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Noetherian Rings

Below A is a commutative ring.

Def. An A-module M is called noetherian if every submodule of M is finitely generated.

The ring A is called noetherian if the A-module A is noetherian; i.e. if every ideal $\mathfrak{a} \subset A$ is f. gen.

Prop. An A-module M is noetherian iff it satisfies the ascending chain condition: if $\{N_i\}_{i \in \mathbb{N}}$ is an ascending chain of submodules of M, then $N_i = N_{i+1}$ for $i \gg 0$.

(i.e. $N_0 \subseteq N_1 \subseteq N_2 \subseteq \ldots$)

→ **Exercise.** Prove this proposition.

Prop. Let $0 \to N_0 \xrightarrow{\varphi_1} N_1 \xrightarrow{\varphi_2} N_2 \to 0$

be an exact seq. of A-modules. TFAE:

(i) N_1 is noetherian.

(ii) N_0 and N_2 are noetherian.

Pf. (i) $\Rightarrow$ (ii): Say $M_0 \subseteq N_0$ and $M_2 \subseteq N_2$ are submods. Then $M_0 \cong \varphi_1(M_0) \subseteq N_1$, so M_0 is f.gen. Since $\varphi_2^{-1}(M_2) \subseteq N_1$ is fin. gen., and $\varphi_2^{-1}(M_2) \to M_2$ is surj, we see that M_2 is f.g.

(ii) $\Rightarrow$ (i): Let $M_1 \subseteq N_1$. Choose finitely many elts. $\tilde{m}_1, \ldots, \tilde{m}_k \in M_0 := \underbrace{\varphi_1^{-1}(M_1)}_{\subseteq N_0}$ that generate it.

and let $m_i := \varphi_1(\tilde{m}_i) \in M_1$. Choose fin. many $\tilde{n}_1, \dots, \tilde{n}_s \in M_2 := \varphi_2(M_1)$ that gen. it, and choose lifts $n_i \in \varphi_2^{-1}(\tilde{n}_i) \subseteq M_1$. Then M_1 is gen. by the elements $m_1, \dots, m_r, n_1, \dots, n_s$. $\square$

<u>Theorem</u>. If A is a noetherian ring, then every finitely generated A-module is noetherian.

<u>proof</u>. The free module A is noeth. by def. Now consider A^n for $n \geq 2$. There is an ex. seq.

$$0 \to A^{n-1} \to A^n \to A \to 0.$$

By induction and the previous prop. we see that A^n is noeth.

Finally let M be some f.g. module. There is an ex. seq.

$$0 \to N \to A^n \to M \to 0$$

for some n. Again using the prop, we conclude that M is noeth. $\square$

Theorem. Let A be a noetherian ring, and let $S \subset A$ be a mult. closed subset. Then the ring A_S is noetherian.

We need a lemma first.

Lemma. Let M be an A-module. Consider the hom. $\lambda : M \to M_S$, $\lambda(m) = \frac{m}{1}$. Let $N \subseteq M_S$ be an A_S-submodule, and define $\tilde{N} := \lambda^{-1}(N)$, which is an A-submodule of M. Then

$$N = A_S \, \lambda(\tilde{N}).$$

Pf. Take $\frac{m}{s} \in N$. Then $\frac{s}{1} \frac{m}{s} = \frac{m}{1} = \lambda(m)$, and $\frac{s}{1} \cdot \frac{m}{s} \in N$. So $m \in \tilde{N}$. But $\frac{m}{s} = \frac{1}{s} \cdot \frac{m}{1} = \frac{1}{s} \cdot \lambda(m)$. $\square$

Already did this for ideals in p. 84.

Pf of thm. Let $g : A \to A_S$ be the can. ring hom. Let $\mathfrak{b} \subset A_S$ be an ideal. Consider

$\mathfrak{a} := g^{-1}(\mathfrak{b})$, which is an ideal of A.

By assumption $\mathfrak{a}$ is fin. gen.; and by Lemma [p. 84] $\mathfrak{b} = A_S \cdot g(\mathfrak{a})$. So $\mathfrak{b}$ is fin. gen. as an A_S-module.

$\square$

Prop. Let A be a noeth. ring, and let $\mathfrak{a} \subseteq A$ be an ideal. Then the ring $B := A/\mathfrak{a}$ is noetherian.

→ Exercise. Prove the prop.

Hilbert Basis Theorem

Theorem. Let A be a noetherian ring. Then the polynomial ring $A[t]$, in a single variable t, is noetherian.

The following terminology is needed for the proof. Let $f(t) = \sum_{i=0}^{n} a_i t^i$ be a polynomial of deg. $n \geq 0$ (so $a_n \neq 0$).
The element $a_n \in A$ is called the leading coefficient of f.

proof. Take an ideal $\mathbb{b}$ in the ring $A[t]$. We have to prove it is fin. gen. Let us assume the contrary (and show a contradiction).
We will construct (*) an infinite sequence $(f_1, f_2, \ldots)$ of nonzero elements of $\mathbb{b}$, satisfying: ⓐ $\deg(f_1) \leq \deg(f_2) \leq \ldots$;
ⓑ Let $\mathbb{b}_{i-1}$ be the ideal gen. by $f_1, \ldots, f_{i-1}$ (if $i=1$ then $\mathbb{b}_0 = 0$). Then $f_i \notin \mathbb{b}_{i-1}$, ⓒ f_i is of minimal degree in $\mathbb{b} - \mathbb{b}_{i-1}$.
The construction: By assumption (‡) $\mathbb{b}_{i-1} \neq \mathbb{b}$. Choose f_i to be an element of minimal degree in $\mathbb{b} - \mathbb{b}_{i-1}$. Note that $\deg(f_i) \geq \deg(f_{i-1})$, if $i \geq 2$, due to the minim. of (◇).
We have a seq. of ideals $\mathbb{b}_1 \subsetneq \mathbb{b}_2 \subsetneq \ldots$
For each i let a_i be the leading coefficient of the poly. $f_i(t)$. Define $\mathfrak{a} \subseteq A$ to be the ideal gen. by $a_1, a_2, \ldots$.

(◇) f_{i-1} in $\mathbb{b} - \mathbb{b}_{i-2} \supseteq \mathbb{b} - \mathbb{b}_{i-1}$.

cont...

(‡) ($\mathbb{b}$ not fin. gen), but $\mathbb{b}_i$ is fin. gen.

This is done (*) by induction on $i \geq 1$.

... cont.

Since A is noetherian, the ideal $\mathfrak{a}$ is fin. gen. Hence $\mathfrak{a} = \sum_{j=1}^{i} A \cdot a_j$ for some $i \geq 1$.
Therefore we have

$$a_{i+1} = \sum_{j=1}^{i} c_j \cdot a_j$$

for some $c_j \in A$.
~~Define~~ the polynomial

$$g(t) := \sum_{j=1}^{i} c_j \cdot f_j(t) \cdot t^{\deg(f_{i+1}) - \deg(f_j)}$$

This is a poly. of $\deg(g) = \deg(f_{i+1})$, with leading coefficient $a_{i+1} \neq 0$. Also $g \in \mathfrak{b}_i$. Now

$$h := f_{i+1} - g \in \mathfrak{b}_{i+1}$$

has $\deg(h) < \deg(f_{i+1})$,$^{(*)}$ and $h \in \mathfrak{b}_{i+1} - \mathfrak{b}_i \subseteq \mathfrak{b} - \mathfrak{b}_i$.$^{(\heartsuit)}$
This contradicts our choice of f_{i+1}. $\square$

(*) f_{i+1} and g have same degree & leading coef.

($\heartsuit$) because $g \in \mathfrak{b}_i$ and $f_{i+1} \notin \mathfrak{b}_i$

Cor. Suppose A is a noetherian ring and B is a finitely generated A-ring. Then B is noetherian.

pf. We can write B as $A[t_1, \dots, t_n]/\mathfrak{b}$ for some $n \geq 1$, and some ideal $\mathfrak{b}$. By Prop. [p. 102] it suffices to prove that the poly. ring $A[t_1, \dots, t_n]$ is noetherian. But this is a consequence of the Hilb. Basis Thm. & induction on n. $\square$

An A-ring B is called <u>essentially finite type</u> if B is a localization $B'_{S'}$ of some fin. type (i.e. fin. gen.) A-ring B', at some m.c. set $S' \subset B'$.

Cor. If A is noetherian and B is an ess. fin. type A-ring, then B is noetherian.

pf. Use last corollary and Thm. [p. 101] $\square$

<u>Example</u> Let $\mathbb{K}$ be a field or $\mathbb{Z}$, let A be a fin. gen. $\mathbb{K}$-ring, and let $\mathfrak{p} \in \operatorname{Spec} A$. Then the local ring $A_{\mathfrak{p}}$ is noetherian.
Reason: $\mathbb{K}$ is noetherian, so A is noeth. by Basis Thm., Finally, $A_{\mathfrak{p}}$ is a localiz. of A.

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