

34.1

Solution to Exer. [p. 27.2].

We are given an  $A$ -module  $M$  and a collection  $\underline{m} = \{m_i\}_{i \in I}$  of elements in it, that has the universal property from the thm. [p. 27.1]. We need to show that  $\underline{m}$  is a basis of  $M$ .

Let  $F := F_{\text{fin}}(I, A)$ , with its standard basis  $\underline{d} = \{d_i\}_{i \in I}$ . By the existence part of the univ. property of  $(M, \underline{m})$ , there is an  $A$ -mod. hom.

$\varphi: M \rightarrow F$  s.t.  $\varphi(m_i) = d_i$  for all  $i \in I$ .

By the existence part of the univ. property of  $(F, \underline{d})$ , there is a hom.  $\psi: F \rightarrow M$  s.t.  $\psi(d_i) = m_i$  for all  $i \in I$ .

Consider the  $A$ -mod. homs

$$\forall \varphi, \text{id}_M: M \rightarrow M.$$

They both satisfy

$$(\varphi \circ \psi)(m_i) = m_i = \text{id}_M(m_i)$$

By the uniqueness part of the univ. prop., we get

$$\boxed{\varphi \circ \psi = \text{id}_M.}$$

of  $(M, \underline{m})$

Similarly we get

$$\boxed{\psi \circ \varphi = \text{id}_F}$$

Hence  $\varphi: M \rightarrow F$  is an isom. It follows that  $\underline{m}$  is a basis of  $M$ .  $\square$

## Exact Sequences

Def. Suppose we are given this data:  
 $A$ -modules  $M_i$  and maps  $\varphi_i: M_i \rightarrow M_{i+1}$ ,  
 indexed by a finite or infinite interval in  $\mathbb{Z}$ .  
 This is called a sequence of  $A$ -modules,  
 and is depicted by this diagram:

$$\mathbb{S} = \left( \dots \rightarrow M_{i-1} \xrightarrow{\varphi_{i-1}} M_i \xrightarrow{\varphi_i} M_{i+1} \rightarrow \dots \right)$$

i.e.  $M_i$  is neither  
initial nor terminal  
in  $\mathbb{S}$

Def. Let  $\mathbb{S}$  be a seq. as above.

(1) Assume that  $\varphi_{i-1}$  &  $\varphi_i$  are defined. We say that  
 $\mathbb{S}$  is exact at  $M_i$  if

$$\text{Im}(\varphi_{i-1}) = \text{Ker}(\varphi_i).$$

(These are submods of  $M_i$ .)

(2) We say that  $\mathbb{S}$  is an exact sequence if  
 it is exact at all  $M_i$  (st.  $M_i$  is not initial or terminal).

~~~~~~~~~

We shall use  $0$  to denote the zero  $A$ -module  
 set.

3.6.

Def. A short exact sequence is an exact sequence of this shape:

$$0 \rightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \rightarrow 0$$

[to remember:]

$$\begin{array}{ccccccc}
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 M_{-2} & M_{-1} & M_0 & M_1 & M_2 & & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
 \varphi_{-2} = 0 & \varphi_{-1} & \varphi_0 & \varphi_1 & \varphi_2 = 0 & & 
 \end{array}$$



Let's talk about short exact seq., with notation as above.

- $\text{Ker}(\varphi) = \text{Im}(0 \rightarrow M') = 0$ , so  $\varphi$  is injective.  
Thus we have an isom  $\varphi: M' \xrightarrow{\cong} \text{Im}(\varphi) \subseteq M$ .  
Let's identify them; i.e. let's assume  $M' = \text{Im}(\varphi)$ .
- $\text{Ker}(\psi) = \text{Im}(\varphi) = M'$ .
- $\text{Im}(\psi) = \text{Ker}(M'' \rightarrow 0) = M''$  so  $\psi$  is surjective.
- We see that  $\psi$  induces an isomorphism

$$\boxed{ \frac{M}{M'} \xrightarrow{\cong} M'' }$$

Note that every injective hom  $\varphi: M' \rightarrow M$ , and every surjective hom  $\psi: M \rightarrow M''$ , can be made into part of a short exact seq.

→ Exercise. Let  $K$  be a field, and let

$$\underline{\underline{0}} = (0 \rightarrow M_0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} \dots \rightarrow M_l \rightarrow 0)$$

be an exact seq. of finite rank  $K$ -modules.

Prove that

$$\sum_{i=0}^l \text{rank}_K(M_i) \cdot (-1)^i = 0.$$

- First try it for  $l \leq 2$ .
- Then use next exercise
- (\*\*) Try to do this for a nonzero ring  $A$  & fin. rank free  $A$ -mods.  $M_i$



→ Exercise Given an exact seq. of  $A$  mods

$$M_{i_0} \rightarrow M_{i_0+1} \rightarrow \dots \rightarrow M_{i_1}$$

show that there <sup>are</sup> modules  $N_{i_0+1}, \dots, N_{i_1}$  and short exact sequences

$$0 \rightarrow N_i \rightarrow M_i \rightarrow N_{i+1} \rightarrow 0$$

for all  $i_0+1 \leq i \leq i_1-1$ .

# Tensor Products

(3P)

Def Let  $M, N, P$  be  $A$ -modules. An  $A$ -bilinear function

is a function  $\beta: M \times N \rightarrow P$  that satisfies these conditions:

- $\beta(m_1 + m_2, n) = \beta(m_1, n) + \beta(m_2, n)$
- $\beta(m, n_1 + n_2) = \beta(m, n_1) + \beta(m, n_2)$
- $\beta(a \cdot m, n) = \beta(m, a \cdot n) = a \cdot \beta(m, n)$

for all  $m, m_1, m_2 \in M$ ;  $n, n_1, n_2 \in N$ ; and  $a \in A$

Example The multiplication  $\cdot: A \times A \rightarrow A$  is  $A$ -bilinear.

Warning: for NC rings the story is more complicated.

Def Let  $M$  &  $N$  be  $A$ -mods. A tensor product of  $M$  &  $N$  is a pair  $(P, \beta)$ , consisting of an  $A$ -mod.  $P$  and an  $A$ -bilinear function.

$$\beta: M \times N \rightarrow P$$

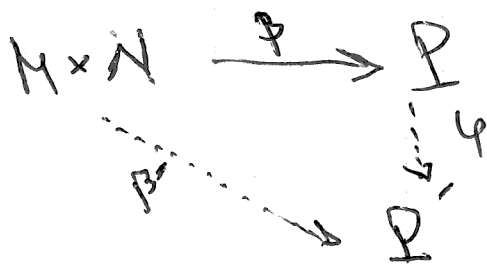
that has the universal property:

(\*) For every  $A$ -bil. func.  $\beta': M \times N \rightarrow P'$  there is a unique  $A$ -mod. hom.  $\psi: P \rightarrow P'$  s.t.

$$\beta' = \psi \circ \beta$$

(39)

The univ. prop. can be expressed as a commutative diagram:



Prop. (Uniqueness) Suppose  $(P, \beta)$  and  $(P', \beta')$  are both tensor products of  $M$  &  $N$ . Then there is a unique isom.  $\psi$  of  $A$ -mods  $P \xrightarrow{\psi} P'$  s.t.  $\beta' = \psi \circ \beta$ .

→ Exers. Prove this prop.

For this reason we can talk about the tensor product.



Thm (Existence). Given  $A$ -mods  $M$  and  $N$ , their tensor product  $(P, \beta)$  exists.

Proof. Let  $F$  be a free  $A$ -module with basis indexed by the set  $I := M \times N$ .

We can take  $F := F_{\text{fin}}(I, A)$ , with the standard

basis  $\{d_i\}_{i \in I} = \underline{d}$ .

special

Inside  $F$  we have these elements:

$$\begin{array}{l}
 \text{---} \quad \underline{d}_{m_1+m_2, n} - (\underline{d}_{m_1, n} + \underline{d}_{m_2, n}) \\
 \text{---} \quad \underline{d}_{m, n_1+n_2} - (\underline{d}_{m, n_1} + \underline{d}_{m, n_2}) \\
 \text{---} \quad \underline{d}_{a \cdot m, n} - a \cdot \underline{d}_{m, n} \\
 \text{---} \quad \underline{d}_{m, a \cdot n} - a \cdot \underline{d}_{m, n}
 \end{array}$$

for all  $m, m_1, m_2 \in M$ ;  $n, n_1, n_2 \in N$ ; and  $a \in A$ .

Let  $K \subseteq F$  be the  $A$ -submodule generated by the special elements  $\underline{d}$ . Define the  $A$ -mod  $\underline{P} := F / K$ .

$$\underline{P} := F / K$$

and the function

$$\begin{aligned}
 \rho: M \times N &\rightarrow \underline{P}, \\
 \rho(m, n) &:= \underline{d}_{m, n} + K.
 \end{aligned}$$

The function  $\beta$  is  $A$ -bilinear, because all conditions are satisfied.

It remains to show that  $(P, \beta)$  has the universal property of a tensor product.

Suppose  $\beta': M \times N \rightarrow P'$  is some  $A$ -bilinear form.

The universal property of a free module says that there's an  $A$ -mod. hom.

$$\tilde{\psi}: F \rightarrow P', \quad \tilde{\psi}(d_{m,n}) = \beta'(m,n).$$

Because  $\beta'$  is  $A$ -bilinear and  $\tilde{\psi}$  is  $A$ -linear,  $\tilde{\psi}$  sends all the special elements  $(a)$  to zero.

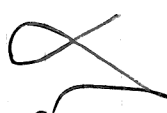
Thus  $\tilde{\psi}(K) = 0$ , and there's an induced  $A$ -lin hom.

$$\psi: P \rightarrow P' \quad \text{st. } \psi(\beta(m,n)) = \beta'(m,n).$$

So  $\psi \circ \beta = \beta'$ .

The uniqueness of this  $\psi$  is because the elements  $\beta(m,n) = d_{m,n} + K$  generate  $P$ .

□



It would seem from the proof that  $P$  is very big. This is just an illusion!  
See ... Cor. [P 4.3]



Notation. Let  $(P, \beta)$  be the tensor product of the  $A$ -mod's  $M$  and  $N$ . We write

$$M \otimes_A N := P$$

and

$$m \otimes n := \beta(m, n) \quad \text{for } m \in M, n \in N.$$

called pure tensor



Prop. The  $A$ -module  $M \otimes_A N$  is generated by the collection  $\{m \otimes n\}_{(m,n) \in M \times N}$ .

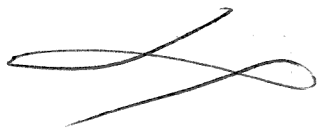
pf. From the proof of the thm,  $M \otimes_A N$  is a quotient of  $F_{\text{fin}}(M \times N, A)$ , and this mod. is gen. by  $\{\delta_{m,n}\}_{(m,n) \in M \times N}$ . □

Exmp. Suppose  $M$  and  $N$  are generated by  $\{m_i\}_{i \in I}$  and  $\{n_j\}_{j \in J}$ . Then  $M \otimes_A N$  is gen. by  $\{m_i \otimes n_j\}_{(i,j) \in I \times J}$ .

→ Exercise. Prove this prop.

Cor. If  $M$  and  $N$  are fin. gen.  $A$ -mods, then so is  $M \otimes_A N$ .

pf Immed. from last prop.  $\square$



Tensor products are usually hard to understand. Often we need to use their univ. properties. This will be done in the next thm.

Exercise.  $A = \mathbb{Z}$ ,  $M = \mathbb{Z}/(m)$ ,  $N = \mathbb{Z}/(n)$ . Calculate  $M \otimes_{\mathbb{Z}} N$ . When is it zero?

Thm. If  $M$  and  $N$  are free  $A$ -modules, with bases  $m = \{m_i\}_{i \in I}$  and  $n = \{n_j\}_{j \in J}$  resp., then  $M \otimes_A N$  is a free  $A$ -mod, with basis  $\{m_i \otimes n_j\}_{(i,j) \in I \times J}$ .

start with  $m = 5$ ,  $n = 6$ .  
hint: 6 is invertible mod 5

pf. ----- to here 15.11

Let  $\varphi: F_{\text{fin}}(I \times J, A) \rightarrow M \otimes_A N$  be the unique  $A$ -lin. hom. s.t.

$$\varphi(d_{i,j}) := m_i \otimes n_j.$$

We will show that  $\varphi$  is an isom, by producing an inverse  $\psi$ .