define $\mathbf{C}^-(A, \mathsf{M})$, $\mathbf{C}^+(A, \mathsf{M})$ and $\mathbf{C}^{\mathrm{b}}(A, \mathsf{M})$ to be full subcategories of $\mathbf{C}(A, \mathsf{M})$ consisting of bounded above, bounded below and bounded complexes respectively.

Likewise we define $\mathbf{K}^-(A, \mathsf{M})$, $\mathbf{K}^+(A, \mathsf{M})$ and $\mathbf{K}^{\mathrm{b}}(A, \mathsf{M})$ to be the corresponding full subcategories of $\mathbf{K}(A, \mathsf{M})$.

Of course

$$\mathbf{C}^{\mathrm{b}}(A, \mathsf{M}) = \mathbf{C}^-(A, \mathsf{M}) \cap \mathbf{C}^+(A, \mathsf{M}),$$

and the same for $\mathsf{K}^{\mathrm{b}}(A, \mathsf{M})$. The subcategories $\mathsf{K}^{\star}(A, \mathsf{M})$, for $\star \in \{-, +, \mathrm{b}\}$, are full pretriangulated subcategory of $\mathsf{K}(A, \mathsf{M})$; this is because the operations of translation and cone preserve the various boundedness conditions.

As the next example shows, sometimes the category $\mathsf{K}^{\star}(A, \mathsf{M})$ can be very degenerate.

Example 7.3.2. Let A be the DG ring $\mathbb{K}[t, t^{-1}]$, the ring of Laurent polynomials in the variable t of degree 1, with the zero differential. If $M = \{M^i\}_{i \in \mathbb{Z}}$ is a nonzero object of $\mathsf{C}(A, \mathsf{M})$, then $M^i \neq 0$ for all i. Therefore the categories $\mathsf{C}^{\star}(A, \mathsf{M})$ and $\mathsf{K}^{\star}(A, \mathsf{M})$ are zero for $\star \in \{-, +, \mathrm{b}\}$.

Let
$$\mathsf{S}^{\star}(A, \mathsf{M}) := \mathsf{K}^{\star}(A, \mathsf{M}) \cap \mathsf{S}(A, \mathsf{M}),$$
the category of quasi-isomorphisms in $\mathsf{K}^{\star}(A, \mathsf{M})$. As already mentioned, Theorem 6.4.3 applies here, so we can localize.

Definition 7.3.3. For $\star \in \{-, +, \mathrm{b}\}$ we define
$$\mathsf{D}^{\star}(A, \mathsf{M}) := \mathsf{K}^{\star}(A, \mathsf{M})_{\mathsf{S}^{\star}(A, \mathsf{M})},$$
the Ore localization of $\mathsf{K}^{\star}(A, \mathsf{M})$ with respect to $\mathsf{S}^{\star}(A, \mathsf{M})$.

Here is another kind of boundedness condition.

Definition 7.3.4. For $\star \in \{-, +, \mathrm{b}\}$ we define $\mathsf{D}(A, \mathsf{M})^{\star}$ to be the full subcategory of $\mathsf{D}(A, \mathsf{M})$ on the complexes M whose cohomology $\mathrm{H}(M)$ is of boundedness type $\star$.

Of course $\mathsf{D}(A, \mathsf{M})^{\star}$ is a full pretriangulated subcategory of $\mathsf{D}(A, \mathsf{M})$.

The next proposition refers to the abelian case only – namely to $\mathsf{D}(\mathsf{M}) = \mathsf{D}(\mathbb{K}, \mathsf{M})$. See Exercise 7.3.12 for a generalization to $\mathsf{D}(A, \mathsf{M})$ for a special sort of DG ring A.

Proposition 7.3.5. *For $\star \in \{-, +, \mathrm{b}\}$ the canonical functor $\mathsf{D}^{\star}(\mathsf{M}) \to \mathsf{D}(\mathsf{M})^{\star}$ is an equivalence of pretriangulated categories.*

Proof. Step 1. Here we prove that $F^- : \mathsf{D}^-(\mathsf{M}) \to \mathsf{D}(\mathsf{M})$ is fully faithful. Let $s : M \to L$ be a quasi-isomorphism with $L \in \mathsf{K}^-(\mathsf{M})$. Say L is concentrated in degrees $\leq i$. Then $\mathrm{H}^j(M) = \mathrm{H}^j(L) = 0$ for all $j > i$. Consider the *smart truncation* of M at i:

$$\operatorname{smt}^{\leq i}(M) := \big(\cdots \to M^{i-2} \xrightarrow{\mathrm{d}} M^{i-1} \xrightarrow{\mathrm{d}} \mathrm{Z}^i(M) \to 0 \to \cdots\big) \tag{7.3.6}$$

where $\mathrm{Z}^i(M) := \operatorname{Ker}(\mathrm{d} : M^i \to M^{i+1})$, the object of i-cocycles, is in degree i. Then $\operatorname{smt}^{\leq i}(M)$ is a subcomplex of M, $\operatorname{smt}^{\leq i}(M) \in \mathsf{K}^-(\mathsf{M})$, and the inclusion $t : \operatorname{smt}^{\leq i}(M) \to M$ is a quasi-isomorphism. According to Proposition 7.2.5, with $\mathsf{K} = \mathsf{K}(\mathsf{M})$ and $\mathsf{K}' = \mathsf{K}^-(\mathsf{M})$, and with condition (r), we see that $F^- : \mathsf{D}^-(\mathsf{M}) \to \mathsf{D}(\mathsf{M})$ is fully faithful.

Step 2. Here we prove that $F^+ : \mathsf{D}^+(\mathsf{M}) \to \mathsf{D}(\mathsf{M})$ is fully faithful. Let $s : L \to M$ be a quasi-isomorphism with $L \in \mathsf{K}^+(\mathsf{M})$. Say L is concentrated in degrees $\geq i$. Then $\mathrm{H}^j(M) = \mathrm{H}^j(L) = 0$ for all $j < i$. Consider the other smart truncation of M at i:

$$\operatorname{smt}^{\geq i}(M) := \big(\cdots \to 0 \to \mathrm{Y}^i(M) \xrightarrow{\mathrm{d}} M^{i+1} \xrightarrow{\mathrm{d}} M^{i+2} \to \cdots\big) \tag{7.3.7}$$

where

$$\mathrm{Y}^i(M) := \mathrm{Coker}(\mathrm{d} : M^{i-1} \to M^i) \tag{7.3.8}$$

is in degree i. Then $\mathrm{smt}^{\geq i}(M)$ is a quotient complex of M, $\mathrm{smt}^{\geq i}(M) \in \mathbf{K}^+(\mathsf{M})$, and the projection $t : M \to \mathrm{smt}^{\geq i}(M)$ is a quasi-isomorphism. According to Proposition 7.2.5, with condition (l), we see that $F^+ : \mathbf{D}^+(\mathsf{M}) \to \mathbf{D}(\mathsf{M})$ is fully faithful.

Step 3. The arguments in step 1 we show that $\mathbf{D}^{\mathrm{b}}(\mathsf{M}) \to \mathbf{D}^+(\mathsf{M})$ is fully faithful. And by step 2, $\mathbf{D}^+(\mathsf{M}) \to \mathbf{D}(\mathsf{M})$ is fully faithful. Therefore $\mathbf{D}^{\mathrm{b}}(\mathsf{M}) \to \mathbf{D}(\mathsf{M})$ is fully faithful.

Step 4. Smart truncation shows that the functor $\mathbf{D}^{\star}(\mathsf{M}) \to \mathbf{D}(\mathsf{M})^{\star}$ is essentially surjective on objects. □

Remark 7.3.9. Most advanced texts write $\mathbf{D}^{\star}(\mathsf{M})$ instead of $\mathbf{D}(\mathsf{M})^{\star}$, and do not use the notation $\mathbf{D}(\mathsf{M})^{\star}$ at all. This is harmless by Proposition 7.3.5.

Remark 7.3.10. The object $\mathrm{Y}^p(M) = \mathrm{Coker}(\mathrm{d}_M^{p-1})$ that appears in formula (7.3.8) does not have a name. The naming conventions would indicate that is should be called the "object of cococycles", because it plays a role that's dual to the role of the object of cocycles $\mathrm{Z}^p(M) = \mathrm{Ker}(\mathrm{d}_M^p)$, and it can't be called "cycles". But the name "cococycles" sounds a bit strange.

Definition 7.3.11. A DG ring A is called *nonpositive* if $A^i = 0$ for all $i > 0$.

Exercise 7.3.12. Let A be a nonpositive DG ring and let M be an abelian category.

(1) Prove that differential on any $M \in \mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ is A^0-linear.
(2) Prove that the smart truncations from formulas (7.3.6) and (7.3.8) are functors from $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ to itself.
(3) Prove Proposition 7.3.5 for $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$.

7.4. **Thick Subcategories of M.** Let M be an abelian category. A *thick abelian subcategory* of M is a full abelian subcategory N that is closed under extensions. Namely if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence in M with $M', M'' \in \mathsf{N}$, then $M \in \mathsf{N}$ too.

Let $\mathbf{D}_{\mathsf{N}}(\mathsf{M})$ be the full subcategory of $\mathbf{D}(\mathsf{M})$ consisting of complexes M such that $\mathrm{H}^i(M) \in \mathsf{N}$ for every i.

Proposition 7.4.1. *If N is a thick abelian subcategory of M then $\mathbf{D}_{\mathsf{N}}(\mathsf{M})$ is a full pretriangulated subcategory of $\mathbf{D}(\mathsf{M})$.*

Proof. Clearly $\mathbf{D}_{\mathsf{N}}(\mathsf{M})$ is closed under translations. Now suppose

$$M' \to M \to M'' \to M[1]$$

is a distinguished triangle in $\mathbf{D}(\mathsf{M})$ such that $M', M \in \mathbf{D}_{\mathsf{N}}(\mathsf{M})$; we have to show that M'' is also in $\mathbf{D}_{\mathsf{N}}(\mathsf{M})$. Consider the exact sequence

$$\mathrm{H}^i(M') \to \mathrm{H}^i(M) \to \mathrm{H}^i(M'') \to \mathrm{H}^{i+1}(M') \to \mathrm{H}^{i+1}(M).$$

The four outer objects belong to N. Since N is a thick abelian subcategory of M it follows that $\mathrm{H}^i(M'') \in \mathsf{N}$. □

Example 7.4.2. Let A be a noetherian commutative ring. The category $\mathsf{Mod}_{\mathsf{f}}\, A$ of finitely generated modules is a thick abelian subcategory of $\mathsf{Mod}\, A$.

Example 7.4.3. Consider $\mathsf{Mod}\,\mathbb{Z} = \mathsf{Ab}$. As above we have the thick abelian subcategory $\mathsf{Ab}_{\mathrm{fgen}} = \mathsf{Mod}_{\mathrm{f}}\,\mathbb{Z}$ of finitely generated abelian groups. There is also the thick abelian subcategory $\mathsf{Ab}_{\mathrm{tors}}$ of torsion abelian groups (every element has a finite order). The intersection of $\mathsf{Ab}_{\mathrm{tors}}$ and $\mathsf{Ab}_{\mathrm{fgen}}$ is the category $\mathsf{Ab}_{\mathrm{fin}}$ of finite abelian groups. This is also thick.

Example 7.4.4. Let X be a noetherian scheme (e.g. an algebraic variety over an algebraically closed field). Consider the abelian category $\mathsf{Mod}\,\mathcal{O}_X$ of $\mathcal{O}_X$-modules. In it there is the thick abelian subcategory $\mathsf{QCoh}\,\mathcal{O}_X$ of quasi-coherent sheaves, and in that there is the thick abelian subcategory $\mathsf{Coh}\,\mathcal{O}_X$ of coherent sheaves.

For a left noetherian ring A we write

$$\mathbf{D}_{\mathrm{f}}(\mathsf{Mod}\,A) := \mathbf{D}_{\mathsf{Mod}_{\mathrm{f}}\,A}(\mathsf{Mod}\,A).$$

Proposition 7.4.5. *Let A be a left noetherian ring and $\star \in \{-, \mathrm{b}\}$. Then the canonical functor*

$$\mathbf{D}^{\star}(\mathsf{Mod}_{\mathrm{f}}\,A) \to \mathbf{D}_{\mathrm{f}}(\mathsf{Mod}\,A)^{\star}$$

is an equivalence of pretriangulated categories.

Proof. Consider the functor

$$F : \mathbf{D}^{-}(\mathsf{Mod}_{\mathrm{f}}\,A) \to \mathbf{D}(\mathsf{Mod}\,A).$$

Suppose $s : M \to L$ is a quasi-isomorphism in $\mathbf{K}(\mathsf{Mod}\,A)$, such that $L \in \mathbf{K}^{-}(\mathsf{Mod}_{\mathrm{f}}\,A)$. Then $M \in \mathbf{D}_{\mathrm{f}}(\mathsf{Mod}\,A)^{-}$. A bit later (in Corollary 10.3.32) we will prove that M admits a free resolution $P \to M$, where P is a bounded above complex of finitely generated free modules. Thus we get a quasi-isomorphism $t : P \to M$ with $P \in \mathbf{K}^{-}(\mathsf{Mod}_{\mathrm{f}}\,A)$. By Proposition 7.2.5 with condition (r) we conclude that F is fully faithful. This also shows that the essential image of F is $\mathbf{D}_{\mathrm{f}}(\mathsf{Mod}\,A)^{-}$.

Next consider the functor

$$G : \mathbf{D}^{\mathrm{b}}(\mathsf{Mod}_{\mathrm{f}}\,A) \to \mathbf{D}^{-}(\mathsf{Mod}_{\mathrm{f}}\,A).$$

Suppose $s : L \to M$ is a quasi-isomorphism in $\mathbf{K}^{-}(\mathsf{Mod}_{\mathrm{f}}\,A)$ with $L \in \mathbf{K}^{\mathrm{b}}(\mathsf{Mod}_{\mathrm{f}}\,A)$. Say $\mathrm{H}(L)$ is concentrated in the integer interval $[d_0, d_1]$. Then $t : M \to \mathrm{smt}^{\geq d_0}(M)$ is a quasi-isomorphism, and $\mathrm{smt}^{\geq d_0}(M) \in \mathbf{K}^{\mathrm{b}}(\mathsf{Mod}_{\mathrm{f}}\,A)$. By Proposition 7.2.5 with condition (l) we conclude that G is fully faithful. Therefore the composition

$$F \circ G : \mathbf{D}^{\mathrm{b}}(\mathsf{Mod}_{\mathrm{f}}\,A) \to \mathbf{D}(\mathsf{Mod}\,A)$$

is fully faithful. Suitable truncations ($\mathrm{smt}^{\geq d_0}$ and $\mathrm{smt}^{\leq d_1}$) show that the essential image of $F \circ G$ is $\mathbf{D}_{\mathrm{f}}(\mathsf{Mod}\,A)^{\mathrm{b}}$. □

7.5. **The Embedding of M in $\mathbf{D}(\mathsf{M})$.** Here again we only consider an abelian category M.

For $M, N \in \mathsf{M}$ there is no difference between $\mathrm{Hom}_{\mathsf{M}}(M, N)$, $\mathrm{Hom}_{\mathbf{C}(\mathsf{M})}(M, N)$ and $\mathrm{Hom}_{\mathbf{K}(\mathsf{M})}(M, N)$. Thus the canonical functors $\mathsf{M} \to \mathbf{C}(\mathsf{M})$ and $\mathsf{M} \to \mathbf{K}(\mathsf{M})$ are fully faithful. The same is true for $\mathbf{D}(\mathsf{M})$, but this requires a proof.

Let $\mathbf{D}(\mathsf{M})^0$ be the full subcategory of $\mathbf{D}(\mathsf{M})$ consisting of complexes whose cohomology is concentrated in degree 0. This is an additive subcategory of $\mathbf{D}(\mathsf{M})$.

Proposition 7.5.1. *The canonical functor $\mathsf{M} \to \mathbf{D}(\mathsf{M})^0$ is an equivalence.*

Proof. Let's denote the canonical functor $\mathsf{M} \to \mathbf{D}(\mathsf{M})^0$ by F. Under the fully faithful embedding $\mathsf{M} \subseteq \mathbf{C}_{\mathrm{str}}(\mathsf{M})$, F is just the restriction of $\tilde{\mathrm{Q}}$.

Since the functor $\mathrm{H}^0 : \mathbf{D}(\mathsf{M}) \to \mathsf{M}$ satisfies $\mathrm{H}^0 \circ F = \mathrm{Id}_{\mathsf{M}}$. This implies that F is faithful.

Next we prove that F is full. Take any objects $M, N \in \mathsf{M}$ and a morphism $q : M \to N$ in $\mathbf{D}(\mathsf{M})$. By Proposition 7.1.7 we know that $q = \tilde{\mathrm{Q}}(a) \circ \tilde{\mathrm{Q}}(s)^{-1}$ for some morphisms $a : L \to N$ and $s : L \to M$ in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, with s a quasi-isomorphism. Let $L' := \mathrm{smt}^{\leq 0}(L)$, as in (7.3.6); so there is a quasi-isomorphism $u : L' \to L$ in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$. Writing $a' := a \circ u$ and $s' := s \circ u$, we see that s' is a quasi-isomorphism, and $q = \tilde{\mathrm{Q}}(a') \circ \tilde{\mathrm{Q}}(s')^{-1}$.

Next let $L'' := \mathrm{smt}^{\geq 0}(L')$, as in (7.3.8); so there is a surjective quasi-isomorphism $v : L' \to L''$ in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$. Because L'' is a complex concentrated in degree 0, we can view it as an object of M. The morphisms a' and s' factor as $a' = a'' \circ v$ and $s' = s'' \circ v$, where $a'' : L'' \to N$ and $s'' : L'' \to M$ are morphisms in M. But s'' is a quasi-isomorphism in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, and so it is actually an isomorphism in M. Therefore we have a morphism $a'' \circ (s'')^{-1}$ in M, and

$$\tilde{\mathrm{Q}}(a'' \circ (s'')^{-1}) = \tilde{\mathrm{Q}}(a'') \circ \tilde{\mathrm{Q}}(s'')^{-1} = \tilde{\mathrm{Q}}(a') \circ \tilde{\mathrm{Q}}(s')^{-1} = q.$$

Finally we have to prove that any $L \in \mathbf{D}(\mathsf{M})^0$ is isomorphic, in $\mathbf{D}(\mathsf{M})$, to a complex L'' that's concentrated in degree 0. But we already showed it in the previous paragraphs. □

Proposition 7.5.2. *Let* M *be an abelian category. Let*

$$0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$$

be a diagram in M*. The following conditions are equivalent*:

(i) *The diagram is an exact sequence.*

(ii) *There is a distinguished triangle*

$$L \xrightarrow{\tilde{\mathrm{Q}}(\phi)} M \xrightarrow{\tilde{\mathrm{Q}}(\psi)} N \xrightarrow{\theta} \mathrm{T}(L)$$

in $\mathbf{D}(\mathsf{M})$.

Exercise 7.5.3. Prove Proposition 7.5.2. (Hint: for the implication (i) ⇒ (ii) you can take $\theta = 0$.)

The last two propositions say that the abelian category M can be recovered from the pretriangulated category $\mathbf{D}(\mathsf{M})$.

8. Derived Functors

As before, $\mathbb{K}$ is a commutative base ring, that shall remain implicit. Let A be a central DG $\mathbb{K}$-ring, and M a $\mathbb{K}$-linear abelian category. The category $\mathbf{C}(A, \mathsf{M})$ of DG A-modules in M was introduced in Subsection 3.7. It is a DG category. The pretriangulated categories $\mathbf{K}(A, \mathsf{M})$ and $\mathbf{D}(A, \mathsf{M})$ were introduced in Subsections 5.4 and 7.1 respectively. There is a triangulated localization functor

$$\mathrm{Q} : \mathbf{K}(A, \mathsf{M}) \to \mathbf{D}(A, \mathsf{M}).$$

Let (B, N) be another pair of DG ring and abelian category. Suppose we are given a DG functor

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N}).$$

Then, according to Theorem 5.4.13, there is an induced triangulated functor

$$(\bar{F}, \bar{\tau}_F) : \mathbf{K}(A, \mathsf{M}) \to \mathbf{K}(B, \mathsf{N})$$

Most triangulated functors that we shall encounter arise this way. For convenience of notation, let us suppress mentioning the translation isomorphism $\bar{\tau}_F$, and let us write F instead of $\bar{F}$.

By postcomposing with the localization functor of $\mathbf{K}(B, \mathsf{N})$ we obtain a triangulated functor

$$\mathrm{Q} \circ F : \mathbf{K}(A, \mathsf{M}) \to \mathbf{D}(B, \mathsf{N}). \tag{8.0.1}$$

Again we denote this triangulated functor by F.

Our goal in this section is to extend F to triangulated functors

$$\mathrm{R}F, \mathrm{L}F : \mathbf{D}(A, \mathsf{M}) \to \mathbf{D}(B, \mathsf{N}).$$

These are the right and left derived functors of F, respectively.

It will be easier to state matters more generally. Thus we shall mostly work in the setup below.

Setup 8.0.2. The following are given:

(1) Pretriangulated categories K and E.
(2) A triangulated functor $F : \mathsf{K} \to \mathsf{E}$.
(3) A denominator set of cohomological origin $\mathsf{S} \subseteq \mathsf{K}$ (see Definition 6.4.2).

Recall that the morphisms in S are called quasi-isomorphisms.

By Proposition 6.4.1 and Theorem 6.4.3, the localization K_{S} exists, and it is a pretriangulated category. The triangulated localization functor is $\mathrm{Q} : \mathsf{K} \to \mathsf{K}_{\mathsf{S}}$.

This setup specializes to (8.0.1) when we take $\mathsf{K} = \mathbf{K}(A, \mathsf{M})$, $\mathsf{S} = \mathbf{S}(A, \mathsf{M})$ and $\mathsf{E} = \mathbf{D}(B, \mathsf{N})$.

Remark 8.0.3. As far as we know, all previous textbooks only consider the special case of the derived functors

$$\mathrm{R}F, \mathrm{L}F : \mathbf{D}(\mathsf{M}) \to \mathbf{D}(\mathsf{N})$$

of a triangulated functor

$$F : \mathbf{K}(\mathsf{M}) \to \mathbf{K}(\mathsf{N}),$$

where M and N are abelian categories. The DG variant is not mentioned at all. However, the definitions and the main existence results, as stated in this section, are virtually the same.

Furthermore, previous textbooks avoid the 2-categorical notation, and that (in our opinion) is a cause for undue difficulties in the presentation.

8.1. **2-Categorical Notation.** In this section we are going to do a lot of work with morphisms of functors (i.e. natural transformations). The language and notation of ordinary category theory that we used so far is not adequate for this purpose. Therefore we will now introduce notation from the theory of 2-*categories*. (We will not give a definition of a 2-category here; but it is basically the data mentioned below, satisfying a few conditions, most of which will be mentioned below too.) In the subsequent sections we will revert to the usual (i.e. 1-categorical) language. For more details on 2-categories the reader can look at [Mac2] or [Ye8, Section 1].

Consider the set $\mathbf{Cat}$ of all categories. The set theoretical aspects are neglected, as explained in Subsection 1.1. (Briefly, the precise solution is this: $\mathbf{Cat}$ is the set of all U-categories; so $\mathbf{Cat}$ is a subset of a bigger Grothendieck universe, say V, and it is a V-category.)

The set $\mathbf{Cat}$ is the set of objects of a 2-category. This means that in $\mathbf{Cat}$ there are two kinds of morphisms: 1-*morphisms* between objects, and 2-*morphisms* between 1-morphisms. There are several kinds of compositions, and these have several properties. All this will be explained below.

Suppose $\mathsf{C}_0, \mathsf{C}_1, \ldots$ are categories, namely objects of $\mathbf{Cat}$. The 1-morphisms between them are the functors. The notation is as usual: $F : \mathsf{C}_0 \to \mathsf{C}_1$ denotes a functor.

Suppose $F, G : \mathsf{C}_0 \to \mathsf{C}_1$ are functors (with the same source and target objects). The 2-morphisms from F to G are the morphisms of functors (i.e. the natural transformations), and the notation is $\eta : F \Rightarrow G$. The double arrow is the distinguishing notation for 2-morphisms. When specializing to an object $M \in \mathsf{C}_0$ we revert to the single arrow notation, namely $\eta_M : F(M) \to G(M)$ is the corresponding morphism in C_1. The diagram depicting this is

$$\begin{tikzcd} \mathsf{C}_0 \arrow[r, bend left, "F", ""{name=U, below}] \arrow[r, bend right, "G"', ""{name=D, above}] & \mathsf{C}_1 \arrow[Rightarrow, from=U, to=D, "\eta"'] \end{tikzcd}$$

We shall refer to such a diagram as a 2-*diagram*.

Each object (category) C has its identity 1-morphism (functor) $\mathrm{Id}_{\mathsf{C}} : \mathsf{C} \to \mathsf{C}$. Each 1-morphism F has its identity 2-morphism (natural transformation) $\mathrm{id}_F : F \Rightarrow F$.

Now we consider compositions. For functors there is nothing new: given functors $F_i : \mathsf{C}_{i-1} \to \mathsf{C}_i$, the composition, that we now call *horizontal composition*, is the functor $F_2 \circ F_1 : \mathsf{C}_0 \to \mathsf{C}_2$. The diagram is

$$\begin{tikzcd} \mathsf{C}_0 \arrow[r, "F_1"] \arrow[rr, bend right, "F_2 \circ F_1"'] & \mathsf{C}_1 \arrow[r, "F_2"] & \mathsf{C}_2 \end{tikzcd}$$

This can be viewed as a commutative 1-diagram, or as a shorthand for the 2-diagram

$$\begin{tikzcd} \mathsf{C}_0 \arrow[r, "F_1"] \arrow[rr, bend right=40, "F_2 \circ F_1"'] & \mathsf{C}_1 \arrow[r, "F_2"] \arrow[d, phantom, "\Downarrow \mathrm{id}"] & \mathsf{C}_2 \\ & {} & \end{tikzcd}$$

in which id is the identity 2-morphism of $F_2 \circ F_1$.

The complication begins with compositions of 2-morphisms. Suppose we are given 1-morphisms $F_i, G_i : \mathsf{C}_{i-1} \to \mathsf{C}_i$ and 2-morphisms $\eta_i : F_i \Rightarrow G_i$. In a diagram:

$$\begin{tikzcd}
\mathsf{C}_0 \arrow[r, bend left, "F_1"{name=U1}] \arrow[r, bend right, "G_1"'{name=D1}] \arrow[from=U1, to=D1, Rightarrow, "\eta_1"'] & \mathsf{C}_1 \arrow[r, bend left, "F_2"{name=U2}] \arrow[r, bend right, "G_2"'{name=D2}] \arrow[from=U2, to=D2, Rightarrow, "\eta_2"'] & \mathsf{C}_2
\end{tikzcd}$$

The *horizontal composition* is the morphism of functors

$$\eta_2 \circ \eta_1 : F_2 \circ F_1 \Rightarrow G_2 \circ G_1.$$

The diagram is

$$\begin{tikzcd}
\mathsf{C}_0 \arrow[r, bend left, "F_2 \circ F_1"{name=U}] \arrow[r, bend right, "G_2 \circ G_1"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\eta_2 \circ \eta_1"'] & \mathsf{C}_2
\end{tikzcd}$$

Exercise 8.1.1. For an object $M \in \mathsf{C}_0$, give an explicit formula for the morphism

$$(\eta_2 \circ \eta_1)_M : (F_2 \circ F_1)(M) \to (G_2 \circ G_1)(M)$$

in the category C_2.

Suppose we are given 1-morphisms $E, F, G : \mathsf{C}_0 \to \mathsf{C}_1$, and 2-morphisms $\zeta : E \Rightarrow F$ and $\eta : F \Rightarrow G$. The diagram depicting this is

$$\begin{tikzcd}
\mathsf{C}_0 \arrow[r, bend left=60, "E"{name=U}] \arrow[r, "F"'{name=M}] \arrow[r, bend right=60, "G"'{name=D}] \arrow[from=U, to=M, Rightarrow, "\zeta"'] \arrow[from=M, to=D, Rightarrow, "\eta"'] & \mathsf{C}_1
\end{tikzcd}$$

The *vertical composition* of ζ and η is the 2-morphism

$$\eta * \zeta : E \to G.$$

Notice the new symbol for this operation. The corresponding diagram is

$$\begin{tikzcd}
\mathsf{C}_0 \arrow[r, bend left, "E"{name=U}] \arrow[r, bend right, "G"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\eta_2 * \eta_1"'] & \mathsf{C}_1
\end{tikzcd}$$

Exercise 8.1.2. For an object $M \in \mathsf{C}_0$, give an explicit formula for the morphism

$$(\eta * \zeta)_M : E(M) \to G(M)$$

in the category C_1.

Something intricate occurs in the situation shown in the next diagram.

$$\begin{tikzcd}
\mathsf{C}_0 \arrow[r, bend left=60, "E_1"] \arrow[r, bend right=60, "G_1"'] \arrow[r, "F_1"'] \arrow[r, phantom, shift left=3ex, "\zeta_1 \Downarrow"] \arrow[r, phantom, shift right=3ex, "\eta_1 \Downarrow"] & \mathsf{C}_1 \arrow[r, bend left=60, "E_2"] \arrow[r, bend right=60, "G_2"'] \arrow[r, "F_2"'] \arrow[r, phantom, shift left=3ex, "\zeta_2 \Downarrow"] \arrow[r, phantom, shift right=3ex, "\eta_2 \Downarrow"] & \mathsf{C}_2
\end{tikzcd}$$

It turns out that

$$(\eta_2 * \zeta_2) \circ (\eta_1 * \zeta_1) = (\eta_2 \circ \eta_1) * (\zeta_2 \circ \zeta_1)$$

as morphisms $E_2 \circ E_1 \Rightarrow G_2 \circ G_1$. This is called the *exchange property*.

Exercise 8.1.3. Prove the exchange property.

Just like general categories, we can talk about pretriangulated categories. There is the 2-category $\mathbf{PTrCat}$ of all pretriangulated categories (over $\mathbb{K}$). The objects here are the pretriangulated categories (K, T); the 1-morphisms are the triangulated functors (F, τ); and the 2-morphisms are the morphisms of triangulated functors η. This is what we are going to use.

8.2. **Some Preliminaries on Triangulated Functors.**

Proposition 8.2.1. *Let $(F, \tau) : \mathsf{K} \to \mathsf{L}$ be a triangulated functor between pretriangulated categories. Assume F is an equivalence (of abstract categories), with quasi-inverse $G : \mathsf{L} \to \mathsf{K}$, and with adjunction isomorphisms $\alpha : G \circ F \xRightarrow{\simeq} \mathrm{Id}_{\mathsf{K}}$ and $\beta : F \circ G \xRightarrow{\simeq} \mathrm{Id}_{\mathsf{L}}$.*

Then there is an isomorphism of functors

$$\nu : G \circ \mathrm{T}_{\mathsf{L}} \xRightarrow{\simeq} \mathrm{T}_{\mathsf{K}} \circ G$$

such that $(G, \nu) : \mathsf{L} \to \mathsf{K}$ is a triangulated functor, and α and β are isomorphisms of triangulated functors.

Proof. It is well-known that G is additive (or in our case, $\mathbb{K}$-linear); but since the proof is so easy, we shall reproduce it. Take any pair of objects $M, N \in \mathsf{L}$. We have to prove that the bijection

$$G_{M,N} : \mathrm{Hom}_{\mathsf{L}}(M, N) \to \mathrm{Hom}_{\mathsf{K}}\big(G(M), G(N)\big)$$

is linear. But

$$G_{M,N} = F^{-1}_{G(M),G(N)} \circ \mathrm{Hom}_{\mathsf{L}}(\beta_M, \beta_N^{-1})$$

as bijections (of sets) between these modules. Since $\alpha^{-1}_{M,N}$ and $F_{G(M),G(N)}$ are $\mathbb{K}$-linear, then so is $G_{M,N}$.

We define the isomorphism of triangulated functors ν by the formula

$$\nu := (\alpha \circ \mathrm{id}_{\mathrm{T}_{\mathsf{K}} \circ G}) * (\mathrm{id}_G \circ \tau \circ \mathrm{id}_G)^{-1} * (\mathrm{id}_{G \circ \mathrm{T}_{\mathsf{L}}} \circ \beta)^{-1},$$

in terms of the 2-categorical notation. This gives rise to a commutative diagram of isomorphisms

$$\begin{tikzcd}
G \circ \mathrm{T}_{\mathsf{L}} \circ F \circ G \arrow[d, Rightarrow, "\mathrm{id} \circ \beta"'] & G \circ F \circ \mathrm{T}_{\mathsf{K}} \circ G \arrow[l, Rightarrow, "\mathrm{id} \circ \tau \circ \mathrm{id}"'] \arrow[d, Rightarrow, "\alpha \circ \mathrm{id}"] \\
G \circ \mathrm{T}_{\mathsf{L}} \arrow[r, Rightarrow, "\nu"] & \mathrm{T}_{\mathsf{K}} \circ G
\end{tikzcd}$$

of additive functors $\mathsf{L} \to \mathsf{K}$. So the pair (G, ν) is a T-additive functor.

The verification that (G, ν) preserves triangles (in the sense of Definition 5.3.1(1)) is done like the proof of the additivity of G, but now using axiom (TR1.a) from Definition 5.2.1 . We leave this as an exercise. $\square$

Exercise 8.2.2. Finish the proof above (the last assertion).

8.3. **Right Derived Functors.**

Definition 8.3.1. Assume Setup 8.0.2. A *right derived functor* of F is a triangulated functor

$$\mathrm{R}F : \mathsf{K}_\mathsf{S} \to \mathsf{E},$$

together with a morphism

$$\eta : F \Rightarrow \mathrm{R}F \circ \mathrm{Q}$$

of triangulated functors $\mathsf{K} \to \mathsf{E}$. The pair $(\mathrm{R}F, \eta)$ must have this universal property:

($\Diamond$) Given any pair (G, θ), consisting of a triangulated functor $G : \mathsf{K}_\mathsf{S} \to \mathsf{E}$ and a morphism of triangulated functors $\theta : F \Rightarrow G \circ \mathrm{Q}$, there is a unique morphism of triangulated functors $\mu : \mathrm{R}F \Rightarrow G$ such that $\theta = (\mu \circ \mathrm{id}_\mathrm{Q}) * \eta$.

Pictorially: there is a 2-diagram

$$\begin{tikzcd}
\mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathsf{E} \\
\mathsf{K}_\mathsf{S} \arrow[ur, "\mathrm{R}F"'] &
\end{tikzcd}$$

For any other pair (G, θ) there is a unique morphism μ that sits in this 2-diagram:

$$\begin{tikzcd}
\mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[dr, phantom, "\Downarrow\eta \;\; \Downarrow\theta"] & \mathsf{E} \\
\mathsf{K}_\mathsf{S} \arrow[ur] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, bend right=20, "\Downarrow\mu"] &
\end{tikzcd}$$

The 1-morphisms in this 2-diagram do not (necessarily) commute; but the diagram of 2-morphisms (with $*$ composition)

$$\begin{tikzcd}
F \arrow[d, Rightarrow, "\eta"'] \arrow[dr, Rightarrow, "\theta"] & \\
\mathrm{R}F \circ \mathrm{Q} \arrow[r, Rightarrow, "\mu \circ \mathrm{id}_\mathrm{Q}"'] & G \circ \mathrm{Q}
\end{tikzcd}$$

is commutative.

Proposition 8.3.2. *If a right derived functor* $(\mathrm{R}F, \eta)$ *exists, then it is unique, up to a unique isomorphism. Namely, if* (G, θ) *is another right derived functor of* F*, then there is a unique isomorphism of triangulated functors* $\mu : \mathrm{R}F \xRightarrow{\simeq} G$ *such that* $\theta = (\mu \circ \mathrm{id}_\mathrm{Q}) * \eta$.

Proof. Despite the apparent complication of the situation, the usual argument for uniqueness of universals (here it is a universal 1-morphism) applies. It shows that the morphism μ from condition ($\Diamond$) is an isomorphism. $\square$

Existence is much harder. Here is a sufficient condition. It is a rephrasing of [RD, Theorem I.5.1], and the proof is basically the same (but we give many more details).

Theorem 8.3.3. *Given Setup* 8.0.2*, assume there is a full pretriangulated subcategory* $\mathsf{J} \subseteq \mathsf{K}$ *with these two properties*:

(a) *If* $\phi : I \to I'$ *is a quasi-isomorphism in* J*, then* $F(\phi) : F(I) \to F(I')$ *is an isomorphism in* E*.*
(b) *Every object* $M \in \mathsf{K}$ *admits a quasi-isomorphism* $\rho : M \to I$ *to some object* $I \in \mathsf{J}$*.*

Then the right derived functor

$$(\mathrm{R}F, \eta) : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$$

exists. Moreover, for any object $I \in \mathsf{J}$ *the morphism*

$$\eta_I : F(I) \to (\mathrm{R}F \circ \mathrm{Q})(I)$$

in E *is an isomorphism.*

Remark 8.3.4. A quasi-isomorphism $\rho : M \to I$ as in condition (b) is supposed to be viewed as a "generalized injective resolution" of M. See Example 8.3.22, where this is made concrete.

We use the letter J for the category of "generalized injective complexes" because the letter I, in this particular font, is too ambiguous.

The proof of the theorem follows some preparation. We we will sometimes suppress the localization functors Q and Q', for the sake of clarity. For instance, given a morphism $s \in \mathsf{S}$, we might say that s is invertible in K_{S}.

Definition 8.3.5. In the situation of Theorem 8.3.3, by a *system of right* J*-resolutions* we mean a pair (I, ρ), where $I : \mathrm{Ob}(\mathsf{K}) \to \mathrm{Ob}(\mathsf{J})$ is a function, and $\rho = \{\rho_M\}_{M \in \mathrm{Ob}(\mathsf{K})}$ is a collection of quasi-isomorphisms $\rho_M : M \to I(M)$ in K. Moreover, if $M \in \mathrm{Ob}(\mathsf{J})$, then $I(M) = M$ and $\rho_M = \mathrm{id}_M$.

Property (b) of Theorem 8.3.3 guarantees that a system of right J-resolutions (I, ρ) exists.

Suppose we made a choice of a system of right J-resolutions. Let us denote by $U : \mathsf{J} \to \mathsf{K}$ the inclusion functor, so $I \circ U$ is the identity on the set $\mathrm{Ob}(\mathsf{J})$. Let us define $F' := F \circ U : \mathsf{J} \to \mathsf{E}$ and $\mathsf{S}' := \mathsf{J} \cap \mathsf{S}$. The localization functor of J is denoted by $\mathrm{Q}' : \mathsf{J} \to \mathsf{J}_{\mathsf{S}'}$. There is a triangulated functor $U_{\mathsf{S}'} : \mathsf{J}_{\mathsf{S}'} \to \mathsf{K}_{\mathsf{S}}$ extending U, and there is equality $\mathrm{Q} \circ U = U_{\mathsf{S}'} \circ \mathrm{Q}'$. These sit in a commutative diagram

$$\begin{tikzcd} \mathsf{J} \arrow[r, "U"] \arrow[d, "\mathrm{Q}'"'] & \mathsf{K} \arrow[d, "\mathrm{Q}"] \\ \mathsf{J}_{\mathsf{S}'} \arrow[r, "U_{\mathsf{S}'}"] & \mathsf{K}_{\mathsf{S}} \end{tikzcd}$$

We know (from Theorem 6.4.3) that the functor F' extends uniquely to a triangulated functor $F'_{\mathsf{S}'} : \mathsf{J}_{\mathsf{S}'} \to \mathsf{E}$. Let $\eta' := \mathrm{id}_{F'}$, which is a 2-morphism

$$\eta' : F' \Rightarrow F'_{\mathsf{S}'} \circ \mathrm{Q}' . \tag{8.3.6}$$

The 2-diagram is:

(8.3.7)
$$\begin{tikzcd} \mathsf{J} \arrow[r, "F'"] \arrow[d, "\mathrm{Q}'"'] \arrow[r, phantom, shift right=3, "\Downarrow \eta'"] & \mathsf{E} \\ \mathsf{J}_{\mathsf{S}'} \arrow[ur, "F'_{\mathsf{S}'}"'] \end{tikzcd}$$

Lemma 8.3.8. *The pair $(F'_{\mathsf{S}'}, \eta')$ is a right derived functor of F'.*

Proof. We need to verify condition ($\Diamond$) of Definition 8.3.1. Say a triangulated functor $G' : \mathsf{J}_{\mathsf{S}'} \to \mathsf{E}$ is given. Because Q' is the identity on objects, the data of a morphism of triangulated functors $\mu' : F'_{\mathsf{S}'} \Rightarrow G'$, namely a collection of morphisms $\mu'_I : F'(I) \to G'(I)$ in E for all $I \in \mathsf{J}$, is the same data as a morphism of triangulated functors

(8.3.9)
$$\theta' := \mu' \circ \mathrm{id}_{\mathrm{Q}'} = (\mu' \circ \mathrm{id}_{\mathrm{Q}'}) * \eta' : F' \Rightarrow G' \circ \mathrm{Q}' .$$

This implies that the function $\mu' \mapsto \theta'$ is injective. Here is the relevant 2-diagram:

(8.3.10)
$$\begin{tikzcd} \mathsf{J} \arrow[r, "F'"] \arrow[d, "\mathrm{Q}'"'] \arrow[r, phantom, shift right=3, "\Downarrow \eta'"] & \mathsf{E} \\ \mathsf{J}_{\mathsf{S}'} \arrow[ur, "F'_{\mathsf{S}'}"] \arrow[ur, bend right, "G'"'] \arrow[ur, phantom, shift right=2, "\Downarrow \mu'"] \arrow[ur, phantom, shift left=4, "\Downarrow \theta'"] \end{tikzcd}$$

We have to prove that the function $\mu' \mapsto \theta'$ is surjective. This amounts to showing that for any morphism $q : I \to J$ in $\mathsf{J}_{\mathsf{S}'}$ there is equality

$$\theta'_J \circ F'_{\mathsf{S}'}(q) = G'(q) \circ \theta'_I$$

of morphisms in E. Let us choose a right fraction presentation $q = a \circ s^{-1}$, with $a : K \to J$ in J and $s : K \to I$ in S'. Because $\theta' : F' \Rightarrow G' \circ \mathrm{Q}'$ is a morphism of functors $\mathsf{J} \to \mathsf{E}$, the solid diagram below

$$\begin{tikzcd} F'(I) \arrow[d, "\theta'_I"'] \arrow[rr, dashed, bend left, "F'_{\mathsf{S}'}(q)"] & F'(K) \arrow[l, "F'(s)"] \arrow[r, "F'(a)"'] \arrow[d, "\theta'_K"'] & F'(J) \arrow[d, "\theta'_J"'] \\ G'(I) \arrow[rr, dashed, bend right, "G'(q)"'] & G'(K) \arrow[l, "G'(s)"'] \arrow[r, "G'(a)"] & G'(J) \end{tikzcd}$$

is commutative. But then, since $F'(s)$ and $G'(s)$ are invertible in E, the whole diagram is commutative. □

Lemma 8.3.11. *The functor $U_{\mathsf{S}'} : \mathsf{J}_{\mathsf{S}'} \to \mathsf{K}_{\mathsf{S}}$ is an equivalence of pretriangulated categories.*

Proof. By the proof of Proposition 7.2.5, with condition (1), together with Proposition 8.2.1. □

Lemma 8.3.12. *Suppose a system of right J-resolutions (I, ρ) has been chosen. Then the function I extends uniquely to a triangulated functor $I : \mathsf{K}_{\mathsf{S}} \to \mathsf{J}_{\mathsf{S}'}$, such that $\mathrm{Id}_{\mathsf{J}_{\mathsf{S}'}} = I \circ U_{\mathsf{S}'}$, and $\rho : \mathrm{Id}_{\mathsf{K}_{\mathsf{S}}} \Rightarrow U_{\mathsf{S}'} \circ I$ is an isomorphism of triangulated functors.*

In other words, the triangulated functor I is a quasi-inverse of $U_{\mathsf{S}'}$. The relevant 2-diagram is this:

$$\begin{tikzcd}
\mathsf{J} \arrow[r, "\mathrm{Q}'"] \arrow[d, "U"'] & \mathsf{J}_{\mathsf{S}'} \arrow[r, "\mathrm{Id}"] \arrow[dr, "U_{\mathsf{S}'}"] & \mathsf{J}_{\mathsf{S}'} \\
\mathsf{K} \arrow[r, "\mathrm{Q}"] & \mathsf{K}_{\mathsf{S}} \arrow[u, "I"] \arrow[r, "\mathrm{Id}"'] \arrow[ur, phantom, "\Uparrow \rho"] & \mathsf{K}_{\mathsf{S}} \arrow[u, "I"']
\end{tikzcd}$$

Proof. By Lemma 8.3.11 the functor $U_{\mathsf{S}'}$ is an equivalence. Take any pair of objects $M, N \in \mathsf{K}$. There is a bijection

$$U_{\mathsf{S}'} : \mathrm{Hom}_{\mathsf{J}_{\mathsf{S}'}}\big(I(M), I(N)\big) \to \mathrm{Hom}_{\mathsf{K}_{\mathsf{S}}}\big(I(M), I(N)\big),$$

and another bijection

$$\mathrm{Hom}(\rho_M^{-1}, \rho_N) : \mathrm{Hom}_{\mathsf{K}_{\mathsf{S}}}(M, N) \to \mathrm{Hom}_{\mathsf{K}_{\mathsf{S}}}\big(I(M), I(N)\big).$$

These bijections say that to any morphism $\psi : M \to N$ in K_{S} there corresponds a unique morphism $I(\psi) : I(M) \to I(N)$ in $\mathsf{J}_{\mathsf{S}'}$, such that

$$U_{\mathsf{S}'}(I(\psi)) \circ \rho_M = \rho_N \circ \psi.$$

An easy calculation shows that $I : \mathsf{K}_{\mathsf{S}} \to \mathsf{J}_{\mathsf{S}'}$ is a functor. Moreover, there is equality of functors $I \circ U_{\mathsf{S}'} = \mathrm{Id}_{\mathsf{J}_{\mathsf{S}'}}$, and an isomorphism of functors $\rho : \mathrm{Id}_{\mathsf{K}_{\mathsf{S}}} \xRightarrow{\simeq} U_{\mathsf{S}'} \circ I$. This says that I is a quasi-inverse of $U_{\mathsf{S}'}$. Therefore, by Proposition 8.2.1, I is a triangulated functor, and ρ is an isomorphism of triangulated functors. □

Lemma 8.3.13. *Under the assumptions of the theorem, let $G : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$ be triangulated functor, and define $G' := G \circ U_{\mathsf{S}'}$. Suppose $\eta' : F' \Rightarrow G' \circ \mathrm{Q}'$ is a morphism of triangulated functors $\mathsf{J} \to \mathsf{E}$. Then there is a unique morphism $\eta : F \Rightarrow G \circ \mathrm{Q}$ of triangulated functors $\mathsf{K} \to \mathsf{E}$ that extends η', namely such that $\eta \circ \mathrm{id}_U = \eta'$.*

Here are the corresponding 2-diagrams:

$$\begin{tikzcd}
\mathsf{J} \arrow[r, "F'"] \arrow[d, "\mathrm{Q}'"'] \arrow[dr, phantom, "\Downarrow \eta'"] & \mathsf{E} \\
\mathsf{J}_{\mathsf{S}'} \arrow[ur, "G'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{J} \arrow[r, "U"] \arrow[d, "\mathrm{Q}'"'] & \mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[dr, phantom, "\Downarrow \eta"] & \mathsf{E} \\
\mathsf{J}_{\mathsf{S}'} \arrow[r, "U_{\mathsf{S}'}"'] & \mathsf{K}_{\mathsf{S}} \arrow[ur, "G"'] &
\end{tikzcd}$$

Here is another way to state the lemma. Let us denote by $\mathrm{Hom}^2_{\mathsf{PTrCat}}(-,-)$ the set of 2-morphisms (morphisms of triangulated functors). Then the operation $\eta \mapsto \eta \circ \mathrm{id}_U$ is a function

$$- \circ \mathrm{id}_U : \mathrm{Hom}^2_{\mathsf{PTrCat}}(F, G \circ \mathrm{Q}) \to \mathrm{Hom}^2_{\mathsf{PTrCat}}(F', G' \circ \mathrm{Q}'),$$

and the lemma asserts that this is a bijection.

Proof. Choose a system of right J-resolutions (I, ρ). For any object $M \in \mathsf{K}$ the morphism ρ_M is invertible in K_S. Hence the morphism

$$G(\rho_M) : G(M) \to G(I(M))$$

is invertible in E. We are given the morphism

$$\eta'_{I(M)} : F'(I(M)) \to G'(I(M))$$

in E. Recall that $F'(I(M)) = F(I(M))$ and $G'(I(M)) = G(I(M))$. Let us define

$$\eta_M := G(\rho_M)^{-1} \circ \eta'_{I(M)} \circ F(\rho_M), \tag{8.3.14}$$

which is a morphism $F(M) \to G(M)$ in E. We get a commutative diagram

$$\begin{tikzcd} F(M) \arrow[r, "\eta_M"] \arrow[d, "F(\rho_M)"'] & G(M) \arrow[d, "G(\rho_M)"] \\ F'(I(M)) \arrow[r, "\eta'_{I(M)}"] & G'(I(M)) \end{tikzcd} \tag{8.3.15}$$

in E.

It is now routine to check that η is a morphism of triangulated functors $F \Rightarrow G \circ \mathrm{Q}$. By construction η extends η'. The uniqueness of η follows from the fact that the diagram (8.3.15) must commute, and thus formula (8.3.14) must hold. □

Proof of Theorem 8.3.3.

Step 1. We choose a system of right J-resolutions (I, ρ). For any object $M \in \mathsf{K}$ we define the object

$$\mathrm{R}F(M) := F(I(M)) \in \mathsf{E} \tag{8.3.16}$$

and the morphism

$$\eta_M := F(\rho_M) : F(M) \to \mathrm{R}F(M) \tag{8.3.17}$$

in E. We still did not say what $\mathrm{R}F$ does to morphisms.

Step 2. For any object $M \in \mathsf{K}$ we have, by construction, $\mathrm{R}F(M) = F'(I(M))$. This means that $\mathrm{R}F = F'_{\mathsf{S}'} \circ I$ on objects. The definition

$$\mathrm{R}F := F'_{\mathsf{S}'} \circ I : \mathsf{K}_\mathsf{S} \to \mathsf{E}\,. \tag{8.3.18}$$

upgrades $\mathrm{R}F$ to a triangulated functor. And there is a commutative diagram of triangulated functors

$$\begin{tikzcd} \mathsf{J} \arrow[r, "\mathrm{Q}'"] \arrow[d, "U"'] \arrow[rr, bend left, "F'"] & \mathsf{J}_{\mathsf{S}'} \arrow[r, "F'_{\mathsf{S}'}"] & \mathsf{E} \\ \mathsf{K} \arrow[r, "\mathrm{Q}"] & \mathsf{K}_\mathsf{S} \arrow[u, "I"'] \arrow[ur, "\mathrm{R}F"'] & \end{tikzcd}$$

Step 3. Recall that we already defined $\eta_M = F(\rho_M)$. In this step we prove that η is a morphism of triangulated functors $\eta : F \to \mathrm{R}F \circ \mathrm{Q}$.

According to Lemma 8.3.13, the morphism of triangulated functors $\eta' : F' \Rightarrow F'_{\mathsf{S}'} \circ \mathrm{Q}'$ from (8.3.6) extends uniquely to a morphism of triangulated functors $\tilde{\eta} : F \Rightarrow \mathrm{R}F \circ \mathrm{Q}$. The 2-diagram is

$$\begin{tikzcd} \mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[r, phantom, shift right=4ex, "\Downarrow \tilde{\eta}"] & \mathsf{E} \\ \mathsf{K}_{\mathsf{S}} \arrow[ur, "\mathrm{R}F"'] & \end{tikzcd}$$

We know that $\eta'_{I(M)} = \mathrm{id}_{F(I(M))}$ and $\mathrm{R}F = F'_{\mathsf{S}'} \circ I$. By construction of the functor I we have $I(\rho_M) = \mathrm{id}_{I(M)}$ in $\mathsf{J}_{\mathsf{S}'}$. Plugging this and $G = \mathrm{R}F$ into formula (8.3.14) we obtain

$$\begin{aligned} \tilde{\eta}_M &= (F'_{\mathsf{S}'}(I(\rho_M))^{-1} \circ \eta'_{I(M)} \circ F(\rho_M) \\ &= (\mathrm{id}_{F(I(M))})^{-1} \circ \mathrm{id}_{F(I(M))} \circ F(\rho_M) = F(\rho_M). \end{aligned}$$

So the morphism $\tilde{\eta}_M$ coincides with η_M. As M varies we get $\tilde{\eta} = \eta$.

Step 4. It remains to verify condition ($\Diamond$) of Definition 8.3.1. Say a pair (G, θ) is given. Define $G' := G \circ U_{\mathsf{S}'}$ and $\theta' := \theta \circ \mathrm{id}_U$. In Lemma 8.3.8 we proved that $(F'_{\mathsf{S}'}, \eta')$ is the right derived functor of F'. Therefore there is a unique morphism $\mu' : F'_{\mathsf{S}'} \Rightarrow G'$ of triangulated functors $\mathsf{J}_{\mathsf{S}'} \to \mathsf{E}$ such that $\mu' \circ \mathrm{id}_{\mathrm{Q}'} = \theta'$. In terms of vertical composition, and using the equality $\eta' = \mathrm{id}_{F'}$, this is

$$(\mu' \circ \mathrm{id}_{\mathrm{Q}'}) * \eta' = \theta'. \tag{8.3.19}$$

In a 2-diagram:

$$\begin{tikzcd}[row sep=huge, column sep=huge] \mathsf{J} \arrow[r, "F'"] \arrow[d, "\mathrm{Q}'"'] \arrow[r, phantom, shift right=4ex, "\Downarrow \eta' \ \Downarrow \theta'"] & \mathsf{E} \\ \mathsf{J}_{\mathsf{S}'} \arrow[ur, "F'_{\mathsf{S}'}"] \arrow[ur, bend right=40, "G'"'] \arrow[ur, phantom, "\Downarrow \mu'", shift right=2ex] & \end{tikzcd}$$

Recall that $F'_{\mathsf{S}'} = \mathrm{R}F \circ U_{\mathsf{S}'}$. The functor $U_{\mathsf{S}'}$ is an equivalence. Hence (like Lemma 8.3.13 but much easier) there is a unique morphism $\mu : \mathrm{R}F \to G$ such that $\mu \circ \mathrm{id}_{U_{\mathsf{S}'}} = \mu'$. We get this 2-diagram:

$$\begin{tikzcd}[row sep=huge, column sep=huge] \mathsf{J} \arrow[r, "U"] \arrow[d, "\mathrm{Q}'"'] & \mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[r, phantom, shift right=4ex, "\Downarrow \eta \ \Downarrow \theta"] & \mathsf{E} \\ \mathsf{J}_{\mathsf{S}'} \arrow[r, "U_{\mathsf{S}'}"'] & \mathsf{K}_{\mathsf{S}} \arrow[ur, "\mathrm{R}F"] \arrow[ur, bend right=40, "G"'] \arrow[ur, phantom, "\Downarrow \mu", shift right=2ex] & \end{tikzcd}$$

We know that

$$\mathrm{id}_{\mathrm{Q}} \circ \mathrm{id}_U = \mathrm{id}_{U_{\mathsf{S}'}} \circ \mathrm{id}_{\mathrm{Q}'} .$$

Hence

$$(\mu \circ \mathrm{id}_{\mathrm{Q}} \circ \mathrm{id}_U) * (\eta \circ \mathrm{id}_U) = (\mu \circ \mathrm{id}_{U_{\mathsf{S}'}} \circ \mathrm{id}_{\mathrm{Q}'}) * \eta' = (\mu' \circ \mathrm{id}'_{\mathrm{Q}}) * \eta'$$

(this is the exchange condition). Taking this with formula (8.3.19), and using the exchange condition once more, we deduce that

$$((\mu \circ \mathrm{id}_{\mathsf{Q}}) * \eta) \circ \mathrm{id}_U = \theta'.$$

The uniqueness in Lemma 8.3.13 now implies that

$$(\mu \circ \mathrm{id}_{\mathsf{Q}}) * \eta = \theta. \tag{8.3.20}$$

Finally we have to establish the uniqueness of μ. Suppose $\tilde{\mu}$ is another morphism $\mathrm{R}F \Rightarrow G$ satisfying (8.3.20). Then $\tilde{\mu}' := \tilde{\mu} \circ \mathrm{id}_{U_{\mathsf{S}'}}$ satisfies (8.3.19). But then, by the uniqueness of μ', we have $\tilde{\mu}' = \mu'$. Therefore (because $U_{\mathsf{S}'}$ is an equivalence) we see that $\tilde{\mu} = \mu$. □

Definition 8.3.21. The construction of the right derived functor $(\mathrm{R}F, \eta)$ in the proof of the theorem above, and specifically formulas (8.3.16) and (8.3.17), is called a *presentation of* $(\mathrm{R}F, \eta)$ *by the system of right* J*-resolutions* (I, ρ).

Of course any other right derived functor of F (perhaps presented by another system of right J-resolutions) is uniquely isomorphic to $(\mathrm{R}F, \eta)$. This is according to Proposition 8.3.2.

In Section 9 we shall give several existence results for the right derived functor

$$(\mathrm{R}F, \eta) : \mathbf{D}^{\star}(A, \mathsf{M}) \to \mathsf{E}$$

of a triangulated functor

$$F : \mathbf{K}^{\star}(A, \mathsf{M}) \to \mathsf{E},$$

under various assumptions on F, A, M and $\star$. These existence results will be based on Theorem 8.3.3: we will prove existence of suitable resolving subcategories $\mathsf{J} \subseteq \mathbf{K}^{\star}(A, \mathsf{M})$. The example below is one such case.

Example 8.3.22. Suppose we start from an additive functor $F : \mathsf{M} \to \mathsf{N}$. We know how to extend it to a DG functor $F : \mathbf{C}^{+}(\mathsf{M}) \to \mathbf{C}^{+}(\mathsf{N})$, and then to a triangulated functor $F : \mathbf{K}^{+}(\mathsf{M}) \to \mathbf{K}^{+}(\mathsf{N})$. By composing with Q we get a triangulated functor $\mathrm{Q} \circ F : \mathbf{K}^{+}(\mathsf{M}) \to \mathbf{D}^{+}(\mathsf{N})$, that we also denote by F for simplicity.

Assume that the abelian category M has enough injectives (this means that any object $M \in \mathsf{M}$ admits an injective resolution). Define J to be the full subcategory of $\mathsf{K} := \mathbf{K}^{+}(\mathsf{M})$ on the bounded below complexes of injective objects; and let $\mathsf{E} := \mathbf{D}^{+}(\mathsf{N})$. We will prove later that properties (a) and (b) of Theorem 8.3.3 hold in this situation. Therefore we have a right derived functor

$$\mathrm{R}F : \mathbf{D}^{+}(\mathsf{M}) \to \mathbf{D}^{+}(\mathsf{N}).$$

In case the functor F is left exact, it has the classical right derived functors $\mathrm{R}^q F : \mathsf{M} \to \mathsf{N}$, $q \geq 0$. Formula (8.3.16) shows that for any $M \in \mathsf{M}$ there is equality $\mathrm{R}^q F(M) = \mathrm{H}^q(\mathrm{R}F(M))$ as objects of N. We will prove that more is true:

$$\mathrm{R}^q F = \mathrm{H}^q \circ \mathrm{R}F$$

as functors $\mathsf{M} \to \mathsf{N}$.

In the situation of Theorem 8.3.3, let $\mathsf{K}^{\dagger}$ be a full pretriangulated subcategory of K. Define $\mathsf{S}^{\dagger} := \mathsf{K}^{\dagger} \cap \mathsf{S}$ and $\mathsf{J}^{\dagger} := \mathsf{K}^{\dagger} \cap \mathsf{J}$. Denote by $V : \mathsf{K}^{\dagger} \to \mathsf{K}$ the inclusion functor, and by $V_{\mathsf{S}^{\dagger}} : \mathsf{K}^{\dagger}_{\mathsf{S}^{\dagger}} \to \mathsf{K}_{\mathsf{S}}$ its localization. Warning: the functor $V_{\mathsf{S}^{\dagger}}$ is not necessarily fully faithful; cf. Proposition 7.2.5.

Proposition 8.3.23. *Assume that every $M \in \mathsf{K}^{\dagger}$ admits a quasi-isomorphism $M \to I$ where $I \in \mathsf{J}^{\dagger}$. Then the pair*

$$(\mathrm{R}F \circ V_{\mathsf{S}^{\dagger}}, \eta \circ \mathrm{id}_V)$$

is a right derived functor of $F \circ V : \mathsf{K}^{\dagger} \to \mathsf{E}$.

Loosely speaking, the proposition says that

$$\mathrm{R}(F \circ V) = \mathrm{R}F \circ V_{\mathsf{S}^{\dagger}}.$$

The proof is an exercise.

Exercise 8.3.24. Prove the last proposition. (Hint: Start by choosing a system of right $\mathsf{J}^{\dagger}$-resolutions of $\mathsf{K}^{\dagger}$. Then extend it to a system of right J-resolutions of K. Now follow the proof of the theorem.)

8.4. **Left Derived Functors.** Left derived functors behave just like right derived functors, except for a change of sides in the target category. Because of this our treatment will be brief: we will state the definitions and the main results, but won't give proofs, beyond a hint here and there.

Definition 8.4.1. Assume Setup 8.0.2. A *left derived functor* of F is a triangulated functor

$$\mathrm{L}F : \mathsf{K}_{\mathsf{S}} \to \mathsf{E},$$

together with a morphism

$$\eta : \mathrm{L}F \circ \mathrm{Q} \Rightarrow F$$

of triangulated functors $\mathsf{K} \to \mathsf{E}$. The pair $(\mathrm{L}F, \eta)$ must have this universal property:

($\Diamond$) Given any pair (G, θ), consisting of a triangulated functor $G : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$ and a morphism of triangulated functors $\theta : G \circ \mathrm{Q} \Rightarrow F$, there is a unique morphism of triangulated functors $\mu : G \Rightarrow \mathrm{L}F$ such that $\theta = \eta * (\mu \circ \mathrm{id}_{\mathrm{Q}})$.

Pictorially: there is a 2-diagram

$$\begin{tikzcd}
\mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] & \mathsf{E} \\
\mathsf{K}_{\mathsf{S}} \arrow[ur, "\mathrm{L}F"'] \arrow[u, phantom, shift left=6ex, "\Uparrow \eta"]
\end{tikzcd}$$

For any other pair (G, θ) there is a unique morphism μ that sits in this 2-diagram:

$$\begin{tikzcd}
\mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] & \mathsf{E} \\
\mathsf{K}_{\mathsf{S}} \arrow[ur, "\mathrm{L}F"'] \arrow[ur, bend right, "G"'] \arrow[u, phantom, shift left=6ex, "\Uparrow \eta"] \arrow[ur, phantom, "\Uparrow \theta \quad \Uparrow \mu"]
\end{tikzcd}$$

The 1-morphisms in this 2-diagram do not (necessarily) commute; but the diagram of 2-morphisms (with $*$ composition)

$$\begin{tikzcd} F & \\ \mathrm{L}F \circ \mathrm{Q} \arrow[u, Rightarrow, "\eta"] & G \circ \mathrm{Q} \arrow[l, Rightarrow, "\mu \circ \mathrm{id}_{\mathrm{Q}}"'] \arrow[ul, Rightarrow, "\theta"'] \end{tikzcd}$$

is commutative.

Proposition 8.4.2. *If a left derived functor* $(\mathrm{L}F, \eta)$ *exists, then it is unique, up to a unique isomorphism. Namely, if* (G, θ) *is another left derived functor of* F*, then there is a unique isomorphism of triangulated functors* $\mu : G \xRightarrow{\simeq} \mathrm{L}F$ *such that* $\theta = \eta * (\mu \circ \mathrm{id}_{\mathrm{Q}})$.

The proof is the same as that of Proposition 8.3.2, with direction of arrows in E reversed.

Theorem 8.4.3. *Given Setup* 8.0.2*, assume there is a full pretriangulated subcategory* $\mathsf{P} \subseteq \mathsf{K}$ *with these two properties*:

(a) *If* $\phi : P \to P'$ *is a quasi-isomorphism in* P*, then* $F(\phi) : F(P) \to F(P')$ *is an isomorphism in* E.
(b) *Every object* $M \in \mathsf{K}$ *admits a quasi-isomorphism* $\rho : P \to M$ *from some object* $P \in \mathsf{P}$.

Then the right derived functor

$$(\mathrm{L}F, \eta) : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$$

exists. Moreover, for any object $P \in \mathsf{P}$ *the morphism*

$$\eta_P : (\mathrm{L}F \circ \mathrm{Q})(P) \to F(P)$$

in E *is an isomorphism.*

The category P is a "generalized category of projectives".

The proof is the same as that of Theorem 8.3.3, with direction of arrows in E reversed.

Definition 8.4.4. In the situation of Theorem 8.4.3, by a *system of left* P*-resolutions* we mean a pair (P, ρ), where $P : \mathrm{Ob}(\mathsf{K}) \to \mathrm{Ob}(\mathsf{P})$ is a function, and $\rho = \{\rho_M\}_{M \in \mathrm{Ob}(\mathsf{K})}$ is a collection of quasi-isomorphisms $\rho_M : P(M) \to M$ in K. Moreover, if $M \in \mathrm{Ob}(\mathsf{P})$, then $P(M) = M$ and $\rho_M = \mathrm{id}_M$.

Property (b) of Theorem 8.4.3 guarantees that a system of left P-resolutions (P, ρ) exists.

Definition 8.4.5. The construction of the left derived functor $(\mathrm{L}F, \eta)$, when proving Theorem 8.4.3 along the lines of Theorem 8.3.3, and specifically the formulas

$$\mathrm{L}F(M) := F(P(M)) \tag{8.4.6}$$

and

$$\eta_M := F(\rho_M) : \mathrm{L}F(M) \to F(M),$$

is called a *presentation of* $(\mathrm{L}F, \eta)$ *by the system of left* P*-resolutions* (P, ρ).

In Section 9 we shall give several existence results for the left derived functor

$$(\mathrm{L}F, \eta) : \mathbf{D}^{\star}(A, \mathsf{M}) \to \mathsf{E}$$

of a triangulated functor

$$F : \mathbf{K}^{\star}(A, \mathsf{M}) \to \mathsf{E},$$

under various assumptions on F, A, M and $\star$. These existence results will be based on Theorem 8.4.3: we will prove existence of suitable resolving subcategories $\mathsf{P} \subseteq \mathbf{K}^{\star}(A, \mathsf{M})$. The example below is one such case.

Example 8.4.7. Suppose we start from an additive functor $F : \mathsf{M} \to \mathsf{N}$. We know how to extend it to a DG functor $F : \mathbf{C}^{-}(\mathsf{M}) \to \mathbf{C}^{-}(\mathsf{N})$, and then to a triangulated functor $F : \mathbf{K}^{-}(\mathsf{M}) \to \mathbf{K}^{-}(\mathsf{N})$. By composing with Q we get a triangulated functor $\mathrm{Q} \circ F : \mathbf{K}^{-}(\mathsf{M}) \to \mathbf{D}^{-}(\mathsf{N})$, that we also denote by F for simplicity.

Assume that the abelian category M has enough projectives (this means that any object $M \in \mathsf{M}$ admits a projective resolution). Define P to be the full subcategory of $\mathsf{K} := \mathbf{K}^{-}(\mathsf{M})$ on the bounded above complexes of projective objects; and let $\mathsf{E} := \mathbf{D}^{-}(\mathsf{N})$. We will prove later that properties (a) and (b) of Theorem 8.4.3 hold in this situation. Therefore we have a left derived functor

$$\mathrm{L}F : \mathbf{D}^{-}(\mathsf{M}) \to \mathbf{D}^{-}(\mathsf{N}).$$

In case the functor F is right exact, it has the classical left derived functors $\mathrm{L}_q F : \mathsf{M} \to \mathsf{N}$, $q \geq 0$. Formula (8.4.6) shows that for any $M \in \mathsf{M}$ there is equality $\mathrm{L}_q F(M) = \mathrm{H}^{-q}(\mathrm{L}F(M))$ as objects of N. We will prove that more is true:

$$\mathrm{L}_q F = \mathrm{H}^{-q} \circ \mathrm{L}F$$

as functors $\mathsf{M} \to \mathsf{N}$.

In the situation of Theorem 8.4.3, let $\mathsf{K}^{\dagger}$ be a full pretriangulated subcategory of K. Define $\mathsf{S}^{\dagger} := \mathsf{K}^{\dagger} \cap \mathsf{S}$ and $\mathsf{P}^{\dagger} := \mathsf{K}^{\dagger} \cap \mathsf{P}$. Denote by $V : \mathsf{K}^{\dagger} \to \mathsf{K}$ the inclusion functor, and by $V_{\mathsf{S}^{\dagger}} : \mathsf{K}^{\dagger}_{\mathsf{S}^{\dagger}} \to \mathsf{K}_{\mathsf{S}}$ its localization. Warning: the functor $V_{\mathsf{S}^{\dagger}}$ is not necessarily fully faithful; cf. Proposition 7.2.5.

Proposition 8.4.8. *Assume that every $M \in \mathsf{K}^{\dagger}$ admits a quasi-isomorphism $P \to M$ where $P \in \mathsf{P}^{\dagger}$. Then the pair*

$$(\mathrm{L}F \circ V_{\mathsf{S}^{\dagger}}, \eta \circ \mathrm{id}_V)$$

is a left derived functor of $F \circ V : \mathsf{K}^{\dagger} \to \mathsf{E}$.

The proof is just like that of Proposition 8.3.23 (which was an exercise...).

9. Resolutions of DG Modules

In this section we are back to the more concrete setting: A is a DG ring, and M is an abelian category (both over a base ring $\mathbb{K}$). We will define *K-projective* and *K-injective* DG modules in $\mathbf{K}(A, \mathsf{M})$. These DG modules form full pretriangulated subcategories of $\mathbf{K}(A, \mathsf{M})$, and are concrete versions of the abstract categories J and P, that played important roles in Subsections 8.3 and 8.4 respectively. For $\mathbf{K}(A)$ we also define *K-flat DG modules*.

9.1. K-Injective DG Modules.

For any i we have an additive functor

$$\mathrm{H}^i : \mathbf{C}_{\mathrm{str}}(A, \mathsf{M}) \to \mathsf{M}\,.$$

There is equality $\mathrm{H}^i = \mathrm{H}^0 \circ \mathrm{T}^i$. The functors H^i pass to the homotopy category, and

$$\mathrm{H}^0 : \mathbf{K}(A, \mathsf{M}) \to \mathsf{M}$$

is a cohomological functor in the sense of Definition 5.3.2.

Definition 9.1.1. A DG module $N \in \mathbf{C}(A, \mathsf{M})$ is called *acyclic* if $\mathrm{H}^i(N) = 0$ for all i.

Definition 9.1.2. A DG module $I \in \mathbf{C}(A, \mathsf{M})$ is called *K-injective* if for every acyclic DG module $N \in \mathbf{C}(A, \mathsf{M})$, the DG $\mathbb{K}$-module $\mathrm{Hom}_{A,\mathsf{M}}(N, I)$ is acyclic.

The definition above characterizes K-injectives as objects of $\mathbf{C}(A, \mathsf{M})$. The next proposition shows that being K-injective is intrinsic to the pretriangulated category $\mathbf{K}(A, \mathsf{M})$, with the cohomological functor H^0 (that tells us which are the acyclic objects).

Proposition 9.1.3. *A DG module $I \in \mathbf{K}(A, \mathsf{M})$ is K-injective if and only if $\mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(N, I) = 0$ for every acyclic DG module $N \in \mathbf{K}(A, \mathsf{M})$.*

Proof. This is because for any integer p we have

$$\mathrm{H}^p\big(\mathrm{Hom}_{A,\mathsf{M}}(N, I)\big) \cong \mathrm{H}^0\big(\mathrm{Hom}_{A,\mathsf{M}}(\mathrm{T}^{-p}(N), I)\big) \cong \mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(\mathrm{T}^{-p}(N), I),$$

and N is acyclic iff $\mathrm{T}^{-p}(N)$ is acyclic. □

The concept of K-injective complex (i.e. a K-injective object of $\mathbf{K}(\mathsf{M})$) was introduced by Spaltenstein [Sp] in 1988. At about the same time other authors (Keller [Kel], Bockstedt-Neeman [BoNe], Bernstein-Lunts [BeLu], ...) discovered this concept independently, with other names (such as *homotopically injective complex*). The texts [BeLu] and [Kel] already talk about DG modules over DG rings.

Remark 9.1.4. When the smart truncation functors exist (e.g. when A is a nonpositive DG ring), it is enough to check for K-injectivity of a DG module $I \in \mathbf{K}^{\star}(A, \mathsf{M})$ against acyclic DG modules $N \in \mathbf{K}^{\star}(A, \mathsf{M})$. Cf. Definition 7.3.11 and Exercise 7.3.12.

Definition 9.1.5. Let $M \in \mathbf{K}(A, \mathsf{M})$. A *K-injective resolution* of M is a quasi-isomorphism $\rho : M \to I$ in $\mathbf{K}(A, \mathsf{M})$, where I is a K-injective DG module.

Remark 9.1.6. In some other texts (and in our Section 10) "resolution" refers to a quasi-isomorphism $\rho : M \to I$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. It usually makes no difference which meaning is used (as long as we know what we are talking about).

In the next section we will prove existence of K-injectives in several contexts. Here is an easy one.

Exercise 9.1.7. Let $I \in \mathsf{K}(\mathsf{M})$ be a complex of injective objects of M, with zero differential. Prove that I is K-injective.

Definition 9.1.8. Let K be a full subcategory of $\mathsf{K}(A, \mathsf{M})$. The full subcategory of K on the K-injective DG modules in it is denoted by $\mathsf{K}_{\mathrm{inj}}$. In other words,

$$\mathsf{K}_{\mathrm{inj}} = \mathsf{K}(A, \mathsf{M})_{\mathrm{inj}} \cap \mathsf{K}\,.$$

Warning: the property of being K-injective is in general not intrinsic to the subcategory K. Cf. Remark 9.1.4.

Proposition 9.1.9. *If K is a full pretriangulated subcategory of $\mathsf{K}(A, \mathsf{M})$, then $\mathsf{K}_{\mathrm{inj}}$ is a full pretriangulated subcategory of K.*

Proof. It suffices to prove that $\mathsf{K}(A, \mathsf{M})_{\mathrm{inj}}$ is a pretriangulated subcategory of $\mathsf{K}(A, \mathsf{M})$. It is easy to see that $\mathsf{K}(A, \mathsf{M})_{\mathrm{inj}}$ is closed under translations. Suppose

$$I \to J \to K \to \mathrm{T}(I)$$

is a distinguished triangle in $\mathsf{K}(A, \mathsf{M})$, with I, J being K-injective DG modules. We have to show that K is also K-injective. Take any acyclic DG module $N \in \mathsf{K}(A, \mathsf{M})$. There is an exact sequence

$$\mathrm{Hom}_{\mathsf{K}(A,\mathsf{M})}(N, J) \to \mathrm{Hom}_{\mathsf{K}(A,\mathsf{M})}(N, K) \to \mathrm{Hom}_{\mathsf{K}(A,\mathsf{M})}(N, \mathrm{T}(I))$$

in $\mathsf{Mod}\,\mathbb{K}$. Because J and $\mathrm{T}(I)$ are K-injectives, Proposition 9.1.3 says that

$$\mathrm{Hom}_{\mathsf{K}(A,\mathsf{M})}(N, J) = 0 = \mathrm{Hom}_{\mathsf{K}(A,\mathsf{M})}(N, \mathrm{T}(I)).$$

Therefore $\mathrm{Hom}_{\mathsf{K}(A,\mathsf{M})}(N, K) = 0$. But N is an arbitrary acyclic DG module, so K is K-injective. □

Example 9.1.10. Let $\star$ be some boundedness condition (namely b, $+$, $-$ or nothing). We know that $\mathsf{K}^{\star}(A, \mathsf{M})$ is a full pretriangulated subcategory of $\mathsf{K}(A, \mathsf{M})$. Hence $\mathsf{K}^{\star}(A, \mathsf{M})_{\mathrm{inj}}$ is a pretriangulated subcategory too.

Definition 9.1.11. Let K be a full pretriangulated subcategory of $\mathsf{K}(A, \mathsf{M})$. We say that K *has enough K-injectives* if any DG module $M \in \mathsf{K}$ admits a K-injective resolution inside K. I.e. there is a quasi-isomorphism $\rho : M \to I$ where $I \in \mathsf{K}_{\mathrm{inj}}$.

Here is the crucial fact regarding K-injectives.

Lemma 9.1.12. *Let K be a full subcategory of $\mathsf{K}(A, \mathsf{M})$. Let $s : I \to M$ be a quasi-isomorphism in K, and assume I is K-injective. Then s has a left inverse, namely there is a morphism $t : M \to I$ in K such that $t \circ s = \mathrm{id}_I$.*

Proof. Since K is a full subcategory of $\mathsf{K}(A, \mathsf{M})$, we can assume that $\mathsf{K} = \mathsf{K}(A, \mathsf{M})$. Consider a distinguished triangle

$$I \xrightarrow{s} M \to N \to \mathrm{T}(I)$$

in $\mathsf{K}(A, \mathsf{M})$ that's built on s. The long exact cohomology sequence tells us that N is an acyclic DG module. So

$$\mathrm{Hom}_{\mathsf{K}(A,\mathsf{M})}(\mathrm{T}^p(N), I) = 0$$

for all p. The exact sequence

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(N, I) \to \mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(M, I) \\ &\quad \to \mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(I, I) \to \mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(\mathrm{T}^{-1}(N), I) \end{aligned}$$

shows that $\phi \mapsto \phi \circ s$ is a bijection

$$\mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(M, I) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(I, I).$$

We take $t : M \to I$ to be the unique morphism in $\mathbf{K}(A, \mathsf{M})$ such that $t \circ s = \mathrm{id}_I$. □

Theorem 9.1.13. *Let A be a DG ring, let M be an abelian category, and let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$. Denote by S the set of quasi-isomorphisms in K. Then the localization functor*

$$\mathrm{Q} : \mathsf{K}_{\mathrm{inj}} \to \mathsf{K}_{\mathsf{S}}$$

is fully faithful.

Proof. Consider any pair of objects $I, J \in \mathsf{K}_{\mathrm{inj}}$. We must prove that the $\mathbb{K}$-module homomorphism

$$\mathrm{Q} : \mathrm{Hom}_{\mathsf{K}}(I, J) \to \mathrm{Hom}_{\mathsf{K}_{\mathsf{S}}}(I, J) \tag{9.1.14}$$

is bijective.

Suppose $q : I \to J$ is a morphism in K_{S}. Let us present q as a left fraction: $q = \mathrm{Q}(s)^{-1} \circ \mathrm{Q}(a)$, where $a : I \to N$ and $s : J \to N$ are morphisms in K, and s is a quasi-isomorphism. By Lemma 9.1.12 s has a left inverse t. We get a morphism $t \circ a : I \to J$ in K, and an easy calculation shows that $\mathrm{Q}(t \circ a) = q$ in K_{S}. This proves surjectivity of (9.1.14).

Now let's prove injectivity of (9.1.14). If $a : I \to J$ is a morphism in K such that $\mathrm{Q}(a) = 0$, then by axiom (LO4) of Ore localization (the left version of axiom (RO4) in Definition 6.2.1), there is a quasi-isomorphism $s : J \to L$ in K such that $s \circ a = 0$ in K. Let t be the left inverse of s. Then $a = t \circ s \circ a = 0$ in K. □

Corollary 9.1.15. *Let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$. If K has enough K-injectives, then the localization functor*

$$\mathrm{Q} : \mathsf{K}_{\mathrm{inj}} \to \mathsf{K}_{\mathsf{S}}$$

is an equivalence.

Proof. By the theorem the functor Q is fully faithful. The extra condition guarantees that Q is essentially surjective on objects. □

Corollary 9.1.16. *Let $\star$ be any boundedness condition. If $\mathbf{K}^\star(A, \mathsf{M})$ has enough K-injectives, then the triangulated functor*

$$\mathrm{Q} : \mathbf{K}^\star(A, \mathsf{M})_{\mathrm{inj}} \to \mathbf{D}^\star(A, \mathsf{M})$$

is an equivalence.

Proof. Since $\mathbf{K}^\star(A, \mathsf{M})$ is a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$, this is a special case of the previous corollary. □

Remark 9.1.17. This result is of tremendous importance, both theoretically and practically. In the theory, it shows that the localized category $\mathbf{D}^\star(A, \mathsf{M})$, which is too big to lie inside the original universe U (see Remark 6.2.16), is equivalent to a U-category. On the practical side, it means that among K-injective objects we do not need fractions to represent morphisms.

Corollary 9.1.18. *Let $\star$ and $\dagger$ be boundedness conditions such that*

$$\mathbf{K}^{\star}(A, \mathsf{M}) \subseteq \mathbf{K}^{\dagger}(A, \mathsf{M}).$$

Assume these categories have enough K-injectives. Then the canonical functor

$$\mathbf{D}^{\star}(A, \mathsf{M}) \to \mathbf{D}^{\dagger}(A, \mathsf{M})$$

is fully faithful.

Proof. Combine Corollary 9.1.16 with the fact that $\mathbf{K}^{\star}(A, \mathsf{M}) \to \mathbf{K}^{\dagger}(A, \mathsf{M})$ is fully faithful. □

Remark 9.1.19. Earlier we only proved that $\mathbf{D}^{\star}(A, \mathsf{M}) \to \mathbf{D}(A, \mathsf{M})$ is fully faithful in special cases (see Proposition 7.3.5 and Exercise 7.3.12).

Corollary 9.1.20. *Let $\phi : I \to J$ be a morphism in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ between K-injective objects. Then ϕ is a homotopy equivalence if and only if it is a quasi-isomorphism.*

Proof. One implication is trivial. For the reverse implication, if ϕ is a quasi-isomorphism then it is an isomorphism in $\mathbf{D}(A, \mathsf{M})$, and by Theorem 9.1.13 for $\mathsf{K} = \mathbf{K}(A, \mathsf{M})$ we see that ϕ is an isomorphism in $\mathbf{K}(A, \mathsf{M})$. □

Here is another useful definition. It is a variant of Definition 8.3.5.

Definition 9.1.21. Let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$, and assume K has enough K-injectives. A *system of K-injective resolutions* in K is a pair (I, ρ), where $I : \mathrm{Ob}(\mathsf{K}) \to \mathrm{Ob}(\mathsf{K}_{\mathrm{inj}})$ is a function, and $\rho = \{\rho_M\}_{M \in \mathrm{Ob}(\mathsf{K})}$ is a collection of quasi-isomorphisms $\rho_M : M \to I(M)$ in K. Moreover, if $M \in \mathrm{Ob}(\mathsf{K}_{\mathrm{inj}})$, then $I(M) = M$ and $\rho_M = \mathrm{id}_M$.

The proposition below is a variant of Lemma 8.3.12.

Proposition 9.1.22. *Suppose a system of K-injective resolutions (I, ρ) has been chosen. Then the function I extends uniquely to a triangulated functor $I : \mathsf{K}_{\mathsf{S}} \to \mathsf{K}_{\mathrm{inj}}$, such that $\mathrm{Id}_{\mathsf{K}_{\mathrm{inj}}} = I \circ \mathrm{Q}|_{\mathsf{K}_{\mathrm{inj}}}$, and $\rho : \mathrm{Id}_{\mathsf{K}_{\mathsf{S}}} \Rightarrow \mathrm{Q} \circ I$ is an isomorphism of triangulated functors.*

Proof. The proof is the same as that of Lemma 8.3.12, except that here we use Corollary 9.1.15. □

The next corollary is a categorical interpretation of the last proposition.

Corollary 9.1.23 (Functorial K-Injective Resolutions)**.** *Let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$, and assume K has enough K-injectives.*

(1) *There are a triangulated functor $I : \mathsf{K} \to \mathsf{K}$ and a morphism of triangulated functors $\rho : \mathrm{Id}_{\mathsf{K}} \to I$, such that for any object $M \in \mathsf{K}$ the object $I(M)$ is K-injective, and the morphism $\rho_M : M \to I(M)$ is a quasi-isomorphism.*
(2) *If (I', ρ') is another such pair, then there is a unique isomorphism of triangulated functors $\zeta : I \xrightarrow{\simeq} I'$ such that $\rho' = \zeta \circ \rho$.*

Exercise 9.1.24. Prove Corollary 9.1.23.

Theorem 9.1.25. *Let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$, and denote by S the set of quasi-isomorphisms in K. Assume K has enough K-injectives. Let E be any pretriangulated category, and let*

$$F : \mathsf{K} \to \mathsf{E}$$

be any triangulated functor. Then F has a right derived functor

$$(\mathrm{R}F, \eta) : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}\,.$$

Furthermore, for any $I \in \mathsf{K}_{\mathrm{inj}}$ the morphism $\eta_I : F(I) \to \mathrm{R}F(I)$ in E is an isomorphism.

Proof. We will use Theorem 8.3.3. In the notation of that theorem, let $\mathsf{J} := \mathsf{K}_{\mathrm{inj}}$. Condition (b) of that theorem holds (this is the "enough K-injectives" assertion). Next, Theorem 9.1.13 implies that any quasi-isomorphism $\phi : I \to J$ in $\mathsf{K}_{\mathrm{inj}}$ is actually an isomorphism. Therefore $F(\phi)$ is an isomorphism in E, and this is condition (a) of Theorem 8.3.3. □

Example 9.1.26. Let A be any DG ring. We will prove later that $\mathbf{K}(A)$ has enough K-injectives. Therefore, given any triangulated functor $F : \mathbf{K}(A) \to \mathsf{E}$ into any pretriangulated category E, the right derived functor

$$(\mathrm{R}F, \eta) : \mathbf{D}(A) \to \mathsf{E}$$

exists.

Suppose we choose a system of K-injective resolutions (I, ρ) in $\mathbf{K}(A)$. Then we get a presentation of $(\mathrm{R}F, \eta)$ as follows: $\mathrm{R}F(M) = F(I(M))$ and $\eta_M = F(\rho_M)$.

9.2. **K-Projective DG Modules.** This subsection is dual to the previous one, and so we will be brief.

Definition 9.2.1. A DG module $P \in \mathbf{C}(A, \mathsf{M})$ is called *K-projective* if for every acyclic DG module $N \in \mathbf{C}(A, \mathsf{M})$, the DG $\mathbb{K}$-module $\mathrm{Hom}_{A,\mathsf{M}}(P, N)$ is acyclic.

Proposition 9.2.2. *A DG module $P \in \mathbf{K}(A, \mathsf{M})$ is K-projective if and only if $\mathrm{Hom}_{\mathbf{K}(A,\mathsf{M})}(P, N) = 0$ for every acyclic DG module $N \in \mathbf{K}(A, \mathsf{M})$.*

The proof is like that of Proposition 9.1.3.

Definition 9.2.3. Let $M \in \mathbf{K}(A, \mathsf{M})$. A *K-projective resolution* of M is a quasi-isomorphism $\rho : P \to M$ in $\mathbf{K}(A, \mathsf{M})$, where P is a K-projective DG module.

Definition 9.2.4. Let K be a full subcategory of $\mathbf{K}(A, \mathsf{M})$. The full subcategory of K on the K-projective DG modules in it is denoted by $\mathsf{K}_{\mathrm{prj}}$. In other words,

$$\mathsf{K}_{\mathrm{prj}} = \mathbf{K}(A, \mathsf{M})_{\mathrm{prj}} \cap \mathsf{K}\,.$$

The same warning after Definition 9.1.8 applies here.

Proposition 9.2.5. *If K is a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$, then $\mathsf{K}_{\mathrm{prj}}$ is a full pretriangulated subcategory of K.*

The proof is like that of Proposition 9.1.9.

Example 9.2.6. Let $\star$ be some boundedness condition (namely b, $+$, $-$ or nothing). Since $\mathbf{K}^{\star}(A, \mathsf{M})$ is a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$, we see that $\mathbf{K}^{\star}(A, \mathsf{M})_{\mathrm{prj}}$ is a pretriangulated subcategory too.

Definition 9.2.7. Let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$. We say that K *has enough K-projectives* if any DG module $M \in \mathsf{K}$ admits a K-projective resolution inside K. I.e. there is a quasi-isomorphism $\rho : P \to M$ where $P \in \mathsf{K}_{\mathrm{prj}}$.

Lemma 9.2.8. *Let K be a full subcategory of $\mathbf{K}(A, \mathsf{M})$. Let $s : M \to P$ be a quasi-isomorphism in K, and assume P is K-projective. Then s has a right inverse; namely there is a morphism $t : P \to M$ in K such that $s \circ t = \mathrm{id}_P$.*

Same proof as that of Lemma 9.1.12.

Theorem 9.2.9. *Let A be a DG ring, let M be an abelian category, and let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$. Denote by S the set of quasi-isomorphisms in K. Then the localization functor*

$$\mathrm{Q} : \mathsf{K}_{\mathrm{prj}} \to \mathsf{K}_{\mathsf{S}}$$

is fully faithful.

The proof is the same as that of Theorem 9.1.13, with reversed arrow. The next corollaries and proposition are also proved like their K-injective counterparts.

Corollary 9.2.10. *Let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$. If K has enough K-projectives, then the localization functor*

$$\mathrm{Q} : \mathsf{K}_{\mathrm{prj}} \to \mathsf{K}_{\mathsf{S}}$$

is an equivalence.

Corollary 9.2.11. *Let $\star$ and $\dagger$ be boundedness conditions such that*

$$\mathbf{K}^{\star}(A, \mathsf{M}) \subseteq \mathbf{K}^{\dagger}(A, \mathsf{M}).$$

Assume these categories have enough K-projectives. Then the canonical functor

$$\mathbf{D}^{\star}(A, \mathsf{M}) \to \mathbf{D}^{\dagger}(A, \mathsf{M})$$

is fully faithful.

Corollary 9.2.12. *Let $\phi : P \to Q$ be a morphism in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ between K-projective objects. Then ϕ is a homotopy equivalence if and only if it is a quasi-isomorphism.*

Definition 9.2.13. Let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$, and assume K has enough K-projectives. A *system of K-projective resolutions* in K is a pair (P, ρ), where $P : \mathrm{Ob}(\mathsf{K}) \to \mathrm{Ob}(\mathsf{K}_{\mathrm{prj}})$ is a function, and $\rho = \{\rho_M\}_{M \in \mathrm{Ob}(\mathsf{K})}$ is a collection of quasi-isomorphisms $\rho_M : P(M) \to M$ in K. Moreover, if $M \in \mathrm{Ob}(\mathsf{K}_{\mathrm{prj}})$, then $P(M) = M$ and $\rho_M = \mathrm{id}_M$.

Proposition 9.2.14. *Suppose a system of K-projective resolutions (P, ρ) has been chosen. Then the function P extends uniquely to a triangulated functor $P : \mathsf{K}_{\mathsf{S}} \to \mathsf{K}_{\mathrm{prj}}$, such that $\mathrm{Id}_{\mathsf{K}_{\mathrm{prj}}} = P \circ \mathrm{Q}|_{\mathsf{K}_{\mathrm{prj}}}$, and $\rho : \mathrm{Q} \circ P \Rightarrow \mathrm{Id}_{\mathsf{K}_{\mathsf{S}}}$ is an isomorphism of triangulated functors.*

Corollary 9.2.15 (Functorial K-Projective Resolutions)**.** *Let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$, and assume K has enough K-projectives.*

1. *There are a triangulated functor $P : \mathsf{K} \to \mathsf{K}$ and a morphism of triangulated functors $\rho : P \to \mathrm{Id}_{\mathsf{K}}$, such that for any object $M \in \mathsf{K}$ the object $P(M)$ is K-projective, and the morphism $\rho_M : P(M) \to M$ is a quasi-isomorphism.*
2. *If (P', ρ') is another such pair, then there is a unique isomorphism of triangulated functors $\zeta : P' \xrightarrow{\simeq} P$ such that $\rho' = \rho \circ \zeta$.*

Theorem 9.2.16. *Let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$, and denote by S the set of quasi-isomorphisms in K. Assume K has enough K-projectives. Let E be any pretriangulated category, and let*

$$F : \mathsf{K} \to \mathsf{E}$$

be any triangulated functor. Then F has a left derived functor

$$(\mathrm{L}F, \eta) : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}\,.$$

Furthermore, for any $P \in \mathsf{K}_{\mathrm{prj}}$ the morphism $\eta_P : \mathrm{L}F(P) \to F(P)$ in E is an isomorphism.

The proof is like that of Theorem 9.1.25.

Example 9.2.17. Let A be any DG ring. We will prove later that $\mathbf{K}(A)$ has enough K-projectives. Therefore, given any triangulated functor $F : \mathbf{K}(A) \to \mathsf{E}$ into any pretriangulated category E, the left derived functor

$$(\mathrm{L}F, \eta) : \mathbf{D}(A) \to \mathsf{E}$$

exists.

Suppose we choose a system of K-projective resolutions (P, ρ) in $\mathbf{K}(A)$. Then we get a presentation of $(\mathrm{L}F, \eta)$ as follows: $\mathrm{L}F(M) = F(P(M))$ and $\eta_M = F(\rho_M)$.

9.3. **K-Flat DG Modules.** Recall that A^{op} is the opposite DG ring. The objects of $\mathbf{C}(A^{\mathrm{op}})$ are the right DG A-modules.

Definition 9.3.1. A DG module $P \in \mathbf{C}(A)$ is called *K-flat* if for every acyclic DG module $N \in \mathbf{C}(A^{\mathrm{op}})$, the DG $\mathbb{K}$-module $N \otimes_A P$ is acyclic.

Proposition 9.3.2. *If $P \in \mathbf{C}(A)$ is K-projective then it is K-flat.*

Proof. Let $\mathbb{K}^*$ be an injective cogenerator of $\mathbf{M}(\mathbb{K}) = \mathsf{Mod}\,\mathbb{K}$. This means that $\mathbb{K}^*$ is an injective $\mathbb{K}$-module, such that any nonzero $\mathbb{K}$-module W admits a nonzero homomorphism $W \to \mathbb{K}^*$. A universal choice is $\mathbb{K}^* = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{K}, \mathbb{Q}/\mathbb{Z})$. It is not hard to see that a DG $\mathbb{K}$-module W is acyclic if and only if $\mathrm{Hom}_{\mathbb{K}}(W, \mathbb{K}^*)$ is acyclic. (Cf. Exercise 10.5.6 for a stronger assertion.)

Take an acyclic complex $N \in \mathbf{C}(A^{\mathrm{op}})$. Then by Hom-tensor adjunction there is an isomorphism of DG $\mathbb{K}$-modules

$$\mathrm{Hom}_{\mathbb{K}}(N \otimes_A P, \mathbb{K}^*) \cong \mathrm{Hom}_A\big(P, \mathrm{Hom}_{\mathbb{K}}(N, \mathbb{K}^*)\big).$$

The right side is acyclic by our assumptions. Hence so is the left side. It follows that $N \otimes_A P$ is acyclic. □

The proof above also gives a hint to the next proposition.

Proposition 9.3.3. *A DG module $P \in \mathbf{K}(A)$ is K-flat iff*

$$\mathrm{Hom}_{\mathbf{K}(A)}\big(P, \mathrm{Hom}_{\mathbb{K}}(N, J)\big) = 0$$

for every acyclic $N \in \mathbf{C}(A^{\mathrm{op}})$ and every injective $J \in \mathsf{Mod}\,\mathbb{K}$.

Exercise 9.3.4. Prove Proposition 9.3.3.

The next proposition will be subsumed later, in Section 12, in a theorem about the left derived tensor bifunctor.

Proposition 9.3.5. *Let K be a full pretriangulated subcategory of $\mathbf{K}(A)$, and denote by S the set of quasi-isomorphisms in K. Assume K has enough K-flat objects. Let B be another central DG $\mathbb{K}$-ring, let $N \in \mathbf{K}(B \otimes_{\mathbb{K}} A^{\mathrm{op}})$, and define*

$$F : \mathsf{K} \to \mathbf{D}(B)$$

to be the triangulated functor $F(M) := \mathrm{Q}(N \otimes_A M)$*, as in Example* 4.4.6 *and Theorem* 5.4.13*. Then* F *has a left derived functor*

$$(\mathrm{L}F, \eta) : \mathsf{K}_{\mathsf{S}} \to \mathbf{D}(B).$$

Furthermore, for any object $P \in \mathsf{K}$ *which is K-flat, the morphism* $\eta_P : \mathrm{L}F(P) \to F(P)$ *in* $\mathbf{D}(B)$ *is an isomorphism.*

Exercise 9.3.6. Prove Proposition 9.3.5. (Hint: look at the proof of Theorem 9.1.25.)

Remark 9.3.7. In view of Proposition 9.3.2, the reader might wonder why we bother with K-flat DG modules. The reason is that on a ringed space $(X, \mathcal{A})$ there are usually very few projective $\mathcal{A}$-modules. But, as we shall prove, there are enough K-flat complexes in $\mathbf{C}(\mathcal{A}) = \mathbf{C}(\mathsf{Mod}\,\mathcal{A})$. This will allow us to have a left derived tensor functor for sheaves.

10. Existence of Resolutions

In this section we continue in the more concrete setting: A is a DG ring, and M is an abelian category (both over a commutative base ring $\mathbb{K}$). We will prove existence of K-projective and K-injective resolutions in several contexts.

10.1. **Direct and Inverse Limits of Complexes.**

We shall have to work with limits in this section. Limits in abstract abelian and DG categories (not to mention pretriangulated categories) are a very delicate issue. We will try to be as concrete as possible, in order to avoid pitfalls and confusion.

Let C be an arbitrary category (not necessarily linear). A *direct system* in C is data

$$\big(\{M_k\}_{k\in\mathbb{N}}, \{\mu_k\}_{k\in\mathbb{N}}\big),$$

where M_k are objects of C, and $\mu_k : M_k \to M_{k+1}$ are morphisms, called transitions. The *direct limit*

$$M = \lim_{k\to} M_k$$

need not exist in C; but if it does, then it is unique up to a unique isomorphism.

By an *inverse system* in the category C we mean data

$$\big(\{M_k\}_{k\in\mathbb{N}}, \{\mu_k\}_{k\in\mathbb{N}}\big),$$

where $\{M_k\}_{k\in\mathbb{N}}$ is a collection of objects, and $\mu_k : M_{k+1} \to M_k$ are morphisms, also called transitions. The *inverse limit*

$$M = \lim_{\leftarrow k} M_k$$

need not exist in C; but if it does, then it is unique up to a unique isomorphism.

Proposition 10.1.1 (Sandwiched Systems)**.** *Let C be a category.*

(1) *Let $\big(\{M_k\}_{k\in\mathbb{N}}, \{\mu_k\}_{k\in\mathbb{N}}\big)$ and $\big(\{N_k\}_{k\in\mathbb{N}}, \{\nu_k\}_{k\in\mathbb{N}}\big)$ be direct systems in C. Assume there are morphisms $\alpha_k : M_k \to N_k$ and $\beta_k : N_k \to M_{k+1}$, such that $\beta_k \circ \alpha_k = \mu_k$ and $\alpha_{k+1} \circ \beta_k = \nu_k$ for all k. If the limit $N = \lim_{k\to} N_k$ exists, then the limit $M = \lim_{k\to} M_k$ also exists, and the canonical morphism $\alpha : M \to N$ is an isomorphism.*

(2) *Let $\big(\{M_k\}_{k\in\mathbb{N}}, \{\mu_k\}_{k\in\mathbb{N}}\big)$ and $\big(\{N_k\}_{k\in\mathbb{N}}, \{\nu_k\}_{k\in\mathbb{N}}\big)$ be inverse systems in C. Assume there are morphisms $\alpha_k : M_k \to N_k$ and $\beta_k : N_k \to M_{k-1}$, such that $\beta_k \circ \alpha_k = \mu_{k-1}$ and $\alpha_{k-1} \circ \beta_k = \nu_{k-1}$ for all k. If the limit $N = \lim_{\leftarrow k} N_k$ exists, then the limit $M = \lim_{\leftarrow k} M_k$ also exists, and the canonical morphism $\alpha : M \to N$ is an isomorphism.*

In other words, sandwiched systems behave the same regarding limits. The direct systems (item (1)) look like this:

$$\begin{tikzcd}
\cdots \arrow[r] & M_k \arrow[r, "\alpha_k"] \arrow[rr, bend right, "\mu_k"'] & N_k \arrow[r, "\beta_k"] \arrow[rr, bend left, "\nu_k"] & M_{k+1} \arrow[r, "\alpha_{k+1}"] \arrow[rr, bend right, "\mu_{k+1}"'] & N_{k+1} \arrow[r, "\beta_{k+1}"] & M_{k+2} \arrow[r] & \cdots
\end{tikzcd}$$

Exercise 10.1.2. Prove Proposition 10.1.1.

Proposition 10.1.3. *Let C be a category.*

(1) *Let* $\{M_k\}_{k\in\mathbb{N}}$ *be a direct system in* C*, and assume the direct limit* $M = \lim_{k\to} M_k$ *exists. Then for any object* $N \in \mathsf{C}$*, the canonical function*

$$\operatorname{Hom}_{\mathsf{C}}(M, N) \to \lim_{\leftarrow k} \operatorname{Hom}_{\mathsf{C}}(M_k, N)$$

is bijective.

(2) *Let* $\{M_k\}_{k\in\mathbb{N}}$ *be an inverse system in* C*, and assume the inverse limit* $M = \lim_{\leftarrow k} M_k$ *exists. Then for any object* $N \in \mathsf{C}$*, the canonical function*

$$\operatorname{Hom}_{\mathsf{C}}(N, M) \to \lim_{\leftarrow k} \operatorname{Hom}_{\mathsf{C}}(N, M_k)$$

is bijective.

Exercise 10.1.4. Prove Proposition 10.1.3.

Now we start talking about limits in the abelian category $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. We have to be careful, because it often not true that limits exist in abelian categories.

Example 10.1.5. Let M be the category of finite abelian groups. The inverse system $\{M_k\}_{k\in\mathbb{N}}$, where $M_k := \mathbb{Z}/(2^k)$, and the transition $\mu_k : M_{k+1} \to M_k$ is the canonical surjection, does not have an inverse limit in M. We can also make $\{M_k\}_{k\in\mathbb{N}}$ into a direct system, in which the transition $\nu_k : M_k \to M_{k+1}$ is multiplication by 2. The direct limit does not exist in M.

Proposition 10.1.6.

(1) *Let* $\{M_k\}_{k\in\mathbb{N}}$ *be a direct system in* $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$*. Assume that for every* i *the direct limit* $\lim_{k\to} M^i_k$ *exists in* M*. Then the direct limit* $M = \lim_{k\to} M_k$ *exists in* $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$*, and in degree* i *it is* $M^i = \lim_{k\to} M^i_k$*.*

(2) *Let* $\{M_k\}_{k\in\mathbb{N}}$ *be an inverse system in* $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$*. Assume that for every* i *the inverse limit* $\lim_{\leftarrow k} M^i_k$ *exists in* M*. Then the inverse limit* $M = \lim_{\leftarrow k} M_k$ *exists in* $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$*, and in degree* i *it is* $M^i = \lim_{\leftarrow k} M^i_k$*.*

Proof. We will only prove item (1); the proof of item (2) is identical. For any integer i define $M^i := \lim_{k\to} M^i_k \in \mathsf{M}$. By the universal property of the direct limit, the differentials $\mathrm{d} : M^i_k \to M^{i+1}_k$ induce differentials $\mathrm{d} : M^i \to M^{i+1}$, and in this way we obtain a complex $M := \{M^i\}_{i\in\mathbb{Z}} \in \mathbf{C}(\mathsf{M})$. Similarly, any element $a \in A^j$ induces morphisms $a : M^i \to M^{i+j}$ in M, and thus M becomes an object of $\mathbf{C}(A, \mathsf{M})$. There are morphisms $M_k \to M$ in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$, and it is easy to see that these make M into a direct limit of the system $\{M_k\}_{k\in\mathbb{N}}$. □

Since limits exist in $\mathsf{M} = \mathsf{Mod}\,\mathbb{K}$, the proposition above says that they exist in $\mathbf{C}(A)$. Similarly they exist in the category $\mathbf{G}(\mathbb{K})$ of graded $\mathbb{K}$-modules.

We say that a direct system $\{M_k\}_{k\in\mathbb{N}}$ in M is *eventually stationary* if $\mu_k : M_k \to M_{k+1}$ are isomorphisms for large k. Similarly we can talk about an eventually stationary inverse system. The limit of an eventually stationary system (direct or inverse) always exists: it is M_k for large enough k.

Proposition 10.1.7.

(1) *Let* $\{M_k\}_{k\in\mathbb{N}}$ *be a direct system in* $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$*. Assume that for each* i *the direct system* $\{M^i_k\}_{k\in\mathbb{N}}$ *in* M *is eventually stationary. Then the direct limit* $M = \lim_{k\to} M_k$ *exists in* $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$*, the direct limit* $\lim_{k\to} \mathrm{H}(M_k)$ *exists in* $\mathbf{G}^0(\mathsf{M})$*, and the canonical morphism*

$$\lim_{k\to} \mathrm{H}(M_k) \to \mathrm{H}(M)$$

in $\mathbf{G}^0(\mathsf{M})$ *is an isomorphism.*

(2) *Let* $\{M_k\}_{k\in\mathbb{N}}$ *be an inverse system in* $\mathbf{C}_{\mathrm{str}}(A,\mathsf{M})$*. Assume that for each* i *the inverse system* $\{M_k^i\}_{k\in\mathbb{N}}$ *in* M *is eventually stationary. Then the inverse limit* $M = \lim_{\leftarrow k} M_k$ *exists in* $\mathbf{C}_{\mathrm{str}}(A,\mathsf{M})$*, the inverse limit* $\lim_{\leftarrow k} \mathrm{H}(M_k)$ *exists in* $\mathbf{G}^0(\mathsf{M})$*, and the canonical morphism*

$$\mathrm{H}(M) \to \lim_{\leftarrow k} \mathrm{H}(M_k)$$

in $\mathbf{G}^0(\mathsf{M})$ *is an isomorphism.*

Proof. (1) As mentioned above, for each i the limit $M^i = \lim_{k\to} M_k^i$ exists in M. By Proposition 10.1.6 the limit $M = \lim_{k\to} M_k$ exists in $\mathbf{C}_{\mathrm{str}}(A,\mathsf{M})$.

Regarding the cohomology: fix an integer i. Take k large enough such that $M_k^{i'} \to M_{k'}^{i'}$ are isomorphisms for all $k \le k'$ and $i-1 \le i' \le i+1$. Then $M_{k'}^{i'} \to M^{i'}$ are isomorphisms in this range, and therefore $\mathrm{H}^i(M_{k'}) \to \mathrm{H}^i(M)$ are isomorphisms for all $k \le k'$. We see that the direct system $\{\mathrm{H}^i(M_k)\}_{k\in\mathbb{N}}$ is eventually stationary, and its direct limit is $\mathrm{H}^i(M)$.

(2) The same. □

When we drop the abstract abelian category M, i.e. when we work with $\mathsf{M} = \mathsf{Mod}\,\mathbb{K} = \mathbf{M}(\mathbb{K})$ and $\mathbf{C}_{\mathrm{str}}(A,\mathsf{M}) = \mathbf{C}_{\mathrm{str}}(A)$, there is no problem of existence of limits. The next proposition says that furthermore "direct limits are exact" in $\mathbf{C}_{\mathrm{str}}(A)$.

Proposition 10.1.8. *Let* $\{M_k\}_{k\in\mathbb{N}}$ *be a direct system in* $\mathbf{C}_{\mathrm{str}}(A)$*. Then the canonical homomorphism*

$$\lim_{k\to} \mathrm{H}(M_k) \to \mathrm{H}(M)$$

in $\mathbf{G}^0(\mathbb{K})$ *is bijective.*

Exercise 10.1.9. Prove Proposition 10.1.8. (Hint: forget the action of A, and work with complexes of abelian groups.)

Exactness of inverse limits tends to be much more complicated than that of direct limits, even for $\mathbb{K}$-modules. We always have to make some condition on the inverse system to have exactness in the limit.

Definition 10.1.10. Let $\big(\{M_k\}_{k\in\mathbb{N}}, \{\mu_k\}_{k\in\mathbb{N}}\big)$ be an inverse system in $\mathbf{M}(\mathbb{K})$. For any $l \ge k$ let $M_{l,k} \subseteq M_k$ be the image of the homomorphism

$$\mathrm{id} \circ \mu_k \circ \cdots \circ \mu_{l-1} : M_l \to M_k.$$

Note that there are inclusions $M_{l+1,k} \subseteq M_{l,k}$, so for fixed k we have an inverse system $\{M_{l,k}\}_{l\ge k}$.

We say that the inverse system $\{M_k\}_{k\in\mathbb{N}}$ has the *Mittag-Leffler* property if for every index k, the inverse system $\{M_{l,k}\}_{l\ge k}$ is eventually stationary.

Example 10.1.11. If the system

$$\big(\{M_k\}_{k\in\mathbb{N}}, \{\mu_k\}_{k\in\mathbb{N}}\big)$$

satisfies one of the following conditions, then it has the Mittag-Leffler property:

(a) The system has surjective transitions.
(b) The system is eventually stationary.
(c) For any $k \in \mathbb{N}$ there exists some $l \ge k$ such that $M_{l,k} = 0$. This is called the *trivial Mittag-Leffler property*, and one says that the system is *pro-zero*.

Theorem 10.1.12 (Mittag-Leffler Argument)**.** *Let $\{M_k\}_{k \in \mathbb{N}}$ be an inverse system in $\mathbf{C}_{\mathrm{str}}(A)$, with inverse limit $M = \lim_{\leftarrow k} M_k$. Assume the system satisfies these two conditions*:

(a) *For every $i \in \mathbb{Z}$ the inverse system $\{M_k^i\}_{k \in \mathbb{N}}$ in $\mathbf{M}(\mathbb{K})$ has the Mittag-Leffler property.*
(b) *For every $i \in \mathbb{Z}$ the inverse system $\{\mathrm{H}^i(M_k)\}_{k \in \mathbb{N}}$ in $\mathbf{M}(\mathbb{K})$ has the Mittag-Leffler property.*

Then the canonical homomorphisms

$$\mathrm{H}^i(M) \to \lim_{\leftarrow k} \mathrm{H}^i(M_k)$$

are bijective.

Proof. We can forget all about the graded A-module structure, and just view this as an inverse system in $\mathbf{C}_{\mathrm{str}}(\mathbb{Z})$, i.e. and inverse system of complexes of abelian groups. Now this is a special case of [KaSc1, Proposition 1.12.4] or [EGA III, Ch. 0_{III}, Proposition 13.2.3]. □

The most useful instance of the ML argument is this:

Corollary 10.1.13. *Let $\{M_k\}_{k \in \mathbb{N}}$ be an inverse system in $\mathbf{C}_{\mathrm{str}}(A)$, with inverse limit $M = \lim_{\leftarrow k} M_k$. Assume the system satisfies these two conditions*:

(a) *For every $i \in \mathbb{Z}$ the inverse system $\{M_k^i\}_{k \in \mathbb{N}}$ has surjective transitions.*
(b) *For every k the DG module M_k is acyclic.*

Then M is acyclic.

Proof. Conditions (a) and (b) here imply conditions (a) and (b) of Theorem 10.1.12, respectively. □

Exercise 10.1.14. Prove Corollary 10.1.13 directly, without resorting to Theorem 10.1.12.

Remark 10.1.15. We will not attempt discussing direct or inverse limits in abstract abelian categories. Such definitions do exist (e.g. for a *Grothendieck abelian category*, cf. [KaSc2, Definition 8.3.24]), but this sort of thing is a source of anxiety (and sometimes of errors).

Before going on, it is good to remember the roles of the objects of cocycles and coboundaries. Let $M \in \mathbf{C}(A, \mathsf{M})$. The object of coboundaries $\mathrm{Z}(M) \subseteq M$ is defined by

$$\mathrm{Z}^i(M) := \mathrm{Ker}\big(\mathrm{d} : M^i \to M^{i+1}\big).$$

The object of cocycles $\mathrm{B}(M) \subseteq M$ is defined by

$$\mathrm{B}^i(M) := \mathrm{Im}\big(\mathrm{d} : M^{i-1} \to M^i\big).$$

Note that $\mathrm{Z}(A)$ is a DG ring with trivial differential, and the objects $\mathrm{Z}(M)$ and $\mathrm{B}(M)$ live in $\mathbf{C}(\mathrm{Z}(A), \mathsf{M})$, with trivial differentials too. There are exact sequences

$$0 \to \mathrm{Z}(M) \to M \xrightarrow{\mathrm{d}} \mathrm{T}(\mathrm{B}(M)) \to 0 \tag{10.1.16}$$

and

$$0 \to \mathrm{B}(M) \to \mathrm{Z}(M) \to \mathrm{H}(M) \to 0 \tag{10.1.17}$$

in $\mathbf{C}_{\mathrm{str}}(\mathrm{Z}(A), \mathsf{M})$.

10.2. **K-Projective Resolutions in $\mathbf{C}^{-}(\mathsf{M})$.** Recall that M is some abelian category, and $\mathbf{C}(\mathsf{M})$ is the DG category of complexes in M. The strict category $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$ is abelian.

A *filtration* on a complex $M \in \mathbf{C}_{\mathrm{str}}(\mathsf{M})$ is a collection $\{F_j(M)\}_{j \geq -1}$ of subobjects of M, such that $F_j(M) \subseteq F_{j+1}(M)$. This is a particular kind of direct system in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$. We say that $M = \lim_{j \to} F_j(M)$ if this limit exists in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, and the canonical morphism $\lim_{j \to} F_j(M) \to M$ is an isomorphism. There are also the subquotients

$$\operatorname{gr}_j^F(M) := F_j(M)/F_{j-1}(M) \in \mathbf{C}_{\mathrm{str}}(\mathsf{M}) \tag{10.2.1}$$

for $j \geq 0$. Sometimes we will be interested in filtrations that have finite length, by which we mean a direct system of subobjects $\{F_j(M)\}_{-1 \leq j \leq k}$ for some $k < \infty$. In this case $\operatorname{gr}_j^F(M)$ is defined only for $0 \leq j \leq k$.

The next definition is inspired by the work of Keller [Kel, Section 3.1].

Definition 10.2.2. Let P be an object of $\mathbf{C}(\mathsf{M})$.

(1) A *semi-projective filtration* on P is a filtration $F = \{F_j(P)\}_{j \geq -1}$ on P as an object of $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, such that:
- $F_{-1}(P) = 0$.
- Each $\operatorname{gr}_j^F(P)$ is a complex of projective objects of M with zero differential.
- $P = \lim_{j \to} F_j(P)$ in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$.

(2) The complex P is called a *semi-projective complex* if it admits some semi-projective filtration.

Theorem 10.2.3. *Let M be an abelian category, and let P be a semi-projective complex in $\mathbf{C}(\mathsf{M})$. Then P is K-projective.*

Proof. Step 1. We start by proving that if $P = \mathrm{T}^k(Q)$, the translation of a projective object $Q \in \mathsf{M}$, then P is K-projective. This is easy: given an acyclic complex $N \in \mathbf{C}(\mathsf{M})$, we have

$$\operatorname{Hom}_{\mathsf{M}}(P, N) = \operatorname{Hom}_{\mathsf{M}}\big(\mathrm{T}^k(Q), N\big) \cong \mathrm{T}^{-k}\big(\operatorname{Hom}_{\mathsf{M}}(Q, N)\big)$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. But $\operatorname{Hom}_{\mathsf{M}}(Q, -)$ is an exact functor $\mathsf{M} \to \mathbf{M}(\mathbb{K})$, so $\operatorname{Hom}_{\mathsf{M}}(Q, N)$ is an acyclic complex.

Step 2. Now P is a complex of projective objects of M with zero differential. This means that

$$P \cong \bigoplus_{k \in \mathbb{Z}} \mathrm{T}^k(Q_k)$$

in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, where each Q_k is a projective object in M. But then

$$\operatorname{Hom}_{\mathsf{M}}(P, N) \cong \prod_{k \in \mathbb{Z}} \operatorname{Hom}_{\mathsf{M}}\big(\mathrm{T}^k(Q_k), N\big).$$

This is an easy case of Proposition 10.1.3. By step 1 and the fact that a product of acyclic complexes in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$ is acyclic (itself an easy case of the Mittag-Leffler argument), we conclude that $\operatorname{Hom}_{\mathsf{M}}(P, N)$ is acyclic.

Step 3. Fix a semi-projective filtration $F = \{F_j(P)\}_{j \geq -1}$ on P. Here we prove that for every j the complex $F_j(P)$ is K-projective. This is done by induction on $j \geq -1$. For $j = -1$ it is trivial. For $j \geq 0$ there is an exact sequence of complexes

$$0 \to F_{j-1}(P) \to F_j(P) \to \operatorname{gr}_j^F(P) \to 0 \tag{10.2.4}$$

in $\mathsf{C}(\mathsf{M})$. In each degree $i \in \mathbb{Z}$ the exact sequence

$$0 \to F_{j-1}(P)^i \to F_j(P)^i \to \mathrm{gr}_j^F(P)^i \to 0$$

in M splits, because $\mathrm{gr}_j^F(P)^i$ is a projective object. Thus the exact sequence (10.2.4) is split exact in the abelian category $\mathsf{G}^0(\mathsf{M})$ of graded objects in M.

Let $N \in \mathsf{C}(\mathsf{M})$ be an acyclic complex. Applying the functor $\mathrm{Hom}_{\mathsf{M}}(-, N)$ to the sequence of complexes (10.2.4) we obtain a sequence

(10.2.5)

$$0 \to \mathrm{Hom}_{\mathsf{M}}\big(\mathrm{gr}_j^F(P), N\big) \to \mathrm{Hom}_{\mathsf{M}}\big(F_j(P), N\big) \to \mathrm{Hom}_{\mathsf{M}}\big(F_{j-1}(P), N\big) \to 0$$

in $\mathsf{C}_{\mathrm{str}}(\mathbb{K})$. Because (10.2.4) is split exact in $\mathsf{G}^0(\mathsf{M})$, the sequence (10.2.5) is split exact in $\mathsf{G}^0(\mathbb{K})$. Therefore (10.2.5) is exact in $\mathsf{C}_{\mathrm{str}}(\mathbb{K})$.

By the induction hypothesis the complex $\mathrm{Hom}_{\mathsf{M}}\big(F_{j-1}(P), N\big)$ is acyclic. By step 1 the complex $\mathrm{Hom}_{\mathsf{M}}\big(\mathrm{gr}_j^F(P), N\big)$ is acyclic. The long exact cohomology sequence associated to (10.2.5) shows that the complex $\mathrm{Hom}_{\mathsf{M}}\big(F_j(P), N\big)$ is acyclic too.

Step 4. We keep the semi-projective filtration $F = \{F_j(P)\}_{j \geq -1}$ from step 3. Take any acyclic complex $N \in \mathsf{C}(\mathsf{M})$. By Proposition 10.1.3 we know that

$$\mathrm{Hom}_{\mathsf{M}}(P, N) \cong \lim_{\leftarrow j} \mathrm{Hom}_{\mathsf{M}}\big(F_j(P), N\big)$$

in $\mathsf{C}_{\mathrm{str}}(\mathbb{K})$. According to step 3 the complexes $\mathrm{Hom}_{\mathsf{M}}\big(F_j(P), N\big)$ are all acyclic. The exactness of the sequences (10.2.5) implies that the inverse system

$$\big\{\mathrm{Hom}_{\mathsf{M}}\big(F_j(P), N\big)\big\}_{j \geq -1}$$

in $\mathsf{C}_{\mathrm{str}}(\mathbb{K})$ has surjective transitions. Now the Mittag-Leffler argument (Corollary 10.1.13) says that the inverse limit complex $\mathrm{Hom}_{\mathsf{M}}(P, N)$ is acyclic. □

Proposition 10.2.6. *Let* M *be an abelian category. If* $P \in \mathsf{C}(\mathsf{M})$ *is a bounded above complex of projectives, then* P *is a semi-projective complex.*

Proof. Say P is nonzero and $\sup(P) = i_1 \in \mathbb{Z}$. For $j \geq -1$ define

$$F_j(P) := \big(\cdots \to 0 \to P^{i_1 - j} \to \cdots \to P^{i_1 - 1} \to P^{i_1} \to \cdots\big) \subseteq P.$$

Then $\{F_j(P)\}_{j \geq -1}$ is a semi-projective filtration on P. □

The next theorem is dual to [RD, Lemma 4.6(1)], in the sense of changing injectives to projectives. (See Theorem 10.4.7 for the injective case.) We give a much more detailed proof.

Theorem 10.2.7. *Let* M *be an abelian category with enough projectives. Any complex* $M \in \mathsf{C}^-(\mathsf{M})$ *admits a quasi-isomorphism* $\rho : P \to M$ *in* $\mathsf{C}^-_{\mathrm{str}}(\mathsf{M})$*, where* P *is a bounded above complex of projectives.*

Proof. After translating M, we can assume that $M^i = 0$ for all $i > 0$. The differential of the complex M is $\mathrm{d}_M^i : M^i \to M^{i+1}$.

We start by choosing an epimorphism $\rho^0 : P^0 \twoheadrightarrow M^0$ in M from some projective object P^0. We get a morphism

$$\delta^0 : M^{-1} \oplus P^0 \to M^0$$

whose components are d_M^{-1} and ρ^0. Next we choose an epimorphism

$$\psi^{-1} : P^{-1} \twoheadrightarrow \mathrm{Ker}(\delta^0)$$

from some projective object P^{-1}. So there is an exact sequence

$$P^{-1} \xrightarrow{\psi^{-1}} M^{-1} \oplus P^0 \xrightarrow{\delta^0} M^0 \to 0.$$

The components of ψ^{-1} are denoted by $\rho^{-1} : P^{-1} \to M^{-1}$ and $\mathrm{d}_P^{-1} : P^{-1} \to P^0$.

Now to the inductive step. Here $i \leq -1$, and we already have objects $P^i, \ldots, P^0$, and morphisms $\rho^i, \ldots, \rho^0$ and $\mathrm{d}_P^i, \ldots, \mathrm{d}_P^{-1}$, that fit into this diagram

$$\begin{tikzcd} & P^i \arrow[r, "\mathrm{d}_P^i"] \arrow[d, "\rho^i"] & P^{i+1} \arrow[r] \arrow[d, "\rho^{i+1}"] & \cdots \arrow[r, "\mathrm{d}_P^0"] & P^0 \arrow[r] \arrow[d, "\rho^0"] & 0 \\ M^{i-1} \arrow[r, "\mathrm{d}_M^{i-1}"] & M^i \arrow[r, "\mathrm{d}_M^i"] & M^{i+1} \arrow[r] & \cdots \arrow[r, "\mathrm{d}_M^0"] & M^0 \arrow[r] & 0 \end{tikzcd} \tag{10.2.8}$$

in M. We still did not prove this diagram is commutative.

Define the morphism

$$\delta^i : M^{i-1} \oplus P^i \to M^i \oplus P^{i+1}$$

to be the one with components $-\mathrm{d}_M^{i-1}$, ρ^i and d_P^i. Expressing direct sums of objects as columns, and letting matrices of morphisms act on them from the left, we have this representation of δ^i :

$$\delta^i = \begin{bmatrix} -\mathrm{d}_M^{i-1} & \rho^i \\ 0 & \mathrm{d}_P^i \end{bmatrix}. \tag{10.2.9}$$

Let us choose an epimorphism

$$\psi^{i-1} : P^{i-1} \twoheadrightarrow \mathrm{Ker}(\delta^i)$$

from a projective object P^{i-1}. We get an exact sequence

$$P^{i-1} \xrightarrow{\psi^{i-1}} M^{i-1} \oplus P^i \xrightarrow{\delta^i} M^i \oplus P^{i+1}. \tag{10.2.10}$$

The components of the morphism ψ^{i-1} are denoted by $\rho^{i-1} : P^{i-1} \to M^{i-1}$ and $\mathrm{d}_P^{i-1} : P^{i-1} \to P^i$. In a matrix representation:

$$\psi^{i-1} = \begin{bmatrix} \rho^{i-1} \\ \mathrm{d}_P^{i-1} \end{bmatrix}.$$

In this way we obtain the slightly bigger diagram

$$\begin{tikzcd} P^{i-1} \arrow[r, "\mathrm{d}_P^{i-1}"] \arrow[d, "\rho^{i-1}"] & P^i \arrow[r, "\mathrm{d}_P^i"] \arrow[d, "\rho^i"] & P^{i+1} \arrow[r] \arrow[d, "\rho^{i+1}"] & \cdots \arrow[r] & P^0 \arrow[r] \arrow[d, "\rho^0"] & 0 \\ M^{i-1} \arrow[r, "\mathrm{d}_M^{i-1}"] & M^i \arrow[r, "\mathrm{d}_M^i"] & M^{i+1} \arrow[r] & \cdots \arrow[r] & M^0 \arrow[r] & 0 \end{tikzcd} \tag{10.2.11}$$

We carry out this construction inductively for all $i \leq -1$, thus obtaining a diagram like (10.2.11) that goes infinitely to the left.

Because $\delta^i \circ \psi^{i-1} = 0$ in (10.2.10), it follows that $\mathrm{d}_P^i \circ \mathrm{d}_P^{i-1} = 0$. Letting $P^i := 0$ for positive i, the collection $P := \{P^i\}_{i \in \mathbb{Z}}$ becomes a complex, with differential $\mathrm{d}_P := \{\mathrm{d}_P^i\}_{i \in \mathbb{Z}}$. The equality $\delta^i \circ \psi^{i-1} = 0$ also implies that

$$\rho^i \circ \mathrm{d}_P^{i-1} = \mathrm{d}_M^{i-1} \circ \rho^{i-1}, \tag{10.2.12}$$

so the collection $\rho := \{\rho^i\}_{i \in \mathbb{Z}}$ is a strict morphism of complexes $\rho : P \to M$.

Let us examine this commutative diagram:

$$\begin{tikzcd}[ampersand replacement=\&]
\& P^{i-1} \arrow[r, "\psi^{i-1}"] \arrow[d, "{(0,\mathrm{id})}"] \& M^{i-1} \oplus P^i \arrow[r, "\delta^i"] \arrow[d, "\mathrm{id}"] \& M^i \oplus P^{i+1} \arrow[d, "\mathrm{id}"] \\
\& M^{i-2} \oplus P^{i-1} \arrow[r, "\delta^{i-1}"] \& M^{i-1} \oplus P^i \arrow[r, "\delta^i"] \& M^i \oplus P^{i+1}
\end{tikzcd} \tag{10.2.13}$$

The top row is exact, because it is (10.2.10). An easy calculation using (10.2.12) shows that $\delta^i \circ \delta^{i-1} = 0$. These two facts combine prove that the bottom row is also exact.

Let $N = \{N^i\}_{i \in \mathbb{Z}}$ be the complex with components $N^i := M^{i-1} \oplus P^i$ for $i \leq -1$, $N^0 := M^0$ and $N^i := 0$ for $i > 0$. The differential $\mathrm{d}_N = \{\mathrm{d}^i_N\}_{i \in \mathbb{Z}}$ is

$$\mathrm{d}^i_N := \delta^i : N^i \to N^{i+1}.$$

As we saw in the paragraph above, the complex N is acyclic. On the other hand, by the definition of the morphisms δ^i in (10.2.9), we see that N is just the standard cone on the strict morphism of complexes

$$\mathrm{T}^{-1}(\rho) : \mathrm{T}^{-1}(P) \to \mathrm{T}^{-1}(M).$$

See Definition 3.9.1. Therefore ρ is a quasi-isomorphism. □

Corollary 10.2.14. *If M is an abelian category with enough projectives, then $\mathbf{C}^-(\mathsf{M})$ has enough K-projectives.*

Proof. Combine Theorem 10.2.7, Proposition 10.2.6 and Theorem 10.2.3. □

For a graded object $N \in \mathbf{G}(\mathsf{M})$ we let

$$\sup(N) := \sup\{i \mid N^i \neq 0\} \subseteq \mathbb{Z} \cup \{\pm\infty\}. \tag{10.2.15}$$

Note that $\sup(N) = -\infty$ if and only if $N = 0$.

Corollary 10.2.16. *Let M be an abelian category with enough projectives, and let $M \in \mathbf{C}(\mathsf{M})$ be a complex with bounded above cohomology. Then M has a K-projective resolution $P \to M$ with $\sup(P) = \sup(\mathrm{H}(M))$.*

Proof. We may assume that $\mathrm{H}(M)$ is not zero. Let $i := \sup(\mathrm{H}(M)) \in \mathbb{Z}$, and take $N := \mathrm{smt}^{\leq i}(M)$, the smart truncation from formula (7.3.6). Then $N \to M$ is a quasi-isomorphism and $\sup(N) = i$. An inspection of the proof of Theorem 10.2.7 shows that the resolution $P \to N$ can be made with $\sup(P) = i$. The composed quasi-isomorphism $P \to M$ has the desired properties. □

Remark 10.2.17. Let M be an abelian category, and let $\mathsf{N} \subseteq \mathsf{M}$ be a thick abelian subcategory. Assume that N has enough M-projectives, namely any object $N \in \mathsf{N}$ admits an epimorphism $P \twoheadrightarrow N$ where $P \in \mathsf{N}$ and it is projective as an object of M. Suppose $M \in \mathbf{C}(\mathsf{M})$ is a complex satisfying these conditions: $\mathrm{H}^i(M) \in \mathsf{N}$ for all i, and $\mathrm{H}^i(M) = 0$ for $i \gg 0$. Then there is a quasi-isomorphism $P \to M$ in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, where the P^i are objects of N that are projective as objects of M, and $\sup(P) = \sup(\mathrm{H}(M))$. See [RD, Lemma I.4.6(3)] for the reverse statement (with injectives). We will prove a slightly less general result for noetherian rings in Example 10.3.33.

10.3. **K-Projective Resolutions in $\mathbf{C}(A)$.** In this subsection A is a DG ring (without any vanishing assumption).

Recall that the translation $\mathrm{T}^{-i}(A)$ is a DG A-module in which the element $\mathrm{t}^{-i}(1)$ is in degree i. This element is a cocycle, and when we forget the differentials, the graded module $\mathrm{T}^{-i}(A)^\natural$ is free over the graded ring $A^\natural$, with basis $\mathrm{t}^{-i}(1)$. Therefore, for any DG A-module M there is a canonical isomorphism

$$\operatorname{Hom}_A\big(\mathrm{T}^{-i}(A), M\big) \cong \mathrm{T}^i(M) \tag{10.3.1}$$

in $\mathbf{C}(\mathbb{K})$, and canonical isomorphisms

$$\operatorname{Hom}_{\mathbf{C}_{\mathrm{str}}(A)}\big(\mathrm{T}^{-i}(A), M\big) \cong \mathrm{Z}^0\big(\operatorname{Hom}_A(\mathrm{T}^{-i}(A), M)\big) \cong \mathrm{Z}^i(M) \tag{10.3.2}$$

in $\mathbf{M}(\mathbb{K})$. (Actually, (10.3.1) is an isomorphism in $\mathbf{C}_{\mathrm{str}}(A)$, but this uses the DG A-bimodule structure of $\mathrm{T}^{-i}(A)$.)

We begin with a definition that is very similar to Definition 10.2.2. Recall the notion of a filtration $F = \{F_j(P)\}_{j \geq -1}$ of a DG module P, and the associated subquotients $\mathrm{gr}_j^F(P)$ from formula (10.2.1).

Definition 10.3.3. Let P be an object of $\mathbf{C}(\mathsf{A})$.

(1) We say that P is a *free DG A-module* if there is an isomorphism

$$P \cong \bigoplus_{s \in S} \mathrm{T}^{-i_s}(A)$$

in $\mathbf{C}_{\mathrm{str}}(A)$, for some indexing set S and some collection of integers $\{i_s\}_{s\in S}$.

(2) A *semi-free filtration* on P is a filtration $F = \{F_j(P)\}_{j \geq -1}$ of P in $\mathbf{C}_{\mathrm{str}}(A)$, such that:
- $F_{-1}(P) = 0$.
- Each $\mathrm{gr}_j^F(P)$ is a free DG A-module.
- $P = \bigcup_j F_j(P)$.

(3) The DG module P is called *semi-free* if it admits some semi-free filtration.

Example 10.3.4. If A is a ring, then a free DG A-module P is a complex of free A-modules with zero differential. A semi-free DG A-module P is also a complex of free A-modules, but there is a differential on it, and there is a subtle condition on P imposed by the existence of a semi-free filtration. If the complex P happens to be bounded above, then it is automatically semi-free, with a filtration like the one in the proof of Proposition 10.2.6.

Exercise 10.3.5. Find a ring A, and a complex P of free A-modules, that is not semi-free. (Hint: Take the ring $A = \mathbb{K}[\epsilon]$ of dual numbers. Find a complex of free A-modules P that is acyclic but not null-homotopic. Now use Theorem 10.3.6 and Corollary 9.2.12 to get a contradiction.)

Theorem 10.3.6. *Let P be an object of $\mathbf{C}(A)$. If P is semi-free, then it is K-projective.*

Proof. It is similar to the proof of Theorem 10.2.3.

Step 1. We start by proving that if $P = \mathrm{T}^{-i}(A)$, a translation of A, then P is K-projective. This is easy: given an acyclic $N \in \mathbf{C}(A)$, we have

$$\operatorname{Hom}_A(P, N) = \operatorname{Hom}_A\big(\mathrm{T}^{-i}(A), N\big) \cong \mathrm{T}^i\big(\operatorname{Hom}_A(A, N)\big) \cong \mathrm{T}^i(N)$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$, and this is acyclic.

Step 2. Now

$$P \cong \bigoplus_{s \in S} \mathrm{T}^{-i_s}(A).$$

Then

$$\mathrm{Hom}_A(P, N) \cong \prod_{s \in S} \mathrm{Hom}_A\big(\mathrm{T}^{-i_s}(A), N\big).$$

By step 1 and the fact that a product of acyclic complexes in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$ is acyclic, we conclude that $\mathrm{Hom}_{\mathsf{M}}(P, N)$ is acyclic.

Step 3. Fix a semi-free filtration $F = \{F_j(P)\}_{j \geq -1}$ of P. Here we prove that for every $j \geq -1$ the DG module $F_j(P)$ is K-projective. This is done by induction on $j \geq -1$. For $j = -1$ it is trivial. For $j \geq 0$ there is an exact sequence

$$0 \to F_{j-1}(P) \to F_j(P) \to \mathrm{gr}_j^F(P) \to 0 \tag{10.3.7}$$

in the abelian category $\mathbf{C}_{\mathrm{str}}(A)$. Because $\mathrm{gr}_j^F(P)$ is a free DG module, it is a projective object in the abelian category $\mathbf{G}^0(A^\natural)$ of graded modules over the graded ring $A^\natural$, gotten by forgetting the differential of A. Therefore the sequence (10.3.7) is split exact in $\mathbf{G}^0(A^\natural)$.

Let $N \in \mathbf{C}(A)$ be an acyclic DG module. Applying the functor $\mathrm{Hom}_A(-, N)$ to the sequence (10.3.7) we obtain a sequence

$$0 \to \mathrm{Hom}_A\big(\mathrm{gr}_j^F(P), N\big) \to \mathrm{Hom}_A\big(F_j(P), N\big) \to \mathrm{Hom}_A\big(F_{j-1}(P), N\big) \to 0 \tag{10.3.8}$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. If we forget differentials this is a sequence in $\mathbf{G}^0(\mathbb{K})$. Because (10.3.7) is split exact in $\mathbf{G}^0(A^\natural)$, it follows that (10.3.8) is split exact in $\mathbf{G}^0(\mathbb{K})$. Therefore (10.3.8) is exact in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$.

By the induction hypothesis the DG $\mathbb{K}$-module $\mathrm{Hom}_A\big(F_{j-1}(P), N\big)$ is acyclic. By step 2 the DG module $\mathrm{Hom}_A\big(\mathrm{gr}_j^F(P), N\big)$ is acyclic. The long exact cohomology sequence associated to (10.3.8) shows that the DG module $\mathrm{Hom}_A\big(F_j(P), N\big)$ is acyclic too.

Step 4. We keep the semi-free filtration $F = \{F_j(P)\}_{j \geq -1}$ from step 3. Take any acyclic $N \in \mathbf{C}(\mathsf{M})$. By Proposition 10.1.3 we know that

$$\mathrm{Hom}_A(P, N) \cong \lim_{\leftarrow j} \mathrm{Hom}_A\big(F_j(P), N\big)$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. According to step 3 the complexes $\mathrm{Hom}_A\big(F_j(P), N\big)$ are all acyclic. The exactness of the sequences (10.3.8) implies that the inverse system

$$\big\{\mathrm{Hom}_A\big(F_j(P), N\big)\big\}_{j \geq -1}$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$ has surjective transitions. Now the Mittag-Leffler argument (Corollary 10.1.13) says that the inverse limit complex $\mathrm{Hom}_A(P, N)$ is acyclic. □

Here is a result similar to Theorem 10.2.7.

Theorem 10.3.9. *Let A be a DG ring. Any $M \in \mathbf{C}(A)$ admits a quasi-isomorphism $\rho : P \to M$ in $\mathbf{C}_{\mathrm{str}}(A)$ from a semi-free DG A-module P.*

Proof. Step 1. In this step we construct a free DG A-module $F_0(P)$ and a homomorphism $F_0(\rho) : F_0(P) \to M$. For any $i \in \mathbb{Z}$ the cohomology $\mathrm{H}^i(M)$ is an $\mathrm{H}^0(A)$-module. Choose a collection of $\mathrm{H}^0(A)$-module generators of $\mathrm{H}^i(M)$, indexed by a set S_0^i. There is a canonical surjection $\mathrm{Z}^i(M) \twoheadrightarrow \mathrm{H}^i(M)$, and we lift these generators to a collection $\{m_s\}_{s \in S^i_0}$ of elements of $\mathrm{Z}^i(M)$. Define the free DG A-module

$$Q^i_0 := \bigoplus_{s \in S^i_0} \mathrm{T}^{-i}(A). \tag{10.3.10}$$

The collection $\{m_s\}_{s \in S^i_0}$ induces a homomorphism

$$\phi^i_0 : Q^i_0 \to M \tag{10.3.11}$$

in $\mathbf{C}_{\mathrm{str}}(A)$, as in formula (10.3.2). Define the free DG A-module

$$F_0(P) := \bigoplus_{i \in \mathbb{Z}} Q^i_0, \tag{10.3.12}$$

and let

$$F_0(\rho) : F_0(P) \to M, \quad F_0(\rho) := \sum_i \phi^i_0 \tag{10.3.13}$$

be the resulting homomorphism in $\mathbf{C}_{\mathrm{str}}(A)$. By construction we see that

$$\mathrm{H}^i(F_0(\rho)) : \mathrm{H}^i(F_0(P)) \to \mathrm{H}^i(M) \tag{10.3.14}$$

is surjective for all i.

Step 2. In this step $j \geq 0$, and we are given the following: a DG A-module $F_j(P)$, a homomorphism $F_j(\rho) : F_j(P) \to M$ in $\mathbf{C}_{\mathrm{str}}(A)$, and a filtration $\{F_{j'}(P)\}_{-1 \leq j' \leq j}$ of $F_j(P)$. These satisfy the following conditions: for all i and all $0 \leq j' \leq j$ the homomorphisms

$$\mathrm{H}^i(F_j(\rho)) : \mathrm{H}^i(F_{j'}(P)) \to \mathrm{H}^i(M) \tag{10.3.15}$$

are surjective; $F_{-1}(P) = 0$; and the DG A-modules $\mathrm{gr}^F_{j'}(P)$ are free for all $0 \leq j' \leq j$.

For any $i \in \mathbb{Z}$ let K^i_j be the kernel of $\mathrm{H}^i(F_j(\rho))$. So there is a short exact sequence

$$0 \to K^i_j \to \mathrm{H}^i(F_j(P)) \xrightarrow{\mathrm{H}^i(F_j(\rho))} \mathrm{H}^i(M) \to 0 \tag{10.3.16}$$

in $\mathbf{M}(\mathrm{H}^0(A))$. Choose a collection of $\mathrm{H}^0(A)$-module generators of K^i_j, indexed by a set S^i_{j+1}. Using the canonical surjection $\mathrm{Z}^i(F_j(P)) \twoheadrightarrow \mathrm{H}^i(F_j(P))$, lift these generators to a collection $\{p_s\}_{s \in S^i_{j+1}}$ of elements of the module of cocycles $\mathrm{Z}^i(F_j(P))$. Define the free DG A-module

$$Q^i_{j+1} := \bigoplus_{s \in S^i_{j+1}} \mathrm{T}^{-i}(A). \tag{10.3.17}$$

The collection of cocycles $\{p_s\}_{s \in S^i_{j+1}}$ induces a homomorphism

$$\phi^i_{j+1} : Q^i_{j+1} \to F_j(P) \tag{10.3.18}$$

in $\mathbf{C}_{\mathrm{str}}(A)$. Next define the free DG A-module

$$Q_{j+1} := \bigoplus_{i \in \mathbb{Z}} Q^i_{j+1} \tag{10.3.19}$$

and the homomorphism

$$\phi_{j+1} : Q_{j+1} \to F_j(P), \quad \phi_{j+1} := \sum_i \phi^i_{j+1} \tag{10.3.20}$$

in $\mathbf{C}_{\mathrm{str}}(A)$.

Now let us define the DG A-module $F_{j+1}(P)$ by attaching Q_{j+1} to $F_j(P)$ along ϕ_{j+1}. Namely, as a graded module we let

$$F_{j+1}(P)^{\natural} := F_j(P)^{\natural} \oplus \mathrm{T}(Q_{j+1})^{\natural}, \tag{10.3.21}$$

and the differential is

$$\mathrm{d}_{F_{j+1}(P)} := \mathrm{d}_{F_j(P)} + \mathrm{d}_{\mathrm{T}(Q_{j+1})} + \phi_{j+1} \circ \mathrm{t}^{-1} .$$

In other words, $F_{j+1}(P)$ is the standard cone on the strict homomorphism ϕ_{j+1}; see Definition 3.9.1. We note that the basis of the free DG module Q^i_{j+1} sits inside $F_{j+1}(P)^{i-1}$.

By construction, $F_j(P)$ is a DG submodule of $F_{j+1}(P)$. Let us denote the inclusion by

$$\mu_j : F_j(P) \rightarrowtail F_{j+1}(P).$$

Because the cocycles in $F_j(P)$ representing K^i_j become coboundaries in $F_{j+1}(P)$, it follows that for any i we have

$$K^i_j \subseteq \operatorname{Ker}\bigl(\mathrm{H}^i(\mu_j) : \mathrm{H}^i(F_j(P)) \to \mathrm{H}^i(F_{j+1}(P))\bigr). \tag{10.3.22}$$

Step 3. In this step we construct the homomorphism $F_{j+1}(\rho)$, continuing from where we left off in step 2. Consider the element $p_s \in \mathrm{Z}^i(F_j(P))$ for some index $s \in S^i_{j+1}$. Because the cohomology class of p_s is in K^i_j, the element $F_j(\rho)(p_s) \in M^i$ is a coboundary. Therefore we can find an element $m_s \in M^{i-1}$ such that $F_j(\rho)(p_s) = \mathrm{d}_M(m_s)$. From (10.3.19) we see that the collection of elements $\{m_s\}_{s \in \coprod_i S^i_{j+1}}$ induces a strict homomorphism of DG modules

$$\rho'_{j+1} : \mathrm{T}(Q_{j+1}) \to M.$$

Define the homomorphism

$$F_{j+1}(\rho) : F_{j+1}(P) \to M$$

to be

$$F_{j+1}(\rho) := F_j(\rho) + \rho'_{j+1}$$

using the direct sum decomposition (10.3.21). It is easy to check that this is a strict homomorphism of DG modules.

Step 4. After going through steps 2 and 3 inductively, we now have a direct system $\{F_j(P)\}_{j \geq -1}$ in $\mathbf{C}_{\mathrm{str}}(A)$, and a direct system of homomorphisms $F_j(\rho) : F_j(P) \to M$. Define the DG A-module

$$P := \lim_{j \to} F_j(P)$$

and the homomorphism

$$\rho := \lim_{j \to} F_j(\rho) : P \to M$$

in $\mathbf{C}_{\mathrm{str}}(A)$. The DG module P has on it the filtration $\{F_j(P)\}$, and it is a semi-free filtration. Indeed, there are isomorphisms $\mathrm{gr}^F_0(P) \cong \bigoplus_{i \in \mathbb{Z}} Q^i_0$ and $\mathrm{gr}^F_{j+1}(P) \cong \mathrm{T}(Q_{j+1})$ for $j \geq 0$.

It remains to prove that ρ is a quasi-isomorphism. We know that the homomorphisms $\mathrm{H}^i(F_j(\rho))$ are surjective for all i and all $j \geq 0$. Define

$$L^i_j := \operatorname{Im}\bigl(\mathrm{H}^i(\mu_j) : \mathrm{H}^i(F_j(P)) \to \mathrm{H}^i(F_{j+1}(P))\bigr) \subseteq \mathrm{H}^i(F_{j+1}(P)).$$

We get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K^i_j \arrow[r, tail, "\mathrm{inc}"] & \mathrm{H}^i(F_j(P)) \arrow[r, two heads, "\mathrm{H}^i(F_j(\rho))"] \arrow[d, two heads, "\alpha^i_j"'] \arrow[dd, bend right=40, "\mathrm{H}^i(\mu_j)"'] & \mathrm{H}^i(M) \arrow[r] & 0 \\
& & L^i_j \arrow[ur, "\beta^i_j"'] \arrow[d, tail, "\mathrm{inc}"'] & & \\
& & \mathrm{H}^i(F_{j+1}(P)) \arrow[uur, two heads, bend right, "\mathrm{H}^i(F_{j+1}(\rho))"'] & &
\end{tikzcd}$$

in $\mathbf{M}(\mathbb{K})$. The top row is an exact sequence (it is (10.3.16)). Because α^i_j is surjective, there is equality

$$\operatorname{Ker}(\beta^i_j) = \alpha^i_j\big(\operatorname{Ker}(\mathrm{H}^i(F_j(\rho)))\big) = \alpha^i_j(K^i_j).$$

But by formula (10.3.22) we know that $\alpha^i_j(K^i_j) = 0$. The conclusion is that

$$\beta^i_j : L^i_j \to \mathrm{H}^i(M) \tag{10.3.23}$$

is an isomorphism. Hence, for every i the direct system $\{L^i_j\}_{j\geq 0}$ has a limit, and the homomorphism

$$\lim_{j\to} L^i_j \to \mathrm{H}^i(M) \tag{10.3.24}$$

is bijective. Now the direct systems $\{L^i_j\}_{j\geq 0}$ and $\big\{\mathrm{H}^i(F_j(P))\big\}_{j\geq 0}$ are sandwiched; so by Proposition 10.1.1(1) we know that the second direct system also has a limit, and the the canonical homomorphism

$$\lim_{j\to} \mathrm{H}^i(F_j(P)) \to \lim_{j\to} L^i_j \tag{10.3.25}$$

is bijective. Finally, according to Proposition 10.1.7 we know that the canonical homomorphism

$$\lim_{j\to} \mathrm{H}^i(F_j(P)) \to \mathrm{H}^i(P) \tag{10.3.26}$$

is bijective. The combination of the bijections (10.3.24), (10.3.25) and (10.3.26) implies that

$$\mathrm{H}^i(\rho) : \mathrm{H}^i(P) \to \mathrm{H}^i(M)$$

is bijective. □

Corollary 10.3.27. *Let A be any DG ring. The category $\mathbf{C}(A)$ has enough K-projectives.*

Proof. Combine Theorems 10.3.6 and 10.3.9. □

The concept of nonpositive DG ring was introduced in Definition 7.3.11.

Corollary 10.3.28. *Assume A is a nonpositive DG ring. For any $M \in \mathbf{C}(A)$ there is a K-projective resolution $P \to M$ with* $\sup(P) = \sup(\mathrm{H}(M))$.

Proof. If $\mathrm{H}(M)$ is unbounded above or zero, the assertion is trivial. So we may assume that $i_1 := \sup(\mathrm{H}(M))$ is an integer. In steps 1 and 2 of the proof of Theorem 10.3.9 we choose the indexing sets S^i_j to be empty whenever this is possible. Namely $S^i_0 = \varnothing$ when $\mathrm{H}^i(M) = 0$, and $S^i_{j+1} = \varnothing$ when $K^i_j = 0$. We claim that with these choices, the inductive construction will satisfy the following extra condition: the homomorphisms

$$\mathrm{H}^i(F_j(\rho)) : \mathrm{H}^i(F_j(P)) \to \mathrm{H}^i(M) \tag{10.3.29}$$

are bijective for all $i \geq i_1+1-j$. This in turn implies that $K^i_j = 0$ for all $i \geq i_1+1-j$. We see that $K^i_j = 0$ and $\mathrm{H}^i(M) = 0$ for all $i \geq i_1 + 1$. Since A is nonpositive, this says that $\sup(F_j(P)) \leq i_1$. Therefore in the limit we get $\sup(P) \leq i_1$.

Let us prove the claim, by induction on $j \geq 0$. For $j = 0$ this is trivial, because both modules in (10.3.29) vanish for $i \geq 1$. Now assume that $j \geq 0$ and the claim holds. So $K^i_j = 0$ for all $i \geq i_1 + 1 - j$. Then, by formula (10.3.21), the DG module $F_{j+1}(P)$ coincides with its submodule $F_j(P)$ in degrees $\geq i_1 - j$. This implies that these DG modules have the same cohomologies in degrees $\geq i_1 - j + 1$, and the same cocycles in degree $i_1 - j$. Thus the homomorphisms $\mathrm{H}^i(F_{j+1}(\rho))$ remain bijective for $i \geq i_1 - j + 1$. These homomorphisms are surjective for all i. But in $F_{j+1}(P)$ there are new coboundaries in degree $i_1 - j$, those coming from $Q^{i_1-j-1}_{j+1}$. These cocycles cause the homomorphism $\mathrm{H}^{i_1-j}(F_{j+1}(\rho))$ to be injective. So the inductive step is completed. □

Definition 10.3.30. Let A be a nonpositive DG ring. A DG A-module P is called *pseudo-finite semi-free* if it admits a semi-free filtration $F = \{F_j(P)\}_{j\geq -1}$ satisfying this extra condition: there are $i_1 \in \mathbb{Z}$ and $r_j \in \mathbb{N}$ such that

$$\mathrm{gr}^F_j(P) \cong \mathrm{T}^{-i_1+j}(A)^{\oplus r_j}$$

in $\mathsf{C}_{\mathrm{str}}(A)$ for all j.

Exercise 10.3.31. Let A be a nonpositive DG ring and let P be a DG A-module. Prove that the following two conditions are equivalent.

(i) P is pseudo-finite semi-free.

(ii) There are numbers $i_1 \in \mathbb{Z}$ and $r_j \in \mathbb{N}$, and an isomorphism

$$P^\natural \cong \bigoplus_{j\geq 0} \mathrm{T}^{-i_1+j}(A^\natural)^{\oplus r_j}$$

in $\mathsf{G}^0(A^\natural)$.

In case A is a ring (i.e. $A^i = 0$ for all $i \neq 0$), prove that these conditions are equivalent to:

(iii) P is a bounded above complex of finitely generated free A-modules.

Corollary 10.3.32. *Assume that A is a nonpositive DG ring, and the ring $\mathrm{H}^0(A)$ is left noetherian. Let M be a DG A-module satisfying these conditions: each $\mathrm{H}^i(M)$ is a finitely generated $\mathrm{H}^0(A)$-module, and $\mathrm{H}^i(M) = 0$ for $i \gg 0$. Then there is a quasi-isomorphism $P \to M$ in $\mathsf{C}_{\mathrm{str}}(A)$ from a pseudo-finite semi-free DG A-module P with $\sup(P) = \sup(\mathrm{H}(M))$.*

Proof. Like in the proof of Corollary 10.3.28, the key to the proof is to economize. Besides the choice of empty indexing sets S^i_j that we imposed there, here we choose all these sets to be finite. This is possible, since the $\mathrm{H}^0(A)$-modules $\mathrm{H}^i(M)$, $\mathrm{H}^i(F_j(P))$ and K^i_j will all be finitely generated. □

Example 10.3.33. A special yet very important case of Corollary 10.3.32 is this: A is a left noetherian ring, and M is a complex of A-modules with bounded above cohomology, such that each $\mathrm{H}^i(M)$ is a finitely generated A-module. Then M has a resolution $P \to M$, where P is a complex of finitely generated free A-modules, and $\sup(P) = \sup(\mathrm{H}(M))$.

10.4. **K-Injective Resolutions in $\mathbf{C}^+(\mathsf{M})$.** In this subsection M is an abelian category, and $\mathbf{C}(\mathsf{M})$ is the category of complexes in M.

In subsection 1.3 we discussed quotients in categories. A *cofiltration* of a complex $I \in \mathbf{C}(\mathsf{M})$ is an inverse system $G = \{G_q(I)\}_{q \geq -1}$ of quotients of I in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$. We say that $I = \lim_{\leftarrow q} G_q(I)$ if this inverse limit exists in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, and the canonical morphism $I \to \lim_{\leftarrow q} G_q(I)$ is an isomorphism. The cofiltration G gives rise to the subquotients

$$\mathrm{gr}^G_q(I) := \mathrm{Ker}\big(G_q(I) \to G_{q-1}(I)\big) \in \mathbf{C}(\mathsf{M}). \tag{10.4.1}$$

Definition 10.4.2. Let I be a complex in $\mathbf{C}(\mathsf{M})$.

(1) A *semi-injective cofiltration* on I is a cofiltration $G = \{G_q(I)\}_{q \geq -1}$ in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$ such that:
   - $G_{-1}(I) = 0$.
   - Each $\mathrm{gr}^G_q(I)$ is a complex of injective objects of M with zero differential.
   - $I = \lim_{\leftarrow q} G_q(I)$.

(2) The complex I is called a *semi-injective complex* if it admits some semi-injective cofiltration.

Theorem 10.4.3. *Let M be an abelian category, and let I be a semi-injective complex in $\mathbf{C}(\mathsf{M})$. Then I is K-injective.*

Proof. The proof is very similar to that of Theorem 10.2.3.

Step 1. We start by proving that if $I = \mathrm{T}^p(J)$, the translation of an injective object $J \in \mathsf{M}$, then I is K-injective. This is easy: given an acyclic complex $N \in \mathbf{C}(\mathsf{M})$, we have

$$\mathrm{Hom}_{\mathsf{M}}(N, I) = \mathrm{Hom}_{\mathsf{M}}\big(N, \mathrm{T}^p(J)\big) \cong \mathrm{T}^p\big(\mathrm{Hom}_{\mathsf{M}}(N, J)\big)$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. But $\mathrm{Hom}_{\mathsf{M}}(-, J)$ is an exact functor $\mathsf{M} \to \mathbf{M}(\mathbb{K})$, so $\mathrm{Hom}_{\mathsf{M}}(N, J)$ is an acyclic complex.

Step 2. Now I is a complex of injective objects of M with zero differential. This means that

$$I \cong \prod_{p \in \mathbb{Z}} \mathrm{T}^p(J_p)$$

in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, where each J_p is an injective object in M. But then

$$\mathrm{Hom}_{\mathsf{M}}(N, I) \cong \prod_{p \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{M}}\big(N, \mathrm{T}^p(J_p)\big).$$

This is an easy case of Proposition 10.1.3(2). By step 1 and the fact that a product of acyclic complexes in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$ is acyclic (itself an easy case of the Mittag-Leffler argument), we conclude that $\mathrm{Hom}_{\mathsf{M}}(N, I)$ is acyclic.

Step 3. Fix a semi-injective cofiltration $G = \{G_q(I)\}_{q \geq -1}$ of I. Here we prove that for every q the complex $G_q(I)$ is K-injective. This is done by induction on q. For $q = -1$ it is trivial. For $q \geq 0$ there is an exact sequence of complexes

$$0 \to \mathrm{gr}_q^G(I) \to G_q(I) \to G_{q-1}(I) \to 0 \tag{10.4.4}$$

in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$. In each degree $p \in \mathbb{Z}$ the exact sequence

$$0 \to \mathrm{gr}_q^G(I)^p \to G_q(I)^p \to G_{q-1}(I)^p \to 0$$

in M splits, because $\mathrm{gr}_q^G(I)^p$ is an injective object. Thus the exact sequence (10.4.4) is split in the category $\mathbf{G}^0(\mathsf{M})$ of graded objects in M.

Let $N \in \mathbf{C}(\mathsf{M})$ be an acyclic complex. Applying the functor $\mathrm{Hom}_{\mathsf{M}}(N, -)$ to the sequence of complexes (10.4.4) we obtain a sequence

$$0 \to \mathrm{Hom}_{\mathsf{M}}\big(N, \mathrm{gr}_q^G(I)\big) \to \mathrm{Hom}_{\mathsf{M}}\big(N, G_q(I)\big) \to \mathrm{Hom}_{\mathsf{M}}\big(N, G_{q-1}(I)\big) \to 0 \tag{10.4.5}$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. Because (10.4.4) is split exact in $\mathbf{G}^0(\mathsf{M})$, the sequence (10.4.5) is split exact in $\mathbf{G}^0(\mathbb{K})$. Therefore (10.4.5) is exact in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$.

By the induction hypothesis the complex $\mathrm{Hom}_{\mathsf{M}}\big(N, G_{q-1}(I)\big)$ is acyclic. By step 2 the complex $\mathrm{Hom}_{\mathsf{M}}\big(N, \mathrm{gr}_q^G(I)\big)$ is acyclic. The long exact cohomology sequence associated to (10.4.5) shows that the complex $\mathrm{Hom}_{\mathsf{M}}\big(N, G_q(I)\big)$ is acyclic too.

Step 4. We keep the semi-injective cofiltration $G = \{G_q(I)\}_{q \geq -1}$ from step 3. Take any acyclic complex $N \in \mathbf{C}(\mathsf{M})$. By Proposition 10.1.3 we know that

$$\mathrm{Hom}_{\mathsf{M}}(N, I) \cong \lim_{\leftarrow q} \mathrm{Hom}_{\mathsf{M}}\big(N, G_q(I)\big)$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. According to step 3 the complexes $\mathrm{Hom}_{\mathsf{M}}\big(N, G_q(I)\big)$ are all acyclic. The exactness of the sequences (10.4.5) implies that the inverse system

$$\big\{\mathrm{Hom}_{\mathsf{M}}\big(N, G_q(I)\big)\big\}_{q \geq -1}$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$ has surjective transitions. Now the Mittag-Leffler argument (Corollary 10.1.13) says that the inverse limit complex $\mathrm{Hom}_{\mathsf{M}}(N, I)$ is acyclic. □

Proposition 10.4.6. *Let M be an abelian category. If I is a bounded below complex of injectives, then I is a semi-injective complex.*

Proof. We can assume that $I \neq 0$. Let p_0 be an integer such that $I^p = 0$ for all $p < p_0$. For $q \geq -1$ let $F_q(I)$ be the subcomplex of I defined by $F_q(I)^p := I^p$ if $p \geq p_0 + q + 1$, and $F_q(I)^p := 0$ otherwise. Then let $G_q(I) := I / F_q(I)$. The cofiltration $G = \{G_q(I)\}_{q \geq -1}$ is semi-injective. □

The next theorem is [RD, Lemma 4.6(1)]. See also [KaSc1, Proposition 1.7.7(i)].

Theorem 10.4.7. *Let M be an abelian category with enough injectives. Any complex $M \in \mathbf{C}^+(\mathsf{M})$ admits a quasi-isomorphism $\rho : M \to I$ in $\mathbf{C}^+_{\mathrm{str}}(\mathsf{M})$ into a bounded below complex of injectives I.*

Proof. The proof is the same as that of Theorem 10.2.7, except for a mechanical reversal of arrows. This is because of the symmetry of the axioms of abelian categories, that exchanges projective and injective objects. □

Exercise 10.4.8. Try to write an explicit proof of Theorem 10.4.7. You will see that this requires a lot of patience. It is a bit like trying to write mirror-flipped text (i.e. text that looks normal when reflected in a mirror).

Remark 10.4.9. One must be careful about using the symmetry of the axioms of abelian categories. It is valid for finitary constructions (as in the proof of Theorem 10.4.7); but it might break down for constructions where limits are involved, such as in the proof of Theorem 10.4.3.

Corollary 10.4.10. *If* M *is an abelian category with enough injectives, then* $\mathbf{C}^{+}(\mathsf{M})$ *has enough K-injectives.*

Proof. Combine Theorems 10.4.7, Proposition 10.4.6 and Theorem 10.4.3. □

For a graded object $N \in \mathbf{G}(\mathsf{M})$ we let

$$\inf(N) := \inf\,\{i \mid N^i \neq 0\} \subseteq \mathbb{Z} \cup \{\pm\infty\}. \tag{10.4.11}$$

Note that $\inf(N) = \infty$ if and only if $N = 0$.

Corollary 10.4.12. *Let* M *be an abelian category with enough injectives, and let* $M \in \mathbf{C}(\mathsf{M})$ *be a complex with bounded below cohomology. Then* M *has a K-injective resolution* $M \to I$ *with* $\inf(I) = \inf(\mathrm{H}(M))$.

Proof. We may assume that $\mathrm{H}(M)$ is nonzero. Let $p := \inf(\mathrm{H}(M)) \in \mathbb{Z}$, and let $N := \mathrm{smt}^{\geq p}(M)$, the smart truncation from formula 7.3.7. So $M \to N$ is a quasi-isomorphism. By proof of Theorem 10.2.7 (with sides flipped) we see that there is a K-injective resolution $N \to I$ with $\inf(I) = \inf(\mathrm{H}(M))$. The composed quasi-isomorphism $M \to I$ is what we are looking for. □

Remark 10.4.13. Let M be an abelian category, and let $\mathsf{N} \subseteq \mathsf{M}$ be a thick abelian subcategory. Assume that N has enough M-injectives, namely any object $N \in \mathsf{N}$ admits a monomorphism $N \rightarrowtail I$ where $I \in \mathsf{N}$ and it is injective as an object of M. Suppose $M \in \mathbf{C}(\mathsf{M})$ is a complex satisfying these conditions: $\mathrm{H}^i(M) \in \mathsf{N}$ for all i, and $\mathrm{H}^i(M) = 0$ for $i \ll 0$. Then there is a quasi-isomorphism $M \to I$ in $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, where the I^p are objects of N that are injective as objects of M, and $\inf(I) = \inf(\mathrm{H}(M))$. See [RD, Lemma I.4.6(3)] for a very sketchy proof.

An important example is this: $(X, \mathcal{O}_X)$ is a noetherian scheme, $\mathsf{M} = \mathsf{Mod}\,\mathcal{O}_X$ and $\mathsf{N} = \mathsf{QCoh}\,\mathcal{O}_X$. According to [RD, Proposition II.7.6] the category N has enough M-injectives.

10.5. **K-Injective Resolutions in $\mathbf{C}(A)$.** Recall that we are working over a nonzero commutative base ring $\mathbb{K}$, and A is a central DG $\mathbb{K}$-ring.

An *injective cogenerator* of the abelian category $\mathbf{M}(\mathbb{K}) = \mathsf{Mod}\,\mathbb{K}$ is an injective $\mathbb{K}$-module $\mathbb{K}^*$ with this property: if M is a nonzero $\mathbb{K}$-module, then $\mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K}^*)$ is nonzero. These always exist. Here are a few examples.

Example 10.5.1. For any nonzero ring $\mathbb{K}$ there is a canonical choice for an injective cogenerator:

$$\mathbb{K}^* := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{K}, \mathbb{Q}/\mathbb{Z}).$$

See proof of Theorem 2.6.10. Usually this a very big module!

Example 10.5.2. Assume $\mathbb{K}$ is a complete noetherian local ring, with maximal ideal $\mathfrak{m}$ and residue field $\Bbbk = \mathbb{K}/\mathfrak{m}$. In this case we would prefer to take the smallest possible injective cogenerator $\mathbb{K}^*$, and this is the injective hull of $\Bbbk$ as a $\mathbb{K}$-module.

Here are some special cases. If $\mathbb{K}$ is a field, then $\mathbb{K}^* = \mathbb{K} = \Bbbk$. If $\mathbb{K} = \widehat{\mathbb{Z}}_p$, the ring of p-adic integers, then $\Bbbk = \mathbb{F}_p$, and $\mathbb{K}^* \cong \widehat{\mathbb{Q}}_p/\widehat{\mathbb{Z}}_p$, which is the p-primary part of $\mathbb{Q}/\mathbb{Z}$. If $\mathbb{K}$ contains some field, then there exists a ring homomorphism $\Bbbk \to \mathbb{K}$

that lifts the canonical surjection $\mathbb{K} \to \Bbbk$. After choosing such a lifting, there is an isomorphism of $\mathbb{K}$-modules

$$\mathbb{K}^* \cong \operatorname{Hom}^{\mathrm{cont}}_{\Bbbk}(\mathbb{K}, \Bbbk),$$

where continuity is for the $\mathfrak{m}$-adic topology on $\mathbb{K}$ and the discrete topology on $\Bbbk$.

In this subsection we fix an injective cogenerator $\mathbb{K}^*$ of $\mathbf{M}(\mathbb{K})$. For any $p \in \mathbb{Z}$ there is the DG $\mathbb{K}$-module $\mathrm{T}^{-p}(\mathbb{K}^*)$, which is concentrated in degree p, and has the trivial differential.

Definition 10.5.3. A DG $\mathbb{K}$-module W is called *cofree* if

$$W \cong \prod_{s \in S} \mathrm{T}^{-p_s}(\mathbb{K}^*)$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$, for some indexing set S and some collection of integers $\{p_s\}_{s \in S}$.

The differential of a cofree DG $\mathbb{K}$-module W is trivial. It is not hard to see that W is a K-injective DG $\mathbb{K}$-module. When we view W as a graded $\mathbb{K}$-module, i.e. as an object of the abelian category $\mathbf{G}^0(\mathbb{K})$, it is injective.

A few more words on the structure of cofree DG $\mathbb{K}$-modules. Let's partition the set S as follows: $S = \coprod_{p \in \mathbb{Z}} S^p$, where $S^p := \{s \in S \mid p_s = p\}$. Then $W^p = \prod_{s \in S^p} \mathbb{K}^*$ as $\mathbb{K}$-modules.

Remark 10.5.4. It will be convenient to blur the distinction between DG modules with zero differentials and graded modules. Specifically, let N be a DG module such that $\mathrm{d}_N = 0$. We are going to identify N with the graded modules $N^\natural$ and $\mathrm{H}(N)$. Typical examples are these: a cofree DG $\mathbb{K}$-module W, and the DG modules $\mathrm{Z}(M)$ and $\mathrm{B}(M)$ arising from any DG module M.

Lemma 10.5.5. *Let M be a DG $\mathbb{K}$-module, let W be a cofree DG $\mathbb{K}$-module, and let $\xi : M \to W$ be a homomorphism in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. If $\mathrm{H}(\xi) : \mathrm{H}(M) \to W$ is the zero homomorphism, then ξ is a coboundary in the DG module $\operatorname{Hom}_{\mathbb{K}}(M, W)$.*

Proof. Because W has zero differential, the homomorphism $\mathrm{H}(\xi)$ is zero iff $\xi|_{\mathrm{Z}(M)} : \mathrm{Z}(M) \to W$ is zero. Consider the exact sequence

$$0 \to \mathrm{Z}(M) \to M^\natural \xrightarrow{\mathrm{t} \circ \mathrm{d}_M} \mathrm{T}(\mathrm{B}(M)) \to 0$$

in $\mathbf{G}^0(\mathbb{K})$. Applying $\operatorname{Hom}_{\mathbb{K}}(-, W)$, and taking only the degree 0 part, we obtain the exact sequence

$$0 \to \operatorname{Hom}_{\mathbb{K}}(\mathrm{B}(M), W)^{-1} \xrightarrow{\operatorname{Hom}(\mathrm{d},\mathrm{id})} \operatorname{Hom}_{\mathbb{K}}(M, W)^0 \to \operatorname{Hom}_{\mathbb{K}}(\mathrm{Z}(M), W)^0 \to 0.$$

We are using the fact that W is injective in $\mathbf{G}^0(\mathbb{K})$. The homomorphism ξ lives in the middle term, and it goes to zero in the right term; hence it comes from some ζ in the left term. Thus $\xi = \zeta \circ \mathrm{d}$ for a degree -1 homomorphism $\zeta : \mathrm{B}(M) \to W$. Again using the fact that W is injective in $\mathbf{G}^0(\mathbb{K})$, and considering the embedding $\mathrm{B}(M) \rightarrowtail M^\natural$, we see that ζ extends to a degree -1 homomorphism $\zeta : M^\natural \to W$. $\square$

Exercise 10.5.6. In the situation of Lemma 10.5.5, prove that there is a canonical isomorphism

$$\operatorname{Hom}_{\mathbb{K}}\big(\mathrm{H}(M), W\big) \cong \mathrm{H}\big(\operatorname{Hom}_{\mathbb{K}}(M, W)\big)$$

in $\mathbf{G}^0(\mathbb{K})$. (Hint: look at the proof of [PSY, Corollary 2.12].)

Lemma 10.5.7. *Let M be a DG $\mathbb{K}$-module with zero differential. There is an injective homomorphism $\chi : M \to W$ into some cofree DG $\mathbb{K}$-module W.*

Proof. It is enough to prove that for any nonzero element $m \in M^p$ there is a homomorphism $\chi_m : M^p \to \mathbb{K}^*$ such that $\chi_m(m) \neq 0$. This is a direct consequence of the fact that $\mathbb{K}^*$ is an injective cogenerator; see the proof of Theorem 2.6.10 for details. □

Definition 10.5.8. Let W be a cofree DG $\mathbb{K}$-module. The *cofree DG A-module coinduced from W* is the DG A-module

$$I_W := \operatorname{Hom}_{\mathbb{K}}(A, W).$$

There is a homomorphism

$$\theta_W : I_W \to W, \quad \theta(\chi) := \chi(1)$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. It is called the *trace*.

Definition 10.5.9. A DG A-module I is called *cofree* if there is an isomorphism $I \cong I_W$ in $\mathbf{C}_{\mathrm{str}}(A)$ for some cofree DG $\mathbb{K}$-module W.

A special cofree DG A-module is $A^* := \operatorname{Hom}_{\mathbb{K}}(A, \mathbb{K}^*)$. Any other cofree DG module I is built from A^*, in the sense that there is an isomorphism

$$I \cong \prod_{s \in S} \mathrm{T}^{-p_s}(A^*)$$

in $\mathbf{C}_{\mathrm{str}}(A)$, using the notation of Definition 10.5.3.

Lemma 10.5.10 (Adjunction)**.** *Let W be a cofree DG $\mathbb{K}$-module, and let M be a DG A-module. The homomorphism*

$$\operatorname{Hom}(\mathrm{id}_M, \theta_W) : \operatorname{Hom}_A(M, I_W) \to \operatorname{Hom}_{\mathbb{K}}(M, W)$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$ is an isomorphism.

Proof. Given $\chi \in \operatorname{Hom}_{\mathbb{K}}(M, W)^p$, let $\phi : M \to I_W$ be the function

$$\phi(m)(a) := (-1)^{ql} \cdot \chi(a \cdot m) \in W$$

for $m \in M^q$ and $a \in A^l$. Then $\phi \in \operatorname{Hom}_A(M, I_W)^p$, and

$$\operatorname{Hom}(\mathrm{id}_M, \theta)(\phi) = \theta \circ \phi = \chi.$$

We see that $\chi \mapsto \phi$ is an inverse of $\operatorname{Hom}(\mathrm{id}_M, \theta)$. □

Recall that $\mathbf{G}^0(A^\natural)$ is the abelian category whose objects are the graded $A^\natural$-modules, and the morphisms are the A-linear homomorphisms of degree 0. The forgetful functor

$$\mathbf{C}_{\mathrm{str}}(A) \to \mathbf{G}^0(A^\natural), \quad M \mapsto M^\natural,$$

is faithful.

Lemma 10.5.11. *Let I be a cofree DG A-module. Then $I^\natural$ is an injective object of $\mathbf{G}^0(A^\natural)$.*

Proof. We can assume that $I = I_W$ for some cofree DG $\mathbb{K}$-module W. For any $M \in \mathbf{G}^0(A^\natural)$ there are isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathbf{G}^0(A^\natural)}\big(M, I_W^\natural\big) &= \operatorname{Hom}_A(M, I_W)^0 \\ &\cong^\heartsuit \operatorname{Hom}_{\mathbb{K}}(M, W)^0 = \prod_{p \in \mathbb{Z}} \operatorname{Hom}_{\mathbb{K}}(M^p, W^p).\end{aligned}$$

The isomorphism $\cong^\heartsuit$ is by Lemma 10.5.10. For every p the functor $\mathbf{G}^0(A^\natural) \to \mathbf{M}(\mathbb{K})$, $M \mapsto M^p$, is exact. Because each W^p is an injective object of $\mathbf{M}(\mathbb{K})$, the functor $\operatorname{Hom}_{\mathbb{K}}(-, W^p)$ is exact. And the product of exact functors into $\mathbf{M}(\mathbb{K})$ is exact. We conclude that the functor $\operatorname{Hom}_{\mathbf{G}^0(A^\natural)}\big(-, I_W^\natural\big)$ is exact. □

The next definition is dual to Definition 10.3.3.

Definition 10.5.12. Let I be an object of $\mathbf{C}(A)$.

(1) A *semi-cofree cofiltration* on I is a cofiltration $G = \{G_q(I)\}_{q \geq -1}$ of I in $\mathbf{C}_{\mathrm{str}}(A)$ such that:
- $G_{-1}(I) = 0$.
- Each $\operatorname{gr}^G_q(I)$ is a cofree DG A-module.
- $I = \lim_{\leftarrow q} G_q(I)$.

(2) The DG A-module I is called a *semi-cofree* if it admits a semi-cofree cofiltration.

Theorem 10.5.13. *Let I be an object of $\mathbf{C}(A)$. If I is semi-cofree, then it is K-injective.*

Proof. The proof is very similar to those of Theorems 10.2.3 and 10.3.6. But because the arguments involve limits, we shall give the full proof.

Step 1. Suppose I is cofree; say $I \cong \prod_{s \in S} \mathrm{T}^{-p_s}(A^*)$. The adjunction formula (Lemma 10.5.10) implies that for any DG A-module N there is an isomorphism

$$\operatorname{Hom}_A(N, I) \cong \prod_{s \in S} \operatorname{Hom}_{\mathbb{K}}\big(\mathrm{T}^{p_s}(N), \mathbb{K}^*\big)$$

of graded $\mathbb{K}$-modules. It follows that if N is acyclic, then so is $\operatorname{Hom}_A(N, I)$.

Step 2. Fix a semi-cofree cofiltration $G = \{G_q(I)\}_{q \geq -1}$ of I. Here we prove that for every $q \geq -1$ the DG module $G_q(I)$ is K-injective. This is done by induction on $q \geq -1$. For $q = -1$ it is trivial. For $q \geq 0$ there is an exact sequence

$$0 \to \operatorname{gr}^F_q(I) \to G_q(I) \to G_{q-1}(I) \to 0 \tag{10.5.14}$$

in the category $\mathbf{C}_{\mathrm{str}}(A)$. Because $\operatorname{gr}^G_q(I)$ is a cofree DG A-module, it is an injective object in the abelian category $\mathbf{G}^0(A^\natural)$; see Lemma 10.5.11. Therefore the sequence (10.5.14) is split exact in $\mathbf{G}^0(A^\natural)$.

Let $N \in \mathbf{C}(A)$ be an acyclic DG module. Applying the functor $\operatorname{Hom}_A(N, -)$ to the sequence (10.5.14) we obtain a sequence

$$0 \to \operatorname{Hom}_A\big(N, \operatorname{gr}^G_q(I)\big) \to \operatorname{Hom}_A\big(N, G_q(I)\big) \to \operatorname{Hom}_A\big(N, G_{q-1}(I)\big) \to 0 \tag{10.5.15}$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. If we forget differentials this is a sequence in $\mathbf{G}^0(\mathbb{K})$. Because (10.5.14) is split exact in $\mathbf{G}^0(A^\natural)$, it follows that (10.5.15) is split exact in $\mathbf{G}^0(\mathbb{K})$. Therefore (10.5.15) is exact in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$.

By the induction hypothesis the DG $\mathbb{K}$-module $\mathrm{Hom}_A\big(N, G_{q-1}(I)\big)$ is acyclic. By step 1 the DG $\mathbb{K}$-module $\mathrm{Hom}_A\big(N, \mathrm{gr}_q^G(I)\big)$ is acyclic. The long exact cohomology sequence associated to (10.5.15) shows that the DG $\mathbb{K}$-module $\mathrm{Hom}_A\big(N, G_q(I)\big)$ is acyclic too.

Step 3. We keep the semi-cofree cofiltration $G = \{G_q(I)\}_{q \geq -1}$ from step 2. Take any acyclic $N \in \mathbf{C}(A)$. By Proposition 10.1.3 we know that

$$\mathrm{Hom}_A(N, I) \cong \lim_{\leftarrow j} \mathrm{Hom}_A\big(N, G_q(I)\big)$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$. According to step 2 the complexes $\mathrm{Hom}_A\big(N, G_q(I)\big)$ are all acyclic. The exactness of the sequences (10.5.15) implies that the inverse system

$$\big\{\mathrm{Hom}_A\big(N, G_q(I)\big)\big\}_{q \geq -1}$$

in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$ has surjective transitions. Now the Mittag-Leffler argument (Corollary 10.1.13) says that the inverse limit complex $\mathrm{Hom}_A(N, I)$ is acyclic. $\square$

Theorem 10.5.16. *Let A be a DG ring. Any DG A-module M admits a quasi-isomorphism $\rho : M \to I$ in $\mathbf{C}_{\mathrm{str}}(A)$ to a semi-cofree DG A-module I.*

We shall need three lemmas before the proof of the theorem.

Lemma 10.5.17. *Let W be a cofree DG $\mathbb{K}$-module, let M be a DG A-module, and let $\chi : \mathrm{H}(M) \to \mathrm{H}(W)$ be a homomorphism in $\mathbf{G}^0(\mathbb{K})$. Then there is a homomorphism $\phi : M \to I_W$ in $\mathbf{C}_{\mathrm{str}}(A)$, such that the diagram*

$$\begin{tikzcd} \mathrm{H}(M) \arrow[r, "\mathrm{H}(\phi)"] \arrow[rr, bend right, "\chi"'] & \mathrm{H}(I_W) \arrow[r, "\mathrm{H}(\theta_W)"] & \mathrm{H}(W) \end{tikzcd}$$

in $\mathbf{G}^0(\mathbb{K})$ is commutative.

Proof. We can assume that

$$W = \prod_{p \in \mathbb{Z}} \prod_{s \in S^p} \mathrm{T}^{-p}(\mathbb{K}^*)$$

for some graded set $S = \coprod_{p \in \mathbb{Z}} S^p$. Then

$$I_W = \prod_{p \in \mathbb{Z}} \prod_{s \in S^p} \mathrm{T}^{-p}(A^*),$$

where $A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K}^*)$ as before. The trace θ_W is a product of translations of the trace $\theta : A^* \to \mathbb{K}^*$. The homomorphism $\chi : \mathrm{H}(M) \to \mathrm{H}(W)$ is a product of $\mathbb{K}$-linear homomorphisms $\chi_s : \mathrm{H}^p(M) \to \mathbb{K}^*$. We see that it suffices to find, for each p and each $s \in S^p$, a homomorphism $\phi_s : M \to \mathrm{T}^{-p}(A^*)$ in $\mathbf{C}_{\mathrm{str}}(A)$, such that $\theta \circ \mathrm{H}^p(\phi_s) = \chi_s$.

Now we consider the simplified situation: $\chi : \mathrm{H}^p(M) \to \mathbb{K}^*$ is a $\mathbb{K}$-linear homomorphism, and we are looking for a homomorphism $\phi : M \to \mathrm{T}^{-p}(A^*)$ in $\mathbf{C}_{\mathrm{str}}(A)$ such that $\theta \circ \mathrm{H}^p(\phi) = \chi$.

For any integer p let

$$\mathrm{Y}^p(M) := \mathrm{Coker}(\mathrm{d}_M^{p-1} : M^{p-1} \to M^p).$$

See Remark 7.3.10. In each degree p there are canonical exact sequences of $\mathbb{K}$-modules

$$0 \to \mathrm{B}^p(M) \to M^p \to \mathrm{Y}^p(M) \to 0 \tag{10.5.18}$$

and

$$0 \to \mathrm{H}^p(M) \to \mathrm{Y}^p(M) \xrightarrow{\mathrm{d}} \mathrm{B}^{p+1}(M) \to 0. \tag{10.5.19}$$

Because $\mathbb{K}^*$ is injective in $\mathbf{M}(\mathbb{K})$, we can extend $\chi : \mathrm{H}^p(M) \to \mathbb{K}^*$ to a homomorphism $\chi : \mathrm{Y}^p(M) \to \mathbb{K}^*$ in $\mathbf{M}(\mathbb{K})$, relative to the embedding $\mathrm{H}^p(M) \rightarrowtail \mathrm{Y}^p(M)$ in (10.5.19). Next we compose with the surjection $M^p \to \mathrm{Y}^p(M)$ in (10.5.18) to obtain a $\mathbb{K}$-linear homomorphism $\chi : M^p \to \mathbb{K}^*$. Note that $\chi \circ \mathrm{d}_M^{p-1} = 0$ by the exact sequence (10.5.18).

We now view χ as a degree $-p$ homomorphism $\chi : M \to \mathbb{K}^*$ in $\mathbf{G}(\mathbb{K})$ that sends all other components of M to zero. As an element of the DG $\mathbb{K}$-module $\mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K}^*)$, χ is a degree $-p$ cocycle. By adjunction (Lemma 10.5.10) we get an element $\psi \in \mathrm{Hom}_A(M, A^*)$, and it is a cocycle of degree $-p$. Then

$$\phi := \mathrm{t}^{-p} \circ \psi : M \to \mathrm{T}^{-p}(A^*)$$

is a homomorphism in $\mathbf{C}_{\mathrm{str}}(A)$ with the desired property. □

Lemma 10.5.20. *In the situation of Lemma* 10.5.17*, let* $N := \mathrm{Cone}(\mathrm{T}^{-1}(\phi))$*, the standard cone on the homomorphism*

$$\mathrm{T}^{-1}(\phi) : \mathrm{T}^{-1}(M) \to \mathrm{T}^{-1}(I_W)$$

in $\mathbf{C}_{\mathrm{str}}(A)$*. Consider the canonical exact sequence*

$$0 \to \mathrm{T}^{-1}(I_W) \to N \xrightarrow{\pi} M \to 0 \tag{10.5.21}$$

in $\mathbf{C}_{\mathrm{str}}(A)$*, shown in formula* (3.9.4)*. Then the composed homomorphism*

$$\chi \circ \mathrm{H}(\pi) : \mathrm{H}(N) \to \mathrm{H}(W)$$

in $\mathbf{G}^0(\mathbb{K})$ *is zero.*

Proof. Passing to the long exact sequence in cohomology of (10.5.21), and then applying $\mathrm{Hom}_{\mathbb{K}}(-, W^p)$, we obtain this long exact sequence:

$$\cdots \to \mathrm{Hom}_{\mathbb{K}}(\mathrm{H}^p(I_W), W^p) \to \mathrm{Hom}_{\mathbb{K}}(\mathrm{H}^p(M), W^p) \to \mathrm{Hom}_{\mathbb{K}}(\mathrm{H}^p(N), W^p) \to \cdots$$

in $\mathbf{M}(\mathbb{K})$. The homomorphism $\chi^p : \mathrm{H}^p(M) \to W^p$ in the middle term comes from $\mathrm{H}^p(\theta_W) : \mathrm{H}^p(I_W) \to W^p$ in the left term. Therefore its image $\chi^p \circ \mathrm{H}^p(\pi) : \mathrm{H}^p(N) \to W^p$ in the right term is zero. □

Lemma 10.5.22. *In the situation of Lemma* 10.5.20*, suppose* $\rho : L \to M$ *is a homomorphism in* $\mathbf{C}_{\mathrm{str}}(A)$ *such that* $\mathrm{H}(\theta_W) \circ \mathrm{H}(\phi) \circ \mathrm{H}(\rho) = 0$*. Then there exists a homomorphism* $\sigma : L \to N$ *in* $\mathbf{C}_{\mathrm{str}}(A)$ *such that* $\pi \circ \sigma = \rho$*.*

See the next commutative diagrams, in $\mathbf{C}_{\mathrm{str}}(A)$ and $\mathbf{G}^0(\mathbb{K})$ respectively.

$$\begin{tikzcd}
 & N \arrow[d, "\pi"] & & \\
L \arrow[ur, dashed, "\sigma"] \arrow[r, "\rho"] \arrow[rrr, bend right, "\xi"'] & M \arrow[r, "\phi"] & I_W \arrow[r, "\theta_W"] & W
\end{tikzcd}$$

$$\begin{tikzcd} & \mathrm{H}(N) \arrow[d, "\mathrm{H}(\pi)"] \\ \mathrm{H}(L) \arrow[ur, dashed, "\mathrm{H}(\sigma)"] \arrow[r, "\mathrm{H}(\rho)"] \arrow[rrr, bend right, "\mathrm{H}(\xi)=0"'] & \mathrm{H}(M) \arrow[r, "\mathrm{H}(\phi)"] & \mathrm{H}(I_W) \arrow[r, "\mathrm{H}(\theta_W)"] & W \end{tikzcd}$$

Proof. It will be convenient to express $N = \mathrm{Cone}(\mathrm{T}^{-1}(\phi))$ in terms of matrices. We will use the equality $\mathrm{T}(\mathrm{T}^{-1}(M)) = M$ to write the graded $A^\natural$-module $N^\natural$ as a column:

$$N^\natural = \begin{bmatrix} \mathrm{T}^{-1}(I_W)^\natural \\ \mathrm{T}(\mathrm{T}^{-1}(M))^\natural \end{bmatrix} = \begin{bmatrix} \mathrm{T}^{-1}(I_W)^\natural \\ M^\natural \end{bmatrix}. \tag{10.5.23}$$

A small calculation, using Definition 3.8.5 and Proposition 3.8.10(1), shows that

$$\mathrm{T}^{-1}(\phi) = \mathrm{t}^{-1}_{\mathrm{T}^{-1}(I_W)} \circ \phi \circ \mathrm{t}_{\mathrm{T}^{-1}(M)} .$$

Note that

$$\mathrm{t}_{\mathrm{T}^{-1}(I_W)} : \mathrm{T}^{-1}(I_W) \to \mathrm{T}(\mathrm{T}^{-1}(I_W)) = I_W$$

is an invertible degree -1 homomorphism, and its inverse

$$\mathrm{t}^{-1}_{\mathrm{T}^{-1}(I_W)} : I_W \to \mathrm{T}^{-1}(I_W)$$

has degree 1. So the differential of N is

$$\mathrm{d}_N = \begin{bmatrix} \mathrm{d}_{\mathrm{T}^{-1}(I_W)} & \mathrm{T}^{-1}(\phi) \circ \mathrm{t}^{-1}_{\mathrm{T}^{-1}(M)} \\ 0 & \mathrm{d}_M \end{bmatrix} = \begin{bmatrix} \mathrm{d}_{\mathrm{T}^{-1}(I_W)} & \mathrm{t}^{-1}_{\mathrm{T}^{-1}(I_W)} \circ \phi \\ 0 & \mathrm{d}_M \end{bmatrix}. \tag{10.5.24}$$

Define $\xi := \theta_W \circ \phi \circ \rho$. This is a homomorphism $\xi : L \to W$ in $\mathbf{C}_{\mathrm{str}}(\mathbb{K})$, and by assumption $\mathrm{H}(\xi) = 0$. According to Lemma 10.5.5, ξ is a coboundary in the DG module $\mathrm{Hom}_{\mathbb{K}}(L, W)$. So there is some $\omega \in \mathrm{Hom}_{\mathbb{K}}(L, W)^{-1}$ such that $\xi = \mathrm{d}(\omega) = \omega \circ \mathrm{d}_L$. Let $\alpha : L \to I_W$ be the unique A-linear homomorphism of degree -1 such that $\theta_W \circ \alpha = \omega$; see Lemma 10.5.10. Define the homomorphism $\sigma : L^\natural \to N^\natural$ in $\mathbf{G}^0(A^\natural)$ to be the column

$$\sigma := \begin{bmatrix} \mathrm{t}^{-1} \circ \alpha \\ \rho \end{bmatrix},$$

where from here to the end of the proof we write $\mathrm{t} := \mathrm{t}_{\mathrm{T}^{-1}(I_W)}$. It is clear that $\pi \circ \sigma = \rho$.

It remains to prove that σ is strict, namely that $\sigma \circ \mathrm{d}_L = \mathrm{d}_N \circ \sigma$. Let us write out these homomorphisms as matrices. We have

$$\sigma \circ \mathrm{d}_L = \begin{bmatrix} \mathrm{t}^{-1} \circ \alpha \circ \mathrm{d}_L \\ \rho \circ \mathrm{d}_L \end{bmatrix}$$

and

$$\mathrm{d}_N \circ \sigma = \begin{bmatrix} \mathrm{d}_{\mathrm{T}^{-1}(I_W)} & \mathrm{t}^{-1} \circ \phi \\ 0 & \mathrm{d}_M \end{bmatrix} \circ \begin{bmatrix} \mathrm{t}^{-1} \circ \alpha \\ \rho \end{bmatrix} = \begin{bmatrix} \mathrm{d}_{\mathrm{T}^{-1}(I_W)} \circ \mathrm{t}^{-1} \circ \alpha + \mathrm{t}^{-1} \circ \phi \circ \rho \\ \mathrm{d}_M \circ \rho \end{bmatrix}.$$

Since ρ is strict, there is equality $\rho \circ \mathrm{d}_L = \mathrm{d}_M \circ \rho$. We need to verify that

$$\mathrm{t}^{-1} \circ \alpha \circ \mathrm{d}_L = \mathrm{d}_{\mathrm{T}^{-1}(I_W)} \circ \mathrm{t}^{-1} \circ \alpha + \mathrm{t}^{-1} \circ \phi \circ \rho$$

as A-linear homomorphisms $L \to \mathrm{T}^{-1}(I_W)$. We are allowed to postcompose with t; so now we have to verify that

$$\alpha \circ \mathrm{d}_L = \mathrm{t} \circ \mathrm{d}_{\mathrm{T}^{-1}(I_W)} \circ \mathrm{t}^{-1} \circ \alpha + \phi \circ \rho$$

as A-linear homomorphisms $L \to I_W$. By adjunction (Lemma 10.5.10) it suffices to verify that they are equal as $\mathbb{K}$-linear homomorphisms after postcomposing with θ_W. But

$$\theta_W \circ \mathrm{t} \circ \mathrm{d}_{\mathrm{T}^{-1}(I_W)} \circ \mathrm{t}^{-1} \circ \alpha = -\theta_W \circ \mathrm{d}_{I_W} \circ \alpha = -\mathrm{d}_W \circ \theta_W \circ \alpha = 0;$$

and

$$\theta_W \circ \phi \circ \rho = \xi = \theta_W \circ \alpha \circ \mathrm{d}_L.$$

□

Proof of Theorem 10.5.16. The proof is morally dual to that of Theorem 10.3.9, but the details are much more complicated. This is the strategy: we will construct an inverse system $\{G_q(I)\}_{q \geq -1}$ in $\mathbf{C}_{\mathrm{str}}(A)$, and an inverse system of homomorphisms $G_q(\rho) : M \to G_q(I)$ in $\mathbf{C}_{\mathrm{str}}(A)$. Then we will prove that the DG module $I := \lim_{\leftarrow q} G_q(I)$ is semi-cofree, and the homomorphism $\lim_{\leftarrow q} G_q(\rho) : M \to I$ is a quasi-isomorphism.

Step 1. In this step we handle $q = 0$. By Lemma 10.5.7 there is an injective homomorphism $\chi : \mathrm{H}(M) \to W$ in $\mathbf{G}^0(\mathbb{K})$ for some cofree DG $\mathbb{K}$-module W. Next, by Lemma 10.5.17, there is a homomorphism $\phi : M \to I_W$ in $\mathbf{C}_{\mathrm{str}}(A)$, such that $\chi = \mathrm{H}(\theta_W) \circ \mathrm{H}(\phi)$.

Define the cofree DG A-module $G_0(I) := I_W$ and the homomorphism

$$G_0(\rho) := \phi : M \to G_0(I).$$

Then the homomorphism

$$\mathrm{H}(G_0(\rho)) : \mathrm{H}(M) \to \mathrm{H}(G_0(I))$$

is injective.

Step 2. In this step $q \geq 0$, and we are given the following: a DG A-module $G_q(I)$, a cofiltration $\{G_{q'}(I)\}_{-1 \leq q' \leq q}$ of $G_q(I)$, and an inverse system of homomorphisms $G_{q'}(\rho) : M \to G_{q'}(I)$ in $\mathbf{C}_{\mathrm{str}}(A)$. These satisfy the following conditions: the homomorphisms

$$\mathrm{H}(G_{q'}(\rho)) : \mathrm{H}(M) \to \mathrm{H}(G_{q'}(I))$$

in $\mathbf{G}^0(\mathbb{K})$ are injective for all $0 \leq q' \leq q$; $G_{-1}(I) = 0$; and the DG A-modules

$$\mathrm{Ker}\big(G_{q'}(I) \to G_{q'-1}(I)\big)$$

are cofree for all $0 \leq q' \leq q$.

Let N be the cokernel of $\mathrm{H}(G_q(\rho))$. So there is a short exact sequence

$$0 \to \mathrm{H}(M) \xrightarrow{\mathrm{H}(G_q(\rho))} \mathrm{H}(G_q(I)) \xrightarrow{\alpha} N \to 0 \tag{10.5.25}$$

in $\mathbf{G}^0(\mathbb{K})$. By Lemma 10.5.7 there is an injective homomorphism $\chi : N \to W$ in $\mathbf{G}^0(\mathbb{K})$ for some cofree DG $\mathbb{K}$-module W. Next, by Lemma 10.5.17, there is a homomorphism $\phi : G_q(I) \to I_W$ in $\mathbf{C}_{\mathrm{str}}(A)$, such that

$$\chi \circ \alpha = \mathrm{H}(\theta_W) \circ \mathrm{H}(\phi)$$

as homomorphisms $\mathrm{H}(G_q(I)) \to W$. Define the DG A-module

$$G_{q+1}(I) := \mathrm{Cone}(\mathrm{T}^{-1}(\phi)),$$

the standard cone on the strict homomorphism $\mathrm{T}^{-1}(\phi)$. There is a canonical exact sequence

$$0 \to \mathrm{T}^{-1}(I_W) \to G_{q+1}(I) \xrightarrow{\mu_q} G_q(I) \to 0 \tag{10.5.26}$$

in $\mathbf{C}_{\mathrm{str}}(A)$. According to Lemma 10.5.20, the homomorphism

$$\chi \circ \alpha \circ \mathrm{H}(\mu_q) : \mathrm{H}(G_{q+1}(I)) \to W$$

in $\mathbf{G}^0(\mathbb{K})$ is zero. Since χ is an injective homomorphism, we conclude that the homomorphism $\alpha \circ \mathrm{H}(\mu_q)$ in the commutative diagram below is zero.

$$\begin{tikzcd}
\mathrm{H}(G_{q+1}(I)) \arrow[d, "\mathrm{H}(\mu_q)"'] \arrow[drr, "0"] \arrow[drrrr, bend left, "0"] & & & & \\
\mathrm{H}(G_q(I)) \arrow[rr, "\alpha"] \arrow[drr, "\mathrm{H}(\phi)"'] & & N \arrow[rr, "\chi"] & & W \\
& & \mathrm{H}(I_W) \arrow[urr, "\mathrm{H}(\theta_W)"'] & &
\end{tikzcd} \tag{10.5.27}$$

Step 3. We continue from step 2. We know from formula (10.5.25) and diagram (10.5.27) that

$$\mathrm{H}(\theta_W) \circ \mathrm{H}(\phi) \circ \mathrm{H}(G_q(\rho)) = 0.$$

According to Lemma 10.5.22 there is a homomorphism

$$G_{q+1}(\rho) : M \to G_{q+1}(I)$$

in $\mathbf{C}_{\mathrm{str}}(A)$ such that the diagram

$$\begin{tikzcd}
& G_{q+1}(I) \arrow[d, "\mu_q"] \\
M \arrow[ur, "G_{q+1}(\rho)"] \arrow[r, "G_q(\rho)"'] & G_q(I)
\end{tikzcd} \tag{10.5.28}$$

in $\mathbf{C}_{\mathrm{str}}(A)$ is commutative.

The next diagram, in $\mathbf{G}^0(\mathbb{K})$, is also commutative, and the bottom row is exact:

$$\begin{tikzcd}
& & \mathrm{H}(G_{q+1}(I)) \arrow[d, "\mathrm{H}(\mu_q)"] \arrow[dr, "0"] & & \\
0 \arrow[r] & \mathrm{H}(M) \arrow[ur, "\mathrm{H}(G_{q+1}(\rho))"] \arrow[r, "\mathrm{H}(G_q(\rho))"'] & \mathrm{H}(G_q(I)) \arrow[r, "\alpha"'] & N \arrow[r] & 0
\end{tikzcd} \tag{10.5.29}$$

Let us define

$$L_q := \mathrm{Im}(\mathrm{H}(\mu_q)) \subseteq \mathrm{H}(G_q(I)).$$

From diagram (10.5.29) we see that the homomorphism

$$\mathrm{H}(G_q(\rho)) : \mathrm{H}(M) \to L_q \tag{10.5.30}$$

in $\mathbf{G}^0(\mathbb{K})$ is bijective. This implies that

$$\mathrm{H}(G_{q+1}(\rho)) : \mathrm{H}(M) \to \mathrm{H}(G_{q+1}(I))$$

in an injective homomorphism, a fact that is needed to keep the induction going.

Step 4. Proceeding with steps 2 and 3 inductively, we obtain an inverse system $\{G_q(I)\}_{q \geq -1}$ of objects in $\mathbf{C}_{\mathrm{str}}(A)$, and an inverse system $G_q(\rho) : M \to G_q(I)$ of homomorphisms in $\mathbf{C}_{\mathrm{str}}(A)$. The DG module $I := \lim_{\leftarrow q} G_q(I)$ comes equipped with the semi-cofree cofiltration $\{G_q(I)\}_{q \geq -1}$, and thus it is semi-cofree.

It remains to prove that the homomorphism

$$\rho := \lim_{\leftarrow q} G_q(\rho) : M \to I$$

is a quasi-isomorphism. From formula (10.5.30) we know that $\mathrm{H}(M) \to \lim_{\leftarrow q} L_q$ is bijective. The inverse systems $\{L_q\}_{q \geq 0}$ and $\{\mathrm{H}(G_q(I))\}_{q \geq 0}$ are sandwiched, so by Proposition 10.1.1(2) the limit of the second inverse system exists, and the canonical homomorphism

$$\lim_{\leftarrow q} L_q \to \lim_{\leftarrow q} \mathrm{H}(G_q(I))$$

is bijective.

Finally, the inverse systems $\{G_q(I)\}_{q \geq 0}$ and $\{\mathrm{H}(G_q(I))\}_{q \geq 0}$ satisfy the ML condition: the first has surjective transitions, and the images of the transitions $\mathrm{H}(G_{q'}(I)) \to \mathrm{H}(G_q(I))$ are stationary for $q' \geq q+1$. Therefore the homomorphism

$$\mathrm{H}(I) \to \lim_{\leftarrow q} \mathrm{H}(G_q(I))$$

is bijective. Putting these facts together, we deduce that ρ is a quasi-isomorphism. □

Corollary 10.5.31. *Let A be any DG ring. The category $\mathbf{C}(A)$ has enough K-injectives.*

Proof. Combine Theorems 10.5.13 and 10.5.16. □

Corollary 10.5.32. *Assume A is a nonpositive DG ring* (*Definition* 7.3.11)*. For any $M \in \mathbf{C}(A)$ there is a K-injective resolution $M \to I$ with* $\inf(I) = \inf(\mathrm{H}(M))$*.*

Proof. For any cofree DG $\mathbb{K}$-module W, the cofree DG A-module I_W has $\inf(I_W) = \inf(W)$. (Assuming that A is nonzero.) Looking at steps 1 and 2 of the proof of Theorem 10.5.16, we see that the DG modules $G_q(I)$ can be chosen such that $\inf(G_q(I)) = \inf(\mathrm{H}(M))$. □

Remark 10.5.33. The proof of Theorem 10.5.16 is quite long and complicated. It would be nice to have a quicker proof.

In Keller's paper [Kel, Section 3.2] there is a slick proof of an even stronger result than Theorem 10.5.16 – but we were unable to understand the details!

cmnt: End of first part (in book)

New Material

11. Recalling Material from Last Year [Temporary]

11.1. **Generalities.** We fix a nonzero commutative base ring $\mathbb{K}$ (e.g. a field or $\mathbb{Z}$). All linear operations are by default $\mathbb{K}$-linear. Thus a ring A is assumed to be $\mathbb{K}$-central; an additive category M is assumed to be $\mathbb{K}$-linear; etc.

The concepts of classical homological algebra: abelian category, additive functor, injective and projective objects, and so on, are all assumed to be familiar.

11.2. **DG Algebra.** Let me quickly go over the important ideas of DG algebra, because they are not so well-known. This is a review of Section 3.

A DG ring is a graded ring $A = \bigoplus_{i \in \mathbb{Z}} A^i$, with a differential d of degree 1, satisfying the graded Leibniz rule

$$\mathrm{d}(a_1 \cdot a_2) = \mathrm{d}(a_1) \cdot a_2 + (-1)^{i_1} \cdot a_1 \cdot \mathrm{d}(a_2)$$

for elements $a_j \in A^{i_j}$.

Over a DG ring A there are left DG modules, right DG modules and DG bimodules. The default is always left modules.

Given DG A-modules M, N, we can form the DG $\mathbb{K}$-module

$$\mathrm{Hom}_A(M, N) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_A(M, N)^i. \tag{11.2.1}$$

The i-th summand consists of degree i homomorphisms that commute, in the graded sense, with the action of A (this is a bit subtle).

If L is a right DG A-module, then

$$L \otimes_A M = \bigoplus_{i \in \mathbb{Z}} (L \otimes_A M)^i$$

is also a DG $\mathbb{K}$-module.

A strict homomorphism of DG A-modules is a homomorphism $\phi : M \to N$ that commutes with the grading, the action of A, and the differentials. Equivalently, ϕ is a 0-cocycle in the DG module $\mathrm{Hom}_A(M, N)$.

Generalizing the notion of DG ring, we get DG categories. A DG category C is a $\mathbb{K}$-linear category, whose Hom modules have a DG structure. I.e. for any pair of objects $M, N \in \mathsf{C}$, the set $\mathrm{Hom}_{\mathsf{C}}(M, N)$ is a DG $\mathbb{K}$-module. The identity automorphism $\mathrm{id}_M = 1_M$ is a degree 0 cocycle. For three objects, the composition is a strict homomorphism of DG $\mathbb{K}$-modules:

$$- \circ - : \mathrm{Hom}_{\mathsf{C}}(M_1, M_2) \otimes_{\mathbb{K}} \mathrm{Hom}_{\mathsf{C}}(M_0, M_1) \to \mathrm{Hom}_{\mathsf{C}}(M_0, M_2).$$

Generalizing the notion of homomorphism of DG rings, we obtain the notion of DG functor

$$F : \mathsf{C} \to \mathsf{D}$$

between a pair of DG categories. If $G : \mathsf{C} \to \mathsf{D}$ is another DG functor, we can talk about a degree i morphism $\eta : F \to G$ of DG functors, and its differential $\mathrm{d}(\eta) : F \to G$, that's a degree $i + 1$ morphism.

To a DG category C we attach two other categories, with the same sets of objects as C. There is the strict category $\mathsf{C}_{\mathrm{str}}$, whose morphisms are the strict morphisms:

$$\mathrm{Hom}_{\mathsf{C}_{\mathrm{str}}}(M, N) := \mathrm{Z}^0(\mathrm{Hom}_{\mathsf{C}}(M, N)).$$

And there is the homotopy category $\mathrm{Ho}(\mathsf{C})$, whose morphisms are the homotopy classes of strict morphisms:

$$\mathrm{Hom}_{\mathrm{Ho}(\mathsf{C})}(M, N) := \mathrm{H}^0(\mathrm{Hom}_{\mathsf{C}}(M, N)).$$

One basic example of a DG category is $\mathbf{C}(A)$, the category of DG A-modules. By definition we take

$$\mathrm{Hom}_{\mathbf{C}(A)}(M, N) := \mathrm{Hom}_A(M, N),$$

the DG module from formula (11.2.1). We have special notation in this context:

$$\mathbf{C}(A)_{\mathrm{str}} := \mathbf{C}_{\mathrm{str}}(A)$$

and

$$\mathrm{Ho}(\mathbf{C}(A)) := \mathbf{K}(A).$$

Another basic example of a DG category is the category $\mathbf{C}(\mathsf{M})$ of complexes over an abelian category M. Its strict category is $\mathbf{C}_{\mathrm{str}}(\mathsf{M})$, and the morphisms here are what is classically called homomorphisms of complexes. The homotopy category is, as usual, denoted by $\mathbf{K}(\mathsf{M})$.

A useful innovation in this course is the merging of these last two types of DG categories into a single entity. Suppose A is a DG ring, and M is an abelian category. For a complex $M = \{M^i\}_{i\in\mathbb{Z}} \in \mathbf{C}(\mathsf{M})$, its set of endomorphisms

$$\mathrm{End}_{\mathsf{C}}(M) := \mathrm{Hom}_{\mathsf{C}}(M, M)$$

is a DG ring (central over $\mathbb{K}$). By definition, a DG A-module in M is a complex $M \in \mathbf{C}(\mathsf{M})$, together with a DG ring homomorphism $A \to \mathrm{End}_{\mathsf{C}}(M)$. There is an obvious (once contemplating this long enough...) notion of degree i A-linear morphism between two such DG modules. In this way we obtain the DG category $\mathbf{C}(A, \mathsf{M})$. Its strict and homotopy categories are $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ and $\mathbf{K}(A, \mathsf{M})$ respectively. Note that $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$ is (secretly) an abelian category.

Just to state the relationship: when $A = \mathbb{K}$ we get

$$\mathbf{C}(A, \mathsf{M}) = \mathbf{C}(\mathsf{M}),$$

and when $\mathsf{M} = \mathbf{M}(\mathbb{K}) = \mathsf{Mod}\,\mathbb{K}$, we get

$$\mathbf{C}(A, \mathsf{M}) = \mathbf{C}(A).$$

11.3. **Translations.** The category $\mathbf{G}(\mathsf{M})$ of graded objects of M has an automorphism called the translation. Given a graded object $M = \{M^i\}_{i\in\mathbb{Z}}$, its translation $\mathrm{T}(M)$ is the graded object whose degree i component is $\mathrm{T}(M)^i := M^{i+1}$. There is a canonical degree -1 morphism

$$\mathrm{t}_M : M \to \mathrm{T}(M)$$

in $\mathbf{G}(\mathsf{M})$, which is the identity after forgetting the grading. This is called the little t operator. Observe that t_M is an isomorphism in $\mathbf{G}(\mathsf{M})$; its inverse t_M^{-1} is of degree $+1$.

If $\phi : M \to N$ is a degree i morphism in $\mathbf{G}(\mathsf{M})$, we let

$$\mathrm{T}(\phi) : \mathrm{T}(M) \to \mathrm{T}(M)$$

be

$$\mathrm{T}(\phi) := \mathrm{t}_N \circ \phi \circ \mathrm{t}_M^{-1}.$$

In this way T is indeed an automorphism of the category $\mathbf{G}(\mathsf{M})$.

Now consider a complex $M \in \mathbf{C}(\mathsf{M})$. Its differential d_M is a degree 1 morphism in $\mathbf{G}(\mathsf{M})$, so we can define

$$\mathrm{d}_{\mathrm{T}(M)} := \mathrm{T}(\mathrm{d}_M),$$

and this is a differential on $\mathrm{T}(M)$.

All this works just as well for DG A-modules in M. We get a DG functor

$$\mathrm{T} : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(A, \mathsf{M}),$$

and it is an automorphism of this DG category. The little t operator is a degree -1 morphism of DG functors

$$\mathrm{t} : \mathrm{Id} \to \mathrm{T},$$

and it is a cocycle.

11.4. **Cones.** In the DG category $\mathbf{C}(A, \mathsf{M})$ there is an intrinsic notion of standard cone. Suppose $\phi : M \to N$ is a strict morphism in $\mathbf{C}(A, \mathsf{M})$. The standard cone of ϕ is the DG module

$$\mathrm{Cone}(\phi) := N \oplus \mathrm{T}(M) = \begin{bmatrix} N \\ \mathrm{T}(M) \end{bmatrix}, \tag{11.4.1}$$

in column notation. The differential $\mathrm{d}_{\mathrm{cone}}$ is the following matrix of degree 1 operators, acting on the column from the left:

$$\mathrm{d}_{\mathrm{cone}} : \begin{bmatrix} \mathrm{d}_N & \phi \circ \mathrm{t}_M^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}(M)} \end{bmatrix}.$$

The standard cone sits inside the standard triangle. This is the diagram

$$M \xrightarrow{\phi} N \xrightarrow{e_\phi} \mathrm{Cone}(\phi) \xrightarrow{p_\phi} \mathrm{T}(M) \tag{11.4.2}$$

in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$, where e_ϕ and p_ϕ are the obvious morphisms.

The standard cone, and also the standard triangle, are functorial in the strict morphism ϕ.

11.5. **DG Functors and Triangles.** We now review Section 4.

DG functors respect all the structure mentioned above. Let me explain. Suppose

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N})$$

is a DG functor. Let us denote by $\mathrm{T}_{A,\mathsf{M}}$ and $\mathrm{T}_{B,\mathsf{N}}$ the two translation functors. There is a strict isomorphism of DG functors

$$\tau_F : F \circ \mathrm{T}_{A,\mathsf{M}} \xrightarrow{\simeq} \mathrm{T}_{B,\mathsf{N}} \circ F \tag{11.5.1}$$

called the translation isomorphism. This is the formula, for any DG module $M \in \mathbf{C}(A, \mathsf{M})$:

$$\tau_{F,M} := \mathrm{t}_{F(M)} \circ F(\mathrm{t}_M)^{-1} : F(\mathrm{T}_{A,\mathsf{M}}(M)) \xrightarrow{\simeq} \mathrm{T}_{B,\mathsf{N}}(F(M)).$$

Next, suppose we are given a strict morphism $\phi : M_0 \to M_1$ in $\mathbf{C}(A, \mathsf{M})$. Then $F(\phi)$ is also a strict morphism. We can form the standard cones $\mathrm{Cone}_{A,\mathsf{M}}(\phi)$ and $\mathrm{Cone}_{B,\mathsf{N}}(F(\phi))$.

It turns out that there is a strict isomorphism

$$\mathrm{cone}(F, \phi) : F(\mathrm{Cone}_{A,\mathsf{M}}(\phi)) \xrightarrow{\simeq} \mathrm{Cone}_{B,\mathsf{N}}(F(\phi)) \tag{11.5.2}$$

in $\mathbf{C}(B, \mathsf{N})$, whose formula is

$$\operatorname{cone}(F, \phi) := \begin{bmatrix} \mathrm{id}_{F(M_1)} & 0 \\ 0 & \tau_{F,M_0} \end{bmatrix}.$$

The following diagram in $\mathbf{C}_{\mathrm{str}}(B, \mathsf{N})$ is commutative:

$$\tag{11.5.3} \begin{tikzcd} F(M_0) \arrow[r, "F(\phi)"] \arrow[d, "="'] & F(M_1) \arrow[r, "F(e_\phi)"] \arrow[d, "="'] & F(\mathrm{Cone}_{A,\mathsf{M}}(\phi)) \arrow[r, "F(p_\phi)"] \arrow[d, "\operatorname{cone}(F,\phi)"'] & F(\mathrm{T}_{A,\mathsf{M}}(M_0)) \arrow[d, "\tau_{F,M_0}"'] \\ F(M_0) \arrow[r, "F(\phi)"] & F(M_1) \arrow[r, "e_{F(\phi)}"] & \mathrm{Cone}_{B,\mathsf{N}}(F(\phi)) \arrow[r, "p_{F(\phi)}"] & \mathrm{T}_{B,\mathsf{N}}(F(M_0)) \end{tikzcd}$$

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11.6. **Pretriangulated Categories and Triangulated Functors.** This is a review of Section 5.

The translation isomorphism introduced above has an abstract version. This is the notion of a T-additive category, which consists of an additive category K, together with an additive automorphism T. Suppose (K, T) and $(\mathsf{K}', \mathrm{T}')$ are T-additive categories. A T-additive functor

$$(F, \tau) : (\mathsf{K}, \mathrm{T}) \to (\mathsf{K}', \mathrm{T}')$$

consists of an additive functor F with an isomorphism of functors

$$\tau : F \circ \mathrm{T} \xrightarrow{\simeq} \mathrm{T}' \circ F.$$

There is a rather obvious notion of composition of T-additive functors. See Definition 5.1.4.

Intrinsic to a T-additive category is the notion of triangle; it is a diagram like this:

$$\tag{11.6.1} L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}(L).$$

A pretriangulated category is a T-additive category (K, T), equipped with a set of triangles, called the distinguished triangles. The set of distinguished triangles must satisfy the following three axioms:

- (TR1) It is closed under isomorphisms; every morphism α sits inside a distinguished triangle like (11.6.1); and every object L sits inside a distinguished triangle with $\alpha = \mathrm{id}_L$ and $N = 0$.
- (TR2) Closure under turning.
- (TR3) Closure under extension (weak functoriality of the cone).

We are deliberately ignoring the octahedral axiom (TR4). This is because it is hard to understand, hard to prove, and unnecessary for our purposes. The "price" for ignoring it is that we only talk about pretriangulated categories – i.e. the prefix "pre" is added everywhere.

Suppose now (K, T) and $(\mathsf{K}', \mathrm{T}')$ are pretriangulated categories. A triangulated functor

$$(F, \tau) : (\mathsf{K}, \mathrm{T}) \to (\mathsf{K}', \mathrm{T}')$$

is a T-additive functor that respects distinguished triangles, in the following sense: for any distinguished triangle (11.6.1) in K, the triangle

$$F(L) \xrightarrow{F(\alpha)} F(M) \xrightarrow{F(\beta)} F(N) \xrightarrow{\tau_L \circ F(\gamma)} \mathrm{T}'(F((L))$$

in K' is distinguished. The composition of triangulated functors is their composition as T-additive functors.

There is a vast source of pretriangulated categories and triangulated functors. For any pair (A, M) the homotopy category $\mathsf{K} := \mathbf{K}(A, \mathsf{M})$ inherits the translation functor T from the DG category $\mathbf{C}(A, \mathsf{M})$, under the canonical full functor

$$\mathrm{P} : \mathbf{C}(A, \mathsf{M}) \to \mathbf{K}(A, \mathsf{M}).$$

By definition, the distinguished triangles in $\mathbf{K}(A, \mathsf{M})$ are those that are isomorphic to the images, under the functor P, of standard triangles. A calculation (Theorem 5.4.4) shows that they satisfy the axioms of pretriangulated category.

We proved (Theorem 5.4.13) that for any DG functor

$$F : \mathbf{C}(A, \mathsf{M}) \to \mathbf{C}(B, \mathsf{N}),$$

the induced T-additive functor

$$(F, \tau_F) : \mathbf{K}(A, \mathsf{M}) \to \mathbf{K}(B, \mathsf{N})$$

is triangulated.

Another source of triangulated functors is by composing other triangulated functors. This will turn out to be of tremendous importance. A mere shadow of this feature is the Grothendieck spectral sequence associated to a composition of functors.

11.7. **Localization of Categories.** Here we review Section 6.

Suppose K is a category, and S is a multiplicatively closed set of morphism in it (just like in a ring). There is always the formal localization of K with respect to S – this is a category K_{S}, with a functor

$$\mathrm{Q} : \mathsf{K} \to \mathsf{K}_{\mathsf{S}},$$

that is the identity on objects, it sends any morphism $s \in \mathsf{S}$ to an isomorphism, and it is initial among all such pairs $(\mathsf{K}_{\mathsf{S}}, \mathrm{Q})$.

The localization is manageable if it has a calculus of fractions, a.k.a. Ore localization. The set S is called a right denominator set if it satisfies the right Ore condition (R1) and the right cancellation condition (R2). We proved in full detail that S is a right denominator set iff $(\mathsf{K}_{\mathsf{S}}, \mathrm{Q})$ is a right Ore localization. The same is true on the left side.

11.8. **The Derived Category.** Now we recall Section 7. We know that the homotopy category $\mathbf{K}(A, \mathsf{M})$ is a pretriangulated category. A morphism $\psi : M \to N$ in $\mathbf{K}(A, \mathsf{M})$ is called a quasi-isomorphism if all the cohomologies

$$\mathrm{H}^i(\psi) : \mathrm{H}^i(M) \to \mathrm{H}^i(N)$$

are isomorphisms (in the category M). The set of quasi-isomorphisms is denoted by $\mathbf{S}(A, \mathsf{M})$.

We proved that $\mathbf{S}(A, \mathsf{M})$ is both a left and right denominator set. The derived category is the localization

$$\mathbf{D}(A, \mathsf{M}) := \mathbf{K}(A, \mathsf{M})_{\mathbf{S}(A, \mathsf{M})}.$$

It is a pretriangulated category, and the localization functor

$$\mathrm{Q} : \mathbf{K}(A, \mathsf{M}) \to \mathbf{D}(A, \mathsf{M})$$

is triangulated.

For a boundedness condition $\star$, that could be $+$, $-$ or b, we denote by $\mathbf{K}^{\star}(A, \mathsf{M})$ the full subcategory of $\mathbf{K}(A, \mathsf{M})$ on the DG modules with this condition. The localization of $\mathbf{K}^{\star}(A, \mathsf{M})$ w.r.t. its quasi-isomorphisms is $\mathbf{D}^{\star}(A, \mathsf{M})$. If the relevant truncation functor exists (this is always so for $\mathbf{K}(\mathsf{M})$), then the functor $\mathbf{K}^{\star}(A, \mathsf{M}) \to \mathbf{K}(A, \mathsf{M})$ is fully faithful.

As before, in the special cases we write $\mathbf{D}(A) := \mathbf{D}(A, \mathsf{Mod}\, \mathbb{K})$ and $\mathbf{D}(\mathsf{M}) := \mathbf{D}(\mathbb{K}, \mathsf{M})$. In this latter case the canonical functor $\mathsf{M} \to \mathbf{D}(\mathsf{M})$, that sends an object M to the complex M concentrated in degree 0, is fully faithful.

11.9. **Derived Functors.** This is a summary of Section 8. Since we want to treat $\mathbf{K}^{\star}(A, \mathsf{M})$ for various boundedness conditions $\star$, we now revert to the more general setting of a pretriangulated category K with a denominator set S of cohomological origin (like the quasi-isomorphisms in $\mathbf{K}(A, \mathsf{M})$).

Setup 11.9.1. The following are given:
- Pretriangulated categories K and E.
- A triangulated functor $F : \mathsf{K} \to \mathsf{E}$.
- A denominator set of cohomological origin $\mathsf{S} \subseteq \mathsf{K}$. The morphisms in it will be called quasi-isomorphisms.

Definition 11.9.2. A right derived functor of F is a pair $(\mathrm{R}F, \eta)$, where

$$\mathrm{R}F : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$$

is a triangulated functor, and

$$\eta : F \Rightarrow \mathrm{R}F \circ \mathrm{Q}$$

is a morphism of triangulated functors $\mathsf{K} \to \mathsf{E}$. The pair $(\mathrm{R}F, \eta)$ must have this universal property:

($\Diamond$) Given any pair (G, θ), consisting of a triangulated functor $G : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$ and a morphism of triangulated functors $\theta : F \Rightarrow G \circ \mathrm{Q}$, there is a unique morphism of triangulated functors $\mu : \mathrm{R}F \Rightarrow G$ such that $\theta = (\mu \circ \mathrm{id}_{\mathrm{Q}}) * \eta$.

Above we used a bit of 2-categorical notation. It is pictured in the following 2-diagrams:

$$\begin{tikzcd}
\mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathsf{E} \\
\mathsf{K}_{\mathsf{S}} \arrow[ur, "\mathrm{R}F"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[dr, phantom, "\Downarrow\eta \ \Downarrow\theta"] & \mathsf{E} \\
\mathsf{K}_{\mathsf{S}} \arrow[ur] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, bend right=20, "\Rightarrow\mu"]
\end{tikzcd}$$

It is quite easy to prove that a right derived functor is unique (up to a unique isomorphism).

Existence rests on the availability of suitable resolutions. Here is the theorem.

Theorem 11.9.3. *Assume there is a full pretriangulated subcategory* $\mathsf{J} \subseteq \mathsf{K}$ *with these two properties*:

(a) *If* $\phi : I \to I'$ *is a quasi-isomorphism in* J*, then* $F(\phi) : F(I) \to F(I')$ *is an isomorphism in* E*.*
(b) *Every object* $M \in \mathsf{K}$ *admits a quasi-isomorphism* $\rho : M \to I$ *to some object* $I \in \mathsf{J}$*.*

Then the right derived functor

$$(\mathrm{R}F, \eta) : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$$

exists. Moreover, for any object $I \in \mathsf{J}$ *the morphism*

$$\eta_I : F(I) \to (\mathrm{R}F \circ \mathrm{Q})(I)$$

in E *is an isomorphism.*

We refer to J as a category of right F-acyclic objects.
Analogously we can talk about left derived functors.

Definition 11.9.4. A *left derived functor* of F is a pair $(\mathrm{L}F, \eta)$, where

$$\mathrm{L}F : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$$

is a triangulated functor, and

$$\eta : \mathrm{L}F \circ \mathrm{Q} \Rightarrow F$$

is a morphism of triangulated functors $\mathsf{K} \to \mathsf{E}$. The pair $(\mathrm{L}F, \eta)$ must have a universal property opposite to the one in Definition 11.9.2.

As for the right derived functor, there is a uniqueness here. And existence relies on the availability of resolutions.

Theorem 11.9.5. *Assume there is a full pretriangulated subcategory* $\mathsf{P} \subseteq \mathsf{K}$ *with these two properties*:

(a) *If* $\phi : P \to P'$ *is a quasi-isomorphism in* P*, then* $F(\phi) : F(P) \to F(P')$ *is an isomorphism in* E*.*
(b) *Every object* $M \in \mathsf{K}$ *admits a quasi-isomorphism* $\rho : P \to M$ *from some object* $P \in \mathsf{P}$*.*

Then the right derived functor

$$(\mathrm{L}F, \eta) : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$$

exists. Moreover, for any object $P \in \mathsf{P}$ *the morphism*

$$\eta_P : (\mathrm{L}F \circ \mathrm{Q})(P) \to F(P)$$

in E *is an isomorphism.*

We refer to P as a category of left F-acyclic objects.

11.10. **Resolutions of DG Modules.** This is a review of Section 9. As we just saw, a sufficient condition for existence of derived functors (left or right) of F is the existence of enough acyclic objects.

In the original book [RD], existence of resolutions was proved for bounded (above or below) complexes, or when the additive functor F was finite dimensional (it was called "way-out" there).

At around 1990 several mathematicians discovered, independently, the secret to unbounded acyclic resolutions. It involves filtrations, and it goes by several names. We prefer the name "K-something resolution", following Spaltenstein.

As before, A is a DG ring and M is an abelian category. A DG module N is called acyclic if $\mathrm{H}^i(N) = 0$ for all i.

Definition 11.10.1. A DG module $I \in \mathbf{C}(A, \mathsf{M})$ is called *K-injective* if for every acyclic DG module $N \in \mathbf{C}(A, \mathsf{M})$, the DG $\mathbb{K}$-module $\mathrm{Hom}_{A,\mathsf{M}}(N, I)$ is acyclic.

It turns out that K-injectives are right F-acyclic for any triangulated functor F.

By K-injective resolution of a DG module M we mean a quasi-isomorphism $M \to I$ into a K-injective DG module I.

For a full pretriangulated subcategory $\mathsf{K} \subseteq \mathbf{K}(A, \mathsf{M})$, we denote by $\mathsf{K}_{\mathrm{inj}}$ the full subcategory of K on the K-injectives in it. It too is pretriangulated.

Theorem 11.10.2. *Let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$, and denote by S the set of quasi-isomorphisms in K. Assume K has enough K-injectives. Let E be any pretriangulated category, and let*

$$F : \mathsf{K} \to \mathsf{E}$$

be any triangulated functor. Then F has a right derived functor

$$(\mathrm{R}F, \eta) : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}\,.$$

Furthermore, for any $I \in \mathsf{K}_{\mathrm{inj}}$ the morphism $\eta_I : F(I) \to \mathrm{R}F(I)$ in E is an isomorphism.

There is a bonus, already proved in [RD] for $\mathbf{K}^{+}(\mathsf{M})$:

Theorem 11.10.3. *Let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$. Denote by S the set of quasi-isomorphisms in K. Then the localization functor*

$$\mathrm{Q} : \mathsf{K}_{\mathrm{inj}} \to \mathsf{K}_{\mathsf{S}}$$

is fully faithful.

Thus, if K has enough K-injectives, the functor Q above is an equivalence of pretriangulated categories.

There is a dual notion, generalizing projective resolutions.

Definition 11.10.4. A DG module $P \in \mathbf{C}(A, \mathsf{M})$ is called *K-projective* if for every acyclic DG module $N \in \mathbf{C}(A, \mathsf{M})$, the DG $\mathbb{K}$-module $\mathrm{Hom}_{A,\mathsf{M}}(P, N)$ is acyclic.

For a full pretriangulated subcategory $\mathsf{K} \subseteq \mathbf{K}(A, \mathsf{M})$, we denote by $\mathsf{K}_{\mathrm{prj}}$ the full subcategory of K on the K-projectives in it. It too is pretriangulated.

Theorem 11.10.5. *Let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$, and denote by S the set of quasi-isomorphisms in K. Assume K has enough K-projectives. Let E be any pretriangulated category, and let*

$$F : \mathsf{K} \to \mathsf{E}$$

be any triangulated functor. Then F has a left derived functor

$$(\mathrm{L}F, \eta) : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}\,.$$

Furthermore, for any $P \in \mathsf{K}_{\mathrm{prj}}$ the morphism $\eta_P : \mathrm{L}F(P) \to F(P)$ in E is an isomorphism.

Once more, for K-projectives there is no need to invert quasi-isomorphisms. This was known in [RD] for $\mathbf{K}^{-}(\mathsf{M})$:

Theorem 11.10.6. *Let K be a full pretriangulated subcategory of $\mathbf{K}(A, \mathsf{M})$. Denote by S the set of quasi-isomorphisms in K. Then the localization functor*

$$\mathrm{Q} : \mathsf{K}_{\mathrm{prj}} \to \mathsf{K}_{\mathsf{S}}$$

is fully faithful.

Thus, if K has enough K-projectives, the functor Q above is an equivalence of pretriangulated categories.

11.11. **Existence of Resolutions.** This is a review of Section 10. We consider four situations where we can prove existence of resolutions. Further situations will be considered later, in geometry.

First, a rephrasing of a semi-classical result from [RD].

Theorem 11.11.1. *If M is an abelian category with enough injectives, and if M is a complex in $\mathbf{C}(\mathsf{M})$ with bounded below cohomology, then M has a K-injective resolution $M \to I$ with $\inf(I) = \inf(\mathrm{H}(M))$.*

This implies:

Corollary 11.11.2. *If M is an abelian category with enough injectives, then $\mathbf{C}^{+}(\mathsf{M})$ has enough K-injectives.*

Next a more recent result (from around 1990).

Theorem 11.11.3. *Let A be any DG ring. The category $\mathbf{C}(A)$ has enough K-injectives.*

Here are two existence results for K-projective resolutions. First, a rephrasing of a semi-classical result from [RD].

Theorem 11.11.4. *If M is an abelian category with enough projectives, and if M is a complex in $\mathbf{C}(\mathsf{M})$ with bounded above cohomology, then M has a K-projective resolution $P \to M$ with $\sup(P) = \sup(\mathrm{H}(M))$.*

This implies:

Corollary 11.11.5. *If M is an abelian category with enough injectives, then $\mathbf{C}^{-}(\mathsf{M})$ has enough K-projectives.*

Finally a more recent result (from around 1990).

Theorem 11.11.6. *Let A be any DG ring. The category $\mathbf{C}(A)$ has enough K-projectives.*

There are notions of K-flat and K-flasque DG modules. We will talk about them in details when we study derived categories in geometry.

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12. Derived Bifunctors

In this section we extend the theory of derived functors to the setting of bifunctors, and study the important special cases of the Hom and tensor bifunctors.

12.1. DG Bifunctors.

We had already talked about bifunctors in Subsection 1.6. That was for categories without further structure. Here we will consider $\mathbb{K}$-linear DG categories, and matters become more complicated.

Definition 12.1.1. Let C_1, C_2 and D be $\mathbb{K}$-linear categories. A *$\mathbb{K}$-linear bifunctor*

$$F : \mathsf{C}_1 \times \mathsf{C}_2 \to \mathsf{D}$$

is a bifunctor such that for any objects $M_i, N_i \in \mathsf{C}_i$ the function

$$F : \operatorname{Hom}_{\mathsf{C}_1}(M_1, N_1) \times \operatorname{Hom}_{\mathsf{C}_2}(M_2, N_2) \to \operatorname{Hom}_{\mathsf{D}}\big(F(M_1, M_2), F(N_1, N_2)\big)$$

is $\mathbb{K}$-bilinear.

Thus, a linear functor F induces, for every quadruple of objects, a $\mathbb{K}$-linear homomorphism

(12.1.2)
$$F : \operatorname{Hom}_{\mathsf{C}_1}(M_1, N_1) \otimes_{\mathbb{K}} \operatorname{Hom}_{\mathsf{C}_2}(M_2, N_2) \to \operatorname{Hom}_{\mathsf{D}}\big(F(M_1, M_2), F(N_1, N_2)\big).$$

We now upgrade this operation to the DG level. In order to treat sign issues properly we make the next definition.

Definition 12.1.3. Let C_1 and C_2 be $\mathbb{K}$-linear DG categories. We define the DG category $\mathsf{C}_1 \otimes_{\mathbb{K}} \mathsf{C}_2$ as follows: the set of objects is

$$\operatorname{Ob}(\mathsf{C}_1 \otimes_{\mathbb{K}} \mathsf{C}_2) := \operatorname{Ob}(\mathsf{C}_1) \times \operatorname{Ob}(\mathsf{C}_2).$$

For any pair of objects

$$(M_1, M_2), (N_1, N_2) \in \operatorname{Ob}(\mathsf{C}_1 \otimes_{\mathbb{K}} \mathsf{C}_2),$$

i.e. $M_i, N_i \in \operatorname{Ob}(\mathsf{C}_i)$, we let

$$\operatorname{Hom}_{\mathsf{C}_1 \otimes_{\mathbb{K}} \mathsf{C}_2}\big((M_1, M_2), (N_1, N_2)\big) := \operatorname{Hom}_{\mathsf{C}_1}(M_1, N_1) \otimes_{\mathbb{K}} \operatorname{Hom}_{\mathsf{C}_2}(M_2, N_2).$$

The formula for the composition is this: given morphisms

$$\phi_i \in \operatorname{Hom}_{\mathsf{C}_i}(L_i, M_i)^{d_i}$$

and

$$\psi_i \in \operatorname{Hom}_{\mathsf{C}_i}(M_i, N_i)^{e_i}$$

for $i = 1, 2$, their tensors are morphisms

$$\phi_1 \otimes \phi_2 \in \operatorname{Hom}_{\mathsf{C}_1 \otimes_{\mathbb{K}} \mathsf{C}_2}\big((L_1, L_2), (M_1, M_2)\big)$$

and

$$\psi_1 \otimes \psi_2 \in \operatorname{Hom}_{\mathsf{C}_1 \otimes_{\mathbb{K}} \mathsf{C}_2}\big((M_1, M_2), (N_1, N_2)\big).$$

Any morphism in $\mathsf{C}_1 \otimes_{\mathbb{K}} \mathsf{C}_2$ is a sum of such tensors. We define the composition to be

$$\begin{aligned}(\psi_1 \otimes \psi_2) \circ (\phi_1 \otimes \phi_2) &:= (-1)^{d_1 \cdot e_2} \cdot (\psi_1 \circ \phi_1) \otimes (\psi_2 \circ \phi_2) \\ &\in \operatorname{Hom}_{\mathsf{C}_1 \otimes_{\mathbb{K}} \mathsf{C}_2}\big((L_1, L_2), (N_1, N_2)\big)^{d_1 + d_2 + e_1 + e_2}.\end{aligned}$$

Example 12.1.4. Suppose C_1 and C_2 are single-object $\mathbb{K}$-linear DG categories. Then $\mathsf{C}_1 \otimes_{\mathbb{K}} \mathsf{C}_2$ is also a single-object $\mathbb{K}$-linear DG category. Denoting this single object by $*$, as the topologists like to do, the endomorphism DG rings satisfy

$$(\mathsf{C}_1 \otimes_{\mathbb{K}} \mathsf{C}_2)(*) = \mathsf{C}_1(*) \otimes_{\mathbb{K}} \mathsf{C}_2(*).$$

See Examples 3.1.6 and 3.3.10.

DG functors between DG categories were introduced in Definition 3.5.1.

Definition 12.1.5. Let C_1, C_2 and D be $\mathbb{K}$-linear DG categories. A *$\mathbb{K}$-linear DG bifunctor*

$$F : \mathsf{C}_1 \times \mathsf{C}_2 \to \mathsf{D}$$

is, by definition, a $\mathbb{K}$-linear DG functor

$$F : \mathsf{C}_1 \otimes_{\mathbb{K}} \mathsf{C}_2 \to \mathsf{D},$$

where $\mathsf{C}_1 \otimes_{\mathbb{K}} \mathsf{C}_2$ is the DG category from Definition 12.1.3.

Warning: due to the signs that odd morphisms acquire, a DG bifunctor F is not a $\mathbb{K}$-linear bifunctor in the sense of Definition 12.1.1. Still, the induced functors on the strict subcategories

$$\operatorname{Str}(F) : \operatorname{Str}(\mathsf{C}_1) \times \operatorname{Str}(\mathsf{C}_2) \to \operatorname{Str}(\mathsf{D})$$

and on the homotopy categories

$$\operatorname{Ho}(F) : \operatorname{Ho}(\mathsf{C}_1) \times \operatorname{Ho}(\mathsf{C}_2) \to \operatorname{Ho}(\mathsf{D})$$

are genuine $\mathbb{K}$-linear bifunctors.

cmnt: Definition 12.1.6 and ?? belong in Section 3

We need to talk about contravariant DG functors.

Definition 12.1.6. Let C and D be DG categories. A *contravariant $\mathbb{K}$-linear DG functor*

$$F : \mathsf{C} \to \mathsf{D}$$

is, by definition, a $\mathbb{K}$-linear DG functor

$$F : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}\,.$$

Here C^{op} is the DG category from Definition 3.4.8.

To make things explicit, a contravariant DG functor F amounts to a function

$$F : \operatorname{Ob}(\mathsf{C}) \to \operatorname{Ob}(\mathsf{D}),$$

together with a strict homomorphism of DG $\mathbb{K}$-modules

$$F : \operatorname{Hom}_{\mathsf{C}}(M, N) \to \operatorname{Hom}_{\mathsf{D}}\big(F(N), F(M)\big)$$

for and pair of objects M, N, such that for any morphisms $\phi \in \operatorname{Hom}_{\mathsf{C}}(L, M)^d$ and $\psi \in \operatorname{Hom}_{\mathsf{C}}(M, N)^e$ there is equality

$$F(\psi \circ \phi) = (-1)^{d \cdot e} \cdot F(\phi) \circ F(\psi) \in \operatorname{Hom}_{\mathsf{D}}\big(F(N), F(L)\big)^{d+e}.$$

And of course $F(\mathrm{id}_M) = \mathrm{id}_{F(M)}$. Once more, such F is not a genuine contravariant functor (because of the signs), but it induces genuine contravariant functors between the strict categories and between the homotopy categories.

Example 12.1.7. Let C be a DG category. The canonical operation $\operatorname{op} : \mathsf{C}^{\mathrm{op}} \to \mathsf{C}$ is a contravariant DG functor.

The definitions above tell us what is a DG bifunctor that is contravariant in the first or the second argument. They also tell us how to treat compositions of contravariant DG functors or bifunctors. And they tell us what are morphisms between contravariant DG functors and between DG bifunctors. The rule is always to write the opposite category in the first argument whenever there is a contravariance, and that puts us in the covariant situation.

Here are the two main examples of DG bifunctors. We give each of them in the commutative version and the noncommutative version (which is very confusing!).

Example 12.1.8. Consider a commutative ring A. The category of complexes of A-modules is the DG category $\mathbf{C}(A)$, and we take $\mathsf{C}_1 = \mathsf{C}_2 = \mathsf{D} := \mathbf{C}(A)$. For any pair of objects $M_1, M_2 \in \mathbf{C}(A)$ there is an object

$$F(M_1, M_2) := M_1 \otimes_A M_2 \in \mathbf{C}(A).$$

This is the usual tensor product of complexes. We define the action of F on morphisms as follows: given

$$\phi_i \in \operatorname{Hom}_{\mathbf{C}(A)}(M_i, N_i)^{k_i} = \operatorname{Hom}_A(M_i, N_i)^{k_i},$$

we let

$$\begin{aligned} F(\phi_1, \phi_2) := \phi_1 \otimes \phi_2 &\in \operatorname{Hom}_A\big(M_1 \otimes_A M_2, N_1 \otimes_A N_2\big)^{k_1+k_2} \\ &= \operatorname{Hom}_{\mathbf{C}(A)}\big(F(M_1, M_2), F(N_1, N_2)\big)^{k_1+k_2}. \end{aligned}$$

The result is a DG bifunctor

$$F : \mathbf{C}(A) \times \mathbf{C}(A) \to \mathbf{C}(A).$$

Example 12.1.9. Consider DG rings A_0, A_1, A_2 (possibly noncommutative, but $\mathbb{K}$-central). Let us define the new DG rings $B_i := A_{i-1} \otimes_{\mathbb{K}} A_i^{\mathrm{op}}$ for $i = 1, 2$. There are corresponding DG categories $\mathsf{C}_i := \mathbf{C}(B_i)$. An object of C_i is just a DG A_{i-1}-A_i-bimodule. Let us also define the DG ring $C := A_0 \otimes_{\mathbb{K}} A_2^{\mathrm{op}}$ and the DG category $\mathsf{D} := \mathbf{C}(C)$. For any pair of objects $M_1 \in \mathsf{C}_1$ and $M_2 \in \mathsf{C}_2$ there is a DG $\mathbb{K}$-module

$$F(M_1, M_2) := M_1 \otimes_{A_1} M_2;$$

see Definition 3.3.19. This has a canonical DG C-module structure:

$$(a_0 \otimes a_2) \cdot (m_1 \otimes m_2) := (-1)^{j_2 \cdot (k_1+k_2)} \cdot (a_0 \cdot m_1) \otimes (m_2 \cdot a_2)$$

for elements $a_i \in A_i^{j_i}$ and $m_i \in M_i^{k_i}$. In this way $F(M_1, M_2)$ becomes an object of D. We define the action of F on morphisms as follows: given

$$\phi_i \in \operatorname{Hom}_{\mathsf{C}_i}(M_i, N_i)^{k_i} = \operatorname{Hom}_{B_i}(M_i, N_i)^{k_i},$$

we let

$$F(\phi_1, \phi_2) := \phi_1 \otimes \phi_2 \in \operatorname{Hom}_{\mathsf{D}}\big(F(M_1, M_2), F(N_1, N_2)\big)^{k_1+k_2}.$$

The result is a DG bifunctor

$$F : \mathsf{C}_1 \times \mathsf{C}_2 \to \mathsf{D}\,.$$

Compare this example to the one-sided construction in Example 4.4.2.

Example 12.1.10. Again we take a commutative ring A, but now our bifunctor F arises from Hom, and so there is contravariance in the first argument. In order to rectify this we work with the opposite category in the first argument. (A certain amount of confusion is unavoidable here!) So we define the DG categories $\mathsf{C}_1 := \mathbf{C}(A)^{\mathrm{op}}$ and $\mathsf{C}_2 = \mathsf{D} := \mathbf{C}(A)$. For any pair of objects $M_1, M_2 \in \mathbf{C}(A)$ there is an object

$$F(M_1, M_2) := \mathrm{Hom}_A(M_1, M_2) \in \mathbf{C}(A).$$

This is the usual Hom complex. We define the action of F on morphisms as follows: given

$$\phi_1 \in \mathrm{Hom}_{\mathsf{C}_1}(M_1, N_1)^{k_1} = \mathrm{Hom}_{\mathbf{C}(A)^{\mathrm{op}}}(M_1, N_1)^{k_1} = \mathrm{Hom}_A(N_1, M_1)^{k_1}$$

and

$$\phi_2 \in \mathrm{Hom}_{\mathsf{C}_2}(M_2, N_2)^{k_2} = \mathrm{Hom}_{\mathbf{C}(A)}(M_2, N_2)^{k_2} = \mathrm{Hom}_A(M_2, N_2)^{k_2}$$

we let

$$\begin{aligned} F(\phi_1, \phi_2) &:= \mathrm{Hom}(\phi_1, \phi_2) \in \mathrm{Hom}_A\big(\mathrm{Hom}_A(M_1, M_2), \mathrm{Hom}_A(N_1, N_2)\big)^{k_1+k_2} \\ &= \mathrm{Hom}_{\mathsf{D}}\big(F(M_1, M_2), F(N_1, N_2)\big)^{k_1+k_2}. \end{aligned}$$

The result is a DG bifunctor

$$F : \mathsf{C}_1 \times \mathsf{C}_2 \to \mathsf{D}\,.$$

Example 12.1.11. Consider DG rings A, A_1, A_2 (possibly noncommutative, but $\mathbb{K}$-central). There is DG bifunctor

$$F := \mathrm{Hom}_A(-,-) : \mathbf{C}(A \otimes_{\mathbb{K}} A_1^{\mathrm{op}})^{\mathrm{op}} \times \mathbf{C}(A \otimes_{\mathbb{K}} A_2^{\mathrm{op}}) \to \mathbf{C}(A_1 \otimes_{\mathbb{K}} A_2^{\mathrm{op}}).$$

The details here are so confusing that we just leave them out. (We will come back to this is Section 17, when discussing noncommutative dualizing complexes).

12.2. **Triangulated Bifunctors.** Recall the notions of T-additive category and pretriangulated category, from Section 5.

Suppose Let $(\mathsf{K}_1, \mathrm{T}_1)$ and $(\mathsf{K}_2, \mathrm{T}_2)$ are T-additive categories (linear over $\mathbb{K}$). There are two induced translation automorphism of the category $\mathsf{K}_1 \times \mathsf{K}_2$:

$$\mathrm{T}_1(M_1, M_2) := \big(\mathrm{T}_1(M_1), M_2\big)$$

and

$$\mathrm{T}_2(M_1, M_2) := \big(M_1, \mathrm{T}_2(M_2)\big)$$

These two functors commute: $\mathrm{T}_2 \circ \mathrm{T}_1 = \mathrm{T}_1 \circ \mathrm{T}_2$.

Definition 12.2.1. Let $(\mathsf{K}_1, \mathrm{T}_1)$, $(\mathsf{K}_2, \mathrm{T}_2)$ and (L, T) be T-additive categories. A *T-additive bifunctor*

$$(F, \tau_1, \tau_2) : (\mathsf{K}_1, \mathrm{T}_1) \times (\mathsf{K}_2, \mathrm{T}_2) \to (\mathsf{L}, \mathrm{T})$$

is made up of an additive bifunctor

$$F : \mathsf{K}_1 \times \mathsf{K}_2 \to \mathsf{L},$$

as in Definition 12.1.1, together with isomorphisms

$$\tau_i : F \circ \mathrm{T}_i \xrightarrow{\simeq} \mathrm{T} \circ F$$

of bifunctors $\mathsf{K}_1 \times \mathsf{K}_2 \to \mathsf{L}$. The condition is that

$$\tau_1 \circ \tau_2 = -\tau_2 \circ \tau_1,$$

as isomorphism

$$F \circ \mathrm{T}_2 \circ \mathrm{T}_1 = F \circ \mathrm{T}_1 \circ \mathrm{T}_2 \xrightarrow{\simeq} \mathrm{T} \circ \mathrm{T} \circ F.$$

In the next exercises we let the reader establish several operations on T-additive bifunctors.

Exercise 12.2.2. In the situation of Definition 12.2.1, suppose

$$(G, \tau) : (\mathsf{L}, \mathrm{T}) \to (\mathsf{L}', \mathrm{T}')$$

is a T-additive functor into a fourth T-additive category $(\mathsf{L}', \mathrm{T}')$. Write the explicit formula for the T-additive bifunctor

$$(G, \tau) \circ (F, \tau_1, \tau_2) : (\mathsf{K}_1, \mathrm{T}_1) \times (\mathsf{K}_2, \mathrm{T}_2) \to (\mathsf{L}', \mathrm{T}').$$

This should be compared to Definition 5.1.4.

Exercise 12.2.3. In the situation of Definition 12.2.1, suppose

$$(F', \tau_1', \tau_2') : (\mathsf{K}_1, \mathrm{T}_1) \times (\mathsf{K}_2, \mathrm{T}_2) \to (\mathsf{L}, \mathrm{T})$$

is another T-additive bifunctor. Write the definition of a morphism of T-additive bifunctors

$$\eta : (F, \tau_1, \tau_2) \to (F', \tau_1', \tau_2').$$

Use Definition 5.1.4 as a template.

Exercise 12.2.4. Give a definition of a T-additive trifunctor. Show that if F and G are T-additive bifunctors, then $G(-, F(-, -))$ and $G(F(-, -), -)$ are T-additive trifunctors (whenever these compositions makes sense).

We now move to pretriangulated categories.

Definition 12.2.5. Let $(\mathsf{K}_1, \mathrm{T}_1)$, $(\mathsf{K}_2, \mathrm{T}_2)$ and (L, T) be pretriangulated categories. A *triangulated bifunctor*

$$(F, \tau_1, \tau_2) : (\mathsf{K}_1, \mathrm{T}_1) \times (\mathsf{K}_2, \mathrm{T}_2) \to (\mathsf{L}, \mathrm{T})$$

is a T-additive bifunctor that respects the pretriangulated structure in each argument. Namely, for any distinguished triangle

$$L_1 \xrightarrow{\alpha_1} M_1 \xrightarrow{\beta_1} N_1 \xrightarrow{\gamma_1} \mathrm{T}_1(L_1)$$

in K_1, and any object $L_2 \in \mathsf{K}_2$, the triangle

$$F(L_1, L_2) \xrightarrow{F(\alpha_1, \mathrm{id})} F(M_1, L_2) \xrightarrow{F(\beta_1, \mathrm{id})} F(N_1, L_2) \xrightarrow{\tau_1 \circ F(\gamma_1, \mathrm{id})} \mathrm{T}(F(L_1, L_2))$$

in L is distinguished; and the same for distinguished triangles in the second argument.

The operations on triangulated bifunctors are the same as those on T-additive bifunctors (see exercises above).

We now connect DG bifunctors and triangulated bifunctors in our favorite setup: DG modules in abelian categories.

Setup 12.2.6. We are given central DG $\mathbb{K}$-rings A_1, A_2, B, $\mathbb{K}$-linear abelian categories $\mathsf{M}_1, \mathsf{M}_2, \mathsf{N}$, and a $\mathbb{K}$-linear DG bifunctor

$$F : \mathbf{C}(A_1, \mathsf{M}_1) \times \mathbf{C}(A_2, \mathsf{M}_2) \to \mathbf{C}(B, \mathsf{N})$$

(Definition 12.1.5).

For any pair of objects (M_1, M_2), with $M_i \in \mathsf{C}(A_i, \mathsf{M}_i)$, there are isomorphisms

$$\tau_{i,M_1,M_2} : F(\mathrm{T}_i(M_1, M_2)) \xrightarrow{\simeq} \mathrm{T}(F(M_1, M_2)) \tag{12.2.7}$$

in $\mathsf{C}(B, \mathsf{N})$, arising from Definition 4.2.1. Let us make it explicit (only for $i = 2$, since the case $i = 1$ is so similar). Fixing the object M_1 we obtain a DG functor

$$G : \mathsf{C}(A_2, \mathsf{M}_2) \to \mathsf{C}(B, \mathsf{N}), \quad G(M_2) := F(M_1, M_2).$$

The isomorphism

$$\tau_{2,M_1,M_2} : G(\mathrm{T}_2(M_2)) \xrightarrow{\simeq} \mathrm{T}(G(M_2))$$

is then

$$\tau_{2,M_1,M_2} = \mathrm{t}_{G(M_2)} \circ G(\mathrm{t}_{M_2})^{-1}.$$

Lemma 12.2.8. *Fix $i \in \{1, 2\}$. Letting the pairs of objects vary, we get an isomorphism*

$$\tau_i : F \circ \mathrm{T}_i \xrightarrow{\simeq} \mathrm{T} \circ F$$

of additive bifunctors

$$\mathsf{C}_{\mathrm{str}}(A_1, \mathsf{M}_1) \times \mathsf{C}_{\mathrm{str}}(A_2, \mathsf{M}_2) \to \mathsf{C}_{\mathrm{str}}(B, \mathsf{N}).$$

Proof. This is an almost immediate consequence of the fact that the little t operators are morphisms of functors (see Theorem 3.8.7(2)), □

These pass to the homotopy categories.

Theorem 12.2.9. *Under Setup* 12.2.6*, the data*

$$(F, \tau_1, \tau_2) : \mathsf{K}(A_1, \mathsf{M}_1) \times \mathsf{K}(A_2, \mathsf{M}_2) \to \mathsf{K}(B, \mathsf{N})$$

is a triangulated bifunctor.

Proof. The only challenge is to prove that (F, τ_1, τ_2) is a T-additive bifunctor; and in that, all we have to prove is that

$$\tau_1 \circ \tau_2 = -\tau_2 \circ \tau_1. \tag{12.2.10}$$

The rest hinges on single-argument considerations, that are handled in Theorems 4.2.3 and 5.4.13.

So let us prove (12.2.10). Choose a pair of objects (M_1, M_2). We have the diagram

(12.2.11)

$$\begin{tikzcd}
& F(\mathrm{T}_1(M_1), \mathrm{T}_2(M_2)) \arrow[dl, "F(\mathrm{id}, \mathrm{t}_{M_2}^{-1})"'] \arrow[dr, "F(\mathrm{t}_{M_1}^{-1}, \mathrm{id})"] & \\
F(\mathrm{T}_1(M_1), M_2) \arrow[d, "\mathrm{t}_{F(\mathrm{T}_1(M_1), M_2)}"'] \arrow[dr, "F(\mathrm{t}_{M_1}^{-1}, \mathrm{id})"] & & F(M_1, \mathrm{T}_2(M_2)) \arrow[dl, "F(\mathrm{id}, \mathrm{t}_{M_2}^{-1})"'] \arrow[d, "\mathrm{t}_{F(M_1, \mathrm{T}_2(M_2))}"] \\
\mathrm{T}(F(\mathrm{T}_1(M_1), M_2)) \arrow[d, "\mathrm{T}(F(\mathrm{t}_{M_1}^{-1}, \mathrm{id}))"'] & F(M_1, M_2) \arrow[dl, "\mathrm{t}_{F(M_1, M_2)}"'] \arrow[dr, "\mathrm{t}_{F(M_1, M_2)}"] & \mathrm{T}(F(M_1, \mathrm{T}_2(M_2))) \arrow[d, "\mathrm{T}(F(\mathrm{id}, \mathrm{t}_{M_2}^{-1}))"] \\
\mathrm{T}(F(M_1, M_2)) \arrow[dr, "\mathrm{T}(\mathrm{t}_{F(M_1, M_2)})"] & & \mathrm{T}(F(M_1, M_2)) \arrow[dl, "\mathrm{T}(\mathrm{t}_{F(M_1, M_2)})"'] \\
& \mathrm{T}(\mathrm{T}(F(M_1, M_2))) &
\end{tikzcd}$$

in $\mathsf{C}(B, \mathsf{N})$. Going from top to bottom on the left edge is the morphism $\tau_1 \circ \tau_2$, and going on the right edge is the morphism $\tau_2 \circ \tau_1$. The bottom diamond is trivially commutative. The two triangles, with common vertex at $F(M_1, M_2)$, are (-1)-commutative, because $\mathrm{t} : \mathrm{Id} \to \mathrm{T}$ is a degree -1 morphism of DG functors. Since they occur on both sides, these signs cancel each other. Finally, the top diamond is (-1)-commutative, because

$$(\mathrm{t}_{M_1}^{-1}, \mathrm{id}) \circ (\mathrm{id}, \mathrm{t}_{M_2}^{-1}) = (\mathrm{t}_{M_1}^{-1}, \mathrm{t}_{M_2}^{-1}) = -(\mathrm{id}, \mathrm{t}_{M_2}^{-1}) \circ (\mathrm{t}_{M_1}^{-1}, \mathrm{id}).$$

□

cmnt: $\boxed{\text{The material below should be moved to Section 5}}$

We now address the contravariant case. Let K be a pretriangulated category. In Proposition 5.2.4 we explained how to make the opposite category K^{op} pretriangulated. This is used in the next two definitions.

Definition 12.2.12. Suppose K and L are pretriangulated categories. A *contravariant triangulated functor* $F : \mathsf{K} \to \mathsf{L}$ is, by definition, a triangulated functor $F : \mathsf{K}^{\mathrm{op}} \to \mathsf{L}$.

Let us provide an explicit formula. For this we need to bring in the translation functors T_{K} and T_{L}, and the translation isomorphism τ. Using Proposition 5.2.4 we see that the triangulated property of F is this: for any distinguished triangle

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \mathrm{T}_{\mathsf{K}}(L)$$

in K, the triangle

$$F(N) \xrightarrow{F(\beta)} F(M) \xrightarrow{F(\alpha)} F(L) \xrightarrow{\tau_N \circ F(-\mathrm{T}_{\mathsf{K}}^{-1}(\gamma))} \mathrm{T}_{\mathsf{L}}(F(N))$$

is a distinguished triangle in L.

For bifunctors there are several options for contravariance.

Definition 12.2.13. Let K_1, K_2 and L be pretriangulated categories. A *triangulated bifunctor that is contravariant in the first or the second argument* is, by definition, a triangulated bifunctor

$$F : \mathsf{K}_1^{\diamondsuit_1} \times \mathsf{K}_2^{\diamondsuit_2} \to \mathsf{L}$$

as in Definition 12.2.5, where the symbols $\diamondsuit_1$ and $\diamondsuit_2$ are either empty or op, as the case may be.

This is nice and clean at first, until we try to employ Theorem 12.2.9 – because we still don't know anything useful about the pretriangulated category $\mathsf{C}(A, \mathsf{M})^{\mathrm{op}}$. This is our next task.

cmnt: $\boxed{\text{following stuff should be moved to an earlier section}}$

Lemma 12.2.14. *Let A be a DG ring and M an abelian category. There is a canonical isomorphism of DG categories*

$$G : \mathsf{C}(A, \mathsf{M})^{\mathrm{op}} \xrightarrow{\simeq} \mathsf{C}(A^{\mathrm{op}}, \mathsf{M}^{\mathrm{op}}).$$

Proof. In [KaSc1, Remark 1.8.11] there is an explicit formula for an isomorphism of categories $G : \mathbf{C}(\mathsf{M})^{\mathrm{op}} \xrightarrow{\simeq} \mathbf{C}(\mathsf{M}^{\mathrm{op}})$. It goes like this. For a complex $M = \{M^i\}_{i \in \mathbb{Z}} \in \mathbf{C}(\mathsf{M})$ they define the complex

$$G(M) = \{G(M)^i\}_{i \in \mathbb{Z}} \in \mathbf{C}(\mathsf{M}^{\mathrm{op}})$$

to have components $G(M)^i := \mathrm{op}(M^{-i})$. The differential $\mathrm{d}_{G(M)} = \{\mathrm{d}^i_{G(M)}\}$ is as follows. The morphism

$$\mathrm{d}^i_{G(M)} : G(M)^i \to G(M)^{i+1}$$

is

$$(-1)^{-i-1} \cdot \mathrm{op}(\mathrm{d}_M^{-i-1}) : \mathrm{op}(M^{-i}) \to \mathrm{op}(M^{-i-1}).$$

It was not mentioned in [KaSc1], but G is in fact an isomorphism of DG categories (i.e. a DG functor that is an isomorphism).

cmnt: this needs to be verified !

For any object $M \in \mathbf{C}(\mathsf{M})$, its endomorphism DG ring in $\mathbf{C}(\mathsf{M})^{\mathrm{op}} \cong \mathbf{C}(\mathsf{M}^{\mathrm{op}})$ is the opposite of its endomorphism DG ring in $\mathbf{C}(\mathsf{M})$. Hence there is a DG ring homomorphism from A^{op} to it. This makes $G(M)$ into a DG A^{op}-module in M^{op}. Lastly we need to check that this A^{op}-module structure is functorial – But that is straightforward. □

cmnt: to here lecture 16 Nov 2016

Remark 12.2.15. Unlike what one might be tempted to think, the lemma above does not say that $\mathbf{C}(A)^{\mathrm{op}}$, the opposite DG category of the category of DG A-modules $\mathbf{C}(A)$, is equivalent to the category $\mathbf{C}(A^{\mathrm{op}})$ of right DG A-modules. What it does say is that

$$\mathbf{C}(A)^{\mathrm{op}} = \mathbf{C}(A, \mathsf{Mod}\,\mathbb{K})^{\mathrm{op}} \cong \mathbf{C}(A^{\mathrm{op}}, (\mathsf{Mod}\,\mathbb{K})^{\mathrm{op}}).$$

On the other hand,

$$\mathbf{C}(A^{\mathrm{op}}) = \mathbf{C}(A^{\mathrm{op}}, \mathsf{Mod}\,\mathbb{K}).$$

But there is never (except for the trivial ring $\mathbb{K}$) an equivalence between $(\mathsf{Mod}\,\mathbb{K})^{\mathrm{op}}$ and $\mathsf{Mod}\,\mathbb{K}$.

Since the homotopy category of $\mathbf{C}(A, \mathsf{M})^{\mathrm{op}}$ is $\mathbf{K}(A, \mathsf{M})^{\mathrm{op}}$, the lemma above gives rise to an isomorphism of additive categories

$$\bar{G} : \mathbf{K}(A, \mathsf{M})^{\mathrm{op}} \to \mathbf{K}(A^{\mathrm{op}}, \mathsf{M}^{\mathrm{op}}). \tag{12.2.16}$$

Now $\mathbf{K}(A, \mathsf{M})^{\mathrm{op}}$ is a pretriangulated category, by virtue of being the opposite of the pretriangulated category $\mathbf{K}(A, \mathsf{M})$. And $\mathbf{K}(A^{\mathrm{op}}, \mathsf{M}^{\mathrm{op}})$ is a pretriangulated category on its own.

Lemma 12.2.17. *There is an isomorphism of additive functors*

$$\tau : \bar{G} \circ \mathrm{T}_{\mathbf{K}(A,\mathsf{M})^{\mathrm{op}}} \xrightarrow{\simeq} \mathrm{T}_{\mathbf{K}(A^{\mathrm{op}},\mathsf{M}^{\mathrm{op}})} \circ \bar{G}$$

such that

$$(\bar{G}, \tau) : \mathbf{K}(A, \mathsf{M})^{\mathrm{op}} \to \mathbf{K}(A^{\mathrm{op}}, \mathsf{M}^{\mathrm{op}}).$$

is a triangulated functor.

Proof.

cmnt: I hope it is true. Needs a proof!

□

Corollary 12.2.18. *Let*

$$F : \mathbf{C}(A_1, \mathsf{M}_1)^{\diamondsuit_1} \times \mathbf{C}(A_2, \mathsf{M}_2)^{\diamondsuit_1} \to \mathbf{C}(B, \mathsf{N})$$

be a DG bifunctor, where symbols $\diamondsuit_1$ and $\diamondsuit_2$ are either empty or op. *Then the induced bifunctor on the homotopy categories*

$$F : \mathbf{K}(A_1, \mathsf{M}_1)^{\diamondsuit_1} \times \mathbf{K}(A_2, \mathsf{M}_2)^{\diamondsuit_1} \to \mathbf{K}(B, \mathsf{N})$$

is a triangulated bifunctor

Proof. Using Lemma 12.2.14 we can get rid of the symbols $\diamondsuit_i$. Then we apply Theorem 12.2.9 to get a triangulated bifunctor, including the data of translation isomorphisms τ_1 and τ_2. Finally we use Lemma 12.2.17 to re-insert the symbols $\diamondsuit_i$. □

12.3. **Right Derived Bifunctors.** We now tackle localized categories. Here, for the sake of simplicity, we shall mostly ignore the translation functors (enough was said about them in the previous subsection).

Setup 12.3.1. The following are given:

(1) Pretriangulated categories K_1, K_2 and E.
(2) A triangulated bifunctor $F : \mathsf{K}_1 \times \mathsf{K}_2 \to \mathsf{E}$.
(3) Denominator sets of cohomological origin $\mathsf{S}_1 \subseteq \mathsf{K}_1$ and $\mathsf{S}_2 \subseteq \mathsf{K}_2$.

cmnt: merge setup with next def?

The morphisms in S_i, for $i = 1, 2$, are referred to as quasi-isomorphisms. The localized category $\mathsf{D}_i := (\mathsf{K}_i)_{\mathsf{S}_i}$ is pretriangulated, and the localization functor $\mathrm{Q}_i : \mathsf{K}_i \to \mathsf{D}_i$ is triangulated. On the product categories we get a functor

$$\mathrm{Q}_1 \times \mathrm{Q}_2 : \mathsf{K}_1 \times \mathsf{K}_2 \to \mathsf{D}_1 \times \mathsf{D}_2 \,.$$

In the next definition we use the 2-categorical notation from Subsection 8.1.

Definition 12.3.2. Under Setup 12.3.1, a *right derived bifunctor* of F is a pair $(\mathrm{R}F, \eta)$, where

$$\mathrm{R}F : \mathsf{D}_1 \times \mathsf{D}_2 \to \mathsf{E}$$

is a triangulated bifunctor, and

$$\eta : F \Rightarrow \mathrm{R}F \circ (\mathrm{Q}_1 \times \mathrm{Q}_2)$$

is a morphism of triangulated bifunctors, such that the following universal property holds:

(R) Given any pair (G, θ), consisting of a triangulated bifunctor

$$G : \mathsf{D}_1 \times \mathsf{D}_2 \to \mathsf{E}$$

and a morphism of triangulated bifunctors $\theta : F \Rightarrow G \circ (\mathrm{Q}_1 \times \mathrm{Q}_2)$, there is a unique morphism of triangulated functors $\mu : \mathrm{R}F \Rightarrow G$ such that $\theta = (\mu \circ \mathrm{id}_{\mathrm{Q}_1 \times \mathrm{Q}_2}) * \eta$.

Here is a diagram showing property (R):.

$$
\begin{tikzcd}
\mathsf{K}_1 \times \mathsf{K}_2 \arrow[r, "F"] \arrow[d, "\mathrm{Q}_1 \times \mathrm{Q}_2"'] \arrow[dr, phantom, "\Downarrow \eta"] & \mathsf{E} \\
\mathsf{D}_1 \times \mathsf{D}_2 \arrow[ur, "\mathrm{R}F"] \arrow[ur, bend right, "G"'] \arrow[r, phantom, "\Downarrow \mu \quad \Downarrow \theta"] &
\end{tikzcd}
\tag{12.3.3}
$$

Proposition 12.3.4. *If a right derived bifunctor exists, then it is unique up to a unique isomorphism.*

Proof. This is just like the proof of Proposition 8.3.2. We leave the small changes up to the reader. □

Existence in general is like Theorem 8.3.3, but more complicated.

Definition 12.3.5. Let K be a pretriangulated category, let $\mathsf{S} \subseteq \mathsf{K}$ be a denominator set of cohomological origin, and let $\mathsf{J} \subseteq \mathsf{K}$ be a full pretriangulated subcategory. We refer to the morphisms in S as quasi-isomorphisms.

(1) Let $M \in \mathsf{K}$. A *right* J*-resolution* of M is a quasi-isomorphism $\rho : M \to I$ to an object $I \in \mathsf{J}$.

(2) We say that K *has enough right* J*-resolutions* if every object $M \in \mathsf{K}$ admits a right J-resolution.

cmnt: $\boxed{\text{this def should be moved to Sec 8}}$

Theorem 12.3.6. *Under Setup* 12.3.1*,*

cmnt*:* $\boxed{\textit{change wording - no setup?}}$

assume there are full pretriangulated subcategories $\mathsf{J}_1 \subseteq \mathsf{K}_1$ *and* $\mathsf{J}_2 \subseteq \mathsf{K}_2$ *with these two properties*:

(a) *Acyclicity*: *if* $\phi_1 : I_1 \to J_1$ *is a quasi-isomorphism in* J_1 *and* $\phi_2 : I_2 \to J_2$ *is a quasi-isomorphism in* J_2*, then*

$$F(\phi_1, \phi_2) : F(I_1, I_2) \to F(J_1, J_2)$$

is an isomorphism in E*.*

(b) *Abundance*: K_1 *has enough right* J_1*-resolutions, and* K_2 *has enough right* J_2*-resolutions.*

Then the right derived bifunctor

$$(\mathrm{R}F, \eta) : \mathsf{D}_1 \times \mathsf{D}_2 \to \mathsf{E}$$

exists. Moreover, for any objects $I_1 \in \mathsf{J}_1$ *and* $I_2 \in \mathsf{J}_2$ *the morphism*

$$\eta_{I_1, I_2} : F(I_1, I_2) \to \mathrm{R}F(I_1, I_2)$$

in E *is an isomorphism.*

In applications we will see that either $\mathsf{J}_1 = \mathsf{K}_1$ or $\mathsf{J}_2 = \mathsf{K}_2$; namely we will only need to resolve in the second or in the first argument, respectively.

The proof of the theorem requires some more work on 2-categorical material. We will therefore interrupt our discussion, and return to the proof of Theorem 12.3.6 in Subsection 12.5.

12.4. **Abstract Derived Functors.**

cmnt: this subsec should be moved to Sec 6, just after Subsec 6.2 ?

cmnt: This subsection, and possibly also subsection 8.1, should be moved to Section 6, just after Subsec 6.2.

Here we deal with right and left derived functors in an abstract setup (as opposed to the triangulated setup).

We first introduce *functor categories*; these will extend our understanding of 2-categorical ideas. All set theoretical issues (sizes of sets) are neglected; the justification is in Subsection 1.1.

Definition 12.4.1. Given categories C and D, let $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ be the category whose objects are the functors $F : \mathsf{C} \to \mathsf{E}$, and the morphisms are the morphisms of functors $\eta : F \to F'$, i.e. the natural transformations.

Remark 12.4.2. In the full-fledged 2-category framework, there is the 2-category $\mathbf{Cat}$. Its objects are the categories. The 1-morphisms are the functors, and the 2-morphisms are the morphisms between functors. Thus using the categories $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ we can talk about part of the structure of $\mathbf{Cat}$, without having to worry about the whole 2-category story.

Suppose $G : \mathsf{C}' \to \mathsf{C}$ and $H : \mathsf{D} \to \mathsf{D}'$ are functors. There is an induced functor

$$\mathsf{F}(G, H) : \mathsf{Fun}(\mathsf{C}, \mathsf{D}) \to \mathsf{Fun}(\mathsf{C}', \mathsf{D}') \tag{12.4.3}$$

defined by $\mathsf{F}(G, H)(F) := H \circ F \circ G$.

Proposition 12.4.4. *If G and H are equivalences, then the functor $\mathsf{F}(G, H)$ in* (12.4.3) *is an equivalence.*

Exercise 12.4.5. Prove Proposition 12.4.4.

Recall that for a category C and a multiplicatively closed set of morphisms $\mathsf{S} \subseteq \mathsf{C}$ we denote by C_S the localization. It comes with the localization functor $\mathrm{Q} : \mathsf{C} \to \mathsf{C}_\mathsf{S}$. See Definition 6.1.2.

For a category E let $\mathsf{E}^\times \subseteq \mathsf{E}$ be the category of isomorphisms; it has all the objects, but its morphisms are just the isomorphisms in E.

Definition 12.4.6. Given categories C and E, a multiplicatively closed set of morphisms $\mathsf{S} \subseteq \mathsf{C}$, and a functor $F : \mathsf{C} \to \mathsf{E}$, we say that F *is localizable to* S if $F(\mathsf{S}) \subseteq \mathsf{E}^\times$. We denote by $\mathsf{Fun}_\mathsf{S}(\mathsf{C}, \mathsf{E})$ the full subcategory of $\mathsf{Fun}(\mathsf{C}, \mathsf{E})$ on the localizable functors.

Here is a useful formulation of the universal property of localization. Recall that a functor is an isomorphism of categories iff it is an equivalence that is bijective on sets of objects.

Proposition 12.4.7. *Let C and E be categories, and let $\mathsf{S} \subseteq \mathsf{C}$ be a multiplicatively closed set of morphisms. Then the functor*

$$\mathsf{F}(\mathrm{Q}, \mathrm{Id}_{\mathsf{E}}) : \mathsf{Fun}(\mathsf{C}_{\mathsf{S}}, \mathsf{E}) \to \mathsf{Fun}_{\mathsf{S}}(\mathsf{C}, \mathsf{E})$$

is an isomorphism of categories.

Exercise 12.4.8. Prove Proposition 12.4.7.

By definition a bifunctor $F : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$ is a functor from the product category $\mathsf{C} \times \mathsf{D}$. See Subsection 1.6. It will be useful to retain both meanings; so we shall write

$$\mathsf{BiFun}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) := \mathsf{Fun}(\mathsf{C} \times \mathsf{D}, \mathsf{E}), \tag{12.4.9}$$

where in the first expression we recall that $\mathsf{C} \times \mathsf{D}$ is a product.

The next proposition describes bifunctors in a non-symmetric fashion.

Proposition 12.4.10. *Let C, D and E be categories. There is an isomorphism of categories*

$$\Xi : \mathsf{Fun}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) \to \mathsf{Fun}(\mathsf{C}, \mathsf{Fun}(\mathsf{D}, \mathsf{E}))$$

with the following formula: for a functor $F : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$, the functor

$$\Xi(F) : \mathsf{C} \to \mathsf{Fun}(\mathsf{D}, \mathsf{E}))$$

is $\Xi(F)(C) := F(C, -)$.

Exercise 12.4.11. Prove Proposition 12.4.10.

Proposition 12.4.12. *Let C and D be categories, and let $\mathsf{S} \subseteq \mathsf{C}$ and $\mathsf{T} \subseteq \mathsf{D}$ be multiplicatively closed sets of morphisms. Then the canonical functor*

$$\Theta : (\mathsf{C} \times \mathsf{D})_{\mathsf{S} \times \mathsf{T}} \to \mathsf{C}_{\mathsf{S}} \times \mathsf{D}_{\mathsf{T}}$$

is an isomorphism of categories.

Proof. The functor Θ is the identity on objects. Thus Θ is an equivalence iff it is an isomorphism. We will produce a functor

$$G : \mathsf{C}_{\mathsf{S}} \times \mathsf{D}_{\mathsf{T}} \to (\mathsf{C} \times \mathsf{D})_{\mathsf{S} \times \mathsf{T}}$$

that is inverse to Θ.

Consider another category E. Invoking Propositions 12.4.10 and 12.4.7 we get a sequence of isomorphisms of categories

$$\mathsf{Fun}(\mathsf{C}_{\mathsf{S}} \times \mathsf{D}_{\mathsf{T}}, \mathsf{E}) \to \mathsf{Fun}(\mathsf{C}_{\mathsf{S}}, \mathsf{Fun}(\mathsf{D}_{\mathsf{T}}, \mathsf{E})) \to \mathsf{Fun}_{\mathsf{S}}(\mathsf{C}, \mathsf{Fun}_{\mathsf{T}}(\mathsf{D}, \mathsf{E})).$$

A short examination shows that the isomorphism Ξ restricts to an isomorphism on the full subcategories

$$\Xi : \mathsf{Fun}_{\mathsf{S} \times \mathsf{T}}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) \to \mathsf{Fun}_{\mathsf{S}}(\mathsf{C}, \mathsf{Fun}_{\mathsf{T}}(\mathsf{D}, \mathsf{E})).$$

Thus we get a commutative diagram of categories

$$\begin{tikzcd}
\mathsf{Fun}(\mathsf{C}_{\mathsf{S}} \times \mathsf{D}_{\mathsf{T}}, \mathsf{E}) \arrow[r] \arrow[dr] & \mathsf{Fun}_{\mathsf{S} \times \mathsf{T}}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) \arrow[d] & \mathsf{Fun}((\mathsf{C} \times \mathsf{D})_{\mathsf{S} \times \mathsf{T}}, \mathsf{E}) \arrow[l] \arrow[dl] \\
& \mathsf{Fun}(\mathsf{C} \times \mathsf{D}, \mathsf{E}) &
\end{tikzcd} \tag{12.4.13}$$

in which the horizontal arrows are isomorphisms of categories.

Now we take $\mathsf{E} := (\mathsf{C} \times \mathsf{D})_{\mathsf{S} \times \mathsf{T}}$, and look at the identity functor Id_{E} as an object in the rightmost category in diagram (12.4.13). There is a unique object G in the leftmost category. It is the inverse of Θ we are looking for. □

Denominator sets were introduced in Definition 6.2.14.

Proposition 12.4.14. *In the situation of Proposition* 12.4.12*, the following conditions are equivalent*:

(i) *The multiplicatively closed sets* $\mathsf{S} \subseteq \mathsf{C}$ *and* $\mathsf{T} \subseteq \mathsf{D}$ *are left* (*resp. right*) *denominator sets.*

(i) *The multiplicatively closed set* $\mathsf{S} \times \mathsf{T} \subseteq \mathsf{C} \times \mathsf{D}$ *is a left* (*resp. right*) *denominator set.*

Exercise 12.4.15. Prove Proposition 12.4.14.

Exercise 12.4.16. Assume the categories C, D and E are $\mathbb{K}$-linear. Let's denote by $\mathsf{AdFun}(\mathsf{C}, \mathsf{D})$ the category of $\mathbb{K}$-linear functors $F : \mathsf{C} \to \mathsf{D}$, and by $\mathsf{AdBiFun}(\mathsf{C} \times \mathsf{D}, \mathsf{E})$ the category of $\mathbb{K}$-linear bifunctors $F : \mathsf{C} \times \mathsf{D} \to \mathsf{E}$. Give linear versions of Propositions 12.4.4, 12.4.7, 12.4.10 and 12.4.12.

cmnt: to here lecture 23 Nov 2016

cmnt: There is a mistake in the proof of Thm 8.3.3. The problem: Lemma 8.3.13. Use Thm 12.4.20 instead.

Definition 12.4.17. Consider a category K and a multiplicatively closed set of morphisms $\mathsf{S} \subseteq \mathsf{K}$, with localization functor $\mathrm{Q} : \mathsf{K} \to \mathsf{K}_{\mathsf{S}}$. Let $F : \mathsf{K} \to \mathsf{E}$ be a functor. A *right derived functor* of F with respect to S is a pair $(\mathrm{R}F, \eta)$, where

$$\mathrm{R}F : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$$

is a functor, and

$$\eta : F \Rightarrow \mathrm{R}F \circ \mathrm{Q}$$

is a morphism of functors, such that the following universal property holds:

(R) Given any pair (G, θ), consisting of a functor $G : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$ and a morphism of functors $\theta : F \Rightarrow G \circ \mathrm{Q}$, there is a unique morphism of functors $\mu : \mathrm{R}F \Rightarrow G$ such that $\theta = (\mu \circ \mathrm{id}_{\mathrm{Q}}) * \eta$.

Here is a 2-diagram showing property (R):

(12.4.18)
$$\begin{tikzcd}
\mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[r, phantom, shift right=4ex, "\Downarrow \eta \quad \Downarrow \theta"] & \mathsf{E} \\
\mathsf{K}_{\mathsf{S}} \arrow[ur, "\mathrm{R}F"] \arrow[ur, bend right=40, "G"'] \arrow[ur, phantom, bend right=20, "\Downarrow \mu"] &
\end{tikzcd}$$

Proposition 12.4.19. *If a right derived functor* $(\mathrm{R}F, \eta)$ *exists, then it is unique, up to a unique isomorphism. Namely, if* (G, θ) *is another right derived functor of* F*, then there is a unique isomorphism of functors* $\mu : \mathrm{R}F \xRightarrow{\simeq} G$ *such that* $\theta = (\mu \circ \mathrm{id}_{\mathrm{Q}}) * \eta$.

Proof. Despite the apparent complication of the situation, the usual argument for uniqueness of universals (here it is a universal 1-morphism) applies. It shows that the morphism μ from condition (R) is an isomorphism. □

Here is a rather general existence result.

Theorem 12.4.20. *In the situation of Definition* 12.4.17*, assume there is a full subcategory* $\mathsf{J} \subseteq \mathsf{K}$ *such the following three conditions hold*:

(a) *The multiplicatively closed set* S *is a left denominator set in* K*.*
(b) *For every object* $M \in \mathsf{K}$ *there is a morphism* $\rho : M \to I$ *in* S*, with target* $I \in \mathsf{J}$*.*
(c) *If* ψ *is a morphism in* $\mathsf{S} \cap \mathsf{J}$*, then* $F(\psi)$ *is an isomorphism in* E*.*

Then the right derived functor

$$(\mathrm{R}F, \eta) : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$$

exists. Moreover, for any object $I \in \mathsf{J}$ *the morphism*

$$\eta_I : F(I) \to \mathrm{R}F(I)$$

in E *is an isomorphism.*

This same result is [KaSc2, Proposition 7.3.2]. However their notation is different: what we call "left denominator set", they call "right multiplicative system".

We need a definition and a few lemmas before giving the proof of the theorem.

Definition 12.4.21. In the situation of Theorem 12.4.20, by a *system of right* J*-resolutions* we mean a pair (I, ρ), where $I : \mathrm{Ob}(\mathsf{K}) \to \mathrm{Ob}(\mathsf{J})$ is a function, and $\rho = \{\rho_M\}_{M \in \mathrm{Ob}(\mathsf{K})}$ is a collection of morphisms $\rho_M : M \to I(M)$ in S. Moreover, if $M \in \mathrm{Ob}(\mathsf{J})$, then $I(M) = M$ and $\rho_M = \mathrm{id}_M$.

Property (b) of Theorem 12.4.20 guarantees that a system of right J-resolutions (I, ρ) exists.

Let us introduce some new notation that will make the proofs more readable:

$$\mathsf{K}' := \mathsf{J}, \quad \mathsf{S}' := \mathsf{J} \cap \mathsf{S}, \quad \mathsf{D} := \mathsf{K}_{\mathsf{S}} \quad \text{and} \quad \mathsf{D}' := \mathsf{K}'_{\mathsf{S}'} . \tag{12.4.22}$$

The inclusion functor is $U : \mathsf{K}' \to \mathsf{K}$, and its localization is $V : \mathsf{D}' \to \mathsf{D}$. These sit in a commutative diagram

$$\begin{tikzcd} \mathsf{K}' \arrow[r, "U"] \arrow[d, "\mathrm{Q}'"'] & \mathsf{K} \arrow[d, "\mathrm{Q}"] \\ \mathsf{D}' \arrow[r, "V"] & \mathsf{D} \end{tikzcd} \tag{12.4.23}$$

Lemma 12.4.24. *The multiplicatively closed set* S' *is a left denominator set in* K'*.*

Proof. We need to verify conditions (LD1) and (LD2) in Definition 6.2.14.

(LD1): Given morphisms $a' : L' \to N'$ in K' and $s' : L' \to M'$ in S', we must find morphisms $b' : M' \to K'$ in K' and $t' : N' \to K'$ in S', such that $t' \circ a' = b' \circ s'$. Because $\mathsf{S} \subseteq \mathsf{K}$ satisfies this condition, we can find morphisms $b : M' \to K$ in K and $t : N' \to K$ in S such that $t \circ a' = b \circ s'$. There is a morphism $\rho : K \to K'$ in S with target $K' \in \mathsf{K}'$. Then the morphisms $t' := \rho \circ t$ and $b' := \rho \circ b$ satisfy $t' \circ a' = b' \circ s'$, and $t' \in \mathsf{S}'$.

(LD2): Given morphisms $a', b' : M' \to N'$ in K' and $s' : L' \to M'$ in S', that satisfy $a' \circ s' = b' \circ s'$, we must find a morphism $t' : N' \to K'$ in S' such that $t' \circ a' = t' \circ b'$. Because $\mathsf{S} \subseteq \mathsf{K}$ satisfies this condition, we can find a morphism $t : N' \to K$ in S such that $t \circ a' = t \circ b'$. There is a morphism $\rho : K \to K'$ in S with target $K' \in \mathsf{K}'$. Then the morphism $t' := \rho \circ t$ has the required property. □

Lemma 12.4.25. *The the functor $V : \mathsf{D}' \to \mathsf{D}$ is an equivalence.*

Proof. This is the same as the proof of Proposition 7.2.5 with condition (r).

cmnt: In Proposition 7.2.5 the labels (r) and (l) have to be flipped. (l) should go with "left denominator"... After the flipping, above has to be "with condition (l)".

□

Lemma 12.4.26. *Suppose a system of right K'-resolutions (I, ρ) has been chosen. Then the function $I : \mathrm{Ob}(\mathsf{K}) \to \mathrm{Ob}(\mathsf{K}')$ extends uniquely to a functor $I : \mathsf{D} \to \mathsf{D}'$, such that $I \circ V = \mathrm{Id}_{\mathsf{D}'}$, and $\rho : \mathrm{Id}_{\mathsf{D}} \Rightarrow V \circ I$ is an isomorphism of functors. Therefore the functor I is a a quasi-inverse of V.*

The relevant 2-diagram is this:

$$\begin{tikzcd}
\mathsf{K}' \arrow[r, "\mathrm{Q}'"] \arrow[d, "U"'] & \mathsf{D}' \arrow[r, "\mathrm{Id}"] \arrow[dr, "V"] & \mathsf{D}' \\
\mathsf{K} \arrow[r, "\mathrm{Q}"] & \mathsf{D} \arrow[u, "I"] \arrow[r, "\mathrm{Id}"'] \arrow[r, phantom, bend left, "\Uparrow \rho"] & \mathsf{D} \arrow[u, "I"']
\end{tikzcd}$$

Recall that in a 2-diagram, an empty polygon means it is commutative, namely it can be filled with $\xRightarrow{\mathrm{id}}$.

Proof. Consider a morphism $\psi : M \to N$ in D. Since $V : \mathsf{D}' \to \mathsf{D}$ is an equivalence, and since $V(I(M)) = I(M)$ and $V(I(N)) = I(N)$, there is a unique morphism

$$I(\psi) : I(M) \to I(N)$$

in D' satisfying

$$V(I(\psi)) := \mathrm{Q}(\rho_N) \circ \psi \circ \mathrm{Q}(\rho_M)^{-1}. \tag{12.4.27}$$

in D.

Let us check that $I : \mathsf{D} \to \mathsf{D}'$ is really a functor. Suppose $\phi : L \to M$ and $\psi : M \to N$ are morphisms in D. Then

$$\begin{aligned}
V\big(I(\psi) \circ I(\phi)\big) &= V(I(\psi)) \circ V(I(\phi)) \\
&= \big(\mathrm{Q}(\rho_N) \circ \psi \circ \mathrm{Q}(\rho_M)^{-1}\big) \circ \big(\mathrm{Q}(\rho_M) \circ \phi \circ \mathrm{Q}(\rho_L)^{-1}\big) \\
&= \mathrm{Q}(\rho_N) \circ (\psi \circ \phi) \circ \mathrm{Q}(\rho_L)^{-1} \\
&= V\big(I(\psi \circ \phi)\big).
\end{aligned}$$

It follows that $I(\psi) \circ I(\phi) = I(\psi \circ \phi)$.

Because $\rho_{M'} : M' \to I(M')$ is the identity for any object $M' \in \mathsf{K}'$, we see that there is equality $I \circ V = \mathrm{Id}_{\mathsf{D}'}$. By the defining formula (12.4.27) of $I(\psi)$ we have a commutative diagram

$$\begin{tikzcd} V(I(M)) \arrow[r, "V(I(\psi))"] & V(I(M)) \\ M \arrow[u, "\mathrm{Q}(\rho_M)"] \arrow[r, "\psi"] & N \arrow[u, "\mathrm{Q}(\rho_N)"'] \end{tikzcd}$$

in D. Hence $\rho : \mathrm{Id}_{\mathsf{D}} \Rightarrow V \circ I$ is an isomorphism of functors. $\square$

Proof of Theorem 12.4.20. Diagram (12.4.23) induces a commutative diagram of categories:

(12.4.28)
$$\begin{tikzcd} \mathsf{Fun}(\mathsf{K}', \mathsf{E}) & \mathsf{Fun}(\mathsf{K}, \mathsf{E}) \arrow[l, "\mathsf{F}(U,\mathrm{Id})"'] \\ \mathsf{Fun}_{\mathsf{S}'}(\mathsf{K}', \mathsf{E}) \arrow[u, "\text{f.f. inc}"'] & \mathsf{Fun}_{\mathsf{S}}(\mathsf{K}, \mathsf{E}) \arrow[l, "\mathsf{F}(U,\mathrm{Id})"', "\text{equiv}"] \arrow[u, "\text{f.f. inc}"] \\ \mathsf{Fun}(\mathsf{D}', \mathsf{E}) \arrow[u, "\mathsf{F}(\mathrm{Q}',\mathrm{Id})", "\text{isom}"'] & \mathsf{Fun}(\mathsf{D}, \mathsf{E}) \arrow[l, "\mathsf{F}(V,\mathrm{Id})"', "\text{equiv}"] \arrow[u, "\text{isom}", "\mathsf{F}(\mathrm{Q},\mathrm{Id})"'] \end{tikzcd}$$

The vertical arrows marked "f.f. incl" are fully faithful inclusions by definition. According to Proposition 12.4.7 the vertical arrows marked "isom" are isomorphisms of categories. And by Lemma 12.4.25 the arrow $\mathsf{F}(V, \mathrm{Id})$ is an equivalence. As a consequence, the arrow $\mathsf{F}(U, \mathrm{Id})$ is also an equivalence.

Step 1. We are given a functor F that is an object of the category in the upper right corner of diagram (12.4.28). Let $F' := F \circ U$; it lives in the the upper left corner of the diagram. But condition (c) says that F' actually belongs to the middle left term in diagram (12.4.28). Because the arrow $\mathsf{F}(\mathrm{Q}', \mathrm{Id})$ is an isomorphism, there is a unique functor $\mathrm{R}F'$ that is an object of the category in the bottom left of diagram (12.4.28). It satisfies $\mathrm{R}F' \circ \mathrm{Q}' = F'$. See next commutative diagram.

(12.4.29)
$$\begin{tikzcd} \mathsf{K}' \arrow[r, "F'"] \arrow[d, "\mathrm{Q}'"'] & \mathsf{E} \\ \mathsf{D}' \arrow[ur, "\mathrm{R}F'"'] & \end{tikzcd}$$

Let $\eta' := \mathrm{id}_{F'}$. We claim that the pair $(\mathrm{R}F', \eta')$ is a right derived functor of F'. Indeed, suppose we are given a pair (G', θ'), where G' is a functor in the bottom left corner of diagram (12.4.28), and $\theta' : F' \Rightarrow G' \circ \mathrm{Q}'$ is a morphism in the top corner of that diagram. See the 2-diagram (12.4.31). Because the function

$$\mathrm{Hom}_{\mathsf{Fun}(\mathsf{D}',\mathsf{E})}(\mathrm{R}F', G') \to \mathrm{Hom}_{\mathsf{Fun}(\mathsf{K}',\mathsf{E})}(F', G' \circ \mathrm{Q}') \tag{12.4.30}$$

is bijective – this is the left edge of diagram (12.4.28) – there is a unique morphism $\mu' : \mathrm{R}F' \Rightarrow G'$ that goes to θ' under (12.4.30).

(12.4.31)
$$\begin{tikzcd}
\mathsf{K}' \arrow[r, "F'"] \arrow[d, "\mathrm{Q}'"'] \arrow[r, phantom, shift right=4ex, "\Downarrow\eta'"] & \mathsf{E} \\
\mathsf{D}' \arrow[ur, "\mathrm{R}F'"] \arrow[ur, bend right, "G'"'] \arrow[ur, phantom, bend right=15, "\Downarrow\mu'"] &
\end{tikzcd}$$
θ'

Step 2. Now we choose a system of right K'-resolutions (I, ρ), in the sense of Definition 12.4.21. By Lemma 12.4.26 we get an equivalence of categories $I : \mathsf{D} \to \mathsf{D}'$, that is a quasi-inverse to V, and an isomorphism of functors $\rho : \mathrm{Id}_{\mathsf{D}} \xrightarrow{\simeq} V \circ I$. See the following 2-diagram (the solid arrows).

(12.4.32)
$$\begin{tikzcd}
\mathsf{K}' \arrow[r, "\mathrm{Q}'"] \arrow[d, "U"'] & \mathsf{D}' \arrow[r, "\mathrm{Id}"] \arrow[dr, "V"] & \mathsf{D}' \arrow[r, "\mathrm{R}F'"] & \mathsf{E} \\
\mathsf{K} \arrow[r, "\mathrm{Q}"'] & \mathsf{D} \arrow[u, "I"] \arrow[r, "\mathrm{Id}"'] \arrow[r, phantom, shift left=3ex, "\Uparrow\rho"] & \mathsf{D} \arrow[u, "I"'] \arrow[ur, dashed, "\mathrm{R}F"'] &
\end{tikzcd}$$

Define the functor

(12.4.33)
$$\mathrm{R}F := \mathrm{R}F' \circ I : \mathsf{D} \to \mathsf{E}\,.$$

It is the dashed arrow in diagram (12.4.32). So the functor $\mathrm{R}F$ lives in the bottom right corner of (12.4.28), and $\mathrm{R}F' = \mathrm{R}F \circ V$.

Step 3. We will now produce a morphism of functors $\eta : F \Rightarrow \mathrm{R}F \circ \mathrm{Q}$. This morphism should live in the category upper right corner of diagram (12.4.28).

Take an object $M \in \mathsf{K}$. There is a morphism $\rho_M : M \to I(M)$ in S, and the target $I(M)$ is an object of K'. Define the morphism

(12.4.34)
$$\eta_M := F(\rho_M) : F(M) \to F(I(M)) = \mathrm{R}F(M)$$

in E. We must prove that the collection of morphisms $\eta = \{\eta_M\}_{M \in \mathsf{K}}$ is a morphism of functors (i.e. a natural transformation). Suppose $\phi : M \to N$ is a morphism in K. We have to show that the diagram

(12.4.35)
$$\begin{tikzcd}
F(M) \arrow[r, "F(\phi)"] \arrow[d, "\eta_M"'] & F(N) \arrow[d, "\eta_N"] \\
\mathrm{R}F(M) \arrow[r, "\mathrm{R}F(\mathrm{Q}(\phi))"] & \mathrm{R}F(N)
\end{tikzcd}$$

in E is commutative.

Now by definition of $\mathrm{R}F$ there is a commutative diagram

(12.4.36)
$$\begin{tikzcd}
\mathrm{R}F(M) \arrow[r, "\mathrm{R}F(\mathrm{Q}(\phi))"] \arrow[d, "="'] & \mathrm{R}F(N) \arrow[d, "="] \\
\mathrm{R}F'(I(M)) \arrow[r, "\mathrm{R}F'(I(\mathrm{Q}(\phi)))"] & \mathrm{R}F'(I(N))
\end{tikzcd}$$

in E. Lemma 12.4.24 tells us that the morphism $I(\mathrm{Q}(\phi))$ in D' can be written as a left fraction

$$I(\mathrm{Q}(\phi)) = \mathrm{Q}'(\psi_1)^{-1} \circ \mathrm{Q}'(\psi_0)$$

of morphisms $\psi_0 \in \mathsf{K}'$ and $\psi_1 \in \mathsf{S}'$. We get a diagram

$$\begin{tikzcd}
M \arrow[rr, "\phi"] \arrow[d, "\rho_M"'] & & N \arrow[d, "\rho_N"] \\
I(M) \arrow[rr, dashed, "I(\mathrm{Q}(\phi))"] \arrow[dr, "\psi_0"'] & & I(N) \arrow[dl, "\psi_1"] \\
& J &
\end{tikzcd} \tag{12.4.37}$$

where the solid arrows are in the category K, the dashed arrow is in D', and the object J belongs to K'. This diagram might fail to be commutative; but after applying Q to it, it becomes a commutative diagram in D. By condition (LO4) of the left Ore localization $\mathsf{Q} : \mathsf{K} \to \mathsf{D}$, there is a morphism $\psi : J \to L$ in S such that

$$\psi \circ \psi_0 \circ \rho_M = \psi \circ \psi_1 \circ \rho_N \circ \phi$$

in K. There is the morphism $\rho_L : L \to I(L)$ in S, whose target $I(L)$ belongs to K'. Thus, after replacing the object J with $I(L)$, the morphism ψ_0 by $\rho_L \circ \psi \circ \psi_0$, and the morphism ψ_1 by $\rho_L \circ \psi \circ \psi_1$, and noting that the latter is a morphism in S', we can now assume that the solid diagram (12.4.37) in K is commutative.

Applying the functor F to the solid commutative diagram (12.4.37) we obtain the solid commutative diagram

$$\begin{tikzcd}
F(M) \arrow[rr, "F(\phi)"] \arrow[d, "F(\rho_M)"'] & & F(N) \arrow[d, "F(\rho_N)"] \\
F'(I(M)) \arrow[rr, dashed, "\mathrm{R}F'(I(\mathrm{Q}(\phi)))"] \arrow[dr, "F'(\psi_0)"'] & & F'(I(N)) \arrow[dl, "F'(\psi_1)"] \\
& F'(J) &
\end{tikzcd} \tag{12.4.38}$$

in E. But the morphism $F'(\psi_1)$ is an isomorphism in E; and

$$\mathrm{R}F'(I(\mathrm{Q}(\phi))) = F'(\psi_1)^{-1} \circ F'(\psi_0)$$

in E. It follows that the top square in (12.4.38) is commutative. Therefore, making use of the commutative diagram (12.4.36), we conclude that diagram (12.4.35) is commutative. So the proof that η is a natural transformation is done.

Step 4. It remains to prove that the pair $(\mathrm{R}F, \eta)$ is a right derived functor of F. Suppose (G, θ) is a pair, where G is a functor in the category in bottom right corner of diagram (12.4.28), and $\theta : F \Rightarrow G \circ \mathrm{Q}$ is a morphism in the top right corner of the diagram. We are looking for a morphism $\mu : \mathrm{R}F \Rightarrow G$ in the bottom right category in diagram (12.4.28) for which $\theta = (\mu \circ \mathrm{id}_{\mathrm{Q}}) * \eta$. Let $G' := G \circ V$, and let $\theta' : F' \Rightarrow G' \circ \mathrm{Q}'$ be the morphism in the top left corner of (12.4.28) corresponding to θ. Because of the equivalence $\mathsf{F}(V, \mathrm{Id})$, finding such μ is the same as finding a morphism $\mu' : \mathrm{R}F' \Rightarrow G'$ in the bottom left category in diagram (12.4.28), satisfying

$$\theta' = (\mu' \circ \mathrm{id}_{\mathrm{Q}'}) * \eta'. \tag{12.4.39}$$

Finally, by step 1 the pair $(\mathrm{R}F', \eta')$ is a right derived functor of F'. This says that there is a unique morphism μ' satisfying (12.4.39). □

Now to left derived functors.

Definition 12.4.40. Consider a category K and a multiplicatively closed set of morphisms $\mathsf{S} \subseteq \mathsf{K}$, with localization functor $\mathrm{Q} : \mathsf{K} \to \mathsf{K}_{\mathsf{S}}$. Let $F : \mathsf{K} \to \mathsf{E}$ be a functor. A *left derived functor* of F with respect to S is a pair $(\mathrm{L}F, \eta)$, where

$$\mathrm{L}F : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$$

is a functor, and

$$\eta : \mathrm{L}F \circ \mathrm{Q} \Rightarrow F$$

is a morphism of functors, such that the following universal property holds:

(L) Given any pair (G, θ), consisting of a functor $G : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$ and a morphism of functors $\theta : G \circ \mathrm{Q} \Rightarrow F$, there is a unique morphism of functors $\mu : G \Rightarrow \mathrm{L}F$ such that $\theta = \eta * (\mu \circ \mathrm{id}_{\mathrm{Q}})$.

Here it is in a 2-diagram:

$$\begin{tikzcd}
\mathsf{K} \arrow[r, "F"] \arrow[d, "\mathrm{Q}"'] \arrow[dr, phantom, "\Uparrow\eta" near end] & \mathsf{E} \\
\mathsf{K}_{\mathsf{S}} \arrow[ur, "\mathrm{L}F"] \arrow[ur, bend right, "G"'] \arrow[ur, phantom, bend right=20, "\Uparrow\mu"] &
\end{tikzcd}$$

Proposition 12.4.41. *If a left derived functor* $(\mathrm{L}F, \eta)$ *exists, then it is unique, up to a unique isomorphism. Namely, if* (G, θ) *is another right derived functor of* F*, then there is a unique isomorphism of functors* $\mu : G \xRightarrow{\simeq} \mathrm{L}F$ *such that* $\theta = \eta * (\mu \circ \mathrm{id}_{\mathrm{Q}})$.

The proof is the same as that of Proposition 12.4.19, only some arrows have to be reversed.

Theorem 12.4.42. *In the situation of Definition* 12.4.40*, assume there is a full subcategory* $\mathsf{P} \subseteq \mathsf{K}$ *such the following three conditions hold*:

(a) *The multiplicatively closed set* S *is a right denominator set in* K.
(b) *For every object* $M \in \mathsf{K}$ *there is a morphism* $\rho : P \to M$ *in* S*, with source* $P \in \mathsf{P}$.
(c) *If* ψ *is a morphism in* $\mathsf{P} \cap \mathsf{S}$*, then* $F(\psi)$ *is an isomorphism in* E.

Then the left derived functor

$$(\mathrm{L}F, \eta) : \mathsf{K}_{\mathsf{S}} \to \mathsf{E}$$

exists. Moreover, for any object $P \in \mathsf{P}$ *the morphism*

$$\eta_P : \mathrm{L}F(P) \to F(P)$$

in E *is an isomorphism.*

The proof is the same as that of Theorem 12.4.20, only some arrows have to be reversed.

For reference we give the next definition.

Definition 12.4.43. In the situation of Theorem 12.4.42, by a *system of left* P-*resolutions* we mean a pair (P, ρ), where $P : \mathrm{Ob}(\mathsf{K}) \to \mathrm{Ob}(\mathsf{P})$ is a function, and $\rho = \{\rho_M\}_{M \in \mathrm{Ob}(\mathsf{K})}$ is a collection of morphisms $\rho_M : P(M) \to M$ in S. Moreover, if $M \in \mathrm{Ob}(\mathsf{P})$, then $P(M) = M$ and $\rho_M = \mathrm{id}_M$.

Property (b) of Theorem 12.4.42 guarantees that a system of left P-resolutions (P, ρ) exists.

12.5. **Right Derived Bifunctors (continued).**

cmnt: reorganize. no splitting of this material

After the interlude on general categories of functors, we return to the triangulated setting.

cmnt: proof of Thm 8.3.3 has to be fixed!!

cmnt: the lemmas below should be imported to Subsec 8.3 for proving Thm 8.3.3

Definition 12.5.1. Let K_1, K_2 and E be $\mathbb{K}$-linear pretriangulated categories. We denote by $\mathsf{TrBiFun}(\mathsf{K}_1 \times \mathsf{K}_2, \mathsf{E})$ the category of $\mathbb{K}$-linear triangulated bifunctors $F : \mathsf{K}_1 \times \mathsf{K}_2 \to \mathsf{E}$.

Implicit in the definition above is that each object of $\mathsf{TrBiFun}(\mathsf{K}_1 \times \mathsf{K}_2, \mathsf{E})$ is a triple (F, τ_1, τ_2). The morphisms in this category are compatible with the translation isomorphism. See Definitions 5.3.1, 5.1.3 and 5.1.5. The category $\mathsf{TrBiFun}$ is $\mathbb{K}$-linear.

Suppose $U_i : \mathsf{K}'_i \to \mathsf{K}_i$ are triangulated functors between pretriangulated categories. We get an induced additive functor

$$\mathsf{F}(U_1 \times U_1, \mathrm{Id}) : \mathsf{TrBiFun}(\mathsf{K}_1 \times \mathsf{K}_2, \mathsf{E}) \to \mathsf{TrBiFun}(\mathsf{K}'_1 \times \mathsf{K}'_2, \mathsf{E}) \tag{12.5.2}$$

with the same formula as in (12.4.3).

Lemma 12.5.3. *If the functors U_1 and U_2 are equivalences, then the functor $\mathsf{F}(U_1 \times U_1, \mathrm{Id})$ in (12.5.2) is an equivalence.*

Proof. This is basically the same as the proof of Proposition 12.4.4 (that itself was an exercise...). The delicate change is that here we have to consider the translation isomorphisms τ_1 and τ_2. But these are controlled by the equivalence

$$\mathsf{F}(U_1 \times U_1, \mathrm{Id}_{\mathsf{E}}) : \mathsf{AdBiFun}(\mathsf{K}_1 \times \mathsf{K}_2, \mathsf{E}) \to \mathsf{AdBiFun}(\mathsf{K}'_1 \times \mathsf{K}'_2, \mathsf{E}).$$

□

Let $\mathsf{S}_i \subseteq \mathsf{K}_i$ be denominator sets of cohomological origin. These are left (and right) denominator sets. We know that the localizations $\mathsf{D}_i := (\mathsf{K}_i)_{\mathsf{S}_i}$ are pretriangulated categories, and the localization functors $\mathrm{Q}_i : \mathsf{K}_i \to \mathsf{D}_i$ are triangulated. See Theorem 6.4.3.

As in Definition 12.4.6 we denote by

$$\mathsf{TrBiFun}_{\mathsf{S}_1 \times \mathsf{S}_2}(\mathsf{K}_1 \times \mathsf{K}_2, \mathsf{E}) \subseteq \mathsf{TrBiFun}(\mathsf{K}_1 \times \mathsf{K}_2, \mathsf{E})$$

the full subcategory on the triangulated bifunctors F such that $F(\mathsf{S}_1 \times \mathsf{S}_2) \subseteq \mathsf{E}^{\times}$.

Lemma 12.5.4. *In the situation above the functor*

$$\mathsf{F}(\mathrm{Q}_1 \times \mathrm{Q}_2, \mathrm{Id}_{\mathsf{E}}) : \mathsf{TrBiFun}(\mathsf{D}_1 \times \mathsf{D}_2, \mathsf{E}) \to \mathsf{TrBiFun}_{\mathsf{S}_1 \times \mathsf{S}_2}(\mathsf{K}_1 \times \mathsf{K}_2, \mathsf{E})$$

is an isomorphism of categories.

Proof. This is basically that same as the proof of Proposition 12.4.7, combined with the isomorphism of pretriangulated categories

$$\mathrm{Q} : (\mathsf{K}_1 \times \mathsf{K}_2)_{\mathsf{S}_1 \times \mathsf{S}_2} \to \mathsf{D}_1 \times \mathsf{D}_2$$

from Proposition 12.4.12. The fine point is that the translation isomorphisms τ_i are controlled by this isomorphism of categories:

$$\mathsf{F}(\mathrm{Q}_1 \times \mathrm{Q}_1, \mathrm{Id}_{\mathsf{E}}) : \mathsf{AdBiFun}(\mathsf{D}_1 \times \mathsf{D}_2, \mathsf{E}) \to \mathsf{AdBiFun}_{\mathsf{S}_1 \times \mathsf{S}_2}(\mathsf{K}_1 \times \mathsf{K}_2, \mathsf{E}).$$

□

We can now give:

Proof of Theorem 12.3.6. It will be convenient to change notation. For $p = 1, 2$ let's define $\mathsf{K}'_p := \mathsf{J}_p$, $\mathsf{S}'_p := \mathsf{K}'_p \cap \mathsf{S}_p$ and $\mathsf{D}'_p := (\mathsf{K}'_p)_{\mathsf{S}'_p}$. The localization functors are $\mathrm{Q}'_p : \mathsf{K}'_p \to \mathsf{D}'_p$. The inclusions are $U_p : \mathsf{K}'_p \to \mathsf{K}_p$, and their localizations are the functors $V_p : \mathsf{D}'_p \to \mathsf{D}_p$. By Lemma 12.4.25 the functors V_p are equivalences.

The situation is depicted in these diagrams. We have this commutative diagram of products of triangulated functors between products of pretriangulated categories:

(12.5.5)
$$\begin{tikzcd}
\mathsf{K}'_1 \times \mathsf{K}'_2 \arrow[r, "U_1 \times U_2"] \arrow[d, "\mathrm{Q}'_1 \times \mathrm{Q}'_2"'] & \mathsf{K}_1 \times \mathsf{K}_2 \arrow[d, "\mathrm{Q}_1 \times \mathrm{Q}_2"] \\
\mathsf{D}'_1 \times \mathsf{D}'_2 \arrow[r, "V_1 \times V_2"] & \mathsf{D}_1 \times \mathsf{D}_2
\end{tikzcd}$$

The arrow $V_1 \times V_2$ is an equivalence. Diagram (12.5.5) induces a commutative diagram of linear categories:

(12.5.6)
$$\begin{tikzcd}
\mathsf{TrBiFun}(\mathsf{K}'_1 \times \mathsf{K}'_2, \mathsf{E}) & \mathsf{TrBiFun}(\mathsf{K}_1 \times \mathsf{K}_2, \mathsf{E}) \arrow[l, "\mathsf{F}(U_1 \times U_2, \mathrm{Id})"'] \\
\mathsf{TrBiFun}_{\mathsf{S}'_1 \times \mathsf{S}'_2}(\mathsf{K}'_1 \times \mathsf{K}'_2, \mathsf{E}) \arrow[u, "\text{f.f. inc}"'] & \mathsf{TrBiFun}_{\mathsf{S}_1 \times \mathsf{S}_2}(\mathsf{K}_1 \times \mathsf{K}_2, \mathsf{E}) \arrow[l, "\mathsf{F}(U_1 \times U_2, \mathrm{Id})"', "\text{equiv}"] \arrow[u, "\text{f.f. inc}"] \\
\mathsf{TrBiFun}(\mathsf{D}'_1 \times \mathsf{D}'_2, \mathsf{E}) \arrow[u, "\mathsf{F}(\mathrm{Q}'_1 \times \mathrm{Q}'_2, \mathrm{Id})", "\text{isom}"'] & \mathsf{TrBiFun}(\mathsf{D}_1 \times \mathsf{D}_2, \mathsf{E}) \arrow[l, "\mathsf{F}(V_1 \times V_2, \mathrm{Id})"', "\text{equiv}"] \arrow[u, "\text{isom}", "\mathsf{F}(\mathrm{Q}_1 \times \mathrm{Q}_2, \mathrm{Id})"']
\end{tikzcd}$$

According to Lemmas 12.5.3 and 12.5.4, the arrows in the diagram above that are marked "isom" or "equiv" are isomorphisms or equivalences, respectively. By definition the arrows marked "f.f. inc" are fully faithful inclusions.

We know that $\mathsf{S}_i \subseteq \mathsf{K}_i$ are left denominator sets. Therefore (see Proposition 12.4.14)

$$\mathsf{S}_1 \times \mathsf{S}_2 \subseteq \mathsf{K}_1 \times \mathsf{K}_2$$

is a left denominator set. Condition (a) of Theorem 12.3.6 says that F sends morphisms in $\mathsf{S}'_1 \times \mathsf{S}'_2$ to isomorphisms in E. Condition (b) there says that there are enough right $\mathsf{K}'_1 \times \mathsf{K}'_2$-resolutions in $\mathsf{K}_1 \times \mathsf{K}_2$.

Thus we are in a position to use the abstract Theorem 12.4.20. It says that there is an abstract right derived functor

$$(\mathrm{R}F, \eta) : \mathsf{D}_1 \times \mathsf{D}_2 \to \mathsf{E}\,.$$

However, going over the proof of Theorem 12.4.20, we see that all constructions there can be made within the triangulated setting, namely in diagram (12.5.6) instead of in diagram (12.4.28). Therefore $\mathrm{R}F$ is an object of the category in the bottom right corner of (12.5.6), and the morphism $\eta : F \Rightarrow \mathrm{R}F \circ \mathrm{Q}$ is in the category in the top right corner of (12.5.6).

cmnt: there might be a general yoga to deduce the above...

□

12.6. **The Bifunctor** RHom**.** Consider a DG ring A and an abelian category M. Like in Example 12.1.10 we get a DG bifunctor

$$F := \mathrm{Hom}_{A,\mathsf{M}}(-,-) : \mathbf{C}(A,\mathsf{M})^{\mathrm{op}} \times \mathbf{C}(A,\mathsf{M}) \to \mathbf{C}(\mathbb{K}).$$

Passing to homotopy categories, and postcomposing with $\mathrm{Q} : \mathbf{K}(\mathbb{K}) \to \mathbf{D}(\mathbb{K})$, we obtain a triangulated bifunctor

$$F = \mathrm{Hom}_{A,\mathsf{M}}(-,-) : \mathbf{K}(A,\mathsf{M})^{\mathrm{op}} \times \mathbf{K}(A,\mathsf{M}) \to \mathbf{D}(\mathbb{K}).$$

Next we pick full pretriangulated subcategories $\mathsf{K}_1, \mathsf{K}_2 \subseteq \mathbf{K}(A,\mathsf{M})$. In practice this choice would be by some boundedness conditions, for instance $\mathsf{K}_2 := \mathbf{C}^+(\mathsf{M})$, cf. Corollary 10.4.10, or $\mathsf{K}_1 := \mathbf{C}^-(\mathsf{M})$, cf. Corollary 10.2.14. We want to construct the right derived bifunctor of the triangulated bifunctor

$$F = \mathrm{Hom}_{A,\mathsf{M}}(-,-) : \mathsf{K}_1^{\mathrm{op}} \times \mathsf{K}_2 \to \mathbf{D}(\mathbb{K}).$$

This is done in the next theorem.

Theorem 12.6.1. *Let $\mathsf{K}_1, \mathsf{K}_2 \subseteq \mathbf{K}(A,\mathsf{M})$ be full pretriangulated subcategories, and let D_i denote the localization of K_i with respect to the quasi-isomorphisms in it. Assume either that K_1 has enough K-projectives, or that K_2 has enough K-injectives.*

Then the triangulated bifunctor

$$\mathrm{Hom}_{A,\mathsf{M}}(-,-) : \mathsf{K}_1^{\mathrm{op}} \times \mathsf{K}_2 \to \mathbf{D}(\mathbb{K})$$

has a right derived bifunctor

$$\mathrm{RHom}_{A,\mathsf{M}}(-,-) : \mathsf{D}_1^{\mathrm{op}} \times \mathsf{D}_2 \to \mathbf{D}(\mathbb{K}).$$

Moreover, if $P_1 \in \mathsf{K}_1$ is K-projective, or if $I_2 \in \mathsf{K}_2$ is K-injective, then the morphism

$$\eta_{P_1,I_2} : \mathrm{Hom}_{A,\mathsf{M}}(P_1, I_2) \to \mathrm{RHom}_{A,\mathsf{M}}(P_1, I_2)$$

in $\mathbf{D}(\mathbb{K})$ is an isomorphism.

Proof. If K_2 has enough K-injectives, then we can take $\mathsf{J}_2 := \mathsf{K}_{2,\mathrm{inj}}$, the full subcategory on the K-injectives inside K_2. And we take $\mathsf{J}_1 := \mathsf{K}_1$. We claim that the conditions of Theorem 12.3.6 are satisfied. Condition (b) is simply the assumption that K_2 has enough K-injectives. As for condition (a): this is Lemma 12.6.2 below.

On the other hand, if K_1 has enough K-projectives, then we can take $\mathsf{J}_1^{\mathrm{op}} := \mathsf{K}_{1,\mathrm{prj}}^{\mathrm{op}}$, where $\mathsf{K}_{1,\mathrm{prj}}$ is the full subcategory on the K-projectives inside K_1. And we take $\mathsf{J}_2 := \mathsf{K}_2$. We claim that the conditions of Theorem 12.3.6 are satisfied for $\mathsf{J}_1^{\mathrm{op}} \subseteq \mathsf{K}_1^{\mathrm{op}}$. Condition (b) is simply the assumption that K_1 has enough K-projectives: a quasi-isomorphism $\rho : P \to M$ in K_1 becomes a quasi-isomorphism $\rho^{\mathrm{op}} : M^{\mathrm{op}} \to P^{\mathrm{op}}$ in $\mathsf{K}_1^{\mathrm{op}}$. As for condition (a): this is Lemma 12.6.2 below.

The last assertion also follows from 12.6.2. $\square$

Lemma 12.6.2. *Suppose $\phi_1 : Q_1 \to P_1$ and $\phi_2 : I_2 \to J_2$ are quasi-isomorphisms in $\mathsf{C}(A, \mathsf{M})$, and either Q_1, P_1 are both K-projective, or I_2, J_2 are both K-injective. Then the homomorphism*

$$\mathrm{Hom}_{A,\mathsf{M}}(\phi_1, \phi_2) : \mathrm{Hom}_{A,\mathsf{M}}(P_1, I_2) \to \mathrm{Hom}_{A,\mathsf{M}}(Q_1, J_2)$$

in $\mathsf{C}(\mathbb{K})$ is a quasi-isomorphism.

Proof. We will only prove the case where Q_1, P_1 are both K-projective; the other case is very similar.

The homomorphism in question factors as follows:

$$\mathrm{Hom}_{A,\mathsf{M}}(\phi_1, \phi_2) = \mathrm{Hom}_{A,\mathsf{M}}(\phi_1, \mathrm{id}_{J_2}) \circ \mathrm{Hom}_{A,\mathsf{M}}(\mathrm{id}_{P_1}, \phi_2).$$

It suffices to prove that each of the factors is a quasi-isomorphism. This can be done by a messy direct calculation, but we will provide an indirect proof that relies on properties of the homotopy category $\mathsf{K} := \mathbf{K}(A, \mathsf{M})$ that were already established.

Let K_2 be the cone on the homomorphism $\phi_2 : I_2 \to J_2$. So K_2 is acyclic. Because P_1 is K-projective it follows that $\mathrm{Hom}_{A,\mathsf{M}}(P_1, K_2)$ is acyclic. Thus for every integer l we have

$$\mathrm{Hom}_{\mathsf{K}}(\mathrm{T}^{-l}(P_1), K_2) \cong \mathrm{H}^l\big(\mathrm{Hom}_{A,\mathsf{M}}(P_1, K_2)\big) = 0. \tag{12.6.3}$$

Next, there is a distinguished triangle

$$I_2 \xrightarrow{\phi_2} J_2 \xrightarrow{\beta_2} K_2 \xrightarrow{\gamma_2} \mathrm{T}(I_2) \tag{12.6.4}$$

in K. Applying the cohomological functor $\mathrm{Hom}_{\mathsf{K}}(\mathrm{T}^{-l}(P_1), -)$ to the distinguished triangle (12.6.4) yields a long exact sequence, as explained in Subsection 5.3. From it we deduce that the homomorphisms

$$\mathrm{Hom}_{\mathsf{K}}(\mathrm{T}^{-l}(P_1), I_2) \to \mathrm{Hom}_{\mathsf{K}}(\mathrm{T}^{-l}(P_1), J_2)$$

are bijective for all l. Using the isomorphisms like (12.6.3) for I_2 and J_2 we see that

$$\mathrm{Hom}_{A,\mathsf{M}}(\mathrm{id}_{P_1}, \phi_2) : \mathrm{Hom}_{A,\mathsf{M}}(P_1, I_2) \to \mathrm{Hom}_{A,\mathsf{M}}(P_1, J_2)$$

is a quasi-isomorphism.

According to Corollary 9.2.12 the homomorphism $\phi_1 : Q_1 \to P_1$ is a homotopy equivalence; so it is an isomorphism in K. Therefore for any integer l the homomorphism

$$\mathrm{Hom}_{\mathsf{K}}(Q_1, \mathrm{T}^l(J_2)) \to \mathrm{Hom}_{\mathsf{K}}(P_1, \mathrm{T}^l(J_2))$$

is bijective. As above we conclude that

$$\mathrm{Hom}_{A,\mathsf{M}}(\phi_1, \mathrm{id}_{J_2}) : \mathrm{Hom}_{A,\mathsf{M}}(Q_1, J_2) \to \mathrm{Hom}_{A,\mathsf{M}}(P_1, J_2)$$

is a quasi-isomorphism. □

Remark 12.6.5. Theorem 12.6.1 should be viewed as a template. It has a variant for $\mathbf{C}(A)$ where A is a commutative ring, as in Example 12.1.10. There are bimodule variants as in Example 12.1.11 and Section 17. And there are geometric versions where the source and target are categories of sheaves – see Section 15.

cmnt: to here lecture 30 Nov 2016

cmnt: no lecture 7 Dec 2016

We end this section with the connection between RHom and morphisms in the derived category.

Definition 12.6.6. Under the assumptions of Theorem 12.6.1, for DG modules $M_1 \in \mathsf{K}_1$ and $M_2 \in \mathsf{K}_2$, and for an integer i, we write

$$\mathrm{Ext}^i_{A,\mathsf{M}}(M_1, M_2) := \mathrm{H}^i\big(\mathrm{RHom}_{A,\mathsf{M}}(M_1, M_2)\big) \in \mathbf{M}(\mathbb{K}).$$

Exercise 12.6.7. Let A be a ring. Prove that for modules $M_1, M_2 \in \mathbf{M}(A)$ the $\mathbb{K}$-module $\mathrm{Ext}^i_A(M_1, M_2)$ defined above is canonically isomorphic to the classical definition.

Corollary 12.6.8. *Under the assumptions of Theorem* 12.6.1*, there is an isomorphism*

$$\mathrm{Ext}^0_{A,\mathsf{M}}(-,-) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{D}(A,\mathsf{M})}(-,-)$$

of additive bifunctors

$$\mathsf{D}_1^{\mathrm{op}} \times \mathsf{D}_2 \to \mathbf{M}(\mathbb{K}).$$

Exercise 12.6.9. Prove Corollary 12.6.8.

12.7. **Left Derived Bifunctors.** The material here is opposite (left vs. right) to that in Subsection 12.3. Because of the similarity, we give only a few details.

The assumptions in the next definition are identical to those in Setup 12.3.1.

Definition 12.7.1. Assume the following are given:

(1) Pretriangulated categories K_1, K_2 and E.
(2) A triangulated bifunctor $F : \mathsf{K}_1 \times \mathsf{K}_2 \to \mathsf{E}$.
(3) Denominator sets of cohomological origin $\mathsf{S}_1 \subseteq \mathsf{K}_1$ and $\mathsf{S}_2 \subseteq \mathsf{K}_2$.

A *left derived bifunctor* of F is a pair $(\mathrm{L}F, \eta)$, where

$$\mathrm{L}F : \mathsf{D}_1 \times \mathsf{D}_2 \to \mathsf{E}$$

is a triangulated bifunctor, and

$$\eta : \mathrm{L}F \circ (\mathrm{Q}_1 \times \mathrm{Q}_2) \Rightarrow F$$

is a morphism of triangulated bifunctors, such that the following universal property holds:

(L) Given any pair (G, θ), consisting of a triangulated bifunctor

$$G : \mathsf{D}_1 \times \mathsf{D}_2 \to \mathsf{E}$$

and a morphism of triangulated bifunctors

$$\theta : G \circ (\mathrm{Q}_1 \times \mathrm{Q}_2) \Rightarrow F,$$

there is a unique morphism of triangulated functors $\mu : G \Rightarrow \mathrm{L}F$ such that

$$\theta = \eta * (\mu \circ \mathrm{id}_{\mathsf{Q}_1 \times \mathsf{Q}_2}).$$

Proposition 12.7.2. *If a left derived bifunctor exists, then it is unique up to a unique isomorphism.*

Proof. This is the opposite of Proposition 12.3.4, and we leave it to the reader to make the adjustments. □

Definition 12.7.3. Let K be a pretriangulated category, let $\mathsf{S} \subseteq \mathsf{K}$ be a denominator set of cohomological origin, and let $\mathsf{P} \subseteq \mathsf{K}$ be a full pretriangulated subcategory.

(1) Let $M \in \mathsf{K}$. A *left* P*-resolution* of M is a morphism $\rho : P \to M$ in S from an object $P \in \mathsf{P}$.
(2) We say that K *has enough left* P*-resolutions* if every object $M \in \mathsf{K}$ admits a left P-resolution.

cmnt: this def should be moved to Sec 8

Theorem 12.7.4. *In the situation of Definition* 12.7.1*, assume there are full pretriangulated subcategories* $\mathsf{P}_1 \subseteq \mathsf{K}_1$ *and* $\mathsf{P}_2 \subseteq \mathsf{K}_2$ *with these two properties*:

(a) *Acyclicity*: *if* $\phi_1 : P_1 \to Q_1$ *is a morphism in* $\mathsf{P}_1 \cap \mathsf{S}_1$ *and* $\phi_2 : P_2 \to Q_2$ *is a quasi-isomorphism in* $\mathsf{P}_2 \cap \mathsf{S}_2$*, then*
$$F(\phi_1, \phi_2) : F(P_1, P_2) \to F(Q_1, Q_2)$$
is an isomorphism in E*.*
(b) *Abundance*: K_1 *has enough left* P_1*-resolutions, and* K_2 *has enough left* P_2*-resolutions.*

Then the left derived bifunctor

$$(\mathrm{L}F, \eta) : \mathsf{D}_1 \times \mathsf{D}_2 \to \mathsf{E}$$

exists. Moreover, for any objects $P_1 \in \mathsf{P}_1$ *and* $P_2 \in \mathsf{P}_2$ *the morphism*

$$\eta_{P_1, P_2} : \mathrm{L}F(P_1, P_2) \to F(P_1, P_2)$$

in E *is an isomorphism.*

Proof. This is the opposite of Theorem 12.3.6, and we leave it to the reader to make the necessary changes in direction. □

In applications we will see that either $\mathsf{P}_1 = \mathsf{K}_1$ or $\mathsf{P}_2 = \mathsf{K}_2$; namely we will only need to resolve in the second or in the first argument, respectively.

Proposition 12.7.5. *If a right derived bifunctor exists, then it is unique up to a unique isomorphism.*

Proof. This is just like the proof of Proposition 8.3.2. We leave the small changes up to the reader. □

Existence in general is like Theorem 8.4.3, but more complicated.

Definition 12.7.6. Let K be a pretriangulated category, let $\mathsf{S} \subseteq \mathsf{K}$ be a denominator set of cohomological origin, and let $\mathsf{P} \subseteq \mathsf{K}$ be a full pretriangulated subcategory.

(1) Let $M \in \mathsf{K}$. A *left* P*-resolution* of M is a quasi-isomorphism $\rho : P \to M$ in S from an object $P \in \mathsf{P}$.
(2) We say that K *has enough left* P*-resolutions* if every object $M \in \mathsf{K}$ admits a left P-resolution.

cmnt: this def should be moved to Sec 8

Theorem 12.7.7. *In the situation of Definition* 12.7.1*, assume there are full pretriangulated subcategories* $\mathsf{P}_1 \subseteq \mathsf{K}_1$ *and* $\mathsf{P}_2 \subseteq \mathsf{K}_2$ *with these two properties*:

(a) *Acyclicity*: *if* $\phi_1 : P_1 \to Q_1$ *is a morphism in* $\mathsf{P}_1 \cap \mathsf{S}_1$ *and* $\phi_2 : P_2 \to Q_2$ *is a morphism in* $\mathsf{P}_2 \cap \mathsf{S}_2$*, then*
$$F(\phi_1, \phi_2) : F(P_1, P_2) \to F(Q_1, Q_2)$$
is an isomorphism in E*.*
(b) *Abundance*: K_1 *has enough left* P_1*-resolutions, and* K_2 *has enough left* P_2*-resolutions.*

Then the left derived bifunctor
$$(\mathrm{L}F, \eta) : \mathsf{D}_1 \times \mathsf{D}_2 \to \mathsf{E}$$
exists. Moreover, for any objects $P_1 \in \mathsf{P}_1$ *and* $P_2 \in \mathsf{P}_2$ *the morphism*
$$\eta_{P_1, P_2} : \mathrm{L}F(P_1, P_2) \to F(P_1, P_2)$$
in E *is an isomorphism.*

In applications we will see that either $\mathsf{P}_1 = \mathsf{K}_1$ or $\mathsf{P}_2 = \mathsf{K}_2$; namely we will only need to resolve in the second or in the first argument, respectively.

Proof. Like that of Theorem 12.3.6. We leave the side changes to the reader. □

12.8. **The Bifunctor $\otimes^{\mathrm{L}}$.** Consider a DG ring A. Like in Example 12.1.9 we get a DG bifunctor
$$F := (- \otimes_A -) : \mathbf{C}(A^{\mathrm{op}}) \times \mathbf{C}(A) \to \mathbf{C}(\mathbb{K}).$$
Passing to homotopy categories, and postcomposing with $\mathrm{Q} : \mathbf{K}(\mathbb{K}) \to \mathbf{D}(\mathbb{K})$, we obtain a triangulated bifunctor
$$F = (- \otimes_A -) : \mathbf{K}(A^{\mathrm{op}}) \times \mathbf{K}(A) \to \mathbf{D}(\mathbb{K}).$$
Next we pick full pretriangulated subcategories $\mathsf{K}_1 \subseteq \mathbf{K}(A^{\mathrm{op}})$ and $\mathsf{K}_2 \subseteq \mathbf{K}(A)$. In practice this choice would be by some boundedness conditions, for instance $\mathsf{K}_1 := \mathbf{C}^{-}(A^{\mathrm{op}})$ or $\mathsf{K}_2 := \mathbf{C}^{-}(A)$, cf. Corollary 10.2.14. We want to construct the left derived bifunctor of the triangulated bifunctor
$$F = (- \otimes_A -) : \mathsf{K}_1 \times \mathsf{K}_2 \to \mathbf{D}(\mathbb{K}).$$
This is done in the next theorem.

Theorem 12.8.1. *Let* $\mathsf{K}_1 \subseteq \mathbf{K}(A^{\mathrm{op}})$ *and* $\mathsf{K}_2 \subseteq \mathbf{K}(A)$ *be full pretriangulated subcategories, and let* D_i *denote the localization of* K_i *with respect to the quasi-isomorphisms in it. Assume that either* K_1 *or* K_2 *has enough K-flat objects.*

Then the triangulated bifunctor
$$(- \otimes_A -) : \mathsf{K}_1 \times \mathsf{K}_2 \to \mathbf{D}(\mathbb{K})$$

has a left derived bifunctor

$$(- \otimes^{\mathrm{L}}_{A} -) : \mathsf{D}_1 \times \mathsf{D}_2 \to \mathbf{D}(\mathbb{K}).$$

Moreover, if either $P_1 \in \mathsf{K}_1$ or $P_2 \in \mathsf{K}_2$ is K-flat, then the morphism

$$\eta_{P_1,P_2} : P_1 \otimes^{\mathrm{L}}_{A} P_2 \to P_1 \otimes_A P_2$$

in $\mathbf{D}(\mathbb{K})$ is an isomorphism.

Note that a DG module $P_1 \in \mathsf{K}_1$ is checked for K-flatness as a right DG A-module; and a DG module $P_2 \in \mathsf{K}_2$ is checked for K-flatness as a left DG A-module.

Proof. If K_2 has enough K-flats, then we can take $\mathsf{P}_2 := \mathsf{K}_{2,\mathrm{flat}}$, the full subcategory on the K-flats inside K_2. And we take $\mathsf{P}_1 := \mathsf{K}_1$. We claim that the conditions of Theorem 12.7.4 are satisfied. Condition (b) is simply the assumption that K_2 has enough K-flats. As for condition (a): this is Lemma 12.8.2 below.

The other case is proved the same way (bur replacing sides). The last assertion also follows from 12.8.2. □

Lemma 12.8.2. *Suppose $\phi_1 : P_1 \to Q_1$ and $\phi_2 : P_2 \to Q_2$ are quasi-isomorphisms in $\mathbf{C}(A^{\mathrm{op}})$ and $\mathbf{C}(A)$ respectively, and either of the conditions below holds*:

(i) *Q_1 and P_1 are both K-flat.*
(ii) *P_2 and Q_2 are both K-flat.*

Then the homomorphism

$$\phi_1 \otimes \phi_2 : P_1 \otimes_A P_2 \to Q_1 \otimes_A Q_2$$

in $\mathbf{C}(\mathbb{K})$ is a quasi-isomorphism.

Proof. We will only prove the lemma under condition (i); the other case is very similar. The homomorphism in question factors as follows:

$$\phi_1 \otimes \phi_2 = (\phi_1 \otimes \mathrm{id}_{P_2}) \circ (\mathrm{id}_{P_1} \otimes \phi_2).$$

It suffices to prove that each of the factors is a quasi-isomorphism. This can be done by a messy direct calculation, but we will provide an indirect proof that relies on properties of the DG categories $\mathbf{C}(A^{\mathrm{op}})$ and $\mathbf{C}(A)$ that were already established.

First we shall prove that $\mathrm{id}_{P_1} \otimes \phi_2$ is a quasi-isomorphism. Let R_2 be the standard cone on the strict homomorphism $\phi_2 : P_2 \to Q_2$. So there is a standard triangle

$$P_2 \xrightarrow{\phi_2} Q_2 \to R_2 \to \mathrm{T}(P_2) \tag{12.8.3}$$

in $\mathbf{C}_{\mathrm{str}}(A)$, and R_2 is acyclic. Applying the DG functor $P_1 \otimes_A -$ to the triangle (12.8.3), and using Theorem 4.3.7, we see that there is a standard triangle

$$P_1 \otimes_A P_2 \xrightarrow{\mathrm{id}_{P_1} \otimes \phi_2} P_1 \otimes_A Q_2 \to P_1 \otimes_A R_2 \to \mathrm{T}(P_1 \otimes_A P_2) \tag{12.8.4}$$

in $\mathbf{C}(\mathbb{K})$. This becomes a distinguished triangle in the pretriangulated category $\mathbf{K}(\mathbb{K})$. Thus there is a long exact sequence in cohomology associated to (12.8.4). Because P_1 is K-flat it follows that $P_1 \otimes_A R_2$ is acyclic. We conclude that $\mathrm{H}^i(\mathrm{id}_{P_1} \otimes \phi_2)$ is bijective for all i.

Now we shall prove that $\phi_1 \otimes \mathrm{id}_{P_2}$ is a quasi-isomorphism. Let $R_1 \in \mathbf{C}(A^{\mathrm{op}})$ be the cone on the homomorphism $\phi_1 : P_1 \to Q_1$. It is both acyclic and K-flat. Using standard triangles like (12.8.3) and (12.8.4) we reduce the problem to showing that $R_1 \otimes_A P_2$ is acyclic. According to Corollary 10.3.27 and Proposition 9.3.2 there is a quasi-isomorphism $\tilde{P}_2 \to P_2$ in $\mathsf{C}(A)$ from some K-flat DG module $\tilde{P}_2$. As already proved in the previous paragraph, since R_1 is K-flat, the homomorphism

$$R_1 \otimes_A \tilde{P}_2 \to R_1 \otimes_A P_2$$

is a quasi-isomorphism. But R_1 is acyclic and $\tilde{P}_2$ is K-flat, and therefore $R_1 \otimes_A \tilde{P}_2$ is acyclic. We conclude that $R_1 \otimes_A P_2$ is acyclic, as required. $\square$

Remark 12.8.5. Theorem 12.8.1 should be viewed as a template. It has a variant for $\mathsf{C}(A)$ where A is a commutative ring, as in Example 12.1.8. There are bimodule variants as in Example 12.1.9 and Section 17. And there are geometric versions where the source and target are categories of sheaves – see Section 15.

13. Dualizing Complexes over Commutative Rings

In this section we finally explain what was outlined, as a motivating discussing, in Subsection 0.1. Dualizing complexes are perhaps the most compelling reason to study derived categories. In the commutative setting of the current section the technicalities are milder than in the geometric setting (Section 16) and the noncommutative setting (Section 17).

We will start with some more technical facts on functors.

cmnt: move them to an earlier location,

Then we will learn about *dualizing complexes* and *residue complexes* over commutative rings, as defined by Grothendieck in [RD] in the 1960's.

The initial plan was to also talk about *Local Duality*, *MGM Equivalence* and *perfect complexes* in this section; but for lack of time and space, these topics will be confined to short remarks. See Remark 13.4.25, ???

cmnt: finish

13.1. **Cohomological Dimension of Functors.** The material here is a refinement of the notion of "way-out functors" from [RD, Section II.7]. It is taken from [Ye10]. As always, there is a fixed base ring $\mathbb{K}$.

cmnt: maybe move this material to an earlier location?

By *generalized integers* we mean elements of the ordered set $\mathbb{Z} \cup \{\pm\infty\}$. Recall that for a subset $S \subseteq \mathbb{Z}$, its infimum is $\inf(S) \in \mathbb{Z} \cup \{\pm\infty\}$, where $\inf(S) = +\infty$ iff $S = \varnothing$. Likewise the supremum is $\sup(S) \in \mathbb{Z} \cup \{\pm\infty\}$, where $\sup(S) = -\infty$ iff $S = \varnothing$. For $i, j \in \mathbb{Z} \cup \{\infty\}$, the expressions $i + j$ and $-i - j$ have obvious values in $\mathbb{Z} \cup \{\pm\infty\}$. And for $i, j \in \mathbb{Z} \cup \{\pm\infty\}$, the expression $i \leq j$ has an obvious meaning.

Let $M = \bigoplus_{i \in \mathbb{Z}} M^i$ be a graded $\mathbb{K}$-module. We write

$$\inf(M) := \inf \{i \mid M^i \neq 0\} \quad \text{and} \quad \sup(M) := \sup \{i \mid M^i \neq 0\}. \tag{13.1.1}$$

The amplitude of M is

$$\operatorname{amp}(M) := \sup(M) - \inf(M) \in \mathbb{N} \cup \{\pm\infty\}. \tag{13.1.2}$$

(For $M = 0$ this reads $\inf(M) = \infty$, $\sup(M) = -\infty$ and $\operatorname{amp}(M) = -\infty$.) Thus M is bounded (resp. bounded above, resp. bounded below) iff $\operatorname{amp}(M) < \infty$ (resp. $\sup(M) < \infty$, resp. $\inf(M) > -\infty$).

Given $i_0 \leq i_1$ in $\mathbb{Z} \cup \{\pm\infty\}$, the *integer interval* with these endpoints is the set of integers

$$[i_0, i_1] := \{i \in \mathbb{Z} \mid i_0 \leq i \leq i_1\}. \tag{13.1.3}$$

There is also the empty integer interval $\varnothing$.

A nonempty integer interval $[i_0, i_1]$ is said to be bounded (resp. bounded above, resp. bounded below) if $i_0, i_1 \in \mathbb{Z}$ (resp. $i_1 \in \mathbb{Z}$, resp. $i_0 \in \mathbb{Z}$). The *length* of this interval is $i_1 - i_0 \in \mathbb{N} \cup \{\infty\}$. Of course the interval has finite length iff it is bounded. We write $-[i_0, i_1] := [-i_1, -i_0]$. Given a second nonempty integer interval $[j_0, j_1]$, we let

$$[i_0, i_1] + [j_0, j_1] := [i_0 + j_0, i_1 + j_1].$$

The the empty integer interval $\varnothing$ is bounded, and its length is $-\infty$. If S is any integer interval, then the sum is the integer interval $S + \varnothing := \varnothing$. And $-\varnothing := \varnothing$.

Definition 13.1.4. Let $M = \bigoplus_{i \in \mathbb{Z}} M^i$ be a graded $\mathbb{K}$-module.

(1) We say that M is concentrated in an integer interval $[i_0, i_1]$ if
$$\{i \in \mathbb{Z} \mid M^i \neq 0\} \subseteq [i_0, i_1].$$

(2) The *concentration* of M is the smallest integer interval $\operatorname{con}(M)$ in which M is concentrated.

In other words, if $M \neq 0$ then
$$i_0 = \inf(M) \leq i_1 = \sup(M),$$
the concentration of M is the interval $\operatorname{con}(M) = [i_0, i_1]$, and the amplitude $\operatorname{amp}(M)$ is the length of $\operatorname{con}(M)$. Furthermore, $\operatorname{con}(M) = \varnothing$ iff $M = 0$.

The next definition is in conflict with Definitions 7.3.3 and 7.3.4; but we already warned that this change will take place (see Remark 7.3.9). The reason for the change: the new definition is more practical.

Definition 13.1.5. Let A be a DG ring and M an abelian category. The expression $\mathbf{D}^{\star}(A, \mathsf{M})$, where "$\star$" is either "+", "−" or "b", refers to the full subcategory of $\mathbf{D}(A, \mathsf{M})$ on the DG modules with bounded below (resp. bounded above, resp. bounded) cohomologies.

Thus, for example, a DG module M belongs to $\mathbf{D}^{\mathrm{b}}(A, \mathsf{M})$ iff $\operatorname{con}(\mathrm{H}(M))$ is a bounded integer interval.

Definition 13.1.6. Let A be a DG ring and M an abelian category. For a DG module $M \in \mathbf{C}(A, \mathsf{M})$ and an integer i, we write
$$M[i] := \mathrm{T}^i(M),$$
the i-th translation of M. This notation applies also to the homotopy category $\mathbf{K}(A, \mathsf{M})$, the derived category $\mathbf{D}(A, \mathsf{M})$, and any other T-additive category.

The notation $M[i]$ makes it difficult to use the little t operator and to talk about translation isomorphisms, but hopefully we won't require them anymore.

Definition 13.1.7. Let A, B be DG rings, let M, N be abelian categories, and let $\mathsf{C} \subseteq \mathbf{D}(A, \mathsf{M})$ be a full additive subcategory.

(1) Let
$$F : \mathsf{C} \to \mathbf{D}(B, \mathsf{N})$$
be an additive functor, and let S be an integer interval. We say that F *has cohomological displacement at most* S if
$$\operatorname{con}\big(\mathrm{H}(F(M))\big) \subseteq \operatorname{con}\big(\mathrm{H}(M)\big) + S$$
for every $M \in \mathsf{C}$.

(2) Let
$$F : \mathsf{C}^{\mathrm{op}} \to \mathbf{D}(B, \mathsf{N})$$
be an additive functor, and let S be an integer interval. We say that F *has cohomological displacement at most* S if
$$\operatorname{con}\big(\mathrm{H}(F(M))\big) \subseteq -\operatorname{con}\big(\mathrm{H}(M)\big) + S$$
for every $M \in \mathsf{C}$.

(3) Let F be as in item (1) or (2). The *cohomological displacement of* F is the smallest integer interval S for which F has cohomological displacement at most S.
(4) Let S be the cohomological displacement of F. The *cohomological dimension* of F is defined to be the length of the integer interval S.

To emphasize the most important case: *the functor F has finite cohomological dimension iff its cohomological displacement is bounded.*

Example 13.1.8. The functor F is the zero functor iff it has cohomological displacement $\varnothing$ and cohomological dimension $-\infty$.

Example 13.1.9. Consider a commutative ring $A = B$, and the abelian categories $\mathsf{M} = \mathsf{N} := \mathbf{M}(\mathbb{K})$. So $\mathbf{D}(A, \mathsf{M}) = \mathbf{D}(B, \mathsf{N}) = \mathbf{D}(A)$. Take $\mathsf{C} := \mathbf{D}(A)$. For the covariant case (item (1) in Definition 13.1.7) take a nonzero projective module P, and let

$$F := \mathrm{RHom}_A(P \oplus P[1], -) : \mathbf{D}(A) \to \mathbf{D}(A).$$

Then F has cohomological displacement $[0, 1]$. For the contravariant case (item (2)) take a nonzero injective module I, and let

$$F := \mathrm{RHom}_A(-, I \oplus I[1]) : \mathbf{D}(A)^{\mathrm{op}} \to \mathbf{D}(A).$$

Then F has cohomological displacement $[-1, 0]$. In both cases the cohomological dimension of F is 1.

Example 13.1.10. Suppose A and B are rings and $F : \mathbf{M}(A) \to \mathbf{M}(B)$ is a left exact additive functor. We get a triangulated functor

$$\mathrm{R}F : \mathbf{D}(A) \to \mathbf{D}(B),$$

and $\mathrm{H}^i(\mathrm{R}F(M)) = \mathrm{R}^i F(M)$ for all $M \in \mathbf{M}(A)$. Taking $\mathsf{C} := \mathbf{M}(A)$, with its canonical embedding into $\mathbf{D}(A)$, we get an additive functor

$$(\mathrm{R}F)|_{\mathbf{M}(A)} : \mathbf{M}(A) \to \mathbf{D}(A).$$

The cohomological dimension of $(\mathrm{R}F)|_{\mathbf{M}(A)}$ equals the usual cohomological dimension of the functor F.

Remark 13.1.11. Assume that in Definition 13.1.7 we take $\mathsf{M} = \mathbf{M}(\mathbb{K})$, $\mathsf{C} = \mathbf{D}(A)$ and F is a triangulated functor. The functor F has bounded below (resp. above) cohomological displacement iff it is way-out right (resp. left), in the sense of [RD, Section I.7].

Definition 13.1.12. Let $\star, \triangle$ be boundedness conditions, and assume the right derived bifunctor

$$\mathrm{RHom}_{A,\mathsf{M}} : \mathbf{D}^{\star}(A, \mathsf{M})^{\mathrm{op}} \times \mathbf{D}^{\triangle}(A, \mathsf{M}) \to \mathbf{D}(\mathbb{K})$$

exists. Let S be an integer interval of length $i \in \mathbb{N} \cup \{\pm\infty\}$.

(1) Let $M \in \mathbf{D}^{\star}(A, \mathsf{M})$, and let $\mathsf{C} \subseteq \mathbf{D}^{\triangle}(A, \mathsf{M})$ be a full additive subcategory. We say that M has *projective concentration* S and *projective dimension* i relative to C if the functor

$$\mathrm{RHom}_{A,\mathsf{M}}(M, -)|_{\mathsf{C}} : \mathsf{C} \to \mathbf{D}(\mathbb{K})$$

has cohomological displacement $-S$.

(2) Let $M \in \mathsf{D}^{\triangle}(A, \mathsf{M})$, and let $\mathsf{C} \subseteq \mathsf{D}^{\star}(A, \mathsf{M})$ be a full additive subcategory. We say that M has *injective concentration* S and *injective dimension* i relative to C if the functor

$$\mathrm{RHom}_{A,\mathsf{M}}(-, M)|_{\mathsf{C}^{\mathrm{op}}} : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}(\mathbb{K})$$

has cohomological displacement S.

(3) If $\mathsf{C} = \mathsf{D}(A, \mathsf{M})$, then we omit the "relative to C" clause.

Example 13.1.13. Continuing with the setup of Example 13.1.9, the DG module $P \oplus P[1]$ (resp. $I \oplus I[1]$) has projective (resp. injective) concentration $[-1, 0]$.

Example 13.1.14. Let A be a DG ring, and consider the free DG module $P := A \in \mathsf{D}(A)$. The functor

$$F := \mathrm{RHom}_A(P, -) : \mathsf{D}(A) \to \mathsf{D}(\mathbb{K})$$

is isomorphic to the forgetful functor, so it has cohomological displacement $[0, 0]$ and cohomological dimension 0. Thus the DG module P has projective concentration $[0, 0]$ and projective dimension 0. Note however that the cohomology $\mathrm{H}(P)$ could be unbounded!

cmnt: $\boxed{\text{next prop should move to Sec 7}}$

Proposition 13.1.15. *Let*

$$0 \to L \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$$

be a short exact sequence in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$. *Then there is a morphism* $\theta : N \to L[1]$ *in* $\mathsf{D}(A, \mathsf{M})$ *such that*

$$L \xrightarrow{\mathrm{Q}(\phi)} M \xrightarrow{\mathrm{Q}(\psi)} N \xrightarrow{\theta} L[1]$$

is a distinguished triangle in $\mathsf{D}(A, \mathsf{M})$.

Proof. We are following the proof of [KaSc1, Proposition 1.7.5]. Let $\tilde{N}$ be the standard cone on ϕ. In matrix notation as in Definition 3.9.1, we have

$$\tilde{N} = \begin{bmatrix} M \\ \mathrm{T}(L) \end{bmatrix} \quad \text{and} \quad \mathrm{d}_{\tilde{N}} = \begin{bmatrix} \mathrm{d}_M & \phi \circ \mathrm{t}^{-1} \\ 0 & \mathrm{d}_{\mathrm{T}(L)} \end{bmatrix}.$$

The object $\tilde{N}$ sits inside the standard triangle

$$L \xrightarrow{\phi} M \xrightarrow{\tilde{\psi}} \tilde{N} \xrightarrow{\tilde{\chi}} \mathrm{T}(L)$$

in $\mathsf{C}_{\mathrm{str}}(A, \mathsf{M})$, where

$$\tilde{\psi} := \begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{\chi} := \begin{bmatrix} 0 & \mathrm{id} \end{bmatrix}$$

in matrix notation. Define the morphism $\gamma = \tilde{N} \to N$ to be $\gamma := \begin{bmatrix} \psi & 0 \end{bmatrix}$. We get a commutative diagram

$$\begin{tikzcd} L \arrow[r, "\phi"] & M \arrow[r, "\tilde{\psi}"] \arrow[dr, "\psi"'] & \tilde{N} \arrow[r, "\tilde{\chi}"] \arrow[d, "\gamma"] & \mathrm{T}(L) \\ & & N & \end{tikzcd}$$

in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. We will prove below that γ is a quasi-isomorphism. Then the morphism $\theta := \mathrm{Q}(\tilde{\chi}) \circ \mathrm{Q}(\gamma)^{-1}$ will work.

It remains to prove that γ is a quasi-isomorphism. Let $\tilde{K}$ be the standard cone on id_L, and let $\tilde{\beta} : \tilde{K} \to \tilde{N}$ be the matrix morphism

$$\begin{bmatrix} \phi & 0 \\ 0 & \mathrm{id} \end{bmatrix} : \begin{bmatrix} L \\ \mathrm{T}(L) \end{bmatrix} \to \begin{bmatrix} M \\ \mathrm{T}(L) \end{bmatrix}.$$

This fits into a short exact sequence

$$0 \to \tilde{K} \xrightarrow{\tilde{\beta}} \tilde{N} \xrightarrow{\gamma} N \to 0$$

in $\mathbf{C}_{\mathrm{str}}(A, \mathsf{M})$. But the DG module $\tilde{K}$ is acyclic, and therefore γ is a quasi-isomorphism. □

The next proposition pertains only to the ring case. To prove it we shall require the following truncation operations. For any complex $M \in \mathbf{C}(A)$ its *stupid truncations* at an integer q are

$$\mathrm{stt}^{\leq q}(M) := \big(\cdots \to M^{q-1} \to M^q \to 0 \to 0 \to \cdots \big) \tag{13.1.16}$$

and

$$\mathrm{stt}^{\geq q}(M) := \big(\cdots \to 0 \to 0 \to M^q \to M^{q+1} \to \cdots \big). \tag{13.1.17}$$

They fit into an exact sequence

$$0 \to \mathrm{stt}^{\geq q}(M) \to M \to \mathrm{stt}^{\leq q-1}(M) \to 0 \tag{13.1.18}$$

in $\mathbf{C}_{\mathrm{str}}(A)$.

cmnt: move all truncation stuff to Sec 3?

Proposition 13.1.19. *Let A be a ring. The following are equivalent for $M \in \mathbf{D}(A)$:*

(i) *M has finite injective dimension.*
(ii) *M has finite injective dimension relative to $\mathbf{M}(A)$.*
(iii) *There is a quasi-isomorphism $M \to I$ in $\mathbf{C}_{\mathrm{str}}(A)$ to a bounded complex of injective A-modules I.*

Proof. (i) ⇒ (ii): This is trivial.

(ii) We may assume that $\mathrm{H}(M)$ is nonzero. Let $[q_0, q_1]$ be the injective concentration of the complex M relative to $\mathbf{M}(A)$, as in Definition 13.1.12; this is a bounded integer interval. Since $M \cong \mathrm{RHom}_A(A, M)$ in $\mathbf{D}(\mathbb{K})$, we see that

$$q_0 = \inf(\mathrm{H}(M)) \leq \sup(\mathrm{H}(M)) \leq q_1.$$

According to Corollary 10.4.12 there is quasi-isomorphism $M \to J$, where J is a complex of injective A-modules and $\inf(J) = q_0$. Take $I := \mathrm{smt}^{\leq q_1}(J)$, the smart truncation from (7.3.6). Then the canonical homomorphism $I \to J$ is a quasi-isomorphism. The complex I is concentrated in the integer interval $[q_0, q_1]$, and $I^q = J^q$ is injective for all $q < q_1$.

Let us prove that $I^{q_1} = \mathrm{Z}^{q_1}(J)$ is also an injective module. Classically we would use a cosyzygy argument. Here we use another trick. Define $I' := \mathrm{stt}^{\leq q_1 - 1}(I)$, so

$$I' = \big(\cdots 0 \to I^{q_0} \to \cdots \to I^{q_1 - 1} \to 0 \to \cdots \big).$$

This is a bounded complex of injectives. Consider the short exact sequence

$$0 \to I^{q_1}[-q_1] \to I \to I' \to 0$$

in $\mathsf{C}_{\mathrm{str}}(A)$. According to Proposition 13.1.15 this gives a distinguished triangle

$$I^{q_1}[-q_1] \to I \to I' \to I^{q_1}[-q_1+1] \tag{13.1.20}$$

in $\mathsf{D}(A)$. Take any A-module N. Applying $\mathrm{RHom}_A(N,-)$ to the distinguished triangle (13.1.20) and then taking cohomologies, we get a long exact sequence

$$\cdots \to \mathrm{Ext}_A^{q+q_1-1}(N, I') \to \mathrm{Ext}_A^{q}(N, I^{q_1}) \to \mathrm{Ext}_A^{q+q_1}(N, I) \to \cdots \tag{13.1.21}$$

in $\mathsf{M}(\mathbb{K})$. For any $q > 0$ the module $\mathrm{Ext}_A^{q+q_1-1}(N, I')$ vanishes trivially. By the definition of the interval $[q_0, q_1]$ and the existence of an isomorphism $M \cong I$ in $\mathsf{D}(A)$, for any $q > 0$ the module $\mathrm{Ext}_A^{q+q_1}(N, I)$ is zero. Hence $\mathrm{Ext}_A^{q}(N, I^{q_1}) = 0$ for all $q > 0$. This proves that the module I^{q_1} injective.

We have quasi-isomorphisms $M \to J$ and $I \to J$. Since I is K-injective, there is a quasi-isomorphism $M \to I$.

(iii) $\Rightarrow$ (i): This is also trivial. □

Exercise 13.1.22. State and prove the analogous result for finite projective dimension of complexes.

In the next definition, A is again a DG ring.

Definition 13.1.23. Let $\star, \triangle$ be boundedness conditions, and assume the left derived bifunctor

$$(- \otimes_A^{\mathrm{L}} -) : \mathsf{D}^{\star}(A^{\mathrm{op}}) \times \mathsf{D}^{\triangle}(A) \to \mathsf{D}(\mathbb{K})$$

exists. Let S be an integer interval of length $i \in \mathbb{N} \cup \{\pm\infty\}$.

1. Let $M \in \mathsf{D}^{\triangle}(A)$, and let $\mathsf{C} \subseteq \mathsf{D}^{\star}(A^{\mathrm{op}})$ be a full additive subcategory. We say that M has *flat concentration* S and *flat dimension* i relative to C if the functor
$$(- \otimes_A^{\mathrm{L}} M)|_{\mathsf{C}} : \mathsf{C} \to \mathsf{D}(\mathbb{K})$$
has cohomological displacement S.
2. If $\mathsf{C} = \mathsf{D}(A^{\mathrm{op}})$, then we omit the "relative to C" clause.

Proposition 13.1.24. *Let A be a ring. The following are equivalent for $M \in \mathsf{D}(A)$:*

- (i) *M has finite flat dimension.*
- (ii) *M has finite flat dimension relative to $\mathsf{M}(A^{\mathrm{op}})$.*
- (iii) *There is a quasi-isomorphism $P \to M$ in $\mathsf{C}_{\mathrm{str}}(A)$ from a bounded complex of flat A-modules P.*

Exercise 13.1.25. Prove Proposition 13.1.24. (The proof is similar to that of Proposition 13.1.19.)

Definition 13.1.26. Suppose the ring A is left noetherian.

1. We denote by $\mathsf{M}_{\mathrm{f}}(A)$ the full subcategory of $\mathsf{M}(A) = \mathsf{Mod}\, A$ on the finite (i.e. finitely generated) modules.
2. We denote by $\mathsf{D}_{\mathrm{f}}(A)$ the full subcategory of $\mathsf{D}(A) = \mathsf{D}(\mathsf{Mod}\, A)$ on the complexes with finite cohomology modules.

Because A is left noetherian, the category $\mathsf{M}_{\mathrm{f}}(A)$ is a thick abelian subcategory of $\mathsf{M}(A)$, and the category $\mathsf{D}_{\mathrm{f}}(A)$ is a pretriangulated subcategory of $\mathsf{D}(A)$. When viewed as a left module, $A \in \mathsf{M}_{\mathrm{f}}(A) \subseteq \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$.

Theorem 13.1.27. *Let A be a left noetherian ring, let N be an abelian category, let*

$$F, G : \mathbf{D}_{\mathrm{f}}(A) \to \mathbf{D}(\mathsf{N})$$

be triangulated functors, and let $\eta : F \to G$ be a morphism of triangulated functors. Assume that the morphism

$$\eta_A : F(A) \to G(A)$$

in $\mathbf{D}(\mathsf{N})$ is an isomorphism.

(1) *If F and G have bounded above cohomological displacements, then*

$$\eta_M : F(M) \to G(M)$$

is an isomorphism for every $M \in \mathbf{D}^-_{\mathrm{f}}(A)$.

(2) *If F and G have bounded cohomological displacements, then η_M is an isomorphism for every $M \in \mathbf{D}_{\mathrm{f}}(A)$.*

We shall require the next lemmas for the proof of the theorem.

Lemma 13.1.28. *Let D be a pretriangulated category, let $F, G : \mathsf{D} \to \mathbf{D}(\mathsf{N})$ be triangulated functors, let $\eta : F \to G$ be a a morphism of triangulated functors, and let*

$$L \xrightarrow{\phi} M \to N \to L[1]$$

be a distinguished triangle in D.

(1) *If the morphisms η_L and η_M are both isomorphisms, then η_N is an isomorphism.*

(2) *Let j be an integer. If $\mathrm{H}^{j-1}(F(N))$, $\mathrm{H}^{j-1}(G(N))$, $\mathrm{H}^j(F(N))$ and $\mathrm{H}^j(G(N))$ are all zero, and if $\mathrm{H}^j(\eta_L)$ is an isomorphism, then $\mathrm{H}^j(\eta_M)$ is an isomorphism.*

Proof. (1) In $\mathbf{D}(\mathsf{N})$ we get the commutative diagram

$$\begin{tikzcd} F(L) \arrow[r] \arrow[d, "\eta_L"] & F(M) \arrow[r] \arrow[d, "\eta_M"] & F(N) \arrow[r] \arrow[d, "\eta_N"] & F(L)[1] \arrow[d, "\eta_L[1]"] \\ G(L) \arrow[r] & G(M) \arrow[r] & G(N) \arrow[r] & G(L)[1] \end{tikzcd} \tag{13.1.29}$$

with horizontal distinguished triangles. According to Proposition 5.3.5, η_N is an isomorphism.

(2) passing to cohomologies in (13.1.29) we have a commutative diagram

$$\begin{tikzcd} \mathrm{H}^{j-1}(F(N)) \arrow[r] \arrow[d, "\mathrm{H}^{j-1}(\eta_N)"] & \mathrm{H}^j(F(L)) \arrow[r, "\mathrm{H}^j(F(\phi))"] \arrow[d, "\mathrm{H}^j(\eta_L)"] & \mathrm{H}^j(F(M)) \arrow[r] \arrow[d, "\mathrm{H}^j(\eta_M)"] & \mathrm{H}^j(F(N)) \arrow[d, "\mathrm{H}^j(\eta_N)"] \\ \mathrm{H}^{j-1}(G(N)) \arrow[r] & \mathrm{H}^j(G(L)) \arrow[r, "\mathrm{H}^j(G(\phi))"] & \mathrm{H}^j(G(M)) \arrow[r] & \mathrm{H}^j(G(N)) \end{tikzcd}$$

The vanishing assumption implies that $\mathrm{H}^j(F(\phi))$ and $\mathrm{H}^j(G(\phi))$ are isomorphisms. Hence $\mathrm{H}^j(\eta_M)$ is an isomorphism. □

Lemma 13.1.30. *Let D be a pretriangulated category, let $F, G : \mathsf{D} \to \mathbf{D}(\mathsf{N})$ be triangulated functors, and let $\eta : F \to G$ be a a morphism of triangulated functors. The following conditions are equivalent for $M \in \mathsf{D}$:*

(i) η_M *is an isomorphism.*
(ii) $\eta_{M[i]}$ *is an isomorphism for every integer* i.
(iii) *The morphism*

$$\mathrm{H}^j(\eta_M) : \mathrm{H}^j(F(M)) \to \mathrm{H}^j(G(M))$$

is an isomorphism for every integer j.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is because both F and G are triangulated functors. The equivalence (i) $\Leftrightarrow$ (iii) is because the functor $\mathrm{H} : \mathbf{D}(\mathsf{N}) \to \mathbf{G}(\mathsf{N})$ is conservative; see Corollary 7.1.8. $\square$

Proof of Theorem 13.1.27. (1) First assume P is a bounded complex of finite free A-modules. Then P is obtained from A by finitely many standard cones and translations. By Lemmas 13.1.28(1) and 13.1.30 it follows that η_P is an isomorphism.

Next let P be a bounded above complex of finite free A-modules. Choose some integer j. Let i_1 be an integer such that the integer interval $[-\infty, i_1]$ contains the cohomological displacements of F and G. Define $P' := \mathrm{stt}^{\leq j-i_1-2}(P)$, the stupid truncation of P below $j - i_1 - 2$; and let $P'' := \mathrm{stt}^{\geq j-i_1-1}(P)$, the complementary stupid truncation. See formulas (13.1.16) and (13.1.17). According to Proposition 13.1.15, the short exact sequence (13.1.18) gives a distinguished triangle

$$P'' \to P \to P' \to P''[1] \tag{13.1.31}$$

in $\mathbf{D}_{\mathrm{f}}(A)$. The complex P'' is a bounded complex of finite free A-modules, so we already know that $\eta_{P''}$ is an isomorphism. Hence $\mathrm{H}^j(\eta_{P''})$ is an isomorphism. On the other hand $\mathrm{H}(P')$ is concentrated in the interval $[-\infty, j - i_1 - 2]$. Therefore $\mathrm{H}^k(F(P')) = \mathrm{H}^k(G(P')) = 0$ for all $k \geq j - 1$. By Lemma 13.1.28(2), $\mathrm{H}^j(\eta_P)$ is an isomorphism. Because j is arbitrary, Lemma 13.1.30 says that η_P is an isomorphism.

Now take an arbitrary $M \in \mathbf{D}^-_{\mathrm{f}}(A)$. By Corollary 10.3.32 and Example 10.3.33 there is a resolution $P \to M$, where P is a bounded above complex of finite free A-modules. Since η_P is an isomorphism, so is η_M.

(2) Now we assume that the functors F and G have finite cohomological dimensions. Take any complex $M \in \mathbf{D}_{\mathrm{f}}(A)$. By Lemma 13.1.30 it suffices to prove that $\mathrm{H}^j(\eta_M)$ is an isomorphism for any integer j.

Let $[i_0, i_1]$ be a bounded integer interval that contains the cohomological displacements of the functors F and G. Define $M'' := \mathrm{smt}^{\leq j-i_0}(M)$, the smart truncation of M below $j - i_0$; and let $M' := \mathrm{smt}^{\geq j-i_0+1}(M)$, the complementary smart truncation. See formulas (7.3.6) and (7.3.7).

cmnt: maybe move the material on smart truncation from Sec 7 to Sec 3...

We obtain a short exact sequence

$$0 \to M'' \to M \to M' \to 0$$

of complexes. The cohomologies satisfy $\mathrm{H}^i(M'') = \mathrm{H}^i(M)$ and $\mathrm{H}^i(M') = 0$ for $i \leq j - i_0$; and $\mathrm{H}^i(M'') = 0$ and $\mathrm{H}^i(M') = \mathrm{H}^i(M)$ for $i \geq j - i_0 + 1$. Therefore we have a distinguished triangle

$$M'' \to M \to M' \to M''[1] \tag{13.1.32}$$

in $\mathbf{D}_{\mathrm{f}}(A)$, and $M'' \in \mathbf{D}^{-}_{\mathrm{f}}(A)$. By part (1) we know that $\eta_{M''}$ is an isomorphism, and therefore also $\mathrm{H}^j(\eta_{M''})$ is an isomorphism. The cohomology $\mathrm{H}(M')$ is concentrated in the interval $[j - i_0 + 1, \infty]$, and therefore the cohomologies $\mathrm{H}(F(M'))$ and $\mathrm{H}(G(M'))$ are concentrated in the interval $[j+1, \infty]$. In particular the objects $\mathrm{H}^{j-1}(F(M'))$, $\mathrm{H}^{j-1}(G(M'))$, $\mathrm{H}^j(F(M'))$ and $\mathrm{H}^j(G(M'))$ are zero. By Lemma 13.1.28(2), $\mathrm{H}^j(\eta_M)$ is an isomorphism. □

Next we give a similar theorem. Recall that if $\mathsf{N}_0 \subseteq \mathsf{N}$ is a thick abelian subcategory, then $\mathbf{D}_{\mathsf{N}_0}(\mathsf{N})$, the full subcategory of $\mathbf{D}(\mathsf{N})$ on the complexes whose cohomologies lie inside N_0, is a pretriangulated subcategory.

Theorem 13.1.33. *Let A be a left noetherian ring, let N be an abelian category, let $\mathsf{N}_0 \subseteq \mathsf{N}$ be a thick abelian subcategory, and let*

$$F : \mathbf{D}_{\mathrm{f}}(A)^{\mathrm{op}} \to \mathbf{D}(\mathsf{N})$$

be a triangulated functor. Assume that $F(A)$ belongs to $\mathbf{D}_{\mathsf{N}_0}(\mathsf{N})$.

(1) *If F has bounded below cohomological displacement, then $F(M)$ belongs to $\mathbf{D}_{\mathsf{N}_0}(\mathsf{N})$ for every $M \in \mathbf{D}^{-}_{\mathrm{f}}(A)$.*
(2) *If F has bounded cohomological displacement, then $F(M)$ belongs to $\mathbf{D}_{\mathsf{N}_0}(\mathsf{N})$ for every $M \in \mathbf{D}_{\mathrm{f}}(A)$.*

Proof. (1) First assume P is a bounded complex of finite free A-modules. Then P is obtained from A by finitely many standard cones and translations. Since $\mathbf{D}_{\mathsf{N}_0}(\mathsf{N})$ is a pretriangulated subcategory and F is a triangulated functor, it follows that $F(P) \in \mathbf{D}_{\mathsf{N}_0}(\mathsf{N})$.

Next let P be a bounded above complex of finite free A-modules. Choose some integer j. We want to prove that $\mathrm{H}^j(F(P)) \in \mathsf{N}_0$. Let i_0 be an integer such that the integer interval $[i_0, \infty]$ contains the cohomological displacement of F. Define $P' := \operatorname{stt}^{\leq -j-1+i_0}(P)$, the stupid truncation of P below $-j-1+i_0$; and let $P'' := \operatorname{stt}^{\geq j+i_0}(P)$, the complementary stupid truncation. We get a distinguished triangle (13.1.31) in $\mathbf{D}_{\mathrm{f}}(A)$. The complex P'' is a bounded complex of finite free A-modules, so we already know that $F(P'') \in \mathbf{D}_{\mathsf{N}_0}(\mathsf{N})$, and in particular $\mathrm{H}^j(F(P'')) \in \mathsf{N}_0$. On the other hand $\mathrm{H}(P')$ is concentrated in the interval $[-\infty, -j-1+i_0]$. Therefore $\mathrm{H}(F(P'))$ is concentrated in the interval $[j+1, \infty]$, and in particular $\mathrm{H}^{j-1}(F(P')) = \mathrm{H}^j(F(P')) = 0$. As we saw in the proof of Lemma 13.1.28(2), $\mathrm{H}^j(F(P'')) \to \mathrm{H}^j(F(P))$ is an isomorphism. The conclusion is that $\mathrm{H}^j(F(P)) \in \mathsf{N}_0$.

Now take an arbitrary $M \in \mathbf{D}^{-}_{\mathrm{f}}(A)$. There is a quasi-isomorphism $P \to M$, where P is a bounded above complex of finite free A-modules. So $F(M) \cong F(P)$, and thus $F(M) \in \mathbf{D}_{\mathsf{N}_0}(\mathsf{N})$.

(2) Now we assume that the functor F has finite cohomological dimension. Take any complex $M \in \mathbf{D}_{\mathrm{f}}(A)$. We want to prove that for any $j \in \mathbb{Z}$ the object $\mathrm{H}^j(F(M))$ lies in N_0.

Let $[i_0, i_1]$ be a bounded integer interval that contains the cohomological displacement of the functor F. Define $M'' := \operatorname{smt}^{\leq -j+1+i_1}(M)$, the smart truncation of M below $-j+1+i_1$; and let $M' := \operatorname{smt}^{\geq -j+2+i_1}(M)$, the complementary smart truncation. As we already noted in the proof of Theorem 13.1.27, there is a distinguished triangle (13.1.32) in $\mathbf{D}_{\mathrm{f}}(A)$. The cohomology of M' is concentrated in the interval $[-j+2+i_1, \infty]$, and therefore the cohomology of $F(M')$ is concentrated in the interval $[-\infty, j-2]$. In particular the objects $\mathrm{H}^{j-1}(F(M'))$ and $\mathrm{H}^j(F(M'))$ are zero. By the proof of Lemma 13.1.28(2), the morphism $\mathrm{H}^j(F(M'')) \to \mathrm{H}^j(F(M))$ is an isomorphism. But $M'' \in \mathbf{D}^-_{\mathrm{f}}(A)$, so as we proved in part (1), its cohomologies are inside N_0. □

Theorems 13.1.27 and 13.1.33 have several obvious modifications, for instance changing the variance of the functor F (replacing the source category by its opposite).

13.2. **Dualizing Complexes.** From here on in this section all rings are commutative noetherian by default.

Let A be a noetherian ring. We have the abelian categories $\mathbf{M}_{\mathrm{f}}(A) \subseteq \mathbf{M}(A)$ as before. But because A is commutative, the Hom bifunctor has another target:

$$\mathrm{Hom}_A(-,-) : \mathbf{M}(A)^{\mathrm{op}} \times \mathbf{M}(A) \to \mathbf{M}(A).$$

Likewise for the right derived bifunctor:

$$\mathrm{RHom}_A(-,-) : \mathbf{D}(A)^{\mathrm{op}} \times \mathbf{D}(A) \to \mathbf{D}(A).$$

When we fix the second argument M, we get an A-linear triangulated functor:

$$\mathrm{RHom}_A(-,M) : \mathbf{D}(A)^{\mathrm{op}} \to \mathbf{D}(A).$$

This is the sort of functor with which we will be concerned.

Let $M \in \mathbf{C}(A)$. The DG A-module

$$\mathrm{Hom}_A(M,M) = \mathrm{End}_A(M)$$

is a central noncommutative DG A-ring; there is a ring homomorphism

$$\alpha_M : A \to \mathrm{Hom}_A(M,M). \tag{13.2.1}$$

When we forget the ring structure, α_M becomes a homomorphism in $\mathbf{C}_{\mathrm{str}}(A)$.

Definition 13.2.2. Given a compex $M \in \mathbf{D}(A)$, the *derived homothety morphism*

$$\alpha^{\mathrm{R}}_M : A \to \mathrm{RHom}_A(M,M)$$

is the morphism in $\mathbf{D}(A)$ with this formula:

$$\alpha^{\mathrm{R}}_M := \eta_{M,M} \circ \mathrm{Q}(\alpha_M).$$

Namely the diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{Q}(\alpha_M)"] \arrow[rr, bend left, "\alpha^{\mathrm{R}}_M"] & \mathrm{Hom}_A(M,M) \arrow[r, "\eta_{M,M}"] & \mathrm{RHom}_A(M,M) \end{tikzcd}$$

in $\mathbf{D}(A)$ is commutative.

Exercise 13.2.3. Prove that if $\rho : M \to I$ is a K-injective resolution, then the diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{Q}(\alpha_I)"] \arrow[drr, "\alpha^{\mathrm{R}}_M"'] & \mathrm{Hom}_A(I,I) \arrow[r, "\eta_{I,I}", "\cong"'] & \mathrm{RHom}_A(I,I) \arrow[d, "\mathrm{RHom}(\mathrm{Q}(\rho),\mathrm{Q}(\rho)^{-1})", "\cong"'] \\ & & \mathrm{RHom}_A(R,R) \end{tikzcd}$$

in $\mathbf{D}(A)$ is commutative.

Exercise 13.2.4. Formulate and prove a version of the previous exercise with a K-projective resolution of M.

Definition 13.2.5. A complex $M \in \mathsf{D}(A)$ is said to have the *derived Morita property* if the derived homothety morphism

$$\alpha_M^{\mathrm{R}} : A \to \mathrm{RHom}_A(M, M)$$

in $\mathsf{D}(A)$ is an isomorphism.

Proposition 13.2.6. *The following conditions are equivalent for a complex $M \in \mathsf{D}(A)$:*

(i) *M has the derived Morita property.*

(ii) *The canonical ring homomorphism*

$$A \to \mathrm{End}_{\mathsf{D}(A)}(M)$$

is a bijective, and

$$\mathrm{Hom}_{\mathsf{D}(A)}(M, M[i]) = 0$$

for all $i \neq 0$.

Exercise 13.2.7. Prove Proposition 13.2.6. (Hint: see Corollary 12.6.8 and the preceeding material.)

Remark 13.2.8. In some texts, a complex M with the derived Morita property is called a *semi-dualizing complex*. This name is only partly justified, because this property occurs in the definition of a dualizing complexes – see Definition 13.2.9 below. However, there is a whole other class of complexes with the derived Morita property – these are the *tilting complexes*. Often these two classes of complexes are disjoint. More on these notions, and their noncommutative variants, will be in Section 17 of the book.

The next definition first appeared in [RD, Section V.2]. The injective dimension of a complex was defined in Definition 13.1.12.

Definition 13.2.9. Let A be a noetherian commutative ring. A complex of A-modules R is called a *dualizing complex* if it has the following three properties:

(i) $R \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$.

(ii) R has finite injective dimension.

(iii) R has the derived Morita property.

Recall that in the traditional literature (e.g. [Mats]), a noetherian ring A is called *regular* if all its local rings $A_{\mathfrak{p}}$ are regular local rings. The *Krull dimension* of A is the dimension of the scheme $\operatorname{Spec} A$; namely the supremum of the lengths of strictly ascending chains of prime ideals in A. In practice we never see regular rings that are not finite dimensional (there are only pretty exotic examples of them). The following convention will simplify matters for us:

Convention 13.2.10. We shall say that a noetherian commutative ring A is *regular* if it has finite Krull dimension, and all its local rings $A_{\mathfrak{p}}$ are regular local rings.

Any field $\mathbb{K}$, and the ring of integers $\mathbb{Z}$, are regular rings. If A is regular, then so is the polynomial ring $A[t_1, \ldots, t_n]$ in $n < \infty$ variables, and also the localization of A at any multiplicatively closed set. See [Mats, Chapter 7].

Example 13.2.11. As prove by Serre, see [Mats, Theorem 19.2], a regular ring A has *finite global cohomological dimension*. This means that there is a number $d \in \mathbb{N}$ such that for any modules $M, N \in \mathsf{M}(A)$ and any $q > d$, the Ext module $\operatorname{Ext}^q_A(M, N)$ vanishes. This implies that any module N has injective dimension $\leq d$ (and also projective dimension $\leq d$).

Taking $R := A$ we see that R satisfies condition (ii) of Definition 13.2.9. The other two conditions hold regardless of the regularity of A. Thus $R = A$ is a dualizing complex over the ring A.

In the Introduction, Subsection 0.1, we used this fact for $A = \mathbb{Z}$.

Definition 13.2.12. Given a dualizing complex $R \in \mathsf{D}(A)$, the *duality functor* associated to it is the triangulated functor

$$D : \mathsf{D}(A)^{\mathrm{op}} \to \mathsf{D}(A), \quad D := \mathrm{RHom}_A(-, R).$$

Let $I, M \in \mathsf{C}(A)$. There is a homomorphism

$$\tilde{\theta}_{M,I} : M \to \operatorname{Hom}_A\bigl(\operatorname{Hom}_A(M, I), I\bigr)$$

in $\mathsf{C}_{\mathrm{str}}(A)$ with formula

$$\tilde{\theta}_{M,I}(m)(\phi) := (-1)^{p \cdot q} \cdot \phi(m)$$

for $m \in M^p$ and $\phi \in \operatorname{Hom}_A(M, I)^q$.

Lemma 13.2.13. *Let R be a dualizing complex over A, with associated duality functor D. There is a unique morphism*

$$\theta : \mathrm{Id} \to D \circ D$$

of triangulated functors from $\mathsf{D}(A)$ to itself, such that if $\rho : R \to I$ is a K-injective resolution, then for any complex $M \in \mathsf{D}(A)$ the diagram

$$\begin{tikzcd}
M \arrow[r, "\mathrm{Q}(\tilde{\theta}_{M,I})"] \arrow[d, "\theta_M"'] & \operatorname{Hom}_A\bigl(\operatorname{Hom}_A(M, I), I\bigr) \arrow[r] & \mathrm{RHom}_A\bigl(\mathrm{RHom}_A(M, I), I\bigr) \arrow[d] \\
D(D(M)) \arrow[rr, "\mathrm{id}"] & & \mathrm{RHom}_A\bigl(\mathrm{RHom}_A(M, R), R\bigr)
\end{tikzcd}$$

in $\mathsf{D}(A)$, in which the unlabeled morphisms are

$$\mathrm{RHom}(\eta^{-1}_{M,I}, \mathrm{id}) \circ \eta_{\operatorname{Hom}_A(M,I),I}$$

and

$$\mathrm{RHom}\bigl(\mathrm{RHom}(\mathrm{id}, \mathrm{Q}(\rho)), \mathrm{Q}(\rho)^{-1}\bigr),$$

is commutative.

Exercise 13.2.14. Prove Lemma 13.2.13.

Here is the first important result regarding dualizing complexes.

Theorem 13.2.15. *Suppose R is a dualizing complex over the noetherian commutative ring A, with associated duality functor D. Then for any complex $M \in \mathsf{D}_{\mathrm{f}}(A)$ the following hold:*

1. *The complex $D(M)$ belongs to $\mathsf{D}_{\mathrm{f}}(A)$.*
2. *The morphism*

$$\theta_M : M \to D(D(M))$$

in $\mathsf{D}(A)$ is an isomorphism.

Proof. (1) Condition (b) of Definition 13.2.9 says that the functor D has finite cohomological dimension. Condition (a) says that $D(A) \in \mathbf{D}_{\mathrm{f}}(A)$. The assertion follows from Theorem 13.1.33, with $\mathsf{N}_0 := \mathbf{M}_{\mathrm{f}}(A)$.

(2) The composition $D \circ D$ is a functor with finite cohomological dimension (at most twice the injective dimension of R). The cohomological dimension of the identity functor is 0 (if $A \neq 0$). By condition (c) of Definition 13.2.9 we know that θ_A is an isomorphism. Now we can use Theorem 13.1.27. □

Corollary 13.2.16. *Under the assumptions of Theorem* 13.2.15*, let $\star$ be one of the boundedness conditions* b, $+$, $-$ *or "empty", and let $-\star$ be the reverse boundedness condition, namely* b, $-$, $+$ *or "empty", respectively. Then the functor*

$$D : \mathbf{D}_{\mathrm{f}}^{\star}(A)^{\mathrm{op}} \to \mathbf{D}_{\mathrm{f}}^{-\star}(A)$$

is an equivalence of pretriangulated categories.

Proof. The previous theorem tells us that D is its own quasi-inverse. The claim about the boundedness holds because D has finite cohomological dimension. □

We saw that dualizing complexes exists over regular rings. This fact is used for the very general existence result below. First a definition and some lemmas.

Definition 13.2.17. Let $f : A \to B$ be a ring homomorphism. We say that f is an *essentially finite type homomorphism* if it factors as $A \to B' \to B$, where $A \to B'$ is finite type, and $B' \to B$ is a localization at some multiplicatively closed set. In this case we also say that B is an *essentially finite type A-ring.*

Example 13.2.18. Let X be a finite type scheme over A, and let $x \in X$ be a point. Then the local ring $\mathcal{O}_{X,x}$ is essentially finite type over A.

A ring homomorphism $A \to B$ gives rise to a forgetful functor $\operatorname{Rest} : \mathbf{M}(B) \to \mathbf{M}(A)$, that in turn determines a DG functor $\operatorname{Rest} : \mathbf{C}(B) \to \mathbf{C}(A)$ and a triangulated functor $\operatorname{Rest} : \mathbf{D}(B) \to \mathbf{D}(A)$. These functors are going to be implicit in the discussion below.

Lemma 13.2.19. *Let $A \to B$ be a ring homomorphism.*

(1) *If $I \in \mathbf{C}(A)$ is K-injective, then $J := \operatorname{Hom}_A(B, I) \in \mathbf{C}(B)$ is K-injective.*
(2) *Given $M \in \mathbf{D}(A)$, let us define*

$$N := \operatorname{RHom}_A(B, M) \in \mathbf{D}(B).$$

Then there is an isomorphism

$$\operatorname{RHom}_B(-, N) \cong \operatorname{RHom}_A(-, M)$$

of triangulated functors $\mathbf{D}(B) \to \mathbf{D}(B)$.

Proof. (1) The is an adjunction calculation. Suppose $L \in \mathbf{C}(B)$ is acyclic. There are isomorphisms

$$\operatorname{Hom}_B(L, J) \cong \operatorname{Hom}_B\big(L, \operatorname{Hom}_A(B, I)\big) \cong \operatorname{Hom}_A(L, I) \tag{13.2.20}$$

in $\mathbf{C}(B)$. Since I is K-injective over A, this complex is acyclic.

(2) Choose a K-injective resolution $M \to I$ in $\mathbf{C}(A)$. Let J be as above. Then $N \to J$ is a K-injective resolution in $\mathbf{C}(B)$. There are isomorphisms of triangulated functors

$$\operatorname{RHom}_A(-, M) \cong \operatorname{Hom}_A(-, I) \tag{13.2.21}$$

and

$$\operatorname{RHom}_B(-, N) \cong \operatorname{Hom}_B(-, J), \tag{13.2.22}$$

where the first functors (13.2.21) are from $\mathsf{D}(A)$ to itself, and the functors (13.2.22) are from $\mathsf{D}(B)$ to itself. But given $L \in \mathsf{C}(B)$ we can view $\operatorname{Hom}_A(L, I)$ as a complex of B-modules, and in this way the functors (13.2.21) become triangulated functors from $\mathsf{D}(B)$ to itself. Formula (13.2.20) shows that the functors (13.2.21) and (13.2.22) are isomorphic. □

Lemma 13.2.23. *Let $A \to B$ be a flat ring homomorphism, let $M \in \mathsf{D}^-_{\mathrm{f}}(A)$, and let $N \in \mathsf{D}^+(A)$. Then there is an isomorphism*

$$\operatorname{RHom}_A(M, N) \otimes_A B \cong \operatorname{RHom}_B(B \otimes_A M, B \otimes_A N)$$

in $\mathsf{D}(B)$. This isomorphism is functorial in M and N.

Proof. First we note that since B is a flat A-module, the functor $- \otimes_A B$ is triangulated (it is its own left derived functor), and it goes $\mathsf{D}(A) \to \mathsf{D}(B)$.

Let's choose a resolution $P \to M$ where P is a bounded above complex of finite free A-modules. After possibly truncating the complex N from below, we can assume it is a bounded below complex of A-modules. There is an isomorphism

$$\operatorname{RHom}_A(M, N) \otimes_A B \cong \operatorname{Hom}_A(P, N) \otimes_A B \tag{13.2.24}$$

in $\mathsf{D}(B)$. We claim that the canonical homomorphism

$$\operatorname{Hom}_A(P, N) \otimes_A B \to \operatorname{Hom}_A(P, N \otimes_A B) \tag{13.2.25}$$

in $\mathsf{C}(B)$ is bijective. This is because of finiteness. To be explicit, in each degree i we have

$$\operatorname{Hom}_A(P, N)^i = \prod_j \operatorname{Hom}_A(P^j, N^{i+j}).$$

This is actually a finite product, because $P^j = 0$ for $j \gg 0$, and $N^{i+j} = 0$ for $j \ll 0$. And for each pair (i, j) the module $\operatorname{Hom}_A(P^j, N^{i+j})$ is a finite product of copies on N^{i+j}, because P^j is a finite free A-module. This shows that

$$\operatorname{Hom}_A(P^j, N^{i+j}) \otimes_A B \cong \operatorname{Hom}_A(P^j, N^{i+j} \otimes_A B).$$

Taking the product on all j we conclude that (13.2.25) is indeed bijective.

Next we apply the usual change of ring adjunction to get the isomorphism

$$\operatorname{Hom}_A(P, B \otimes_A N) \cong \operatorname{Hom}_B(B \otimes_A P, B \otimes_A N) \tag{13.2.26}$$

in $\mathsf{C}(B)$. Since $B \otimes_A P \to B \otimes_A M$ is a K-projective resolution over B, there is an isomorphism

$$\operatorname{Hom}_B(B \otimes_A P, B \otimes_A N) \cong \operatorname{RHom}_B(B \otimes_A M, B \otimes_A N) \tag{13.2.27}$$

in $\mathsf{D}(B)$.

Combining the isomorphisms (13.2.24), (13.2.25), (13.2.26) and (13.2.27) gives us the desired isomorphism. The functoriality is clear. □

Lemma 13.2.28. *Let I be an A-module. The following conditions are equivalent*:

(i) *I is injective.*
(ii) *For any finite A-module M the module $\operatorname{Ext}^1_A(M, I)$ is zero.*

Exercise 13.2.29. Prove Lemma 13.2.28. (Hint: use the Baer criterion Theorem 2.6.7.)

Lemma 13.2.30. *The injective dimension of a complex $N \in \mathbf{D}(A)$ equals the cohomological dimension of the functor*

$$\mathrm{RHom}_A(-,N)|_{\mathbf{M}_{\mathrm{f}}(A)^{\mathrm{op}}} : \mathbf{M}_{\mathrm{f}}(A)^{\mathrm{op}} \to \mathbf{D}(A).$$

Proof. By definition the injective dimension of N, say d, is the cohomological dimension of the functor

$$\mathrm{RHom}_A(-,N) : \mathbf{D}(A)^{\mathrm{op}} \to \mathbf{D}(A).$$

Let d' be the cohomological dimension of the functor $\mathrm{RHom}_A(-,N)|_{\mathbf{M}_{\mathrm{f}}(A)^{\mathrm{op}}}$. Obviously the inequality $d \geq d'$ holds. For the reverse inequality we may assume that $\mathrm{H}(N)$ is nonzero and $d' < \infty$. This implies that there are integers $q_1 = q_0 + d'$ such that for any $M \in \mathbf{M}_{\mathrm{f}}(A)$ there is an inclusion

$$\mathrm{con}\big(\mathrm{RHom}_A(M,N)\big) \subseteq [q_0, q_1].$$

In particular, for $M = A$, we get $\mathrm{con}(\mathrm{H}(N)) \subseteq [q_0, q_1]$. Let $N \to J$ be an injective resolution in $\mathbf{C}(A)$ with $\inf(J) = q_0$. Take $I := \mathrm{smt}^{\leq q_1}(J)$, the smart truncation from (7.3.6). The proof of Proposition 13.1.19, plus Lemma 13.2.28, show that $N \to I$ is an injective resolution. But then

$$\mathrm{RHom}_A(-,N) \cong \mathrm{Hom}_A(-,I),$$

so this functor has cohomological displacement in the interval $[q_0, q_1]$, that has length d'. □

Proposition 13.2.31. *Let $A \to B$ be a finite ring homomorphism, and let R_A be a dualizing complex over A. Then the complex*

$$R_B := \mathrm{RHom}_A(B, R_A) \in \mathbf{D}(B)$$

is a dualizing complex over B.

Proof. Consider the functors $D_A := \mathrm{RHom}_A(-,R_A)$ and $D_B := \mathrm{RHom}_B(-,R_B)$. As explained in the proof of Lemma 13.2.19(2), they are isomorphic as functors from $\mathbf{D}(B)$ to itself. Since $R_B = D_A(B)$ and $B \in \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$, by Corollary 13.2.16 we have $R_B \in \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. But then also $R_B \in \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(B)$. Next, because $D_B(L) \cong D_A(L)$ for any $L \in \mathbf{D}(B)$, this implies that the cohomological dimension of D_B is at most that of D_A, which is finite. We see that the injective dimension of the complex R_B is finite. Lastly, there is an isomorphism $D_B \circ D_B \cong D_A \circ D_A$ as functors from $\mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(B)$ to itself, and hence $\theta : \mathrm{Id} \to D_B \circ D_B$ is an isomorphism. Applying this to the object $B \in \mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(B)$ we see that

$$\alpha = \theta_B : B \to (D_B \circ D_B)(B)$$

is an isomorphism. So R_B has the derived Morita property. The conclusion is that R_B is a dualizing complex over B. □

Proposition 13.2.32. *Let $A \to B$ be a localization ring homomorphism, and let R_A be a dualizing complex over A. Then the complex*

$$R_B := B \otimes_A R_A \in \mathbf{D}(B)$$

is a dualizing complex over B.

Proof. It is clear that $R_B \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(B)$. By Lemma 13.2.30, to compute the injective dimension of R_B it is enough to look at $\mathrm{RHom}_B(M, R_B)$ for $M \in \mathsf{M}_{\mathrm{f}}(B)$. We can find a finite A-submodule $M' \subseteq M$ such that $B \cdot M' = M$; and then $M \cong B \otimes_A M'$. Lemma 13.2.23 tells us that

$$\mathrm{RHom}_B(M, R_B) \cong \mathrm{RHom}_A(M', R_A) \otimes_A B.$$

We conclude that the injective dimension of R_B is at most that of R_A, which is finite. Lastly, by the same lemma we get an isomorphism

$$\mathrm{RHom}_B(R_B, R_B) \cong \mathrm{RHom}_A(R_A, R_A) \otimes_A B,$$

and it is compatible with the morphisms from B. Thus R_B has the derived Morita property. □

Theorem 13.2.33. *Let $\mathbb{K}$ be a regular ring, and let A be an essentially finite type $\mathbb{K}$-ring. Then A has a dualizing complex.*

Proof. The ring homomorphism $\mathbb{K} \to A$ can be factored as $\mathbb{K} \to A_{\mathrm{pl}} \to A_{\mathrm{ft}} \to A$, where $A_{\mathrm{pl}} = \mathbb{K}[t_1, \ldots, t_n]$ is a polynomial ring, $A_{\mathrm{pl}} \to A_{\mathrm{ft}}$ is surjective, and $A_{\mathrm{ft}} \to A$ is a localization. (The subscripts stand for "polynomial" and "finite type" respectively.) According to [Mats, Theorem 19.5] the ring A_{pl} is regular; so, as shown in Example 13.2.18, the complex $R_{\mathrm{pl}} := A_{\mathrm{pl}}$ is a dualizing complex over A_{pl}.

Define

$$R_{\mathrm{ft}} := \mathrm{RHom}_{A_{\mathrm{pl}}}(A_{\mathrm{ft}}, R_{\mathrm{pl}}) \in \mathsf{D}(A_{\mathrm{ft}}).$$

By Proposition 13.2.31 this is a dualizing complex over A_{ft}. Finally define

$$R := A \otimes_{A_{\mathrm{ft}}} R_{\mathrm{ft}} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A).$$

By Proposition 13.2.32 this is dualizing complex over A. □

The proof of Theorem 13.2.33 might give the impression that A could have a lot of nonisomorphic dualizing complexes. This is not quite true, as we now prove.

Theorem 13.2.34. *Let A be a noetherian ring with connected spectrum, and let R and R' be dualizing complexes over A. Then there is a rank 1 projective A-module L and an integer d, such that $R' \cong R \otimes_A L[d]$ in $\mathsf{D}(A)$.*

Some lemmas first.

Lemma 13.2.35 (Künneth Trick)**.** *Let $M, M' \in \mathsf{D}^-(A)$, and let $i, i' \in \mathbb{Z}$ be such that $\sup(\mathrm{H}(M)) \leq i$ and $\sup(\mathrm{H}(M')) \leq i'$. Then*

$$\mathrm{H}^{i+i'}(M \otimes^{\mathrm{L}}_A M') \cong \mathrm{H}^i(M) \otimes_A \mathrm{H}^{i'}(M').$$

Exercise 13.2.36. Prove Lemma 13.2.35.

Lemma 13.2.37 (Projective Truncation Trick)**.** *Let $M \in \mathsf{D}(A)$, with $i_1 := \sup(\mathrm{H}(M)) \in \mathbb{Z}$. Assume the A-module $P := \mathrm{H}^{i_1}(M)$ is projective. Then there is an isomorphism*

$$M \cong \mathrm{smt}^{\leq i_1 - 1}(M) \oplus P[-i_1]$$

in $\mathsf{D}(A)$.

Exercise 13.2.38. Prove Lemma 13.2.37. (Hint: first replace M with $\mathrm{smt}^{\leq i_1}(M)$. Then prove that P is a direct summand of M^{i_1}.))

By a *principal open set* in $\operatorname{Spec} A$ we mean a set of the form $\operatorname{Spec} A_s$, where A_s is the localization of A at the element $s \in A$. Note that

$$\operatorname{Spec} A_s = \{\mathfrak{p} \in \operatorname{Spec} A \mid s \notin \mathfrak{p}\}.$$

Lemma 13.2.39. *Let $M, M' \in \mathsf{M}_{\mathrm{f}}(A)$, and let $\mathfrak{p} \subseteq A$ be a prime ideal.*

(1) *If $M_{\mathfrak{p}} \neq 0$ and $M'_{\mathfrak{p}} \neq 0$ then $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} M'_{\mathfrak{p}} \neq 0$.*
(2) *If $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} M'_{\mathfrak{p}} \cong A_{\mathfrak{p}}$ then $M_{\mathfrak{p}} \cong M'_{\mathfrak{p}} \cong A_{\mathfrak{p}}$.*
(3) *If $M_{\mathfrak{p}} \cong A_{\mathfrak{p}}$, then there is a principal open neighborhood $\operatorname{Spec} A_s$ of $\mathfrak{p}$ in $\operatorname{Spec} A$ such that $M_s \cong A_s$ as A_s-modules.*

Exercise 13.2.40. Prove Lemma 13.2.39. (Hint: use the Nakayama Lemma.)

Here is a pretty difficult technical lemma.

Lemma 13.2.41. *In the situation of the theorem, let $M, M' \in \mathsf{D}^-_{\mathrm{f}}(A)$ satisfy $M \otimes^{\mathrm{L}}_A M' \cong A$ in $\mathsf{D}(A)$. Then $M \cong L[d]$ in $\mathsf{D}(A)$ for some rank 1 projective A-module L and an integer d.*

Proof. For any prime $\mathfrak{p} \subseteq A$ let $M_{\mathfrak{p}} := A_{\mathfrak{p}} \otimes_A M$, and define

$$e_{\mathfrak{p}} := \sup(\mathrm{H}(M_{\mathfrak{p}})) \in \mathbb{Z} \cup \{-\infty\}.$$

Define the number $e'_{\mathfrak{p}}$ similarly.

Fix one prime $\mathfrak{p}$. Since

$$M_{\mathfrak{p}} \otimes^{\mathrm{L}}_{A_{\mathfrak{p}}} M'_{\mathfrak{p}} \cong A_{\mathfrak{p}} \tag{13.2.42}$$

is nonzero, it follows that $\mathrm{H}(M_{\mathfrak{p}}) \neq 0$ and $\mathrm{H}(M'_{\mathfrak{p}}) \neq 0$. So $e_{\mathfrak{p}}, e'_{\mathfrak{p}} \in \mathbb{Z}$, and $\mathrm{H}^{e_{\mathfrak{p}}}(M_{\mathfrak{p}})$, $\mathrm{H}^{e'_{\mathfrak{p}}}(M'_{\mathfrak{p}})$ are nonzero finite $A_{\mathfrak{p}}$-modules. By Lemma 13.2.39(1) we know that

$$\mathrm{H}^{e_{\mathfrak{p}}}(M_{\mathfrak{p}}) \otimes_{A_{\mathfrak{p}}} \mathrm{H}^{e'_{\mathfrak{p}}}(M'_{\mathfrak{p}}) \neq 0.$$

According to Lemma 13.2.35 we have

$$\mathrm{H}^{e_{\mathfrak{p}}}(M_{\mathfrak{p}}) \otimes_{A_{\mathfrak{p}}} \mathrm{H}^{e'_{\mathfrak{p}}}(M'_{\mathfrak{p}}) \cong \mathrm{H}^{(e_{\mathfrak{p}}+e'_{\mathfrak{p}})}(M_{\mathfrak{p}} \otimes^{\mathrm{L}}_{A_{\mathfrak{p}}} M'_{\mathfrak{p}}) \cong \mathrm{H}^{(e_{\mathfrak{p}}+e'_{\mathfrak{p}})}(A_{\mathfrak{p}}).$$

But $A_{\mathfrak{p}}$ is concentrated in degree 0; this forces $e_{\mathfrak{p}} + e'_{\mathfrak{p}} = 0$ and

$$\mathrm{H}^{e_{\mathfrak{p}}}(M_{\mathfrak{p}}) \otimes_{A_{\mathfrak{p}}} \mathrm{H}^{e'_{\mathfrak{p}}}(M'_{\mathfrak{p}}) \cong A_{\mathfrak{p}}$$

in $\mathsf{D}(A_{\mathfrak{p}})$. By Lemma 13.2.39(2) we now see that

$$\mathrm{H}^{e_{\mathfrak{p}}}(M_{\mathfrak{p}}) \cong \mathrm{H}^{e'_{\mathfrak{p}}}(M'_{\mathfrak{p}}) \cong A_{\mathfrak{p}}. \tag{13.2.43}$$

According to Lemma 13.2.37 there are isomorphisms

$$M_{\mathfrak{p}} \cong A_{\mathfrak{p}}[-e_{\mathfrak{p}}] \oplus \operatorname{smt}^{\leq e_{\mathfrak{p}}-1}(M_{\mathfrak{p}}) \tag{13.2.44}$$

and

$$M'_{\mathfrak{p}} \cong A_{\mathfrak{p}}[-e'_{\mathfrak{p}}] \oplus \operatorname{smt}^{\leq e'_{\mathfrak{p}}-1}(M'_{\mathfrak{p}})$$

in $\mathsf{D}(A_{\mathfrak{p}})$. These, with (13.2.42), give an isomorphism

$$\big(A_{\mathfrak{p}}[-e_{\mathfrak{p}}] \oplus \operatorname{smt}^{\leq e_{\mathfrak{p}}-1}(M_{\mathfrak{p}})\big) \otimes_{A_{\mathfrak{p}}} \big(A_{\mathfrak{p}}[-e'_{\mathfrak{p}}] \oplus \operatorname{smt}^{\leq e'_{\mathfrak{p}}-1}(M'_{\mathfrak{p}})\big) \cong A_{\mathfrak{p}}. \tag{13.2.45}$$

The left side of (13.2.45) is the direct sum of four objects. Passing to the cohomology of (13.2.45) we see that

$$N := \mathrm{H}\big(\operatorname{smt}^{\leq e_{\mathfrak{p}}-1}(M_{\mathfrak{p}})[-e'_{\mathfrak{p}}]\big)$$

is a direct summand of $A_{\mathfrak{p}}$. But, since $e'_{\mathfrak{p}} + e_{\mathfrak{p}} = 0$, the graded module N is concentrated in the degree interval $[\infty, -1]$. It follows that $N = 0$. Therefore, by (13.2.44), we deduce that

$$M_{\mathfrak{p}} \cong A_{\mathfrak{p}}[-e_{\mathfrak{p}}]. \tag{13.2.46}$$

The calculation above works for any prime $\mathfrak{p}$. From (13.2.46) we get

$$A_{\mathfrak{p}} \otimes_A \mathrm{H}^i(M) \cong \mathrm{H}^i(M_{\mathfrak{p}}) \cong \begin{cases} A_{\mathfrak{p}} & \text{if } i = e_{\mathfrak{p}}, \\ 0 & \text{otherwise.} \end{cases} \tag{13.2.47}$$

We now use Lemma 13.2.39(3) to deduce that for any prime $\mathfrak{p}$ there is an open neighborhood $U_{\mathfrak{p}}$ of $\mathfrak{p}$ in $\operatorname{Spec} A$ such that $\mathrm{H}^{e_{\mathfrak{p}}}(M_{\mathfrak{q}}) \cong A_{\mathfrak{q}}$ for all $\mathfrak{q} \in U_{\mathfrak{p}}$. This implies, by equation (13.2.47), that $e_{\mathfrak{q}} = e_{\mathfrak{p}}$. Therefore $\mathfrak{p} \mapsto e_{\mathfrak{p}}$ is a locally constant function $\operatorname{Spec} A \to \mathbb{Z}$. We assumed that $\operatorname{Spec} A$ is connected, and this implies that this is a constant function, say $e_{\mathfrak{p}} = -d$ for some integer d.

Define $L := \mathrm{H}^{-d}(M) \in \mathbf{M}_{\mathrm{f}}(A)$. Using truncation we see that $M \cong L[d]$ in $\mathbf{D}(A)$. We know that $L_{\mathfrak{p}} \cong A_{\mathfrak{p}}$ for all primes $\mathfrak{p}$. Finally, Lemma 13.2.23 says that the A-module L is projective. □

Remark 13.2.48. Lemma 13.2.41 is actually true in much greater generality: the ring A does not have to be noetherian, and we do not have to assume that the complexes M and M' have bounded above or finite cohomology. The proof is harder. See [Ye10, Theorem 6.13].

Proof of Theorem 13.2.34. Define the duality functors $D := \mathrm{RHom}_A(-, R)$ and $D' := \mathrm{RHom}_A(-, R')$; these are finite dimensional contravariant triangulated functors from $\mathbf{D}_{\mathrm{f}}(A)$ to itself. And define $F := D' \circ D$ and $F' := D \circ D'$, that are finite dimensional covariant triangulated functors from $\mathbf{D}_{\mathrm{f}}(A)$ to itself. Let

$$M := F(A) = (D'(D(A)) = \mathrm{RHom}_A(R, R') \tag{13.2.49}$$

and

$$M' := F'(A) = (D(D'(A)) = \mathrm{RHom}_A(R', R).$$

These are objects of $\mathbf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$.

For any object $N \in \mathbf{D}(A)$ there is a morphism

$$\psi_N : N \otimes^{\mathrm{L}}_A \mathrm{RHom}_A(R, R') \to \mathrm{RHom}_A\big(\mathrm{RHom}_A(N, R), R'\big)$$

defined as follows: we choose a K-projective resolution $P \to N$ and a K-injective resolution $R' \to I'$. Then ψ_N is represented by the obvious homomorphism of complexes

$$P \otimes_A \mathrm{Hom}_A(R, I') \to \mathrm{Hom}_A\big(\mathrm{Hom}_A(P, R), I'\big).$$

As N changes, ψ_N is a morphism of triangulated functors

$$\psi : - \otimes^{\mathrm{L}}_A M \to D' \circ D = F.$$

For $N = A$ the morphism ψ_A is an isomorphism, by equation (13.2.49). The functor F has finite cohomological dimension, and the functor $- \otimes^{\mathrm{L}}_A M$ has bounded above cohomological displacement. According to Theorem 13.1.27, the morphism ψ_N is an isomorphism for any $N \in \mathbf{D}^{-}_{\mathrm{f}}(A)$. In particular this is true for $N := M'$. So, using Theorem 13.2.15, we obtain

$$M' \otimes^{\mathrm{L}}_A M \cong (D' \circ D)(M') \cong (D' \circ D \circ D \circ D')(A) \cong A.$$

According to Lemma 13.2.41 there is an isomorphism $M \cong L[d]$. Finally, using the isomorphism ψ_R, we get

$$R \otimes_A L[d] \cong F(R) = D'(D(R)) \cong D'(A) = R'.$$

□

What if $\operatorname{Spec} A$ has more than one connected component? A decomposition of $\operatorname{Spec} A$ into open-closed subschemes

$$\operatorname{Spec} A = \operatorname{Spec} A_1 \sqcup \cdots \sqcup \operatorname{Spec} A_r$$

corresponds to a decomposition of A into a product of rings:

$$A = A_1 \times \cdots \times A_r. \tag{13.2.50}$$

The noetherian property implies that $\operatorname{Spec} A$ has only finitely many connected components. If each $\operatorname{Spec} A_i$ in (13.2.50) is connected and nonempty, then this is called the *connected component decomposition of A*.

Let $\mathsf{K}_1, \ldots, \mathsf{K}_r$ be pretriangulated categories. The product category $\mathsf{K} := \prod_{i=1}^{r} \mathsf{K}_i$ has a pretriangulated structure such that the functors $\mathsf{K}_i \to \mathsf{K} \to \mathsf{K}_i$ are triangulated.

Proposition 13.2.51. *Given a ring isomorphism $A \cong \prod_{i=1}^{r} A_i$, the functor*

$$M \mapsto \prod_i A_i \otimes_A M$$

is an equivalence of pretriangulated categories

$$\mathsf{D}(A) \to \prod_i \mathsf{D}(A_i).$$

Exercise 13.2.52. Prove Proposition 13.2.51.

Corollary 13.2.53. *Let R and R' be dualizing complexes over A, and let* (13.2.50) *be the connected component decomposition of A. Then there is an isomorphism*

$$R' \cong R \otimes_A \big(L_1[d_1] \oplus \cdots \oplus L_r[d_r]\big)$$

in $\mathsf{D}(A)$, where each L_i is a rank 1 projective A_i-module, and each d_i is an integer. Furthermore, the modules L_i are unique up to isomorphism, and the integers d_i are unique.

Exercise 13.2.54. Prove Corollary 13.2.53.

Remark 13.2.55. A rank 1 projective A-module L is also called an *invertible A-module*. This is because L is invertible for the tensor product. Recall that the group of isomorphism classes of invertible A-modules is the *commutative Picard group* $\operatorname{Pic}_A(A)$.

The *commutative derived Picard group* $\operatorname{DPic}_A(A)$ is the abelian group $\operatorname{Pic}_A(A) \times \mathbb{Z}^r$ that classifies dualizing complexes over A, as in Corollary 13.2.53.

Now assume that A is *noncommutative*, and flat central over a commutative ring $\mathbb{K}$. There are noncommutative versions of dualizing complexes and of "invertible" complexes, that are called *tilting complexes*. The latter form the nonabelian group $\operatorname{DPic}_{\mathbb{K}}(A)$, and it classifies noncommutative dualizing complexes. See [Ric1], [Ric2], [Kel], [Ye4] and [RoZi]. We hope to get to this material in Section 17 of the book.

Remark 13.2.56. The lack of uniqueness of dualizing complexes has always been a source of difficulty. A certain uniqueness or functoriality is needed, already for proving existence of dualizing complexes on schemes.

In [RD] Grothendieck utilized local and global duality in order to formulate a suitable uniqueness of dualizing complexes. This approach was very cumbersome (even without providing details!)

Since then there have been a few approaches in the literature to attack this difficulty. Generally speaking, these approaches came in two flavors:

- Representablity: this started with Deligne's Appendix to [RD], and continued most notably in the work of Neeman, and of Lipman et al. See [Ne2], [Li1] and their references.
- Explicit Constructions: mostly in the early work of Lipman et al., including [Li1] and [LNS], and in the work of Yekutieli [Ye2], and [Ye3] and [Ye6]. references.

In the Section ??? of the book we will present *rigid dualizing complexes*, for which there is a built-in functoriality.

13.3. **More on Injective Resolutions.** We start with a few facts about injective modules over rings that are neither commutative nor noetherian. Sources for this material are [Rot] and [Lam].

Definition 13.3.1.

(1) Let M be an A-module. A submodule $N \subseteq M$ is called an *essential submodule* if for every nonzero submodule $L \subseteq M$, the intersection $N \cap L$ is nonzero. In this case we also say that M is an *essential extension* of N.
(2) An *essential monomorphism* is a monomorphism $\phi : N \rightarrowtail M$ whose image is an essential submodule of M.
(3) Let M be an A-module. An *injective hull* (or *injective envelope*) of M is an injective module I, together with an essential monomorphism $M \rightarrowtail I$.

Proposition 13.3.2. *Any A-module M admits an injective hull.*

Proof. See [Lam, Section 3.D]. □

There is a weak uniqueness result for injective hulls.

Proposition 13.3.3. *Let M be an A-module, and suppose $\phi : M \rightarrowtail I$ and $\phi' : M \rightarrowtail I'$ are monomorphisms into injective modules.*

(1) *If ϕ is essential, then there is a monomorphism $\psi : I \xrightarrow{\simeq} I'$ such that $\psi \circ \phi = \phi'$.*
(2) *If ϕ' is also essential, then ψ above is an isomorphism.*

Exercise 13.3.4. Prove Proposition 13.3.3.

In classical homological algebra we talk about the minimal injective resolution of a module. Let us recall it. We start with taking the injective hull $M \rightarrowtail I^0$. This gives an exact sequence

$$0 \to M \to I^0 \to M^1 \to 0,$$

where M^1 is the cokernel. Then we take the injective hull $M^1 \rightarrowtail I^1$, and this gives a longer exact sequence

$$0 \to M \to I^0 \to I^1 \to M^2 \to 0,$$

etc. We want to generalize this idea to complexes.

Definition 13.3.5.

(1) A *minimal injective complex* of A-modules is a bounded below complex of injective modules I, such that for every integer q the submodule of cocycles $\mathrm{Z}^q(I) \subseteq I^q$ is essential.

(2) Let $M \in \mathbf{D}^+(A)$. A *minimal injective resolution* of M is a quasi-isomorphism $M \to I$ into a minimal injective complex I.

Proposition 13.3.6. *Let $M \in \mathbf{D}^+(A)$.*

(1) *There exists a minimal injective resolution $\phi : M \to I$.*

(2) *If $\phi' : M \to I'$ is another minimal injective resolution, then there is an isomorphism $\psi : I \to I'$ in $\mathbf{C}_{\mathrm{str}}(A)$ such that $\phi' = \psi \circ \phi$.*

(3) *If M has finite injective dimension, then it has a bounded minimal injective resolution I.*

Proof. We know that there is a quasi-isomorphism $M \to J$ where J is a bounded below complex of injective modules. For any q let E^q be an injective hull of $\mathrm{Z}^q(I)$. By Proposition 13.3.3(1) we can assume that E^q sits inside J^q like this: $\mathrm{Z}^q(I) \subseteq E^q \subseteq J^q$. Since E^q is injective, we can decompose J^q into a direct sum: $J^q \cong E^q \oplus K^q$. The homomorphism $\mathrm{d}_J^q : K^q \to J^{q+1}$ is a monomorphism since $K^q \cap \mathrm{Z}^q(I) = 0$. And the image $\mathrm{d}_J^q(K^q)$ is contained in E^{q+1}. Thus $\mathrm{d}_J^q(K^q)$ is a direct summand of E^{q+1}, and this shows that the quotient

$$I^{q+1} := E^{q+1}/\mathrm{d}_J^q(K^q) \cong J^{q+1}/(K^{q+1} \oplus \mathrm{d}_J^q(K^q))$$

is an injective module. The canonical surjection of graded modules $\pi : J \to I$ is a homomorphism of complexes, with kernel the acyclic complex

$$\bigoplus_q \big(K^q[-q] \xrightarrow{\mathrm{d}_J^q} \mathrm{d}_J^q(K^q)[-q-1]\big).$$

Therefore π is a quasi-isomorphism. A short calculation shows that I is a minimal injective complex, i.e. $\mathrm{Z}^q(I) \subseteq I^q$ is essential.

(2) See next exercise. (We will not need this fact.)

(3) According to Proposition 13.1.19, the complex J that appears in item (1) can be chosen to be bounded. □

Exercise 13.3.7. Prove Proposition 13.3.6(2).

Remark 13.3.8. There is a more general version of minimal injective complex: it is a K-injective complex I consisting of injective modules, such that each $\mathrm{Z}^q(I) \subseteq I^q$ is essential. See [Kr, Appendix B].

Remark 13.3.9. Important: the isomorphisms ψ in Propositions 13.3.3 and 13.3.6 are not unique (see next exercise). We will see below (in Subsection ???? that a rigid residue complex is a minimal injective complex that has no nontrivial rigid automorphisms.

Exercise 13.3.10. Take $A := \mathbb{K}[[t]]$, the power series ring over a field $\mathbb{K}$. Let $M := A/(t)$, the trivial module (the residue field viewed as an A-module).

(1) Find the minimal injective resolution

$$0 \to M \to I^0 \to I^1 \to 0.$$

(2) Find nontrivial automorphisms of the complex I in $\mathsf{C}_{\mathrm{str}}(A)$ that fix the submodule $M \subseteq I^0$.

Now we add the noetherian condition.

Proposition 13.3.11. *Assume A is a left noetherian ring. Let $\{I_z\}_{z \in Z}$ be a collection of injective A-modules. Then $I = \bigoplus_{z \in Z} I_z$ is an injective A-module.*

Exercise 13.3.12. Prove Proposition 13.3.11. (Hint: use the Baer criterion.)

From here all rings here are noetherian commutative. For them much more can be said.

Recall that a module M is called *indecomposable* is it not the direct sum of two nonzero modules.

Definition 13.3.13. Let $\mathfrak{a} \subseteq A$ be an ideal.

(1) Let M be an A-module. The $\mathfrak{a}$-*torsion submodule* of M is the submodule $\Gamma_{\mathfrak{a}}(M)$ consisting of the elements that are annihilated by powers of $\mathfrak{a}$. Thus
$$\Gamma_{\mathfrak{a}}(M) = \lim_{i \to} \operatorname{Hom}_A(A/\mathfrak{a}^i, M) \subseteq M.$$
(2) If $\Gamma_{\mathfrak{a}}(M) = M$ then M is called an $\mathfrak{a}$-*torsion module*.

Perhaps the most important theorem about injective modules over noetherian commutative rings is the following structural result due to Matlis [Matl] from 1958. See also [Ste, Section IV.4], [Lam, Sections 3.F and 3.I], [Mats, Section 18] and [BrSh].

Theorem 13.3.14 (Matlis)**.** *Let A be a noetherian commutative ring.*

(1) *Let $\mathfrak{p} \subseteq A$ be a prime ideal, and let $J(\mathfrak{p})$ be the injective hull of the $A_{\mathfrak{p}}$-module $\boldsymbol{k}(\mathfrak{p})$. Then, as an A-module, $J(\mathfrak{p})$ is injective, indecomposable and $\mathfrak{p}$-torsion.*
(2) *Suppose I is an indecomposable injective A-module. Then $I \cong J(\mathfrak{p})$ for a unique prime ideal $\mathfrak{p} \subseteq A$.*
(3) *Every injective A module I is a direct sum of indecomposable injective A-modules.*

Theorem 13.3.14 tells us that any injective A-module I can be written as a direct sum

$$I \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec} A} J(\mathfrak{p})^{\oplus \mu_{\mathfrak{p}}} \tag{13.3.15}$$

for a collection of cardinal numbers $\{\mu_{\mathfrak{p}}\}_{\in \operatorname{Spec} A}$, called the *Bass numbers.* General counting tricks can show that the multiplicity $\mu_{\mathfrak{p}}$ is an invariant of I. But we can be more precise:

Proposition 13.3.16. *Assume A is a noetherian commutative ring. Let I be an injective A-module, with decomposition* (13.3.15)*. Then for any $\mathfrak{p}$ there is equality*
$$\mu_{\mathfrak{p}} = \operatorname{rank}_{\boldsymbol{k}(\mathfrak{p})}\big(\operatorname{Hom}_{A_{\mathfrak{p}}}\big(\boldsymbol{k}(\mathfrak{p}), A_{\mathfrak{p}} \otimes_A I\big)\big).$$

Proof. Consider another prime $\mathfrak{q}$. If $\mathfrak{q} \not\subseteq \mathfrak{p}$ then there is an element $a \in \mathfrak{q} - \mathfrak{p}$, and then a is both invertible and locally nilpotent on $A_{\mathfrak{p}} \otimes_A J(\mathfrak{q})$. This implies that $A_{\mathfrak{p}} \otimes_A J(\mathfrak{q}) = 0$. On the other hand, if $\mathfrak{q} \subseteq \mathfrak{p}$, then $A_{\mathfrak{p}} \otimes_A J(\mathfrak{q}) \cong J(\mathfrak{q})$. Therefore
$$A_{\mathfrak{p}} \otimes_A I \cong \bigoplus_{\mathfrak{q} \subseteq \mathfrak{p}} J(\mathfrak{p})^{\oplus \mu_{\mathfrak{p}}}.$$

Next, if $\mathfrak{q} \subsetneq \mathfrak{p}$, then there is an element $b \in \mathfrak{p} - \mathfrak{q}$, and it is both invertible and zero on the module

$$\operatorname{Hom}_{A_{\mathfrak{p}}}\big(\boldsymbol{k}(\mathfrak{p}), J(\mathfrak{q})\big).$$

The implication is that this module is zero.

Finally, if $\mathfrak{q} = \mathfrak{p}$ then we have

$$\operatorname{Hom}_{A_{\mathfrak{p}}}\big(\boldsymbol{k}(\mathfrak{p}), J(\mathfrak{p})\big) \cong \operatorname{Hom}_{A_{\mathfrak{p}}}\big(\boldsymbol{k}(\mathfrak{p}), \boldsymbol{k}(\mathfrak{p})\big) \cong \boldsymbol{k}(\mathfrak{p}),$$

because the inclusion $\boldsymbol{k}(\mathfrak{p}) \subseteq J(\mathfrak{p})$ is essential.

Putting all these cases together we see that

$$\operatorname{Hom}_{A_{\mathfrak{p}}}\big(\boldsymbol{k}(\mathfrak{p}), A_{\mathfrak{p}} \otimes_A I\big) \cong \boldsymbol{k}(\mathfrak{p})^{\oplus \mu_{\mathfrak{p}}}$$

as $\boldsymbol{k}(\mathfrak{p})$-modules. □

13.4. **Residue Complexes.** In this subsection A is a noetherian commutative ring. Here we introduce residue complexes (called residual complexes in [RD]). Most of the material is taken from the original [RD]. In Example 13.4.12 we will see the relation between the geometry of $\operatorname{Spec} A$ and the structure of dualizing complexes over A (continuing Example 0.1.8 from the Introduction). Example 13.4.23 will explain the relation to residues in the classical sense.

Lemma 13.4.1. *Let R be a dualizing complex over A and let $\mathfrak{p} \subseteq A$ be a prime ideal. There is an integer d such that*

$$\operatorname{Ext}^i_{A_{\mathfrak{p}}}(\boldsymbol{k}(\mathfrak{p}), R_{\mathfrak{p}}) \cong \begin{cases} \boldsymbol{k}(\mathfrak{p}) & \text{if } i = -d, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Proposition 13.2.32, $R_{\mathfrak{p}}$ is a dualizing complex over the local ring $A_{\mathfrak{p}}$. And by Proposition 13.2.31,

$$S := \operatorname{RHom}_{A_{\mathfrak{p}}}(\boldsymbol{k}(\mathfrak{p}), R_{\mathfrak{p}})$$

is a dualizing complex over the residue field $\boldsymbol{k}(\mathfrak{p})$. Since $\boldsymbol{k}(\mathfrak{p})$ is a regular ring, it is also a dualizing complex over itself. Theorem 13.2.34 tells us that $S \cong \boldsymbol{k}(\mathfrak{p})[d]$ in $\mathsf{D}(\boldsymbol{k}(\mathfrak{p}))$ for some integer d. □

Definition 13.4.2. The number d in the lemma above is called the *dimension of* $\mathfrak{p}$ *relative to* R, and is denoted by $\dim_R(\mathfrak{p})$.

Let us recall a few notions regarding the combinatorics of prime ideals in a ring A. A prime ideal $\mathfrak{q}$ is an *immediate specialization* of another prime $\mathfrak{p}$ if $\mathfrak{p} \subsetneq \mathfrak{q}$, and there is no other prime $\mathfrak{p}'$ such that $\mathfrak{p} \subsetneq \mathfrak{p}' \subsetneq \mathfrak{q}$. In other words, if the dimension of the local ring $A_{\mathfrak{q}}/\mathfrak{p}_{\mathfrak{q}}$ is 1.

A *chain of prime ideals* in A is a sequence $(\mathfrak{p}_0, \ldots, \mathfrak{p}_n)$ of primes such that $\mathfrak{p}_i \subsetneq \mathfrak{p}_{i+1}$ for all i. The number n is the *length* of the chain. The chain is called *saturated* if for each i the prime $\mathfrak{p}_{i+1}$ is an immediate specialization of $\mathfrak{p}_i$.

Theorem 13.4.3. *Let R be a dualizing complex over A and let $\mathfrak{p}, \mathfrak{q} \subseteq A$ be prime ideals. Assume that $\mathfrak{q}$ is an immediate specialization of $\mathfrak{p}$. Then*

$$\dim_R(\mathfrak{q}) = \dim_R(\mathfrak{p}) - 1.$$

To prove this theorem we need a baby version of local cohomology: codimension 1 only.

Let $\mathfrak{a}$ be an ideal in A. The torsion functor $\Gamma_{\mathfrak{a}}$ has a right derived functor $\mathrm{R}\Gamma_{\mathfrak{a}}$. For any complex $M \in \mathbf{D}(A)$, the module $\mathrm{H}^p_{\mathfrak{a}}(M) := \mathrm{H}^p(\mathrm{R}\Gamma_{\mathfrak{a}}(M))$ is called the *p-th* *cohomology of M with supports in* $\mathfrak{a}$. In case A is a local ring and $\mathfrak{m}$ is its maximal ideal, then $\mathrm{H}^p_{\mathfrak{m}}(M)$ is also called the *local cohomology of* M.

Now suppose $\mathfrak{a}$ is a principal ideal in A, generated by an element a. Let $A_a = A[a^{-1}]$ be the localized ring. For any A-module M we write $M_a = A_a \otimes_A M$. There is a canonical exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(M) \to M \to M_a. \tag{13.4.4}$$

Lemma 13.4.5. *Let $\mathfrak{a} = (a)$ be a principal ideal in A.*

(1) *For any injective module I the sequence*

$$0 \to \Gamma_{\mathfrak{a}}(I) \to I \to I_a \to 0$$

is exact.

(2) *For any $M \in \mathbf{D}^+(A)$ and any there is a long exact sequence of A-modules*

$$\cdots \to \mathrm{H}^p_{\mathfrak{a}}(M) \to \mathrm{H}^p(M) \to \mathrm{H}^p(M_a) \to \mathrm{H}^{p+1}_{\mathfrak{a}}(M) \to \cdots .$$

Proof. (1) Let $J(\mathfrak{q})$ be an indecomposable injective A-module. According to Theorem 13.3.14(1), if $a \in \mathfrak{q}$ then $\Gamma_{\mathfrak{a}}(J(\mathfrak{q})) = J(\mathfrak{q})$ and $J(\mathfrak{q})_a = 0$. But if $a \notin \mathfrak{q}$ then $J(\mathfrak{q}) = J(\mathfrak{q})_a$ and $\Gamma_{\mathfrak{a}}(J(\mathfrak{q})) = 0$. By Theorem 13.3.14 we see that any injective module I breaks up into a direct sum $I = \Gamma_{\mathfrak{a}}(I) \oplus I_a$, and this proves that the sequence is split exact.

(2) Choose a resolution $M \to I$ by a bounded below complex of injectives. We obtain an exact sequence of complexes as in item (1). The long exact sequence in cohomology

$$\cdots \to \mathrm{H}^p(\Gamma_{\mathfrak{a}}(I)) \to \mathrm{H}^p(I) \to \mathrm{H}^p(I_a) \to \mathrm{H}^{p+1}(\Gamma_{\mathfrak{a}}(I)) \to \cdots$$

is what we want. □

Lemma 13.4.6. *Suppose A is an integral domain, with fraction field K, such that $A \neq K$. Then K is not a finite A-module.*

Proof. Let $a \in A$ be a nonzero element that is not invertible. Then

$$A \subsetneq a^{-1} \cdot A \subsetneq a^{-2} \cdot A \subsetneq \cdots \subseteq K$$

is an infinite ascending sequence of A-submodules in K. □

Lemma 13.4.7. *For any ideal $\mathfrak{a}$ and any $M \in \mathbf{D}(A)$ there is an isomorphism of A-modules*

$$\mathrm{H}^p_{\mathfrak{a}}(M) \cong \lim_{k \to} \mathrm{Ext}^p_A(A/\mathfrak{a}^k, M).$$

Proof. Choose a K-injective resolution $M \to I$. Then, using the fact that cohomology commutes with direct limits, we have

$$\begin{aligned} \mathrm{H}^p_{\mathfrak{a}}(M) &\cong \mathrm{H}^p(\Gamma_{\mathfrak{a}}(I)) \cong \mathrm{H}^p\big(\lim_{k \to} \mathrm{Hom}_A(A/\mathfrak{a}^k, I)\big) \\ &\cong \lim_{k \to} \mathrm{H}^p\big(\mathrm{Hom}_A(A/\mathfrak{a}^k, I)\big) \cong \lim_{k \to} \mathrm{Ext}^p_A(A/\mathfrak{a}^k, M). \end{aligned}$$

□

Lemma 13.4.8. *Assume A is local, with maximal ideal $\mathfrak{m}$. Let R be a dualizing complex over A, and let $d := \dim_R(\mathfrak{m})$. Then for any $i \neq -d$ the local cohomology $\mathrm{H}^i_{\mathfrak{m}}(R)$ vanishes.*

See Remark 13.4.25 for more about $\mathrm{H}^{-d}_{\mathfrak{m}}(R)$.

Proof. We know that

$$\operatorname{Ext}^i_A(\boldsymbol{k}(\mathfrak{m}), R) \cong \begin{cases} \boldsymbol{k}(\mathfrak{m}) & \text{if } i = -d, \\ 0 & \text{otherwise.} \end{cases}$$

Let N be a finite length A-module. Since N is gotten from the residue field $\boldsymbol{k}(\mathfrak{m})$ by finitely many extensions, induction on the length of N shows that $\operatorname{Ext}^i_A(N, R) = 0$ for all $i \neq -d$. This holds in particular for $N := A/\mathfrak{m}^k$. Now use Lemma 13.4.7. □

Proof of Theorem 13.4.3. After replacing A with $A_{\mathfrak{q}}/\mathfrak{p}_{\mathfrak{q}}$, we can assume that $\mathfrak{p} = 0$ and $A = A_{\mathfrak{q}}$. Thus A is a 1-dimensional local integral domain, with only two primes ideals: $0 = \mathfrak{p}$ and the maximal ideal $\mathfrak{q}$. Take any nonzero element $a \in \mathfrak{q}$. Then the localization A_a is the field of fractions of A, i.e. $A_a = \boldsymbol{k}(\mathfrak{p})$. On the other hand, letting $\mathfrak{a} := (a) \subseteq A$, the quotient $A/\mathfrak{a}$ is a finite length A-module, so the ideal $\mathfrak{a}$ is $\mathfrak{q}$-primary, and $\Gamma_{\mathfrak{a}} = \Gamma_{\mathfrak{q}}$.

Define $d := \dim_R(\mathfrak{q})$ and $e := \dim_R(\mathfrak{p})$. By Lemma 13.4.5 there is an exact sequence of A-modules

$$\cdots \to \mathrm{H}^{-e}_{\mathfrak{a}}(R) \to \mathrm{H}^{-e}(R) \xrightarrow{\phi} \mathrm{H}^{-e}(R_a) \to \mathrm{H}^{-e+1}_{\mathfrak{a}}(R) \to \cdots .$$

Since $a \neq 0$ there are equalities $A_a = A_{\mathfrak{p}} = \operatorname{Frac}(A) = \boldsymbol{k}(\mathfrak{p})$. Then $\mathrm{H}^{-e}(R_a) \cong \boldsymbol{k}(\mathfrak{p})$, and this is not a finite A-module by Lemma 13.4.6. On the other hand the A-module $\mathrm{H}^{-e}(R)$ is finite. We conclude that homomorphism ϕ is not surjective, and thus $\mathrm{H}^{-e+1}_{\mathfrak{a}}(R) \neq 0$. But $\mathrm{H}^{-e+1}_{\mathfrak{a}}(R) = \mathrm{H}^{-e+1}_{\mathfrak{q}}(R)$, so according to Lemma 13.4.8 we must have $-e + 1 = -d$. Thus $e = d + 1$ as claimed. □

Corollary 13.4.9. *If A has a dualizing complex, then the Krull dimension of A is finite. More precisely, if R is a dualizing complex over A, then* $\dim(A)$ *is at most the injective dimension of R.*

Proof. Let $[i_0, i_1]$ be the injective concentration of the complex R. See Definition 13.1.12. This is a bounded interval. Since

$$\operatorname{Ext}^i_{A_{\mathfrak{p}}}(\boldsymbol{k}(\mathfrak{p}), R_{\mathfrak{p}}) \cong \operatorname{Ext}^i_A(A/\mathfrak{p}, R)_{\mathfrak{p}},$$

we see that the number $\dim_R(\mathfrak{p}) \in [i_0, i_1]$.

Let $(\mathfrak{p}_0, \ldots, \mathfrak{p}_n)$ be a chain of prime ideals in A. Because A is noetherian, we can squeeze more primes into this chain, until after finitely many steps it becomes saturated. According to Theorem 13.4.3 we have

$$n = \dim_R(\mathfrak{p}_0) - \dim_R(\mathfrak{p}_n).$$

Therefore $n \leq i_1 - i_0$. □

Definition 13.4.10. The ring A is called *catenary* if for any pair of primes $\mathfrak{p} \subseteq \mathfrak{q}$ there is a number $n_{\mathfrak{p},\mathfrak{q}}$ such that for any saturated chain $(\mathfrak{p}_0, \ldots, \mathfrak{p}_n)$ with $\mathfrak{p}_0 = \mathfrak{p}$ and $\mathfrak{p}_n = \mathfrak{q}$, there is equality $n = n_{\mathfrak{p},\mathfrak{q}}$.

Corollary 13.4.11. *If A has a dualizing complex, then it is catenary.*

Proof. Let R be a dualizing complex over A. The proof of the previous corollary shows that the number

$$n_{\mathfrak{p},\mathfrak{q}} = \dim_R(\mathfrak{p}) - \dim_R(\mathfrak{q})$$

has the desired property. □

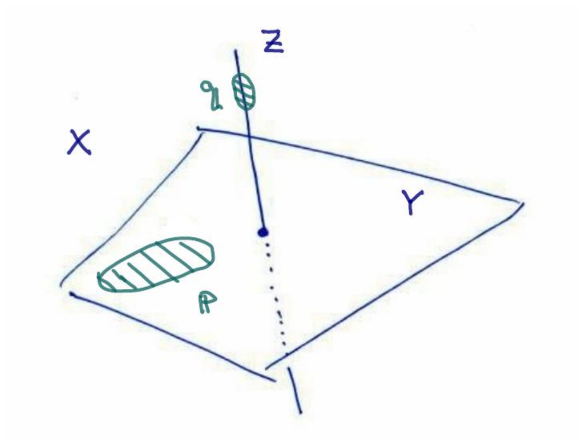

FIGURE 8. An algebraic variety X that is connected but not equidimensional: it has irreducible components Y and Z of dimensions 2 and 1 respectively. The generic points $\mathfrak{p} \in Y$ and $\mathfrak{q} \in Z$ are shown.

Example 13.4.12. This is a continuation of Example 0.1.8 from the Introduction. Consider the ring

$$A = \mathbb{R}[t_1, t_2, t_3]/(t_3 \cdot t_1,\, t_3 \cdot t_2).$$

The affine algebraic variety

$$X = \operatorname{Spec} A \subseteq \mathbf{A}^3_{\mathbb{R}}$$

is shown in figure 8. It is the union of a plane Y and a line Z, meeting at the origin.

Since the ring A is finite type over the field $\mathbb{R}$, it has a dualizing complex R. We will now prove that there is some integer i s.t. $\mathrm{H}^i(R)$ and $\mathrm{H}^{i+1}(R)$ are nonzero.

Define the prime ideals $\mathfrak{m} := (t_1, t_2, t_3)$, $\mathfrak{q} := (t_1, t_2)$ and $\mathfrak{p} := (t_3)$. Thus $\mathfrak{m}$ is the origin, $\mathfrak{q}$ is the generic point of the line $Z = \operatorname{Spec} A/\mathfrak{q}$, and $\mathfrak{p}$ is the generic point of the plane $Y = \operatorname{Spec} A/\mathfrak{p}$. By translating R as needed, we cam assume that $\dim_R(\mathfrak{m}) = 0$. Since $\mathfrak{m}$ is an immediate specialization of $\mathfrak{q}$, Theorem 13.4.3 tells us that $\dim_R(\mathfrak{q}) = 1$. Similarly, since any line in Y passing through the origin gives rise to a saturated chain $(\mathfrak{p}, \mathfrak{q}', \mathfrak{m})$, we see that $\dim_R(\mathfrak{p}) = 2$.

Since $\mathfrak{q}$ is the generic point of Z, its local ring is the residue field: $A_{\mathfrak{q}} = \boldsymbol{k}(\mathfrak{q})$. We know that $\dim_R(\mathfrak{q}) = 1$. Hence

$$\boldsymbol{k}(\mathfrak{q}) \cong \operatorname{Ext}^{-1}_{A_{\mathfrak{q}}}(\boldsymbol{k}(\mathfrak{q}), R_{\mathfrak{q}}) = \operatorname{Ext}^{-1}_{A_{\mathfrak{q}}}(A_{\mathfrak{q}}, R_{\mathfrak{q}}) \cong \operatorname{Ext}^{-1}_{A}(A, R)_{\mathfrak{q}} \cong \mathrm{H}^{-1}(R)_{\mathfrak{q}}.$$

Therefore $\mathrm{H}^{-1}(R) \neq 0$. A similar calculation involving $\mathfrak{p}$ shows that $\mathrm{H}^{-2}(R) \neq 0$.

Example 13.4.13. Let A be a local ring, with maximal ideal $\mathfrak{m}$ and residue field $\boldsymbol{k}(\mathfrak{m})$. Recall that A is called *Gorenstein* if the free module A has finite injective dimension. The ring A is called called *Cohen-Macaulay* if its depth is equal to its dimension, where the depth of A is the minimal integer i such that $\operatorname{Ext}^i_A(\boldsymbol{k}(\mathfrak{m}), A) \neq 0$. It is known that Gorenstein implies Cohen-Macaulay. See [Mats] for details.

As is our usual practice (cf. Convention 13.2.10) we shall say that a noetherian commutative ring A is Cohen-Macaulay (resp. Gorenstein) if it has finite Krull dimension, and all its local rings $A_{\mathfrak{p}}$ are Cohen-Macaulay (resp. Gorenstein) local rings, as defined above.

Assume A has a connected spectrum, and let R be a dualizing complex over A. Grothendieck showed in [RD, Section V.9] that A is a Cohen-Macaulay ring iff $R \cong L[d]$ for some finite module L and some integer d; the proof is not easy. It is however pretty easy to prove (using Theorem 13.2.34) that A is a Gorenstein ring iff $R \cong L[d]$ for some invertible module L and some integer d.

There is a lot more to say about the relation between the CM (Cohen-Macaulay) property and duality. See Remark 13.4.27

Recall that for any $\mathfrak{p} \in \operatorname{Spec} A$ we denote by $J(\mathfrak{p})$ the corresponding indecomposable injective module.

Definition 13.4.14. A *residue complex* over A is a complex of A-module $\mathcal{K}$ having these properties:

(i) $\mathcal{K}$ is a dualizing complex.

(ii) For any integer d there is an isomorphism of A-modules

$$\mathcal{K}^{-d} \cong \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec} A \\ \dim_{\mathcal{K}}(\mathfrak{p}) = d}} J(\mathfrak{p}) \ .$$

The reason we like residue complexes is this:

Theorem 13.4.15. *Suppose $\mathcal{K}$ and $\mathcal{K}'$ are residue complexes over A that have the same dimension function. Then the homomorphism*

$$\mathrm{Q} : \operatorname{Hom}_{\mathbf{C}_{\mathrm{str}}(A)}(\mathcal{K}, \mathcal{K}') \to \operatorname{Hom}_{\mathbf{D}(A)}(\mathcal{K}, \mathcal{K}')$$

is bijective.

In more words: for any morphism $\psi : \mathcal{K} \to \mathcal{K}'$ in $\mathbf{D}(A)$ there is a unique strict homomorphism of complexes $\phi : \mathcal{K} \to \mathcal{K}'$ such that $\psi = \mathrm{Q}(\phi)$.

Proof. Since the complex $\mathcal{K}'$ is K-injective, by Theorem ????

cmnt: creat new thm just before Cor 9.1.13

we know that the homomorphism

$$\mathrm{Q} : \operatorname{Hom}_{\mathbf{K}(A)}(\mathcal{K}, \mathcal{K}') \to \operatorname{Hom}_{\mathbf{D}(A)}(\mathcal{K}, \mathcal{K}')$$

is bijective. And by definition the homomorphism

$$\mathrm{P} : \operatorname{Hom}_{\mathbf{C}_{\mathrm{str}}(A)}(\mathcal{K}, \mathcal{K}') \to \operatorname{Hom}_{\mathbf{K}(A)}(\mathcal{K}, \mathcal{K}')$$

is surjective. It remains to prove that

$$\operatorname{Hom}_A(\mathcal{K}, \mathcal{K}')^{-1} = 0,$$

i.e. here are no nonzero degree -1 homomorphisms $\gamma : \mathcal{K} \to \mathcal{K}'$.

The residue complexes $\mathcal{K}$ and $\mathcal{K}'$ decompose into indecomposable summands by the formula in property (ii) of Definition 13.4.14. A homomorphism $\gamma : \mathcal{K} \to \mathcal{K}'$ of degree -1 is nonzero iff at least one of its components

$$\gamma_{\mathfrak{p},\mathfrak{q}} : J(\mathfrak{p}) \to J(\mathfrak{q})$$

is nonzero, for some $J(\mathfrak{p}) \subseteq \mathcal{K}^{-i}$ and $J(\mathfrak{q}) \subseteq \mathcal{K}'^{\,-i-1}$. Denoting by dim the dimension function of both these dualizing complexes, we have $\dim(\mathfrak{p}) = i$ and $\dim(\mathfrak{q}) = i + 1$. But the lemma below says that $\mathfrak{q}$ has to be a specialization of

$\mathfrak{p}$. Therefore, as in the proof of Corollary 13.4.9, there is an inequality in the oppose direction: $\dim(\mathfrak{p}) \geq \dim(\mathfrak{q})$. We see that it is impossible to have a nonzero degree -1 homomorphism $\gamma : \mathcal{K} \to \mathcal{K}'$. □

Lemma 13.4.16. *Let $\mathfrak{p}, \mathfrak{q}$ be prime ideals. If there is a nonzero homomorphism $\gamma : J(\mathfrak{p}) \to J(\mathfrak{q})$, then $\mathfrak{q}$ is a specialization of $\mathfrak{p}$.*

Proof. Assume $\mathfrak{q}$ is not a specialization of $\mathfrak{p}$; i.e. $\mathfrak{p} \subsetneq \mathfrak{q}$. So there is an element $a \in \mathfrak{p} - \mathfrak{q}$. Let $\gamma : J(\mathfrak{p}) \to J(\mathfrak{q})$ be a homomorphism, and consider the module $N := \gamma(J(\mathfrak{p})) \subseteq J(\mathfrak{q})$. Since $J(\mathfrak{p})$ is $\mathfrak{p}$-torsion, the element a acts on N locally-nilpotently. On the other hand, $J(\mathfrak{q})$ is a module over $A_{\mathfrak{q}}$, so a acts invertibly on $J(\mathfrak{q})$, and hence it has zero annihilator in N. The conclusion is that $N = 0$. □

Here is a general existence theorem.

Theorem 13.4.17. *Suppose the ring A has a dualizing complex R. Let $R \to \mathcal{K}$ be a minimal injective resolution of R. Then $\mathcal{K}$ is a residue complex over A.*

The proof is after two lemmas.

Lemma 13.4.18. *Let $S \subseteq A$ be a multiplicatively closed set, with localization A_S. For any A-module M we write $M_S := A_S \otimes_A M$.*

1. *If I is an injective A-module, then I_S is an injective A_S-module.*
2. *If I is an injective A-module and $M \subseteq I$ is an essential A-submodule, then $M_S \subseteq I_S$ is an essential A_S-submodule.*
3. *If I is a minimal injective complex of A-modules, then I_S is a minimal injective complex of A_S-modules,*

Proof. (1) By Theorem 13.3.14 there is a direct sum decomposition $I \cong I' \oplus I''$, where

$$I' \cong \bigoplus_{\mathfrak{p} \cap S = \varnothing} J(\mathfrak{p})^{\oplus \mu_{\mathfrak{p}}} \quad \text{and} \quad I'' \cong \bigoplus_{\mathfrak{p} \cap S \neq \varnothing} J(\mathfrak{p})^{\oplus \mu_{\mathfrak{p}}}.$$

If $\mathfrak{p} \cap S = \varnothing$ then $J(\mathfrak{p}) \cong J(\mathfrak{p})_S$ is an injective A_S-module; and if $\mathfrak{p} \cap S \neq \varnothing$ then $J(\mathfrak{p})_S = 0$. We see that $I_S \cong I'$ is an injective A_S-module.

(2) Denote by $\lambda : I \to I_S$ the canonical homomorphism. Under the decomposition $I \cong I' \oplus I''$ above, $\lambda|_{I'} : I' \to I_S$ is an isomorphism.

Let L be a nonzero A_S-submodule of I_S. Since λ is split, we can lift it to a submodule $L' \subseteq I' \subseteq I$, such that $\lambda : L' \to L$ is bijective. Because $M \subseteq I$ is essential, we know that $M \cap L'$ is nonzero. But $M \cap L' \subseteq I'$, so $\lambda(M \cap L')$ is a nonzero submodule of L. Yet $M \cap L' \subseteq M$, so $\lambda(M \cap L') \subseteq \lambda(M) \subseteq M_S$. Therefore $M_S \cap L \neq 0$.

(3) By part (1) the complex I_S is a bounded below complex of injective A_S-modules. Exactness of localization shows that $\mathrm{Z}^n(I_S) = \mathrm{Z}^n(I)_S$ inside I^n_S; so by part (2) the inclusion $\mathrm{Z}^n(I_S) \rightarrowtail I^n_S$ is essential. □

Lemma 13.4.19. *Let $\mathfrak{a} \subseteq A$ be an ideal, and define $B := A/\mathfrak{a}$.*

1. *If I is an injective A-module, then $J := \mathrm{Hom}_A(B, I)$ is an injective B-module.*
2. *Let I and J be as above. If $M \subseteq I$ is an essential A-submodule, then $N := \mathrm{Hom}_A(B, M)$ is an essential B-submodule of J.*
3. *If I is a minimal injective complex of A-modules, then $J := \mathrm{Hom}_A(B, I)$ is a minimal injective complex of B-modules,*

Proof. (1) This is imemdiate from adjunction.

(2) We identity J and N with the submodules of I and M respectively that are the annihilators of $\mathfrak{a}$. Let $L \subseteq J$ be a nonzero B-submodule. Then L is a nonzero A-submodule of I. Because M is essential, the intersection $L \cap M$ is nonzero. But $L \cap M$ is annihilated by $\mathfrak{a}$, so it sits inside N, and in fact $L \cap M = L \cap N$.

(3) By part (1) the complex J is a bounded below complex of injective B-modules. Left exactness of $\operatorname{Hom}_A(B, -)$ shows that $\mathrm{Z}^n(J) = \operatorname{Hom}_A(B, \mathrm{Z}^n(I))$ inside J^n; so by part (2) the inclusion $\mathrm{Z}^n(J) \rightarrowtail J^n$ is essential. □

Proof of Theorem 13.4.17. Since $\mathcal{K} \cong R$ in $\mathbf{D}(A)$ it follows that $\mathcal{K}$ is a dualizing complex. To show that $\mathcal{K}$ has property (ii) of Definition 13.4.14 we have to count multiplicities. For any $\mathfrak{p}$ and d let $\mu_{\mathfrak{p},d}$ be the multiplicity of $J(\mathfrak{p})$ in $\mathcal{K}^{-d}$, so that

$$\mathcal{K}^{-d} \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec} A} J(\mathfrak{p})^{\oplus \mu_{\mathfrak{p},d}}.$$

We have to prove that

$$\mu_{\mathfrak{p},d} = \begin{cases} 1 & \text{if } \dim_{\mathcal{K}}(\mathfrak{p}) = d, \\ 0 & \text{otherwise.} \end{cases} \tag{13.4.20}$$

Now by Lemma 13.4.18(3) the complex $\mathcal{K}_{\mathfrak{p}} = A_{\mathfrak{p}} \otimes_A \mathcal{K}$ is a minimal injective complex of $A_{\mathfrak{p}}$-modules. Because $\mathcal{K}_{\mathfrak{p}}$ is K-injective over $A_{\mathfrak{p}}$ we get

$$\operatorname{Ext}^{-d}_{A_{\mathfrak{p}}}(\boldsymbol{k}(\mathfrak{p}), R_{\mathfrak{p}}) \cong \mathrm{H}^{-d}\big(\operatorname{Hom}_{A_{\mathfrak{p}}}(\boldsymbol{k}(\mathfrak{p}), \mathcal{K}_{\mathfrak{p}})\big)$$

as $\boldsymbol{k}(\mathfrak{p})$-modules. By Lemma 13.4.19(3) the complex $\operatorname{Hom}_{A_{\mathfrak{p}}}(\boldsymbol{k}(\mathfrak{p}), \mathcal{K}_{\mathfrak{p}})$ is a minimal injective complex of $\boldsymbol{k}(\mathfrak{p})$-modules. It is easy to see (and we leave this verification to the reader) that a minimal injective complex over a field must have trivial differential. Therefore

$$\mathrm{H}^{d}\big(\operatorname{Hom}_{A_{\mathfrak{p}}}(\boldsymbol{k}(\mathfrak{p}), \mathcal{K}_{\mathfrak{p}})\big) \cong \operatorname{Hom}_{A_{\mathfrak{p}}}(\boldsymbol{k}(\mathfrak{p}), \mathcal{K}_{\mathfrak{p}}^{-d}).$$

Now by arguments like in the proof of Lemma 13.4.18(1) we know that

$$\operatorname{Hom}_{A_{\mathfrak{p}}}(\boldsymbol{k}(\mathfrak{p}), J(\mathfrak{q})_{\mathfrak{p}}) \cong \begin{cases} \boldsymbol{k}(\mathfrak{p}) & \text{if } \mathfrak{q} = \mathfrak{p}, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\operatorname{Hom}_{A_{\mathfrak{p}}}(\boldsymbol{k}(\mathfrak{p}), \mathcal{K}_{\mathfrak{p}}^{-d}) \cong \boldsymbol{k}(\mathfrak{p})^{\oplus \mu_{\mathfrak{p},d}}.$$

We see that

$$\operatorname{rank}_{\boldsymbol{k}(\mathfrak{p})}\big(\operatorname{Ext}^{-d}_{A_{\mathfrak{p}}}(\boldsymbol{k}(\mathfrak{p}), R_{\mathfrak{p}})\big) = \mu_{\mathfrak{p},d}.$$

But by Definition 13.4.2 this number satisfies (13.4.20). □

Corollary 13.4.21. *If $\mathcal{K}$ is a residue complex over A then it is a minimal injective complex.*

Proof. Let $\phi : \mathcal{K} \to \mathcal{K}'$ be a minimal injective resolution of $\mathcal{K}$. According to Theorem 13.4.17, $\mathcal{K}'$ is also a residue complex. Now $\mathrm{Q}(\phi) : \mathcal{K} \to \mathcal{K}'$ is an isomorphism in $\mathbf{D}(A)$, so by Theorem 13.4.15 we know that $\phi : \mathcal{K} \to \mathcal{K}'$ is an isomorphism in $\mathbf{C}_{\mathrm{str}}(A)$. □

Exercise 13.4.22. Find a direct proof of Corollary 13.4.21, without resorting to Theorems 13.4.17 and 13.4.15. (Hint: look at the proof of Proposition 13.3.6.)

We end this section with an example and two remarks.

Example 13.4.23. Take an algebraically closed field $\mathbb{K}$ (e.g. $\mathbb{K} = \mathbb{C}$), and let $A := \mathbb{K}[t]$, the ring of polynomials in a variable t. So $\operatorname{Spec} A = \mathbf{A}^1_{\mathbb{K}}$, the affine line over $\mathbb{K}$.

Grothendieck's full duality theory from [RD], or alternatively the theory of *rigid complexes*, that we will talk about later in the book, both say that the "correct" dualizing complex over A is $R_A := \Omega^1_{A/\mathbb{K}}[1]$. Here $\Omega^1_{A/\mathbb{K}}$ is the module of differential 1-forms, which is a free A-module of rank 1 with basis $\mathrm{d}t$. The corresponding residue complex (the minimal injective resolution of the complex R_A, see Theorem 13.4.17) is the *rigid residue complex* $\mathcal{K}_A$ of A.

There is a very explicit way to write the complex $\mathcal{K}_A$, and it relies on the notion of *algebraic residues*. This is, presumably, the source of the name "residue complex".

Consider a maximal ideal $\mathfrak{m} \subseteq A$. It is generated by $t - \lambda$ for some $\lambda \in \mathbb{K}$. The $\mathfrak{m}$-adic completion of the local ring $A_\mathfrak{m}$ is denoted by $\widehat{A}_\mathfrak{m}$, and it a power series ring: $\widehat{A}_\mathfrak{m} = \mathbb{K}[[t - \lambda]]$.

Let L denote the field of fractions of A, so $L = \mathbb{K}(t)$, the field of rational functions. The completion $\widehat{L}_\mathfrak{m}$ of L at a maximal ideal $\mathfrak{m} = (t - \lambda)$ is the Laurent series field $\mathbb{K}((t - \lambda))$.

A rational differential form is an element

$$\alpha = f(t) \cdot \mathrm{d}t \in \Omega^1_{L/\mathbb{K}} = L \otimes_A \Omega^1_{A/\mathbb{K}},$$

where $f(t) \in L$. Any maximal ideal $\mathfrak{m} = (t-\lambda) \subseteq A$ determines a residue functional

$$\operatorname{Res}_\mathfrak{m} : \Omega^1_{L/\mathbb{K}} \to \mathbb{K}.$$

The formula is this: for $\alpha = f(t) \cdot \mathrm{d}t$ we express $f(t)$ as a Laurent series in $\widehat{L}_\mathfrak{m}$, say

$$f(t) = \sum_j \mu_j \cdot (t - \lambda)^j,$$

with coefficients $\mu_j \in \mathbb{K}$. Then

$$\operatorname{Res}_\mathfrak{m}(\alpha) := \mu_{-1}.$$

More on this can be found in [Har, Theorem III.7.14.1].

The rational differential form $\alpha \in \Omega^1_{L/\mathbb{K}}$ defines a continuous functional

$$\partial_\mathfrak{m}(\alpha) : \widehat{A}_\mathfrak{m} \to \mathbb{K}, \quad \partial_\mathfrak{m}(\alpha)(a) := \operatorname{Res}_\mathfrak{m}(a \cdot \alpha).$$

We get an A-linear homomorphism

$$\partial_\mathfrak{m} : \Omega^1_{L/\mathbb{K}} \to \operatorname{Hom}^{\mathrm{cont}}_{\mathbb{K}}(\widehat{A}_\mathfrak{m}, \mathbb{K}).$$

By Matlis theory the module

$$J(\mathfrak{m}) := \operatorname{Hom}^{\mathrm{cont}}_{\mathbb{K}}(\widehat{A}_\mathfrak{m}, \mathbb{K})$$

is the indecomposable injective A-module associated to $\mathfrak{m}$. And, letting $\mathfrak{p} := (0)$, the module $J(\mathfrak{p}) := \Omega^1_{L/\mathbb{K}}$ is the generic indecomposable injective module.

A careful calculation shows that the sequence of A-modules

$$0 \to \Omega^1_{A/\mathbb{K}} \to \Omega^1_{L/\mathbb{K}} \xrightarrow{\sum_\mathfrak{m} \partial_\mathfrak{m}} \bigoplus_\mathfrak{m} \operatorname{Hom}^{\mathrm{cont}}_{\mathbb{K}}(\widehat{A}_\mathfrak{m}, \mathbb{K}) \to 0$$

is exact. Therefore the rigid residue complex of A is

$$\mathcal{K}_A = \big(\mathcal{K}_A^{-1} \xrightarrow{\partial} \mathcal{K}_A^0\big) = \Big(\Omega^1_{L/\mathbb{K}} \xrightarrow{\sum_{\mathfrak{m}} \partial_{\mathfrak{m}}} \bigoplus_{\mathfrak{m}} \operatorname{Hom}^{\mathrm{cont}}_{\mathbb{K}}(\widehat{A}_{\mathfrak{m}}, \mathbb{K})\Big).$$

A similar (but much more complicated) description is possible for any finite type $\mathbb{K}$-scheme; this was done in [Ye2].

Remark 13.4.24. Here is a brief explanation of *Matlis duality*. For more details see [RD, Section V.5], [Mats, Theorem 18.6] or [BrSh, Section 10.2]. Assume A is a complete local ring with maximal ideal $\mathfrak{m}$. As usual, the category of finite A-modules is $\mathbf{M}_{\mathrm{f}}(A)$. There is also the category $\mathbf{M}_{\mathrm{a}}(A)$ of artinian A-modules. These are full abelian subcategories of $\mathbf{M}(A)$. Note that these subcategories are characterized by dual properties: the objects of $\mathbf{M}_{\mathrm{f}}(A)$ are noetherian, i.e. they satisfy the ascending chain condition; and the objects of $\mathbf{M}_{\mathrm{a}}(A)$ satisfy the descending chain condition.

Consider the indecomposable injective module $J(\mathfrak{m})$. The functor $D := \operatorname{Hom}_A(-, J(\mathfrak{m}))$ is exact of course. Matlis duality asserts that

$$D : \mathbf{M}_{\mathrm{f}}(A)^{\mathrm{op}} \to \mathbf{M}_{\mathrm{a}}(A)$$

is an equivalence, with quasi-inverse D.

Remark 13.4.25. We now provide a brief discussion of *local duality*, based on [RD, Section V.6]. (There is a weaker variant of this result, for modules instead of complexes, that can be found in [BrSh, Theorem 11.2.6].) Again A is local, with maximal ideal $\mathfrak{m}$. Let R be a dualizing complex over A. By translating R we can assume that $\dim_R(\mathfrak{m}) = 0$. Lemma 13.4.8 tells us that $\mathrm{H}^i_{\mathfrak{m}}(R) = 0$ for all $i \neq 0$. A calculation, that relies on Matlis duality, shows that $\mathrm{H}^0_{\mathfrak{m}}(R) \cong J(\mathfrak{m})$, the indecomposable injective corresponding to $\mathfrak{m}$.

Let us fix an isomorphism $\beta : \mathrm{H}^0_{\mathfrak{m}}(R) \xrightarrow{\simeq} J(\mathfrak{m})$. This induces a morphism

$$\theta_M : \mathrm{R}\Gamma_{\mathfrak{m}}(M) \to \operatorname{Hom}_A\big(\operatorname{RHom}_A(M, R), J(\mathfrak{m})\big), \tag{13.4.26}$$

functorial in $M \in \mathbf{D}^+(A)$. The Local Duality Theorem [RD, Theorem V.6.2] says that θ_M is an isomorphism if $M \in \mathbf{D}^+_{\mathrm{f}}(A)$.

Here is a modern take on this theorem: we can construct the morphism θ_M for any $M \in \mathbf{D}(A)$. Let's replace R by the residue complex $\mathcal{K}$ (the minimal injective resolution of R). Then β is just a module isomorphism $\beta : \mathcal{K}^0 \xrightarrow{\simeq} J(\mathfrak{m})$. For any complex M we choose a K-injective resolution $M \to I(M)$. Then θ_M is represented by the homomorphism

$$\Gamma_{\mathfrak{m}}(I(M)) \to \operatorname{Hom}_A\big(\operatorname{Hom}_A(I(M), \mathcal{K}), \mathcal{K}^0\big)$$

in $\mathbf{C}_{\mathrm{str}}(A)$ that sends an element $u \in \Gamma_{\mathfrak{m}}(I(M))^p$ and a homomorphism

$$\phi \in \operatorname{Hom}_A(I(M), \mathcal{K})^{-p}$$

to $\phi(u) \in \mathcal{K}^0$.

We know that the functors appearing in equation (13.4.26) have finite cohomological dimensions. Since $A \in \mathbf{D}^+_{\mathrm{f}}(A)$, the local duality theorem from [RD] tells us that θ_A is an isomorphism. Now we can apply Theorem 13.1.27 to conclude that θ_M is an isomorphism for any $M \in \mathbf{D}_{\mathrm{f}}(A)$.

Remark 13.4.27. Here is more on the CM property and duality. Let A be a noetherian ring with connected spectrum. Assume A has a dualizing complex R, and corresponding dimension function $\dim_R$.

Consider a complex $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. In [RD] Grothendieck defines M to be a *CM complex with respect to* R if for any prime ideal $\mathfrak{p} \subseteq A$ and every $i \neq -\dim_R(\mathfrak{p})$ the local cohomology satisfies $\mathrm{H}^i_{\mathfrak{p}}(M_{\mathfrak{p}}) = 0$. Notice that this is a property of the sheaf $\mathcal{M}$ (the sheafification of the module M) on the topological space $X := \operatorname{Spec} A$.

It is proved in [RD] that when A is a regular ring, $R = A$, and M is a finite A-module, then M is a CM module (in the conventional sense) iff it is a CM complex.

In [YeZh2] we proved that the following are equivalent for a complex $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$:

(i) The complex M is CM w.r.t. R.
(ii) The complex $\mathrm{RHom}_A(M, R)$ has only one nonzero cohomology module.

It follows that the CM complexes form an abelian subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$. In fact, they are the heart of a perverse t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$, and hence they deserve to be called *perverse finite A-modules*. Geometrically, on the scheme $X := \operatorname{Spec} A$, the CM complexes inside $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ form a stack of abelian categories, and so they are *perverse coherent sheaves*. All this is explained in [YeZh2, Section 6].

14. Rigid Dualizing Complexes over Commutative Rings

cmnt: start of course IV

As we saw in the previous section, a dualizing complex R over a noetherian commutative ring A is not unique. This was the source of major difficulties in [RD], first for gluing dualizing complexes on schemes, and then for producing the trace morphisms.

In 1997, M. Van den Bergh [VdB] discovered the idea of *rigidity* for dualizing complexes. This was done in the context of noncommutative ring theory: A is a noncommutative noetherian ring, central over a base field $\mathbb{K}$. The theory of noncommutative rigid dualizing complexes was developed further in several papers of Zhang and Yekutieli, among them [YeZh1] and [YeZh2]. Some of this material will be discussed in Section 17 of the book.

Here we will deal with the commutative side only, which turns out to be extremely powerful. Before explaining it, let us first observe that this is one of the rare cases in which an idea originating from noncommutative algebra had significant impact in commutative algebra.

In this section we define rigid dualizing complexes, and prove their existence and uniqueness, in the following context: $\mathbb{K}$ is a regular noetherian commutative ring (e.g. a field or the ring of integers $\mathbb{Z}$), and A is a flat essentially finite type commutative $\mathbb{K}$-ring. We then introduce the functorial properties of rigid dualizing complexes: rigid traces and rigid localization morphisms. After that we pass to rigid residue complexes. For them we also define the ind-rigid trace morphisms. These concepts will allow us (in Section 16) to geometrize all the above – namely to produce rigid residue complexes of essentially finite type $\mathbb{K}$-schemes, and to manipulate them effectively. The material here is based on several papers of Zhang and Yekutieli, including them [YeZh1], [YeZh2], [YeZh3], [YeZh4], [Ye11] and [Ye13].

The theory of rigid dualizing complexes does not really require the flatness assumption (of A over $\mathbb{K}$). In the papers [YeZh3] and [YeZh4] the authors had already developed this theory without flatness, using flat DG ring resolutions. This is a much more difficult theory, and in fact there were a few crucial mistakes in these two papers. These mistakes were discovered by Avramov, Iyengar, Lipman and Nayak in the paper [AILN], and one error was corrected there. The remaining mistakes have since been rectified (in [Ye11] and [Ye13]). See Remark 14.1.25 below.

14.1. **The Squaring Operation and Rigid Complexes.** In this subsection we work in the following setup:

Setup 14.1.1. A is a nonzero commutative ring, and B is a flat commutative A-ring.

Consider the enveloping ring $B \otimes_A B$. It comes equipped with a few ring homomorphisms:

$$B \xrightarrow{\eta_i} B \otimes_A B \xrightarrow{\epsilon} B, \tag{14.1.2}$$

where $\eta_0(b) := b \otimes 1$, $\eta_1(b) := 1 \otimes b$, and $\epsilon(b_0 \otimes b_1) := b_0 \cdot b_1$. We view B as a module over $B \otimes_A B$ via ϵ. Of course $\epsilon \circ \eta_i = \mathrm{id}_B$.

Remark 14.1.3. It will be helpful to consider a $(B \otimes_A B)$-module M as an A-central B-B-bimodule, where the left B-action on M is through η_0, and the right action is through η_1. This is the noncommutative point of view. To be precise, if B had been a noncommutative central A-ring, then the enveloping ring would have been $B \otimes_A B^{\mathrm{op}}$. More on this in Section 17.

Suppose we are given B-modules M_0 and M_1. Then the tensor product $M_0 \otimes_A M_1$ is a $(B \otimes_A B)$-module. In this way we get an additive bifunctor

$$(- \otimes_A -) : \mathsf{M}(B) \times \mathsf{M}(B) \to \mathsf{M}(B \otimes_A B).$$

Passing to complexes, and then to homotopy categories, we obtain a triangulated bifunctor

$$(- \otimes_A -) : \mathsf{K}(B) \times \mathsf{K}(B) \to \mathsf{K}(B \otimes_A B). \tag{14.1.4}$$

Lemma 14.1.5. *The bifunctor* (14.1.4) *has a left derived bifunctor*

$$(- \otimes^{\mathrm{L}}_A -) : \mathsf{D}(B) \times \mathsf{D}(B) \to \mathsf{D}(B \otimes_A B).$$

If either M_0 or M_1 is a complex of B-modules that is K-flat over A, then the morphism

$$\eta_{M_0,M_1} : M_0 \otimes^{\mathrm{L}}_A M_1 \to M_0 \otimes_A M_1$$

in $\mathsf{D}(B \otimes_A B)$ is an isomorphism.

Proof. This is a variant of Theorem 12.8.1. We know by Corollary 10.3.27 and Proposition 9.3.2 that any complex $M \in \mathsf{C}(B)$ admits a K-flat resolution $P \to M$. Because B is flat over A, the complex P is also K-flat over A. By Theorem 12.7.7 the left derived functor $- \otimes^{\mathrm{L}}_A -$ exists, and the condition on η_{M_0,M_1} holds. □

Remark 14.1.6. The innocuous looking Lemma 14.1.5 is actually of tremendous importance. Without the flatness of $A \to B$ we could do very little homological algebra of bimodules. Getting around the lack of flatness requires the use of flat DG ring resolutions, as explained in Remark 14.1.25.

Any module $L \in \mathsf{M}(B)$ has an action by $B \otimes_A B$ coming from the homomorphism ϵ in (14.1.2). Consider now a module $N \in \mathsf{M}(B \otimes_A B)$. The abelian group N has two possible B-module structures, coming from the homomorphisms η_i. Thus the abelian group $\mathrm{Hom}_{B \otimes_A B}(L, N)$ has three possible B-module structures: there is one action from the B-module structure on L, and there are two from the B-module structures on N. The next easy lemma is crucial.

Lemma 14.1.7. *The three B-module structures on* $\mathrm{Hom}_{B \otimes_A B}(L, N)$ *coincide.*

Exercise 14.1.8. Prove the lemma.

We are mostly interested in the B-module $L = B$. As the module N changes, we get an additive functor

$$\operatorname{Hom}_{B \otimes_A B}(B, -) : \mathbf{M}(B \otimes_A B) \to \mathbf{M}(B).$$

Passing to complexes, and then to homotopy categories, we get a triangulated functor

$$\operatorname{Hom}_{B \otimes_A B}(B, -) : \mathbf{K}(B \otimes_A B) \to \mathbf{K}(B).$$

This has a right derived functor

$$\operatorname{RHom}_{B \otimes_A B}(B, -) : \mathbf{D}(B \otimes_A B) \to \mathbf{D}(B), \tag{14.1.9}$$

that is calculated by K-injective resolutions. Namely if $N \in \mathbf{C}(B \otimes_A B)$ is a K-injective complex, then the morphism

$$\eta_N : \operatorname{Hom}_{B \otimes_A B}(B, N) \to \operatorname{RHom}_{B \otimes_A B}(B, N)$$

in $\mathbf{D}(B)$ is an isomorphism.

By composing the bifunctor $(- \otimes^{\mathrm{L}}_A -)$ from Lemma 14.1.5 and the functor $\operatorname{RHom}_{B \otimes_A B}(B, -)$ from (14.1.9) we obtain a triangulated bifunctor

$$\operatorname{RHom}_{B \otimes_A B}(B, - \otimes^{\mathrm{L}}_A -) : \mathbf{D}(B) \times \mathbf{D}(B) \to \mathbf{D}(B). \tag{14.1.10}$$

Definition 14.1.11. Under Setup 14.1.1, the *squaring operation* is the functor

$$\operatorname{Sq}_{B/A} : \mathbf{D}(B) \to \mathbf{D}(B)$$

defined as follows:

(1) For a complex $M \in \mathbf{D}(B)$, its square is the complex
$$\operatorname{Sq}_{B/A}(M) := \operatorname{RHom}_{B \otimes_A B}(B, M \otimes^{\mathrm{L}}_A M) \in \mathbf{D}(B).$$

(2) For a morphism $\phi : M \to N$ in $\mathbf{D}(B)$, its square is the morphism
$$\operatorname{Sq}_{B/A}(\phi) := \operatorname{RHom}_{B \otimes_A B}(B, \phi \otimes^{\mathrm{L}}_A \phi) : \operatorname{Sq}_{B/A}(M) \to \operatorname{Sq}_{B/A}(N)$$
in $\mathbf{D}(B)$.

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It will be good to have an explicit formulation of the squaring operation. Let us first choose a K-projective resolution $\sigma : P \to M$ in $\mathbf{C}(B)$. Note that P is unique up to homotopy equivalence. Since B is flat over A, the complex P is K-flat over A. We get an isomorphism

$$\operatorname{pres}_P : P \otimes_A P \xrightarrow{\simeq} M \otimes^{\mathrm{L}}_A M \tag{14.1.12}$$

in $\mathbf{D}(B \otimes_A B)$ that we call a *presentation*. It is uniquely characterized by the commutativity of the diagram

$$\begin{tikzcd}
M \otimes^{\mathrm{L}}_A M \arrow[r, "\eta_{M,M}"] & M \otimes_A M \\
P \otimes^{\mathrm{L}}_A P \arrow[u, "\mathrm{Q}(\sigma) \otimes^{\mathrm{L}}_A \mathrm{Q}(\sigma)", "\cong"'] \arrow[r, "\eta_{P,P}", "\cong"'] & P \otimes_A P \arrow[u, "\mathrm{Q}(\sigma \otimes_A \sigma)"'] \arrow[ul, "\operatorname{pres}_P"', "\cong"]
\end{tikzcd}$$

in $\mathbf{D}(B \otimes_A B)$.

Next we choose a K-injective resolution $\rho : P \otimes_A P \to I$ in $\mathbf{C}(B \otimes_A B)$. It is unique up to homotopy equivalence. The resolution ρ gives rise to an isomorphism

$$\operatorname{pres}_I : \operatorname{Hom}_{B\otimes_A B}(B, I) \xrightarrow{\simeq} \operatorname{RHom}_{B\otimes_A B}(B, P \otimes_A P) \tag{14.1.13}$$

in $\mathbf{D}(B)$ such that the diagram

$$\begin{tikzcd}
\operatorname{Hom}_{B\otimes_A B}(B, P \otimes_A P) \arrow[r, "\eta_{M,P\otimes_A P}"] \arrow[d, "\mathrm{Q}(\operatorname{Hom}_{B\otimes_A B}(B,\rho))"'] & \operatorname{RHom}_{B\otimes_A B}(B, P \otimes_A P) \arrow[d, "\operatorname{RHom}_{B\otimes_A B}(B,\mathrm{Q}(\rho))", "\cong"'] \\
\operatorname{Hom}_{B\otimes_A B}(B, I) \arrow[r, "\eta_{M,I}", "\cong"'] \arrow[ur, "\operatorname{pres}_I", "\cong"'] & \operatorname{RHom}_{B\otimes_A B}(B, I)
\end{tikzcd}$$

is commutative.

The combination of the presentations pres_P and pres_I gives an isomorphism

$$\operatorname{pres}_{P,I} : \operatorname{Hom}_{B\otimes_A B}(B, I) \xrightarrow{\simeq} \operatorname{Sq}_{B/A}(M) \tag{14.1.14}$$

in $\mathbf{D}(B)$, that we also call a presentation.

Let $\phi : M \to N$ be a morphism in $\mathbf{D}(B)$. The morphism $\operatorname{Sq}_{B/A}(\phi)$ can also be made explicit using presentations. For that we need to choose a K-projective resolution $\sigma_N : Q \to N$ in $\mathbf{C}(B)$, and a K-injective resolution $\rho_N : Q \otimes_A Q \to J$ in $\mathbf{C}(B \otimes_A B)$. These provide us with a presentation

$$\operatorname{pres}_{Q,J} : \operatorname{Hom}_{B\otimes_A B}(M, J) \xrightarrow{\simeq} \operatorname{Sq}_{B/A}(N).$$

There are homomorphisms $\tilde{\phi} : P \to Q$ in $\mathbf{C}_{\mathrm{str}}(B)$, and $\chi : I \to J$ in $\mathbf{C}_{\mathrm{str}}(B \otimes_A B)$, both unique up to homotopy, such that the diagrams

$$\begin{tikzcd}
P \arrow[r, "\mathrm{Q}(\sigma)", "\cong"'] \arrow[d, "\mathrm{Q}(\tilde{\phi})"'] & M \arrow[d, "\phi"] \\
Q \arrow[r, "\mathrm{Q}(\sigma_N)", "\cong"'] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
M \otimes^{\mathrm{L}}_A M \arrow[d, "\phi\otimes^{\mathrm{L}}_A\phi"'] & P \otimes_A P \arrow[l, "\operatorname{pres}_P"', "\cong"] \arrow[r, "\mathrm{Q}(\rho)", "\cong"'] \arrow[d, "\mathrm{Q}(\tilde{\phi}\otimes_A\tilde{\phi})"'] & I \arrow[d, "\mathrm{Q}(\chi)"] \\
N \otimes^{\mathrm{L}}_A N & Q \otimes_A Q \arrow[l, "\operatorname{pres}_Q"', "\cong"] \arrow[r, "\mathrm{Q}(\rho_N)", "\cong"'] & J
\end{tikzcd}$$

in $\mathbf{D}(C)$ and $\mathbf{D}(B \otimes_A B)$ respectively are commutative. See Subsections 9.1 and 9.2. Then the diagram

$$\begin{tikzcd}
\operatorname{Hom}_{B\otimes_A B}(B, I) \arrow[r, "\operatorname{pres}_{P,I}", "\cong"'] \arrow[d, "\mathrm{Q}(\operatorname{Hom}_{B\otimes_A B}(\mathrm{id}_B,\chi))"'] & \operatorname{Sq}_{B/A}(M) \arrow[d, "\operatorname{Sq}_{B/A}(\phi)"] \\
\operatorname{Hom}_{B\otimes_A B}(B, J) \arrow[r, "\operatorname{pres}_{Q,J}", "\cong"'] & \operatorname{Sq}_{B/A}(N)
\end{tikzcd} \tag{14.1.15}$$

in $\mathbf{D}(B)$ is commutative.

The squaring operation is not an additive functor. In fact, it is a *quadratic functor*:

Theorem 14.1.16. *Let $\phi : M \to N$ be a morphism in $\mathsf{D}(B)$ and let $b \in B$. Then*
$$\operatorname{Sq}_{B/A}(b \cdot \phi) = b^2 \cdot \operatorname{Sq}_{B/A}(\phi),$$
as morphisms $\operatorname{Sq}_{B/A}(M) \to \operatorname{Sq}_{B/A}(N)$ in $\mathsf{D}(B)$.

Proof. We shall use presentations. Let $\tilde{\phi} : P \to Q$ be a homomorphism in $\mathsf{C}_{\mathrm{str}}(B)$ that represents ϕ, as above. Then the homomorphism
$$b \cdot \tilde{\phi} : P \to Q$$
$\mathsf{C}_{\mathrm{str}}(B)$ represents $b \cdot \phi$. Tensoring we get a homomorphism
$$(b \cdot \tilde{\phi}) \otimes_A (b \cdot \tilde{\phi}) : P \otimes_A P \to Q \otimes_A Q$$
$\mathsf{C}_{\mathrm{str}}(B \otimes_A B)$. But
$$(b \cdot \tilde{\phi}) \otimes_A (b \cdot \tilde{\phi}) = (b \otimes b) \cdot (\tilde{\phi} \otimes_A \tilde{\phi}).$$
Hence on the K-injectives we get the homomorphism
$$(b \otimes b) \cdot \chi : I \to J$$
$\mathsf{C}_{\mathrm{str}}(B \otimes_A B)$. We conclude that
$$\operatorname{Hom}_{B \otimes_A B}(\mathrm{id}_B, (b \otimes b) \cdot \chi) : \operatorname{Hom}_{B \otimes_A B}(B, I) \to \operatorname{Hom}_{B \otimes_A B}(B, J)$$
represents $\operatorname{Sq}_{B/A}(b \cdot \phi)$. Finally, by Lemma 14.1.7 we know that
$$\operatorname{Hom}_{B \otimes_A B}(\mathrm{id}_B, (b \otimes b) \cdot \chi) = \operatorname{Hom}_{B \otimes_A B}(b^2 \cdot \mathrm{id}_B, \chi) = b^2 \cdot \operatorname{Hom}_{B \otimes_A B}(\mathrm{id}_B, \chi).$$
□

Definition 14.1.17. Let $M \in \mathsf{D}(B)$. A *rigidifying isomorphism* for M over B relative to A is an isomorphism
$$\rho : M \xrightarrow{\simeq} \operatorname{Sq}_{B/A}(M)$$
in $\mathsf{D}(B)$.

Definition 14.1.18. A *rigid complex* over B relative to A is a pair (M, ρ), consisting of a complex $M \in \mathsf{D}(B)$ and a rigidifying isomorphism
$$\rho : M \xrightarrow{\simeq} \operatorname{Sq}_{B/A}(M)$$
in $\mathsf{D}(B)$.

Definition 14.1.19. Suppose (M, ρ) and (N, σ) are rigid complexes over B relative to A. A *morphism of rigid complexes*
$$\phi : (M, \rho) \to (N, \sigma)$$
is a morphism $\phi : M \to N$ in $\mathsf{D}(B)$, such that the diagram
$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\phi"'] & \operatorname{Sq}_{B/A}(M) \arrow[d, "\operatorname{Sq}_{B/A}(\phi)"] \\ N \arrow[r, "\sigma"] & \operatorname{Sq}_{B/A}(N) \end{tikzcd}$$
in $\mathsf{D}(B)$ is commutative.

The category of rigid complexes over B relative to A is denoted by $\mathsf{D}(B)_{\mathrm{rig}/A}$.

Recall that a complex $M \in \mathbf{D}(B)$ has the derived Morita property if the derived homothety morphism

$$\alpha_M^{\mathrm{R}} : B \to \mathrm{RHom}_B(M, M)$$

in $\mathbf{D}(B)$ is an isomorphism.

Theorem 14.1.20. *Let (M, ρ) be a rigid complex over B relative to A. If M has the derived Morita property, then the only automorphism of (M, ρ) in $\mathbf{D}(B)_{\mathrm{rig}/A}$ is the identity.*

Proof. Let

$$\phi : (M, \rho) \xrightarrow{\simeq} (M, \rho)$$

be an automorphism in $\mathbf{D}(B)_{\mathrm{rig}/A}$. By Proposition 13.2.6, there is a unique invertible element $b \in B$ such that $\phi = b \cdot \mathrm{id}_M$, as morphisms $M \to M$ in $\mathbf{D}(B)$.

Next, according to Theorem 14.1.16, we have

$$\mathrm{Sq}_{B/A}(\phi) = \mathrm{Sq}_{B/A}(b \cdot \mathrm{id}_M) = b^2 \cdot \mathrm{Sq}_{B/A}(\mathrm{id}_M).$$

Plugging this into the diagram in Definition 14.1.19 we get a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\rho", "\cong"'] \arrow[d, "b \cdot \mathrm{id}_M"'] & \mathrm{Sq}_{B/A}(M) \arrow[d, "b^2 \cdot \mathrm{id}_M"] \\ M \arrow[r, "\rho", "\cong"'] & \mathrm{Sq}_{B/A}(M) \end{tikzcd}$$

in $\mathbf{D}(B)$. Once more using Proposition 13.2.6 we see that $b^2 = b$. Because b is an invertible element, it follows that $b = 1$. Thus $\phi = \mathrm{id}_M$. □

Example 14.1.21. Assume $B = A$, and take $M := B$. Then $B \otimes_A B \cong B$, $M \otimes_A^{\mathrm{L}} M \cong M$, and there are canonical isomorphisms

$$\mathrm{Sq}_{B/A}(M) = \mathrm{RHom}_{B \otimes_A B}(B, M \otimes_A^{\mathrm{L}} M) \cong \mathrm{Hom}_B(B, M) \cong M.$$

Thus the pair (M, id) belongs to $\mathbf{D}(B)_{\mathrm{rig}/A}$. Furthermore, the complex M has the derived Morita property, so Theorem 14.1.20 applies.

To the reader who might object to this as being a ridiculously stupid example, we say that in all important situations, there is exactly one object in $\mathbf{D}(B)_{\mathrm{rig}/A}$ (up to unique isomorphism, according to Theorem 14.1.20). And it is induced, in a suitable sense, from the one in the example above. See Subsection 14.5.

The next two exercises provide rigid complexes that are far from trivial. These exercises will reappear later, as steps to produce rigid complexes over A relative to $\mathbb{K}$, where $\mathbb{K}$ is a regular ring and A is an essentially finite type $\mathbb{K}$-ring.

Exercise 14.1.22. Take $B := A[t_1, \ldots, t_n]$, the polynomial ring in n variables.

(1) Prove that

$$\mathrm{Ext}^i_{B \otimes_A B}(B, B \otimes_A B) \cong \begin{cases} B & \text{if } \; i = n \\ 0 & \text{if } \; i \neq n. \end{cases}$$

(Hint: Let I be the kernel of the multiplication homomorphism $B \otimes_A B \to B$. Show that I is generated by the sequence $(c_1, \ldots, c_n)$, where $c_j := t_j \otimes 1 - 1 \otimes t_j$. Then show this is a regular sequence. Now use the Koszul complex associated to this sequence as a free resolution of B over $B \otimes_A B$.

(2) Conclude from (1) that the complex $B[n] \in \mathsf{D}(B)$ is rigid relative to A; namely that there is a rigidifying isomorphism
$$\rho : B[n] \xrightarrow{\simeq} \mathrm{Sq}_{B/A}(B[n]).$$

(3) Let $\boldsymbol{\omega}_{B/A} := \Omega^n_{B/A}$ be the module of degree n differential forms. It is the n-th exterior power of $\Omega^1_{B/A}$, so it is a free B module of rank 1, with basis $\mathrm{d}(t_1) \wedge \cdots \wedge \mathrm{d}(t_1)$. Prove that the complex $\boldsymbol{\omega}_{B/A}[n]$ has a *canonical rigidifying isomorphism*. I.e. there is a rigidifying isomorphism
$$\rho : \boldsymbol{\omega}_{B/A}[n] \xrightarrow{\simeq} \mathrm{Sq}_{B/A}(\boldsymbol{\omega}_{B/A}[n])$$
in $\mathsf{D}(B)$ that is invariant under A-ring automorphisms of B. (Hint: Compare the action of an automorphism f of B on $\boldsymbol{\omega}_{B/A} = \Omega^n_{B/A}$ and on the last term of the Koszul complex. The Jacobian determinant of f will apear in various powers. This calculation is quite explicit in [RD, Section III.7], where it is called the *fundamental local isomorphism.*)

Exercise 14.1.23. Let B be a finite projective A-ring. Define
$$\boldsymbol{\omega}_{B/A} := \mathrm{Hom}_A(B, A) \in \mathsf{M}(B).$$
Prove that the complex $\boldsymbol{\omega}_{B/A}$ has a *canonical rigidifying isomorphism.* I.e. there is a rigidifying isomorphism
$$\rho : \boldsymbol{\omega}_{B/A} \xrightarrow{\simeq} \mathrm{Sq}_{B/A}(\boldsymbol{\omega}_{B/A})$$
in $\mathsf{D}(B)$ that is invariant under A-ring automorphisms of B.

Remark 14.1.24. The squaring operation is related to *Hochschild cohomology.* Assume for simplicity that A is a field and M is a B-module. Then for each i the cohomology
$$\mathrm{H}^i(\mathrm{Sq}_{B/A}(M)) = \mathrm{Ext}^i_{B \otimes_A B}(B, M \otimes_A M)$$
is the i-th Hochschild cohomology with values in the B-bimodule $M \otimes_A M$. For more on this material see [AILN], [Sha1] and [Sha2].

Remark 14.1.25. It is possible to avoid the assumption that B is flat over A. This is done by choosing a DG ring $\tilde{B}$ that is K-flat as a DG A-module, and a DG ring quasi-isomorphism $\tilde{B} \to B$ over A. Such resolutions always exist. Then we take
$$\mathrm{Sq}_{B/A}(M) := \mathrm{RHom}_{\tilde{B} \otimes_A \tilde{B}}(B, M \otimes^{\mathrm{L}}_A M). \tag{14.1.26}$$
This was the construction used by Zhang and Yekutieli in the paper [YeZh3].

Unfortunately there was a serious error in [YeZh3]: we did not prove that formula (14.1.26) does not depend on the resolution $\tilde{B}$. This error was discovered, and corrected, by Avramov, Iyengar, Lipman and Nayak in in their paper [AILN].

There were ensuing errors in [YeZh3] regarding the functoriality of the squaring operation in the ring B (this will be studied in Subsection 14.3 below). The paper [AILN] did not treat such functoriality at all, and the construction and proofs were corrected only in our recent paper [Ye11]. It in worthwhile to mention that the correct proofs (both in [AILN] and [Ye11]) rely on *noncommutative DG rings* and DG bimodules over them.

Because the non-flat case is so much more complicated, we have decided not to reproduce it in the book. The interested reader can look up the research papers

[Ye11], [Ye13], [Ye14] and [Ye15], the survey article [Ye6], and the lecture notes [Ye12].

A general treatment of derived categories of bimodules, based on K-flat DG ring resolutions, is in the paper [Ye16].

cmnt: $\boxed{\text{to here } 22/03/2017}$

14.2. **Adjunctions.** Before we can tackle the functorial behavior of the squaring operation, we need some more basic facts relating the derived categories $\mathsf{D}(A)$ and $\mathsf{D}(B)$ in the presence of a ring homomorphism $A \to B$. In this subsection all rings are commutative.

cmnt: maybe this should be moved to an earlier location in the book?

Suppose $u : A \to B$ is a ring homomorphism. The restriction (or forgetful) functor

$$\operatorname{Rest}_u : \mathsf{M}(B) \to \mathsf{M}(A)$$

sends a B-module N to the same abelian group, made into an A-module via u. This functor extends to a DG functor on complexes:

$$\operatorname{Rest}_u : \mathsf{C}(B) \to \mathsf{C}(A). \tag{14.2.1}$$

Because it is an exact functor, it extends to derived categories:

$$\operatorname{Rest}_u : \mathsf{D}(B) \to \mathsf{D}(A).$$

We shall usually supress this functor when the meaning is clear, in order to reduce clutter.

For any A-module M there are functorial isomorphisms

$$A \otimes_A M \xrightarrow{\simeq} M$$

and

$$\operatorname{Hom}_A(A, M) \xrightarrow{\simeq} M$$

in $\mathsf{M}(A)$. These isomorphisms extend to the derived category: for any complex of A-modules M there are functorial isomorphisms

$$A \otimes^{\mathrm{L}}_A M \xrightarrow{\simeq} M \tag{14.2.2}$$

and

$$\operatorname{RHom}_A(A, M) \xrightarrow{\simeq} M \tag{14.2.3}$$

in $\mathsf{D}(A)$. Again, to reduce clutter, we will use these canonical isomorphisms implicitly.

Definition 14.2.4. A ring homomorphism $u : A \to B$ is called a *localization homomorphism* if there is an isomorphism of A-rings $B \cong A[S^{-1}] = A_S$ some multiplicatively closed set $S \subseteq A$.

Note that a localization ring homomorphism is flat.

Definition 14.2.5. Let $u : A \to B$ be a ring homomorphism, and let $M \in \mathsf{D}(A)$ and $N \in \mathsf{D}(B)$ be complexes.

(1) A morphism

$$\theta : N \to M$$

in $\mathsf{D}(A)$ is called a *backward* (or *trace*) *morphism over* u.

(2) A morphism

$$\lambda : M \to N$$

in $\mathsf{D}(A)$ is called a *forward morphism over* u. In case the ring homomorphism u is a localization homomorphism, we also call λ a *localization morphism over* u.

The concepts of forward and backward morphisms make sense also in the categories $\mathsf{M}(-)$, $\mathsf{C}(-)$, $\mathsf{C}_{\mathrm{str}}(-)$ and $\mathsf{K}(-)$.

The standard adjunction formula give rise to a bifunctorial bijection (an isomorphism of A-modules in fact)

$$\mathrm{badj}_{u,M,N} : \mathrm{Hom}_{\mathsf{M}(A)}(N, M) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{M}(B)}\big(N, \mathrm{Hom}_A(B, M)\big) \tag{14.2.6}$$

for $M \in \mathsf{M}(A)$ and $N \in \mathsf{M}(B)$. We refer to this isomorphism as *backward adjunction*, since it takes a backward morphism $\theta : N \to M$ in $\mathsf{M}(A)$ to the morphism

$$\mathrm{badj}_{u,M,N}(\theta) : N \to \mathrm{Hom}_A(B, M)$$

in $\mathsf{M}(B)$.

cmnt: do we actually use backward adjunction?

Likewise, there is a bifunctorial bijection

$$\mathrm{fadj}_{u,M,N} : \mathrm{Hom}_{\mathsf{M}(A)}(M, N) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{M}(B)}\big(B \otimes_A M, N\big) \tag{14.2.7}$$

for $M \in \mathsf{M}(A)$ and $N \in \mathsf{M}(B)$. We refer to this isomorphism as *forward adjunction*, since it takes a forward morphism $\lambda : M \to N$ in $\mathsf{M}(A)$ to the morphism

$$\mathrm{fadj}_{u,M,N}(\lambda) : B \otimes_A M \to N \tag{14.2.8}$$

in $\mathsf{M}(B)$.

The backward and forward adjunctions extend to derived categories:

Proposition 14.2.9. *Let $u : A \to B$ be a ring homomorphism.*

(1) *There is a unique isomorphism*

$$\mathrm{dbadj}_{u,M,N} : \mathrm{Hom}_{\mathsf{D}(A)}(N, M) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{D}(B)}\big(N, \mathrm{RHom}_A(B, M)\big)$$

called derived backward adjunction*, which is functorial in $M \in \mathsf{D}(A)$ and $N \in \mathsf{D}(B)$, and such that the diagram*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}_{\mathrm{str}}(A)}\big(N, M\big) \arrow[r, "\mathrm{dbadj}"] \arrow[d, "\mathrm{Q}"'] & \mathrm{Hom}_{\mathsf{C}_{\mathrm{str}}(B)}\big(N, \mathrm{Hom}_B(C, M)\big) \arrow[d, "\Theta_{\mathrm{b}} \circ \mathrm{Q}"] \\
\mathrm{Hom}_{\mathsf{D}(A)}\big(N, M\big) \arrow[r, "\mathrm{badj}"] & \mathrm{Hom}_{\mathsf{D}(B)}\big(N, \mathrm{RHom}_B(C, M)\big)
\end{tikzcd}$$

is commutative.

(2) *There is a unique isomorphism*

$$\mathrm{dfadj}_{u,M,N} : \mathrm{Hom}_{\mathsf{D}(A)}(M, N) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{D}(B)}\big(B \otimes^{\mathrm{L}}_A M, N\big)$$

called derived forward adjunction*, which is functorial in $M \in \mathsf{D}(A)$ and $N \in \mathsf{D}(B)$, and such that the diagram*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}_{\mathrm{str}}(A)}\big(M, N\big) \arrow[r, "\mathrm{dfadj}"] \arrow[d, "\mathrm{Q}"'] & \mathrm{Hom}_{\mathsf{C}_{\mathrm{str}}(B)}\big(B \otimes_A M, N\big) \arrow[d, "\Theta_{\mathrm{f}} \circ \mathrm{Q}"] \\
\mathrm{Hom}_{\mathsf{D}(A)}\big(M, N\big) \arrow[r, "\mathrm{fadj}"] & \mathrm{Hom}_{\mathsf{D}(B)}\big(B \otimes^{\mathrm{L}}_A M, N\big)
\end{tikzcd}$$

is commutative.

Exercise 14.2.10. Prove Proposition 14.2.9. Give precise formulas for the morphisms Θ_{b} and Θ_{f}. (Hint: in item (1) (resp. (2)), look what happens when M is K-injective (resp. K-projective).)

Definition 14.2.11. Let $u : A \to B$ be a ring homomorphism, and let $M \in \mathbf{D}(B)$ and $N \in \mathbf{D}(C)$ be complexes.

(1) A backward morphism $\theta : N \to M$ in $\mathbf{D}(A)$ over u is called *a nondegenerate backward morphism* if the corresponding morphism

$$\operatorname{dbadj}_{u,M,N}(\theta) : N \to \operatorname{RHom}_A(B, M)$$

in $\mathbf{D}(B)$ is an isomorphism.

(2) A forward morphism $\lambda : M \to N$ in $\mathbf{D}(A)$ over u is called *nondegenerate forward morphism* if the corresponding morphism

$$\operatorname{dfadj}_{u,M,N}(\lambda) : B \otimes^{\mathrm{L}}_A M \to N$$

in $\mathbf{D}(C)$ is an isomorphism.

Example 14.2.12. Given $u : A \to B$ and $M \in \mathbf{D}(B)$, let $N := \operatorname{RHom}_A(B, M) \in \mathbf{D}(B)$. The identity morphism $\mathrm{id}_N : N \to N$ in $\mathbf{D}(B)$ corresponds by adjunction to a trace morphism

$$\operatorname{Tr}_{u,M} : N \to M$$

in $\mathbf{D}(B)$. Since

$$\operatorname{dbadj}_{u,M,N}(\operatorname{Tr}_{u,M}) = \mathrm{id}_N,$$

we see that $\operatorname{Tr}_{u,M}$ is a nondegenerate trace morphism.

Example 14.2.13. Given $u : A \to B$ and $M \in \mathbf{D}(A)$, let $N := B \otimes^{\mathrm{L}}_A M \in \mathbf{D}(B)$. The identity morphism $\mathrm{id}_N : N \to N$ in $\mathbf{D}(B)$ corresponds by adjunction to a forward morphism

$$\mathrm{q}_{v,M} : M \to N \tag{14.2.14}$$

in $\mathbf{D}(A)$. Since

$$\operatorname{dfadj}_{u,M,N}(\mathrm{q}_{u,M}) = \mathrm{id}_N,$$

we see that $\mathrm{q}_{u,M}$ is a nondegenerate forward morphism.

Example 14.2.15. If $A = B$ and $u = \mathrm{id}_A$, then backward and forward morphisms over u are just morphisms in $\mathbf{D}(A)$. Nondegenerate (backward or forward) morphisms are just isomorphisms in $\mathbf{D}(A)$.

We end this subsection with two useful propositions, both borrowed from [YeZh3]. They will be needed later on.

The catch in the next proposition is that the complex P of flat A-module is bounded *below*, not above.

Proposition 14.2.16. *Let P and N be bounded below complexes of A-modules. Assume that each P^i is a flat A-module, and that N has finite flat dimension over A. Then the canonical morphism $P \otimes^{\mathrm{L}}_A N \to P \otimes_A N$ in $\mathbf{D}(A)$ is an isomorphism.*

Proof. Choose a bounded flat resolution $Q \to N$ over A. We have to show that $P \otimes_A Q \to P \otimes_A N$ is a quasi-isomorphism. Let L be the cone on the quasi-isomorphism $Q \to N$. It is enough to show that the complex $P \otimes_A L$ is acyclic. We note that L is a bounded below acyclic complex and P is a bounded below complex of flat modules. To prove that $\mathrm{H}^i(P \otimes_A L) = 0$ for any given i we might as well replace P with its stupid truncation

$$P' := \operatorname{stt}^{\leq j_1}(P) = (\cdots \to P^{j_1 - 1} \to P^{j_1} \to 0 \to \cdots)$$

for $j_1 \gg i$. Now P' is K-flat, so $P' \otimes_A L$ is acyclic. □

Proposition 14.2.17. *Let $A \to B \to C$ be ring homomorphisms, and let $L \in \mathbf{D}(C)$, $M \in \mathbf{D}(B)$ and $N \in \mathbf{D}(A)$ be complexes. There is a morphism*

$$\Psi_{L,M,N} : \operatorname{RHom}_B(L, M) \otimes^{\mathrm{L}}_A N \to \operatorname{RHom}_B(L, M \otimes^{\mathrm{L}}_A N)$$

in $\mathbf{D}(C)$, which is functorial in these complexes.

Moreover, if conditions (a) *and* (b) *below hold, then $\Psi_{L,M,N}$ is an isomorphism.*

(a) *The ring B is noetherian.*
(b) *The restriction of L to B is in $\mathbf{D}^-_{\mathrm{f}}(B)$, the complex M is in $\mathbf{D}^+(B)$, and the complex N has has finite flat dimension over A.*

Proof. Let $\rho : M \to I$ be a K-injective resolution in $\mathbf{C}(B)$, let $\sigma : P \to N$ be a K-flat resolution in $\mathbf{C}(A)$, and let $\tau : I \otimes_A P \to J$ be a K-injective resolution in $\mathbf{C}(B)$. There is an obvious homomorphism

$$\tilde{\Psi}_{L,I,P} : \operatorname{Hom}_B(L, I) \otimes_A P \to \operatorname{Hom}_B(L, I \otimes_A P) \tag{14.2.18}$$

in $\mathbf{C}_{\mathrm{str}}(C)$. Its formula is

$$\tilde{\Psi}(\psi \otimes p)(l) := \pm \tau(\psi(l) \otimes p)$$

for homogeneous elements $\psi \in \operatorname{Hom}_B(L, I)$, $p \in P$ and $l \in L$, and with the Koszul sign rule. There also the homomorphism

$$\operatorname{Hom}_B(L, \tau) : \operatorname{Hom}_B(L, I \otimes_A P) \to \operatorname{Hom}_B(L, J). \tag{14.2.19}$$

The composition

$$\operatorname{Hom}_B(L, \tau) \circ \tilde{\Psi}_{L,I,P} : \operatorname{Hom}_B(L, I) \otimes_A P \to \operatorname{Hom}_B(L, J) \tag{14.2.20}$$

represents a morphism $\Psi_{L,M,N}$ in $\mathbf{D}(C)$, and this is functorial in the complexes L, M, N.

Now suppose conditions (a) and (b) hold. It suiffices to prove that for a good choice of resolutions, the homomorphism in (14.2.20) is a quasi-isomorphism. for this we might as well forget C, and work in $\mathbf{C}_{\mathrm{str}}(B)$.

By smart truncation we can assume that M is a bounded below complex of B-modules. Because B is noetherian and $L \in \mathbf{D}^-_{\mathrm{f}}(B)$, according to Corollary 10.3.32 there is a quasi-isomorphism $\pi : Q \to L$, where Q is a bounded above complex of finite free B-modules. Since N has finite flat dimension, we can assume that P is a bounded complex of flat A-modules.

Consider the next commutative diagram in $\mathsf{C}_{\mathrm{str}}(B)$.

$$\begin{tikzcd}
\mathrm{Hom}_B(L,I)\otimes_A P \arrow[r, "\tilde{\Psi}_{L,I,P}"] \arrow[d, "\mathrm{Hom}_B(\pi,\mathrm{id})\otimes_A \mathrm{id}", "\mathrm{qi}"'] & \mathrm{Hom}_B(L, I\otimes_A P) \arrow[r, "\mathrm{Hom}_B(\mathrm{id},\tau)"] \arrow[d, "\mathrm{Hom}_B(\pi,\mathrm{id})"] & \mathrm{Hom}_B(L,J) \arrow[d, "\mathrm{Hom}_B(\pi,\mathrm{id})"'] \\
\mathrm{Hom}_B(Q,I)\otimes_A P \arrow[r, "\tilde{\Psi}_{Q,I,P}"] & \mathrm{Hom}_B(Q, I\otimes_A P) \arrow[r, "\mathrm{Hom}_B(\mathrm{id},\tau)", "\mathrm{qi}"'] & \mathrm{Hom}_B(Q,J) \\
\mathrm{Hom}_B(Q,M)\otimes_A P \arrow[r, "\tilde{\Psi}_{Q,M,P}"] \arrow[u, "\mathrm{Hom}_B(\mathrm{id},\rho)\otimes_A \mathrm{id}"', "\mathrm{qi}"] & \mathrm{Hom}_B(Q, M\otimes_A P) \arrow[u, "\mathrm{Hom}_B(\mathrm{id},\rho\otimes_A \mathrm{id})"', "\mathrm{qi}"] &
\end{tikzcd}$$

The homomorphisms marked "qi" are quasi-isomorphisms. The various boundedness conditions on the complexes Q, M, P imply that in each degree i we have a finite sums (as opposed to infinite products)

$$(\mathrm{Hom}_B(Q,M)\otimes_A P)^i = \bigoplus_{j,k} \mathrm{Hom}_B(Q^j, M^k)\otimes_A P^{i-k+j}$$

and

$$(\mathrm{Hom}_B(Q,M\otimes_A P))^i = \bigoplus_{j,k} \mathrm{Hom}_B(Q^j, M^k\otimes_A P^{i-k+j}).$$

Because each Q^j is finite free, there is an isomorphism

$$\mathrm{Hom}_B(Q^j, M^k)\otimes_A P^{i-k+j} \xrightarrow{\simeq} \mathrm{Hom}_B(Q^j, M^k\otimes_A P^{i-k+j}).$$

Therefore $\tilde{\Psi}_{Q,M,P}$ is an isomorphism in $\mathsf{C}_{\mathrm{str}}(B)$. $\square$

14.3. **Functoriality of the Squaring Operation.**

cmnt: stuff moved from here

We now return to the flatness setup. In this subsection we assume:

Setup 14.3.1. A is a commutative ring, and the rings B, C, D, B', C', B'' are flat commutative A-rings.

To simplify notation we are going to borrow the "enveloping" notation from noncommutative ring theory. This is the content of the next definition.

Definition 14.3.2. Suppose $u : B \to C$ is an A-ring homomorphism.

(1) We write $B^{\mathrm{en}} := B \otimes_A B$, $C^{\mathrm{en}} := C \otimes_A C$ and $u^{\mathrm{en}} := u \otimes_A u$. Thus $u^{\mathrm{en}} : B^{\mathrm{en}} \to C^{\mathrm{en}}$ is a homomorphism between flat A-rings.

(2) Suppose $\theta : N \to M$ is a trace homomorphism in $\mathbf{C}_{\mathrm{str}}(B)$ over u (see Definition 14.2.5(1)). We write $M^{\mathrm{en}} := M \otimes_A M$, $N^{\mathrm{en}} := N \otimes_A N$ and $\theta^{\mathrm{en}} := \theta \otimes_A \theta$. Thus $\theta^{\mathrm{en}} : N^{\mathrm{en}} \to M^{\mathrm{en}}$ is a trace homomorphism in $\mathbf{C}_{\mathrm{str}}(B^{\mathrm{en}})$ over u^{en}.

(3) Suppose $\theta : N \to M$ is a forward homomorphism in $\mathbf{C}_{\mathrm{str}}(B)$ over u (see Definition 14.2.5(2)). We write $M^{\mathrm{en}} := M \otimes_A M$, $N^{\mathrm{en}} := N \otimes_A N$ and $\lambda^{\mathrm{en}} := \lambda \otimes_A \lambda$. Thus $\lambda^{\mathrm{en}} : M^{\mathrm{en}} \to N^{\mathrm{en}}$ is a forward homomorphism in $\mathbf{C}_{\mathrm{str}}(B^{\mathrm{en}})$ over u^{en}.

Let $u : B \to C$ be an A-ring homomorphism, and let $\theta : N \to M$ be a trace morphism in $\mathbf{D}(B)$ over u. We choose a K-projective resolution $P \to M$ in $\mathbf{C}(B)$, and then a K-injective resolution $P^{\mathrm{en}} \to I$ in $\mathbf{C}(B^{\mathrm{en}})$. These give us a presentation $\mathrm{pres}_{P,I}$ of $\mathrm{Sq}_{B/A}(M)$; see formula (14.1.14). Similarly we choose a K-projective resolution $Q \to N$ in $\mathbf{C}(C)$, and then a K-injective resolution $Q^{\mathrm{en}} \to J$ in $\mathbf{C}(C^{\mathrm{en}})$. These give us a presentation $\mathrm{pres}_{Q,J}$ of $\mathrm{Sq}_{C/A}(N)$.

Next let us choose a K-projective resolution $\tilde{Q} \to Q$ of Q in $\mathbf{C}(B)$. The trace morphism $\theta : N \to M$ in $\mathbf{D}(B)$ is represented by a homomorphism $\tilde{\theta} : \tilde{Q} \to P$ in $\mathbf{C}_{\mathrm{str}}(B)$. Namely the diagram

(14.3.3)
$$
\begin{tikzcd}
\tilde{Q} \arrow[r, "\cong"] \arrow[dr, "\mathrm{Q}(\tilde{\theta})"'] & Q \arrow[r, "\cong"] & N \arrow[d, "\theta"] \\
 & P \arrow[r, "\cong"] & M
\end{tikzcd}
$$

in $\mathbf{D}(B)$ is commutative.

Since B and C are flat over A, the complexes $P, Q, \tilde{Q}$ are all K-flat over A. We obtain the solid diagram

(14.3.4)
$$
\begin{tikzcd}
\tilde{Q}^{\mathrm{en}} \arrow[r, "\mathrm{qis}"] \arrow[dr, "\tilde{\theta}^{\mathrm{en}}"'] & Q^{\mathrm{en}} \arrow[r, "\mathrm{qis}"] & J \arrow[d, dashed, "\chi"] \\
 & P^{\mathrm{en}} \arrow[r, "\mathrm{qis}"] & I
\end{tikzcd}
$$

in $\mathbf{C}_{\mathrm{str}}(B^{\mathrm{en}})$, in which the arrows marked "qis" are quasi-isomorphism. Since I is K-injective, there is a homomorphism $\chi : J \to I$ that makes this diagram commutative up to homotopy. This induces a homomorphism

(14.3.5)
$$\mathrm{Hom}_{u^{\mathrm{en}}}(u, \chi) : \mathrm{Hom}_{C^{\mathrm{en}}}(C, J) \to \mathrm{Hom}_{B^{\mathrm{en}}}(B, I)$$

in $\mathsf{C}_{\mathrm{str}}(B)$.

Proposition 14.3.6. *Let $u : B \to C$ be homomorphism between flat A-rings, and let $\theta : N \to M$ be a trace morphism in $\mathsf{D}(B)$ over u. There is a unique trace morphism*

$$\mathrm{Sq}_{u/A}(\theta) : \mathrm{Sq}_{C/A}(N) \to \mathrm{Sq}_{B/A}(M)$$

in $\mathsf{D}(B)$ over u, called the square of θ, *that has the following property*:

($\diamond$) *For any choices $P, Q, \tilde{Q}, \tilde{\theta}, I, J, \chi$ as above, the diagram*

$$\begin{tikzcd}
\mathrm{Hom}_{C^{\mathrm{en}}}(C, J) \arrow[r, "\mathrm{pres}_{Q,J}"] \arrow[d, "\mathrm{Q}(\mathrm{Hom}_{u^{\mathrm{en}}}(u,\chi))"'] & \mathrm{Sq}_{C/A}(N) \arrow[d, "\mathrm{Sq}_{u/A}(\theta)"] \\
\mathrm{Hom}_{B^{\mathrm{en}}}(B, I) \arrow[r, "\mathrm{pres}_{P,I}"] & \mathrm{Sq}_{B/A}(M)
\end{tikzcd}$$

in $\mathsf{D}(B)$ is commutative.

Proof. This is because the complexes $P, Q, \tilde{Q}, I, J$ are unique up to homotopy equivalence, and the homomorphisms $\tilde{\theta}, \chi$ are unique up to homotopy. □

Theorem 14.3.7 (Trace Functoriality)**.** *We are given this input*:

- *Homomorphisms of flat A-rings $u : B \to C$ and $v : C \to D$.*
- *Complexes $M \in \mathsf{D}(B)$, $N \in \mathsf{D}(C)$ and $L \in \mathsf{D}(D)$.*
- *A trace morphism $\theta : N \to M$ in $\mathsf{D}(B)$ over u, and a trace morphism $\zeta : L \to N$ in $\mathsf{D}(C)$ over v.*

Then the following hold:

(1) *There is equality*

$$\mathrm{Sq}_{u/A}(\theta) \circ \mathrm{Sq}_{v/A}(\zeta) = \mathrm{Sq}_{v \circ u/A}(\theta \circ \zeta)$$

of trace morphisms $\mathrm{Sq}_{D/A}(L) \to \mathrm{Sq}_{B/A}(M)$ in $\mathsf{D}(B)$ over $v \circ u$.

(2) *If $C = B$ and $u = \mathrm{id}_B$, then*

$$\mathrm{Sq}_{u/A}(\theta) = \mathrm{Sq}_{B/A}(\theta),$$

where the latter is the morphism from Definition 14.1.11(2)*.*

Proof. (1) Say we choose a presentation $\mathrm{pres}_{R,K}$ of $\mathrm{Sq}_{D/A}(L)$. Then there is a homomorphism $\xi : J \to K$ such that $\mathrm{Hom}_{v^{\mathrm{en}}}(v, \xi)$ represents $\mathrm{Sq}_{v/A}(\zeta)$, as in Proposition 14.3.6. Due to the uniqueness up to homotopy of these choices, the homomorphism

$$\mathrm{Hom}_{(v \circ u)^{\mathrm{en}}}(v \circ u, \chi \circ \xi)$$

represents $\mathrm{Sq}_{v \circ u/A}(\theta \circ \zeta)$. But

$$\mathrm{Hom}_{(v \circ u)^{\mathrm{en}}}(v \circ u, \chi \circ \xi) = \mathrm{Hom}_{u^{\mathrm{en}}}(u, \chi) \circ \mathrm{Hom}_{v^{\mathrm{en}}}(v, \xi).$$

(2) Clear. □

Now consider a localization homomorphism $v : B \to B'$ of A-rings. Suppose we are given complexes $M \in \mathsf{D}(B)$ and $M' \in \mathsf{D}(B')$, and a localization morphism $\lambda : M \to M'$ in $\mathsf{D}(B)$ over v. Let's choose a K-projective resolution $P \to M$ in $\mathsf{C}(B)$, and then a K-injective resolution $\rho : P^{\mathrm{en}} \to I$ in $\mathsf{C}(B^{\mathrm{en}})$. Likewise let's choose a K-projective resolution $P' \to M'$ in $\mathsf{C}(B')$, and then a K-injective resolution $\rho' : P'^{\,\mathrm{en}} \to I'$ in $\mathbf{C}(B'^{\,\mathrm{en}})$. These choices give us presentations $\operatorname{pres}_{P,I}$ and $\operatorname{pres}_{P',I'}$ of $\operatorname{Sq}_{B/A}(M)$ and $\operatorname{Sq}_{B'/A}(M')$ respectively.

Because P is K-projective, there is a homomorphism $\tilde{\lambda} : P \to P'$ in $\mathbf{C}_{\mathrm{str}}(B)$ that makes the diagram

$$\begin{tikzcd}
P \arrow[r, "\cong"] \arrow[d, "\mathrm{Q}(\tilde{\lambda})"'] & M \arrow[d, "\lambda"] \\
P' \arrow[r, "\cong"] & M'
\end{tikzcd}$$

in $\mathbf{D}(B)$ commutative. On bimodules we get a homomorphism

$$\tilde{\lambda}^{\mathrm{en}} : P^{\mathrm{en}} \to P'^{\,\mathrm{en}}$$

in $\mathbf{C}_{\mathrm{str}}(B^{\mathrm{en}})$.

We have the following solid diagram in $\mathbf{C}_{\mathrm{str}}(B'^{\,\mathrm{en}})$:

(14.3.8)
$$\begin{tikzcd}
B'^{\,\mathrm{en}} \otimes_{B^{\mathrm{en}}} P^{\mathrm{en}} \arrow[r, "\mathrm{id} \otimes \rho"] \arrow[d, "\mathrm{fadj}_{v^{\mathrm{en}}}(\tilde{\lambda}^{\mathrm{en}})"'] & B'^{\,\mathrm{en}} \otimes_{B^{\mathrm{en}}} I \arrow[d, dashed, "\tilde{\xi}"] \\
P'^{\,\mathrm{en}} \arrow[r, "\rho'"] & I'
\end{tikzcd}$$

where $\mathrm{fadj}(\tilde{\lambda}^{\mathrm{en}})$ is the forward adjunction from (14.2.8). Since $v^{\mathrm{en}} : B^{\mathrm{en}} \to B'^{\,\mathrm{en}}$ is flat, the homomorphism $\mathrm{id} \otimes \rho$ above is a quasi-isomorphism. On the other hand the complex I' is K-injective. Therefore there is a homomorphism

$$\tilde{\xi} : B'^{\,\mathrm{en}} \otimes_{B^{\mathrm{en}}} I \to I'$$

in $\mathbf{C}_{\mathrm{str}}(B'^{\,\mathrm{en}})$ that makes the diagram (14.3.8) commutative up to homotopy. By forward adjunction, $\tilde{\xi} = \mathrm{fadj}_{v^{\mathrm{en}}}(\xi)$ for a unique homomorphism

(14.3.9)
$$\xi : I \to I'$$

in $\mathbf{C}_{\mathrm{str}}(B^{\mathrm{en}})$. We obtain a diagram

(14.3.10)
$$\begin{tikzcd}
P^{\mathrm{en}} \arrow[r, "\rho"] \arrow[d, "\tilde{\lambda}^{\mathrm{en}}"'] & I \arrow[d, "\xi"] \\
P'^{\,\mathrm{en}} \arrow[r, "\rho'"] & I'
\end{tikzcd}$$

in $\mathbf{C}_{\mathrm{str}}(B^{\mathrm{en}})$. Since (14.3.8) is commutative up to homotopy, the same is true for (14.3.10).

The homomorphism ξ induces a homomorphism

(14.3.11)
$$\mathrm{Hom}_{B^{\mathrm{en}}}(B, \xi) : \mathrm{Hom}_{B^{\mathrm{en}}}(B, I) \to \mathrm{Hom}_{B^{\mathrm{en}}}(B, I')$$

in $\mathbf{C}_{\mathrm{str}}(B)$. By the forward adjunction formula (14.2.7) there is an isomorphism

$$\mathrm{fadj}_{v^{\mathrm{en}}, B, I'} : \mathrm{Hom}_{B^{\mathrm{en}}}\big(B, I'\big) \xrightarrow{\simeq} \mathrm{Hom}_{B'^{\,\mathrm{en}}}(B'^{\,\mathrm{en}} \otimes_{B^{\mathrm{en}}} B, I').$$

But $B \to B'$ is a localization, so there are unique B-ring isomorphisms

$$B'^{\,\mathrm{en}} \otimes_{B^{\mathrm{en}}} B = (B' \otimes_A B') \otimes_{B \otimes_A B} B \cong B' \otimes_B B' \cong B'.$$

Therefore in this particular situation we get an isomorphism

(14.3.12)
$$\mathrm{fadj}_{v^{\mathrm{en}}, B, I'} : \mathrm{Hom}_{B^{\mathrm{en}}}\big(B, I'\big) \xrightarrow{\simeq} \mathrm{Hom}_{B'^{\,\mathrm{en}}}(B', I')$$

in $\mathsf{C}_{\mathrm{str}}(B)$.

Proposition 14.3.13. *Let $v : B \to B'$ be a localization homomorphism between flat A-rings, and let $\lambda : M \to M'$ be a localization morphism in $\mathsf{D}(B)$ over v. There is a unique localization morphism*

$$\mathrm{Sq}_{v/A}(\lambda) : \mathrm{Sq}_{B/A}(M) \to \mathrm{Sq}_{B'/A}(M')$$

in $\mathsf{D}(B)$ over v, called the square of λ, *that has the following property*:

(†) *For any choices of resolutions and homomorphisms as above, the diagram*

$$\begin{tikzcd}
\mathrm{Hom}_{B^{\mathrm{en}}}(B, I) \arrow[r, "\mathrm{pres}_{P,I}"] \arrow[d, "\mathrm{Hom}_{B^{\mathrm{en}}}(B,\xi)"'] & \mathrm{Sq}_{B/A}(M) \arrow[dd, "\mathrm{Sq}_{v/A}(\lambda)"] \\
\mathrm{Hom}_{B^{\mathrm{en}}}(B, I') \arrow[d, "\mathrm{fadj}_{v^{\mathrm{en}},B,I'}"'] & \\
\mathrm{Hom}_{B'^{\mathrm{en}}}(B', I') \arrow[r, "\mathrm{pres}_{P',I'}"] & \mathrm{Sq}_{B'/A}(M')
\end{tikzcd}$$

in $\mathsf{D}(B)$ is commutative.

Proof. The reason is that the choices made are unique up to homotopy. □

In case $B' = C = B$, $v = u = \mathrm{id}_B$ and $\lambda = \theta$, there is an apparent conflict between the morphisms $\mathrm{Sq}_{v/A}(\lambda)$ from Proposition 14.3.13 and $\mathrm{Sq}_{u/A}(\theta)$ from Proposition 14.3.6. This apparent conflict is removed by part (2) of Theorem 14.3.14 below, in conjunction with part (2) of Theorem 14.3.7.

Theorem 14.3.14 (Localization Functoriality)**.** *We are given this input*:

- *Localization homomorphisms $v : B \to B'$ and $v' : B' \to B''$ between flat A-rings.*
- *Complexes $M \in \mathsf{D}(B)$, $M' \in \mathsf{D}(B')$ and $M'' \in \mathsf{D}(B'')$.*
- *A localization morphism $\lambda : M \to M'$ in $\mathsf{D}(B)$ over v, and a localization morphism $\lambda' : M' \to M''$ in $\mathsf{D}(B')$ over u'.*

Then the following hold:

(1) *There is equality*

$$\mathrm{Sq}_{v'/A}(\lambda') \circ \mathrm{Sq}_{v/A}(\lambda) = \mathrm{Sq}_{v' \circ v/A}(\lambda' \circ \lambda)$$

of localization morphisms $\mathrm{Sq}_{B/A}(M) \to \mathrm{Sq}_{B''/A}(M'')$ *in $\mathsf{D}(B)$ over $v' \circ v$.*

(2) *If $B' = B$ and $v = \mathrm{id}_B$, then*

$$\mathrm{Sq}_{v/A}(\lambda) = \mathrm{Sq}_{B/A}(\lambda),$$

where the latter is the morphism from Definition 14.1.11(2).

Proof. This is similar to the proof of Theorem 14.3.7. We leave the details to the reader. □

Exercise 14.3.15. Give a detailed proof of Theorem 14.3.14.

cmnt: $\boxed{\text{to here 29/03/2017}}$

The next result relates the two type of functorialities of the squaring operation.

Theorem 14.3.16 (Compatibility of Traces and Localizations)**.** *We are given a commutative diagram of homomorphisms between flat A-rings*

$$\begin{tikzcd} B \arrow[r, "u"] \arrow[d, "v"'] & C \arrow[d, "w"] \\ B' \arrow[r, "u'"] & C' \end{tikzcd}$$

in which v is a localization, and

$$u' \otimes_B w : B' \otimes_B C \to C'$$

is an isomorphism (i.e. the diagram is cocartesian). We are also given this information:

- *Complexes $M \in \mathbf{D}(B)$, $N \in \mathbf{D}(C)$, $M' \in \mathbf{D}(B')$ and $N' \in \mathbf{D}(C')$.*
- *A trace morphism $\theta : N \to M$ in $\mathbf{D}(B)$ over u.*
- *A localization morphism $\lambda : M \to M'$ in $\mathbf{D}(B)$ over v.*
- *A trace morphism $\theta' : N' \to M'$ in $\mathbf{D}(B')$ over u'.*
- *A localization morphism $\mu : N \to N'$ in $\mathbf{D}(C)$ over w.*

These morphisms are required to render the diagram

$$\begin{tikzcd} M \arrow[d, "\lambda"'] & N \arrow[l, "\theta"'] \arrow[d, "\mu"] \\ M' & N' \arrow[l, "\theta'"] \end{tikzcd}$$

in $\mathbf{D}(B)$ commutative.

Then the diagram

$$\begin{tikzcd} \mathrm{Sq}_{B/A}(M) \arrow[d, "\mathrm{Sq}_{v/A}(\lambda)"'] & \mathrm{Sq}_{C/A}(N) \arrow[l, "\mathrm{Sq}_{u/A}(\theta)"'] \arrow[d, "\mathrm{Sq}_{w/A}(\mu)"] \\ \mathrm{Sq}_{B'/A}(M') & \mathrm{Sq}_{C'/A}(N') \arrow[l, "\mathrm{Sq}_{u'/A}(\theta')"] \end{tikzcd}$$

in $\mathbf{D}(B)$ is commutative.

Proof. By the forward adjunction formula (14.2.7), the given morphisms in $\mathbf{D}(B)$ fit into a larger commutative diagram

(14.3.17)
$$\begin{tikzcd} M \arrow[d, "\mathrm{q}_{v,M}"'] & N \arrow[l, "\theta"'] \arrow[d, "\mathrm{q}_{w,N}"] \\ B' \otimes_B M \arrow[d, "\mathrm{fadj}_v(\lambda)"'] & B' \otimes_B N \arrow[l, "\mathrm{id} \otimes \theta"'] \arrow[d, "\mathrm{fadj}_w(\mu)"] \\ M' & N' \arrow[l, "\theta'"] \end{tikzcd}$$

in which the bottom square is in the category $\mathbf{D}(B')$. Applying the squaring to (14.3.17) we obtain a diagram

(14.3.18)
$$\begin{tikzcd}
\mathrm{Sq}_{B/A}(M) \arrow[d, "\mathrm{Sq}_{v/A}(\mathrm{q}_{v,M})"'] & \mathrm{Sq}_{C/A}(N) \arrow[l, "\mathrm{Sq}_{u/A}(\theta)"'] \arrow[d, "\mathrm{Sq}_{w/A}(\mathrm{q}_{w,N})"] \\
\mathrm{Sq}_{B'/A}(B' \otimes_B M) \arrow[d, "\mathrm{Sq}_{B'/A}(\mathrm{fadj}_v(\lambda))"'] & \mathrm{Sq}_{C'/A}(B' \otimes_B N) \arrow[l, "\mathrm{Sq}_{u'/A}(\mathrm{id} \otimes \theta)"'] \arrow[d, "\mathrm{Sq}_{C'/A}(\mathrm{fadj}_w(\mu))"] \\
\mathrm{Sq}_{B'/A}(M') & \mathrm{Sq}_{C'/A}(N') \arrow[l, "\mathrm{Sq}_{u'/A}(\theta')"']
\end{tikzcd}$$

The bottom square is commutative by Theorem 14.3.7. It remains to prove that the top square is commutative.

Thus we can assume that $M' = B' \otimes_B M$, $N' = C' \otimes_C N \cong B' \otimes_B N$, $\lambda = \mathrm{q}_{v,M}$, $\mu = \mathrm{q}_{w,N}$ and $\theta' = \mathrm{id} \otimes \theta$. Let us choose resolutions $P \to M$, $Q \to N$ and $\tilde{Q} \to Q$ as we did before Proposition 14.3.6. Letting $P' := B' \otimes_B P$, $Q' := B' \otimes_B Q$ and $\tilde{Q}' := B' \otimes_B \tilde{Q}$, these are resolutions of M' and N' respectively. Choose a homomorphism $\tilde{\theta} : \tilde{Q} \to P$ that represents θ, as in diagram (14.3.3). Then

$$\tilde{\theta}' := \mathrm{id}_{B'} \otimes \tilde{\theta} : \tilde{Q}' \to P'$$

represents θ'. There is a diagram

(14.3.19)
$$\begin{tikzcd}
P \arrow[d, "\mathrm{q}_{v,P}"'] & \tilde{Q} \arrow[l, "\tilde{\theta}"'] \arrow[d, "\mathrm{q}_{v,\tilde{Q}}"'] & Q \arrow[l] \arrow[d, "\mathrm{q}_{w,Q}"'] \\
P' & \tilde{Q}' \arrow[l, "\tilde{\theta}'"'] & Q' \arrow[l]
\end{tikzcd}$$

in $\mathbf{C}_{\mathrm{str}}(B)$ that's commutative up to homotopy. The unmarked arrows are quasi-isomorphisms.

Now we pass to bimodules. As before we choose K-injective resolutions $P^{\mathrm{en}} \to I$ in $\mathbf{C}(B^{\mathrm{en}})$, $Q^{\mathrm{en}} \to J$ in $\mathbf{C}(C^{\mathrm{en}})$, $P'^{\,\mathrm{en}} \to I'$ in $\mathbf{C}(B'^{\,\mathrm{en}})$ and $Q'^{\,\mathrm{en}} \to J'$ in $\mathbf{C}(C'^{\,\mathrm{en}})$. Consider the following complicated diagram in $\mathbf{C}_{\mathrm{str}}(B^{\mathrm{en}})$:

(14.3.20)
$$\begin{tikzcd}
I \arrow[d, "\xi_I"'] & P^{\mathrm{en}} \arrow[l] \arrow[d, "\mathrm{q}_{v^{\mathrm{en}},P^{\mathrm{en}}}"'] & \tilde{Q}^{\mathrm{en}} \arrow[l, "\tilde{\theta}^{\mathrm{en}}"'] \arrow[d, "\mathrm{q}_{v^{\mathrm{en}},\tilde{Q}^{\mathrm{en}}}"'] & Q^{\mathrm{en}} \arrow[l] \arrow[r] \arrow[d, "\mathrm{q}_{w^{\mathrm{en}},Q^{\mathrm{en}}}"'] & J \arrow[d, "\xi_J"] \arrow[llll, bend right, "\chi"'] \\
I' & P'^{\,\mathrm{en}} \arrow[l] & \tilde{Q}'^{\,\mathrm{en}} \arrow[l, "\tilde{\theta}'"'] & Q'^{\,\mathrm{en}} \arrow[l] \arrow[r] & J' \arrow[llll, bend left, "\chi'"]
\end{tikzcd}$$

The unmarked arrows are quasi-isomorphisms. The top and bottom half-moons are two versions of diagram (14.3.4), and they are commutative up to homotopies. The two squares in the middle are the bimodule version of of diagram (14.3.19), and too they are commutative up to homotopies. The two squares on the extreme left and right are two versions of (14.3.10), so they are commutative up to homotopies. Therefore the diagram

(14.3.21)
$$\begin{tikzcd} I \arrow[d, "\xi_I"'] & J \arrow[l, "\chi"'] \arrow[d, "\xi_J"] \\ I' & J' \arrow[l, "\chi'"'] \end{tikzcd}$$

in $\mathbf{C}_{\mathrm{str}}(B^{\mathrm{en}})$, that is the outer boundary of (14.3.20), is commutative up to homotopy.

Finally, applying $\mathrm{Hom}_{-^{\mathrm{en}}}(-,-)$ to the diagram (14.3.21) we obtain the diagram

$$\begin{tikzcd} \mathrm{Hom}_{B^{\mathrm{en}}}(B, I) \arrow[d, "\mathrm{Hom}_{B^{\mathrm{en}}}(B,\xi_I)"'] & \mathrm{Hom}_{C^{\mathrm{en}}}(C, J) \arrow[l, "\mathrm{Hom}_{u^{\mathrm{en}}}(u,\chi)"'] \arrow[d, "\mathrm{Hom}_{C^{\mathrm{en}}}(C,\xi_J)"] \\ \mathrm{Hom}_{B^{\mathrm{en}}}(B, I') \arrow[d, "\mathrm{fadj}_{v^{\mathrm{en}},B,I'}"'] & \mathrm{Hom}_{C^{\mathrm{en}}}(C, J') \arrow[l, "\mathrm{Hom}_{u^{\mathrm{en}}}(u,\chi')"'] \arrow[d, "\mathrm{fadj}_{w^{\mathrm{en}},C,J'}"] \\ \mathrm{Hom}_{B'^{\mathrm{en}}}(B', I') & \mathrm{Hom}_{C'^{\mathrm{en}}}(C', J') \arrow[l, "\mathrm{Hom}_{u'^{\mathrm{en}}}(u',\chi')"'] \end{tikzcd}$$

in $\mathbf{C}_{\mathrm{str}}(B)$. It is commutative up to homotopy. By Proposition 14.3.13, the outer boundary of this diagram represents the diagram

$$\begin{tikzcd} \mathrm{Sq}_{B/A}(M) \arrow[d, "\mathrm{Sq}_{v/A}(\mathrm{q}_{v,M})"'] & \mathrm{Sq}_{C/A}(N) \arrow[l, "\mathrm{Sq}_{u/A}(\theta)"'] \arrow[d, "\mathrm{Sq}_{w/A}(\mathrm{q}_{w,N})"] \\ \mathrm{Sq}_{B'/A}(B' \otimes_B M) & \mathrm{Sq}_{C'/A}(B' \otimes_B N) \arrow[l, "\mathrm{Sq}_{u'/A}(\mathrm{id} \otimes \theta)"'] \end{tikzcd}$$

in $\mathbf{D}(B)$, and therefore this last diagram is commutative. $\square$

cmnt: leave the cup product until later - need it only for residue thm

14.4. **Interlude: DG Ring Resolutions.**

cmnt: this material belongs way back in the book...

For establishing the functoriality of rigid complexes we need to use DG rings a bit. (Not nearly as deeply as what is outlined in Remark 14.1.25.

Suppose A and B are DG rings. A homomorphism of DG rings $u : A \to B$ induces a homomorphism of graded rings $\mathrm{H}(u) : \mathrm{H}(A) \to \mathrm{H}(B)$; cf. Example 3.3.18. The DG ring homomorphism f is called a *quasi-isomorphism of DG rings* if $\mathrm{H}(u)$ is an isomorphism.

A DG ring homomorphism $u : A \to B$ induces a DG functor

$$\mathrm{Rest}_u : \mathbf{C}(B) \to \mathbf{C}(A)$$

called restriction, that was already encountered in Subsection 14.2. Since Rest_u is exact, it passes to a triangulated functor

$$\mathrm{Rest}_u : \mathbf{D}(B) \to \mathbf{D}(A).$$

There is also the *induction* functor

$$\mathrm{Ind}_u : \mathbf{C}(A) \to \mathbf{C}(B), \quad \mathrm{Ind}_u(M) := B \otimes_A M.$$

It has a left derived functor

$$\mathrm{LInd}_u : \mathbf{D}(A) \to \mathbf{D}(A), \quad \mathrm{LInd}_u(M) := B \otimes^{\mathrm{L}}_A M,$$

which is a triangulated functor.

Here is a fundamental result. We do not know where is was first proved.

Theorem 14.4.1. *Let $u : A \to B$ be a homomorphism of DG rings.*

(1) *The functor LInd_f is a left adjoint to Rest_f. That is to say, for any $M \in \mathbf{D}(A)$ and $N \in \mathbf{D}(B)$ there is a bijection*

$$\mathrm{dfadj}_u : \mathrm{Hom}_{\mathbf{D}(A)}(M, \mathrm{Rest}_u(N)) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{D}(B)}(\mathrm{LInd}_u(M), N),$$

and it is functorial in M and N.

(2) *If u is a quasi-isomorphism, then Rest_u and LInd_u are equivalences of triangulated categories.*

Proof. (1) This is Proposition 14.2.9(2), stated a bit differently.

(2) This is [YeZh3, Proposition 1.4(1)].

cmnt: maybe write full proof here?

□

Definition 14.4.2. A DG ring A is called a *commutative DG ring* if it has the following properties:

(a) Nonpositivity: $A^i = 0$ for all $i > 0$.

(b) Strong commutativity:

$$b \cdot a = (-1)^{i \cdot j} \cdot a \cdot b$$

for all $a \in A^i$ and $b \in A^j$, and $a^2 = 0$ if i is odd.

This definition is taken from [Ye11]. In [YeZh3] the term "super-commutative" was used instead of "strongly commutative".

By *nonpositive graded set* we mean a set X that is partitioned into subsets $X = \coprod_{i \leq 0} X^i$. The elements of X^i are said to have degree i.

Given a nonpositive graded set X, we can form the noncommutative polynomial ring $\mathbb{Z}\langle X \rangle$ in X over $\mathbb{Z}$. As a graded $\mathbb{Z}$-module, $\mathbb{Z}\langle X \rangle$ is free with basis the collection of monomials

$$\{x_1 \cdots x_l\}_{x_1,\ldots,x_l \in X}.$$

The degree of a monomial $x_1 \cdots x_l$, with $x_p \in X^{i_p}$, is $i_1 + \cdots + i_l$. The multiplication in $\mathbb{Z}\langle X \rangle$ is defined by

$$(x_1 \cdots x_l) \cdot (x_{l+1} \cdots x_m) := x_1 \cdots x_m.$$

The *commutative polynomial ring* in X over $\mathbb{Z}$ is the quotient ring

$$\mathbb{Z}[X] := \mathbb{Z}\langle X \rangle / I,$$

where I is the two-sided ideal of $\mathbb{Z}\langle X \rangle$ generated by the elements

$$y \cdot x - (-1)^{i \cdot j} \cdot x \cdot y$$

for all $x \in X^i$ and $y \in X^j$, and $x \cdot x$ if i is odd.

Recall that for a DG object M, the graded object gotten by forgetting the differential is denoted by $M^\natural$.

Definition 14.4.3. Let $A \to \tilde{B}$ be a homomorphism between commutative DG rings. We say that $\tilde{B}$ is a *semi-free commutative DG ring over A* if there is an isomorphism of graded $A^\natural$-rings

$$\tilde{B}^\natural \cong A^\natural \otimes_{\mathbb{Z}} \mathbb{Z}[X]$$

for some nonpositive graded set X.

Definition 14.4.4. Let $f : A \to B$ be a homomorphism of commutative DG rings. A *semi-free commutative DG ring resolution* of B over A is a semi-free commutative DG ring $\tilde{B}$ over A, together with a surjective quasi-isomorphism of DG A-rings $\tilde{B} \to B$.

Theorem 14.4.5. *Let $f : A \to B$ be a homomorphism of commutative DG rings. There exists a semi-free commutative DG ring resolution $\tilde{B} \to B$ of B over A.*

We will not use this general theorem, but rather the slightly different Theorem 14.4.8 below. A proof of Theorem 14.4.5 can be found in [YeZh3, Proposition 1.7(1)] and [Ye11, Theorem 3.21(1)].

Example 14.4.6. Take $A := \mathbb{Z}$ and $B := Z/(6)$. The Koszul complex $\tilde{B} := \mathrm{K}(\mathbb{Z}, 6)$ from Example 3.3.9 is a semi-free commutative DG ring resolution of B over A.

Definition 14.4.7. Let $f : A \to B$ be a homomorphism of commutative DG rings. A *K-projective commutative DG ring resolution* of B over A is a commutative DG ring $\tilde{B}$ over A, which is K-projective as a DG A-module, together with a surjective quasi-isomorphism of DG A-rings $\tilde{B} \to B$.

Of course a semi-free commutative DG ring resolution is K-projective too. But often (and unlike Example 14.4.6) we can't produce semi-free commutative DG ring resolutions with suitable finiteness properties.

Theorem 14.4.8. *Let $A \to B$ be a homomorphism of commutative rings. Assume A is noetherian and B is finite over A. Then there exists a K-projective commutative DG ring resolution $v : \tilde{B} \to B$ of B over A, such that each $\tilde{B}^i$ is a finite free A-module.*

Proof. This is [YeZh3, Proposition 1.7(3)], but we will give the whole proof here.

The strategy is this: we will construct an ascending sequence of commutative DG A- rings $\{F_j(\tilde{B})\}_{j \geq 0}$, together with DG A-ring homomorphisms $F_j(v) : F_j(\tilde{B}) \to B$. Then

$$\tilde{B} := \lim_{j \to} F_j(\tilde{B})$$

and

$$u := \lim_{j \to} F_j(u)$$

will have the desired properties.

We start by choose a finite collection $\{b_x\}_{x \in X^0}$ of elements of B that generate it as an A-ring. We consider the finite set X^0 to be of degree 0. Because $A \to B$ is finite, each $b_x \in B$ satisfies some monic polynomial $f_x(t) \in A[t]$. Define the ring $F_0(\tilde{B})$ to be

$$F_0(\tilde{B}) := A[X^0]/(\{f_x(b_x)\}_{x \in X^0}).$$

This ring is finite free as an A-module, and there is a surjection of A-rings

$$F_0(u) : F_0(\tilde{B}) \to B.$$

Next consider the ideal $\mathfrak{b}_0 := \operatorname{Ker}(F_0(u))$. Since the ring $F_0(\tilde{B})$ is noetherian, this ideal is generated by finitely many elements $\{b_x\}_{x \in X^{-1}}$. The finite indexing set X^{-1} is given degree -1. We let

$$F_1(\tilde{B}) := F_0(\tilde{B})[X^{-1}],$$

the commutative polynomial ring on the odd set X^{-1}. We then define a differential d on $F_1(\tilde{B})$ by $\mathrm{d}(x) := b_x$ for $x \in X^{-1}$. And we define a homomorphism

$$F_1(u) : F_1(\tilde{B}) \to B$$

that extends $F_0(u)$ by the formula $F_1(u)(x) := 0$ for $x \in X^{-1}$. So $F_1(u)$ is a DG ring homomorphism, and

$$\mathrm{H}^0(F_1(u)) : \mathrm{H}^0(F_1(\tilde{B})) \to \mathrm{H}^0(B) = B$$

is bijective.

We continue inductively. At the j-th step ($j \geq 1$) we have a DG ring

cmnt: $\boxed{\text{finish proof}}$

□

14.5. **Functoriality of Rigid Complexes.**

14.6. **Rigid Dualizing Complexes.**

14.7. **Rigid Residue Complexes.**

15. Derived Categories in Geometry [later]

15.1. Recalling Facts on Ringed Spaces.

15.2. K-Flat Resolutions in $\mathsf{C}(\mathcal{A})$ [later].

15.3. K-Injective Resolutions in $\mathsf{C}(\mathcal{A})$ [later].

15.4. K-Flasque Resolutions in $\mathsf{C}(\mathcal{A})$ [later].

15.5. Standard Derived Functors in Geometry [later].

15.6. Survey: Poincaré-Verdier Duality [later, optional].

15.7. Survey: Applications to Birational Geometry [later, optional].

16. Residues and Duality in Algebraic Geometry [later]

16.1. Dualizing Complexes on Schemes [later].

16.2. Rigid Residue Complexes on Schemes [later].

16.3. The Residue Theorem [later].

16.4. Grothendieck Duality for Proper Maps [later].

16.5. Perverse Coherent Sheaves on Schemes [later, optional].

16.6. Survey of Related Material [later, optional].

17. Derived Categories in Noncommutative Algebra [later]

17.1. Noncommutative Dualizing Complexes [later].

17.2. Noncommutative Tilting Complexes [later].

17.3. The Noncommutative Derived Picard Group of a Ring [later].

17.4. Derived Morita Theory [later].

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