8 Nov 2017

Given a collection $\underline{m} = \{m_i\}_{i\in I}$ in M, the hom.

(†) $F_{fin}(I, A) \to M$, $\underline{a} \mapsto \underline{a} \cdot \underline{m}$

from page 22 sends

$$\delta_i \mapsto m_i.$$

__Prop__ The collection $\underline{m}$ in Def. (p. 23) is a basis (resp. generating, resp. lin. indep.) iff the hom. (†) above is bijective (resp. surjective resp. injective.)

The proof is trivial.

Any module M has a generating collection — just take $\mu = \{m_i\}_{i\in I}$ to be $I := M$ and $\mu := id$

__Def__ An A-module M is called __finitely generated__ if it has some finite generating collection.

(Universal Property) 27.1

Thm Let M be a free A-module with basis $\underline{\underline{m}} = \{m_i\}_{i\in I}$, and let N be some A module with coll. $\underline{\underline{n}} = \{n_i\}_{i\in I}$

univ. property of $(M, \underline{\underline{m}})$ — There is a unique A-module hom

$$\varphi : M \to N$$

s.t. $\boxed{(*)\ \varphi(m_i) = n_i}$ for all $i \in I$.

pf. By Prop. (p. 27) we can assume that $M = F_{fin}(I, A)$ and $m_i = \delta_i$.
The unique hom. φ is then

$$\varphi(\underline{\underline{a}}) := \underline{\underline{a}} \cdot \underline{\underline{n}} \qquad \text{for } \underline{\underline{a}} \in F_{fin}(I, A).$$

Indeed, $\varphi(\delta_i) = \delta_i \cdot \underline{\underline{n}} = n_i$, so the condition $(*)$ holds. For uniqueness: if $\psi : F_{fin}(I, A)$ also sats $(*)$, then

$$\psi(\underline{\underline{a}}) = \psi\Big(\sum_{i\in \operatorname{supp}(\underline{\underline{a}})} a_i \cdot \delta_i\Big) = \sum_{i\in \operatorname{supp}(\underline{\underline{a}})} a_i \cdot \psi(\delta_i)$$

$$= \sum_{i\in\cdots} a_i n_i = \underline{\underline{a}} \cdot \underline{\underline{n}} = \varphi(\underline{\underline{a}}).$$

So $\psi = \varphi$. Δ

Later we will see that $I \mapsto F_{fin}(I, A)$ is a functor $\underline{\text{Sets}} \to \underline{\text{Mod}}\ A$, that has an abstract characterization.

(27.2)

→ Exercise. Prove the converse of the theorem:
Let M be an A-module, and let $\underline{m} = \{m_i\}_{i \in I}$ be a collection in M that has the universal property from the thm. Then $\underline{m}$ is a basis of M.

Thm If K is a field, then every K-module is free.

You had seen a proof for fin. gen. mods before. In general we need the axiom of choice.

Proof Given a K-module M, let's take a set I with cardinality $|I| > |M|$.
Consider the following set S; it consists of collections $\mu = \{m_j\}_{j \in J}$ in M, indexed by subsets $J \subseteq I$, that are *linearly independent*. We put a partial ordering on S as follows: for $\mu : J \to M$ and $\mu' : J' \to M$, we define $\mu' \leq \mu$ if $J' \subseteq J$, and $\mu|_{J'} = \mu'$.

The set S is nonempty, because it contains $\mu : \emptyset \to K$.

If $S' \subseteq S$ is a chain (an ordered subset), then it has a supremum in S': the union $\mu := \bigcup_{\mu' \in S'} \mu' : J = \bigcup J' \to M$ is a function. It is linearly independent, since any linear rel. $\underline{a} \cdot \mu = 0$ for $\underline{a} \in F_{fin}(J, M)$ will have $\mathrm{Supp}(\underline{a}) \subseteq J'$ for some $\mu' : J' \to M$ in S', and μ' is lin. indep.
So $\underline{a} = 0$.

...

(cont)

By Zorn's Lemma there is a <u>maximal</u> element $\mu : J \to M$ in $\mathcal{S}$.

The lin. indep. implies that μ is injective as a function; so $|J| \leq |M| < I$, and thus $J \subsetneq I$.

We claim that μ generates M, and hence it is a basis. Otherwise there is some element $m^+ \in M$ such that

$$m^+ \neq \underline{\underline{a}} \cdot \mu$$

for all $\underline{\underline{a}} \in F_{fin}(J, A)$. Define

$$\mu^+ : J^+ \to M \quad \text{by} \quad J^+ := J \cup \{j^+\}$$

for some $j^+ \in I - J$, and

$$\mu^+(j) := \begin{cases} \mu(j) & \text{if } j \in J \\ m^+ & \text{if } j = j^+. \end{cases}$$

Then, like in the finite proof, μ^+ is lin. indep; so $\mu^+ \in \mathcal{S}$. But $\mu^+ > \mu$.

$\square$

Thm Let K be a field, let M be a K-module, and let $\underline{m} = \{m_i\}_{i \in I}$ and $\underline{n} = \{n_j\}_{j \in J}$ be two bases of M.
Then $|I| = |J|$.

proof. If I or J is finite, then the old proof shows that $|I| = |J|$. (either by matrix mult. or by "replacement lemma")
We write $\mathrm{rank}_K(M) := |I|$.
Now assume both I and J are infinite.
We denote by $\mathrm{Fin}(I)$ the set of finite subsets of I. A fact from set theory:

$$|\mathrm{Fin}(I)| = |I|.$$

Define a function

$$g : J \to \mathrm{Fin}(I)$$

as follows: for $j \in J$ we write

$$n_j = \underline{a} \cdot \underline{m}, \quad \underline{a} \in F_{fin}(I, K),$$

and then we let

$$g(j) := \mathrm{Supp}(\underline{a}) \in \mathrm{Fin}(I).$$

For any finite subset $I' \in \mathrm{Fin}(I)$, the set

$$\{ j \in J \mid g(j) = I' \} = g^{-1}(I')$$

is finite: its cardinality is $\leq |I'|$, because $\{n_j\}_{j \in g^{-1}(I')}$ is a lin. indep. coll. inside the K-mod. M' generated by $\{m_i\}_{i \in I'}$, and $\mathrm{rank}_K(M') = |I'|$.

(fiber = preimage of an element)

So
$$g : J \to \mathrm{Fin}(I)$$
is a function with finite fibers. It follows that
$$|J| \leq |\mathrm{Fin}(I)| = |I|$$
(The calc. is
$$|J| = \sum_{I' \in \mathrm{Fin}(I)} |g^{-1}(I')| \leq |\mathrm{Fin}(I)| \cdot \aleph_0 = |\mathrm{Fin}(I)| = |I|)$$
By symmetry we also have
$$|I| \leq |J|.$$
So $|I| = |J|$. $\square$

Lemma. Let A be an A-mod, $\mathfrak{a} \subseteq A$ an ideal and $\underline{m} = \{m_i\}_{i \in I}$ a generating coll. in M. Then every elt n of $\mathfrak{a} \cdot M$ can be written as
$$n = \underline{a} \cdot \underline{m}$$
for some $\underline{a} \in F_{\mathrm{fin}}(I, \mathfrak{a})$

→ Exer. prove the lemma.

(32)

Thm Let A be a nonzero ring, let M be a free A-module, and let
$\underline{m} = \{m_i\}_{i \in I}$ and $\underline{n} = \{n_j\}_{j \in J}$
be basis of M. Then
$$|I| = |J|.$$

Pf. Let $\mathfrak{m} \subseteq A$ be a maximal ideal, with residue field $K := A/\mathfrak{m}$. Define the K-mod.
$$\bar{M} := M/\mathfrak{m} \cdot M \quad \text{and the collection}$$

$\pi: M \to \bar{M}$ is the can. surjection

$$\underline{\bar{m}} := \{\bar{m}_i\}_{i \in I}, \quad \text{with } \bar{m}_i := \pi(m_i) \in \bar{M}.$$

We will prove that $\underline{\bar{m}}$ is a basis of $\bar{M}$. The same arguments will show that $\underline{\bar{n}} = \{\bar{n}_j\}_{j \in J}$ is a basis of $\bar{M}$. By the previous thm we know that $|I| = |J|$.

- It is obvious that $\underline{\bar{m}}$ generates $\bar{M}$.

- Linear independence: Say $\underline{\bar{a}} = \{\bar{a}_i\}_{i \in I}$ is a fin. supp. coll. in $\bar{A} = K$ s.t.
$$\underline{\bar{a}} \cdot \underline{\bar{m}} = 0.$$
Here $a_i \in A$ are representatives. Thus
$$\underline{a} \cdot \underline{m} \in \operatorname{Ker}(M \xrightarrow{\pi} \bar{M}) = \mathfrak{m} \cdot M.$$

(cont.) By the lemma there's some $\underline{\underline{b}} \in F_{fin}(I, \mathfrak{a})$ s.t.

$$\underline{\underline{a}} \cdot \underline{\underline{m}} = \underline{\underline{b}} \cdot \underline{\underline{m}}.$$

But $\underline{\underline{m}}$ is lin. indep., so

$$\underline{\underline{a}} = \underline{\underline{b}}.$$

Thus $a_i \in \mathfrak{a}$, $\bar{a}_i = 0$ and $\bar{\underline{\underline{a}}} = 0$.

□

Def. Let A be a nonzero ring and M a free A-module. The rank of M, with notation $\mathrm{rank}_A(M)$, is the cardinality $|I|$ of any basis $\underline{\underline{m}} = \{m_i\}_{i \in I}$ of M.

The last thm justifies this def.

– to here 8.11 –

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→ **Exercise**. Here A is some (nonzero comm.) ring.

(1) Let $f: X \to Y$ be a function between sets. Show that there's a unique A-mod. hom.

$$\int_f : F_{fin}(X, A) \to F_{fin}(Y, A)$$

s.t.

$$\int_f (\delta_x) = \delta_{f(x)} \quad \text{for } x \in X.$$

(2) Given sets X, Y and elements

$\varphi \in F_{fin}(X, A)$ $\quad \psi \in F_{fin}(Y, A)$,

show that the function

$$\varphi \boxtimes \psi : X \times Y \to A, \quad (x,y) \mapsto \varphi(x) \cdot \psi(y)$$

belongs to $F_{fin}(X \times Y, A)$.

(3) Take $X = Y := \mathbb{N}$. Consider the addition function

$$+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}.$$

Define the "multiplication" function

$\boxed{B := F_{fin}(\mathbb{N}, A), \text{ and}}$

$$\cdot : B \times B \to B, \quad \varphi \cdot \psi := \int_+ (\varphi \boxtimes \psi).$$

Show that B is a comm. ring, and that $A \to B$, $a \mapsto a \cdot \delta_0$ is a ring hom.

(4) Try to identify the ring B, as something familiar.