Example If $A = K$ is a field, then K-modules are vector spaces.

Example If $M = A \oplus A$, the direct sum of two copies of A (we will talk about direct sums later), and we view elements of M as columns:

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \;\middle|\; a_1, a_2 \in A \right\},$$

then

$$\mathrm{End}_A(M) \cong \mathrm{Mat}_{2\times 2}(A),$$

the ring of 2×2 matrices, acting on columns from the left by matrix multiplication.

The image of the ring hom. $A \to \mathrm{End}_A(M)$ is the scalar matrices.

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Def. Let M be an A-mod. A submodule of M is a subset $N \subseteq M$ s.t. $0 \in N$, and N is closed under the operations $+$ and $a\cdot$, $a \in A$.

Of course $(N, 0, +, \cdot)$ is an A-module.

Prop. Let $N \subseteq M$ be an A-submodule. The quotient abelian group M/N has an A-mod. structure with mult.: $a \cdot \bar{m} := \overline{a \cdot m}$, $m \in M$.

Exer. (opt) prove this.

Def. Let $\varphi: M \to N$ be an A-module hom.

(1) The kernel of φ is

$$\mathrm{Ker}(\varphi) := \{ m \in M \mid \varphi(m) = 0_N \} \subseteq M$$

(2) The image of φ is

$$\mathrm{Im}(\varphi) := \varphi(M) \subseteq N.$$

Prop. Let $\varphi: M \to N$ be an A-mod. hom.

1) $\mathrm{Ker}(\varphi)$ is a submodule of M.

2) $\mathrm{Im}(\varphi)$ " " " " N.

3) There is a canonical A-mod. isom.

$$M/\mathrm{Ker}(\varphi) \xrightarrow{\sim} \mathrm{Im}(\varphi)$$

Exer. (opt) prove this prop.

Of course if $M' \subseteq M$ is a submodule, the inclusion $\varphi: M' \hookrightarrow M$ is an injective mod. hom. Let $\bar{M} := M/M'$. Then the projection $\psi: M \to \bar{M}$ is a surjective mod. hom.

<u>Def</u> Let A be a ring, $\mathfrak{a} \subseteq A$ an ideal and M an A-module. Define

$$\mathfrak{a} \cdot M := \left\{ \sum_{i=0}^{k} a_i \cdot m_i \;\middle|\; \begin{array}{l} k \geq 0;\ a_1, \dots, a_k \in \mathfrak{a}; \\ m_1, \dots, m_k \in M \end{array} \right\}$$

$$\subseteq M.$$

<u>Prop.</u> Under the assumptions of the def. above:

(1) $\mathfrak{a} \cdot M$ is an A-submodule of M.

(2) Let $\bar{A} := A/\mathfrak{a}$ and $\bar{M} := M/(\mathfrak{a} \cdot M)$. Then $\bar{M}$ is an $\bar{A}$-module, with mult.

$$\bar{a} \cdot \bar{m} := \overline{a \cdot m} \quad \text{for } a \in A \ \& \ m \in M.$$

<u>Exer.</u> prove the prop.

If $f : A \to B$ is a ring hom, and N is a B-module, then we can make N into an A-module like this:

$$a \cdot n := f(a) \cdot n, \quad \text{for } a \in A \text{ and } n \in N.$$

This can be seen by direct calculation, or using the prop on pages 13.1, 13.2, with these ring homs:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "h = g \circ f"'] & B \arrow[r, "g"] & \mathrm{End}_{\mathbb{Z}}(N) \end{tikzcd}$$

In the context of p. 16, the can. hom

$$\pi : M \to \bar{M} = M / (\mathfrak{a} \cdot M)$$

is a surjective A-mod. hom.

Finitely Supported Functions

We begin with some constructions of "functional analysis" flavor.

Let I be a set and M an abelian group. We denote by $F(I,M)$ the set of <u>functions</u> $\mu: I \to M$.

This is the same as the cartesian <u>power</u>:

$$F(I,M) = \prod_{i \in I} M = M^I .$$

Another way to view $\mu: I \to M$ is a <u>collection</u> of elements of M, indexed by I:

$$\underline{m} = \{m_i\}_{i \in I}, \quad \text{where } m_i := \mu(i) \in M.$$

"boldface m"

<u>Example</u> If $I = \mathbb{N}$, then a collection of elts indexed by $\mathbb{N}$ is a <u>sequence</u> in M.

<u>Exa</u>. If $I = \{1, 2, \dots, r\}$, then $F(I,M) = M^r$, the set of r-tuples.

The set $F(I,M)$ is an abelian group

- The zero elt. is the const. fun. 0.
- The addition is $(\mu+\nu)(i) := \mu(i)+\nu(i)$

If A is a ring and M is an A-module, then $F(I,M)$ is an A-module, with multiplication

$$(a\cdot\mu)(i) := a\cdot\mu(i) \in M.$$

→ Exercise. Let A be a ring, M an A-mod, and I a set.

(1) Show that $F(I,A)$ is a ring.

(2) " " $F(I,M)$ is an $F(I,A)$-module.

(3) " " there's a ring hom. $A \to F(I,A)$.

<u>Def</u> The <u>support</u> of $\mu \in F(I, M)$ is the set

$$\mathrm{Supp}(\mu) := \{ i \in I \,/\, \mu(i) \neq 0 \}.$$

<u>Def</u> We denote by $F_{fin}(I, M)$ the set of functions $\mu : I \to M$ with <u>finite support</u>

It is clear that

$$F_{fin}(I, M) \subseteq F(I, M)$$

is an A-submodule. ←

<u>Def</u> Let $\mu = \underline{\underline{m}} = \{m_i\}_{i \in I}$ be a finitely supported collection in M.

The <u>sum</u> of $\mu = \underline{\underline{m}}$ is

$$\sum_I \underline{\underline{m}} = \sum_{i \in I} m_i := \sum_{i \in \mathrm{Supp}(\underline{\underline{m}})} m_i \in M.$$

This makes sense, since $\mathrm{Supp}(\underline{\underline{m}})$ is a finite set.

If $\underline{\underline{m}}$ is the empty collection, ie.

$\underline{\underline{m}} = \{m_i\}_{i \in I}$ with $I = \emptyset$, then

$$\sum_I \underline{\underline{m}} := 0.$$

<u>Proposition</u>

The operation

$$\sum_I : F_{fin}(I, M) \to M$$

is an A-mod. homomorphism, i.e.:

$$\sum_I (\mu + \nu) = \sum_I \mu + \sum_I \nu$$

and

$$\sum (a \cdot \mu) = a \cdot \sum_I \mu$$

for all $\mu, \nu \in F_{fin}(I, M)$ and $a \in A$.

<u>Exer</u> (opt) Prove this.

Given a finitely supported collection

$$\underline{\underline{a}} = \{a_i\}_{i \in I} \in F_{fin}(I, A)$$

and any collection

$$\underline{\underline{m}} = \{m_i\}_{i \in I} \in F(I, M),$$

the collection $\{a_i \cdot m_i\}_{i \in I}$ is fin. supp.

$\{a_i\}$ $\qquad\qquad\qquad\qquad$ $\{m_i\}$

<u>Def</u> For $\underline{\underline{a}} \in F_{fin}(I, A)$ and $\underline{\underline{m}} \in F(I, M)$ we let

$$\underline{\underline{a}} \cdot \underline{\underline{m}} := \sum_{i \in I} a_i \cdot m_i \in M.$$

<u>Prop.</u> Given $\underline{\underline{m}} = \{m_i\} \in F(I, M)$, the function

$$F_{fin}(I, A) \to M, \quad \underline{\underline{a}} \mapsto \underline{\underline{a}} \cdot \underline{\underline{m}},$$

is an A-module hom.

<u>Exer</u> (opt) prove this.

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Free Modules

Def. Let A be a nonzero ring, let M be an A-module, and let

$$\underline{\underline{m}} = \{m_i\}_{i \in I}$$

be a collection of elements of M, indexed by a set I.

(1) We say that $\underline{\underline{m}}$ *generates* M if for every elt. $n \in M$ there *exists* a finitely supported collection $\underline{\underline{a}}$ in A s.t. indexed by I

$$\underline{\underline{a}} \cdot \underline{\underline{m}} = n.$$

(2) We say that $\underline{\underline{m}}$ is *linearly independent* if the *only* fin. supp. coll. $\underline{\underline{a}} \in F_{fin}(I, A)$ s.t.

$$\underline{\underline{a}} \cdot \underline{\underline{m}} = 0$$

is $\underline{\underline{a}} = 0$.

(3) We say that $\underline{\underline{m}}$ is a *basis* of M if it is both generating and linearly independent.

<u>Prop</u> Let $\varphi: M \to N$ be an A-module isomorphism, let $\underline{\underline{m}} = \{m_i\}_{i \in I}$ be a coll. in M, and let

$$\underline{\underline{n}} = \{n_i\}_{i \in I}$$

be the coll. in N defined by $n_i := \varphi(m_i)$.

Then: $\underline{\underline{m}}$ is a generating coll. (resp. lin. indep. coll., resp. basis) of M iff $\underline{\underline{n}}$ is a generating coll. (resp. lin. indep. coll., resp. basis) of N.

→ <u>Exer</u> prove this.

<u>Def</u>. Let A be a nonzero ring. An A-module M is called <u>free</u> if it has a basis.

<u>Prop</u> Under Def p. 23, $\underline{\underline{m}}$ is a basis of M iff for every $n \in M$ there is a unique $\underline{\underline{a}} \in F_{fin}(I, A)$ s.t.

$$n = \underline{\underline{a}} \cdot \underline{\underline{m}}.$$

→ <u>Exercise</u>. prove this.

Example. A a nonzero ring, $\mathfrak{a} \subseteq A$ an ideal, and $M := A/\mathfrak{a}$. The A-mod M is free iff $\mathfrak{a} = \{0\}$ or $\mathfrak{a} = A$.

- if $\mathfrak{a} = 0$ then $M \cong A$. The module A is free with basis the seq. $\underline{m} = (1)$ of length 1. Indeed, any $a = a \cdot 1$; and $a \cdot 1 = 0 \Rightarrow a = 0$.

- if $\mathfrak{a} = A$ then $M \cong \{0\}$. This is free with basis the empty sequence $()$.

Conversely, if M is free, then it has some basis

$$\underline{m} = \{m_i\}_{i \in I}.$$

- If $I = \emptyset$ then $M = \{0\}$, since it is gen. by the empty collection (see bottom p. 20). Then $A/\mathfrak{a} = \{0\}$, so $\mathfrak{a} = A$.

- If $I \neq \emptyset$, take some $i \in I$. For any $a \in \mathfrak{a}$ we have

$$a \cdot m_i = 0, \quad \text{since } M \text{ is an } A/\mathfrak{a}\text{-mod.}$$

Thus, for $\underline{a} := a\delta_i \in F_{fin}(I, A)$ we have $\underline{a} \cdot \underline{m}$. By lin. indep. of $\underline{m}$ we have $\underline{a} = 0$. So $a = 0$. Hence $\mathfrak{a} = \{0\}$.

∞

We will see later that $|I| = 1$ in the last case.

Q 6

<u>Example</u> For any set I the A-module $F_{fin}(I, A)$ is free. A basis is $\underline{\delta} := \{\delta_i\}_{i \in I}$,

where $\delta_i(j) := \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$

Why?

- <u>generating</u>: take $\underline{a} \in F_{fin}(I, A)$.
Then (*) $\underline{a} = \sum_{i \in Supp(\underline{a})} a_i \cdot \delta_i = \underline{a} \cdot \underline{\delta}$.

- <u>lin. indep.</u>: Say $\underline{a} \in F_{fin}(I, A)$ has $\underline{a} \cdot \underline{\delta} = 0$. By (*) we have $\underline{a} = 0$.

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1.11.17