34.1

Solution to Exer. [p. 27.2].

We are given an A-module M and a collection $\underline{m} = \{m_i\}_{i \in I}$ of elements in it, that has the universal property from the thm. [p. 27.1]. We need to show that $\underline{m}$ is a basis of M.

Let $F := F_{fin}(I, A)$, with its standard basis $\underline{\delta} = \{\delta_i\}_{i \in I}$. By the existence part of the univ. property of $(M, \underline{m})$, there is an A-mod. hom.
$$\varphi : M \to F \quad \text{s.t. } \varphi(m_i) = \delta_i \text{ for all}$$
$i \in I$. By the existence part of the univ. property of $(F, \underline{\delta})$, there is is a hom. $\psi : F \to M$ s.t. $\psi(\delta_i) = m_i$ for all $i \in I$.

Consider the A-mod homs
$$\psi \circ \varphi, \ \mathrm{id}_M : M \to M.$$
They both satisfy
$$(\psi \circ \varphi)(m_i) = m_i = \mathrm{id}_M(m_i)$$
By the uniqueness part of the univ. prop. of $(M, \underline{m})$ we get
$$\boxed{\psi \circ \varphi = \mathrm{id}_M.}$$
Similarly we get $\boxed{\varphi \circ \psi = \mathrm{id}_F}$

Hence $\varphi : M \to F$ is an isom. It follows that $\underline{m}$ is a basis of M. $\square$

08/11/2017

Exact Sequences

<u>Def.</u> Suppose we are given this data: A-modules M_i and hom's $\varphi_i : M_i \to M_{i+1}$, indexed by a finite or infinite <u>interval</u> in $\mathbb{Z}$. This is called a <u>sequence</u> of A-modules, and is depicted by this diagram:

$$\underline{\underline{S}} = \left(\cdots \longrightarrow M_{i-1} \xrightarrow{\varphi_{i-1}} M_i \xrightarrow{\varphi_i} M_{i+1} \to \cdots \right)$$

i.e. M_i is neither initial nor terminal in $\underline{\underline{S}}$

<u>Def.</u> Let $\underline{\underline{S}}$ be a seq. as above.

(1) Assume that φ_{i-1} & φ_i are defined. We say that $\underline{\underline{S}}$ is <u>exact</u> at M_i if

$$\mathrm{Im}(\varphi_{i-1}) = \mathrm{Ker}(\varphi_i).$$

(These are submods of M_i.)

(2) We say that $\underline{\underline{S}}$ is an <u>exact sequence</u> if it exact at all M_i (s.t. M_i is not initial or terminal).

We shall use 0 to denote the zero A-module $\{0\}$.

3.6.

Def. A short exact sequence is an exact sequence of this shape:

$$0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$$

[to enumerate: M_{-2}, M_{-1}, M_0, M_1, M_2; $\varphi_{-2}=0$, φ_{-1}, φ_0, $\varphi_1=0$]

Let's talk about short exact seq, with notation as above.

- $\mathrm{Ker}(\varphi) = \mathrm{Im}(0 \to M') = 0$, so φ is injective. Thus we have an isom $\varphi: M' \xrightarrow{\cong} \mathrm{Im}(\varphi) \subseteq M$. Let's identify them; i.e. let's assume $M' = \mathrm{Im}(\varphi)$.
- $\mathrm{Ker}(\psi) = \mathrm{Im}(\varphi) = M'$.
- $\mathrm{Im}(\psi) = \mathrm{Ker}(M'' \to 0) = M''$. So ψ is surjective.
- We see that ψ induces an isomorphism

$$\boxed{\frac{M}{M'} \xrightarrow{\cong} M''.}$$

Note that every injective hom $\varphi: M' \to M$, and every surjective hom $\psi: M \to M''$, can be made into part of a short exact seq.

→ Exercise. Let k be a field, and let

$$\underline{\underline{S}} = (0 \to M_0 \xrightarrow{\varphi_0} M_1 \xrightarrow{\varphi_1} \cdots \longrightarrow M_\ell \to 0)$$

be an exact seq. of finite rank k-modules.

Prove that

$$\sum_{i=0}^{\ell} \mathrm{rank}_k(M_i) \cdot (-1)^i = 0.$$

- First try it for $\ell \leq 2$.
- Then use next exercise

(**) Try to do this for a nonzero ring A & fin. rank free A-mods. M_i

∞

→ Exercise Given an exact seq. of A mods

$$M_{i_0} \to M_{i_0+1} \to \cdots \to M_{i_1}$$

show that there are modules $N_{i_0+1}, \ldots, N_{i_1}$, and short exact sequences

$$0 \to N_i \to M_i \to N_{i+1} \to 0$$

for all $i_0+1 \leq i \leq i_1 - 1$.

Tensor Products

Def Let M, N, P be A-modules. An A-bilinear function

$$\beta : M \times N \to P$$

is a function sats. these conditions:

- $\beta(m_1 + m_2, n) = \beta(m_1, n) + \beta(m_2, n)$
- $\beta(m, n_1 + n_2) = \beta(m, n_1) + \beta(m, n_2)$
- $\beta(a \cdot m, n) = \beta(m, a \cdot n) = a \cdot \beta(m, n)$

for all $m, m_1, m_2 \in M$; $n, n_1, n_2 \in N$; and $a \in A$

Example The multiplication

$$\cdot : A \times A \to A$$

is A-bilinear.

Warning: for NC rings the story is more complicated.

Def Let M & N be A-mods. A tensor product of M & N is a pair (P, β), consisting of an A-mod. P and an A-bilinear funct.

$$\beta : M \times N \to P$$

that has this universal property:

$(*)$ For every A-bil. funct. $\beta' : M \times N \to P'$ there is a unique A-mod. hom. $\varphi : P \to P'$ s.t.

$$\beta' = \varphi \circ \beta .$$

The univ. prop. can be expressed as a commutative diagram:

$$\begin{tikzcd} M \times N \arrow[r, "\beta"] \arrow[dr, dashed, "\beta'"'] & P \arrow[d, dotted, "\varphi"] \\ & P' \end{tikzcd}$$

Prop. (Uniqueness) Suppose (P, β) and (P', β') are both tensor products of M & N. Then the A-modules
there is a unique isom. of A-mods
$$\varphi : P \xrightarrow{\sim} P'$$
s.t.
$$\beta' = \varphi \circ \beta.$$

→ Exer. Prove this prop.

For this reason we can talk about the tensor product.

Thm (Existence). Given A-mods M and N, their tensor product (P, β) exists.

Proof. Let F be a free A-module with basis indexed by the set $I := M \times N$.
We can take $F := F_{fin}(I, A)$, with the standard basis $\{\delta_i\}_{i \in I} =: \underline{\delta}$.

Inside F we have these special elements:

$$(\diamond) \begin{cases} - & \delta_{m_1+m_2, n} - (\delta_{m_1, n} + \delta_{m_2, n}) \\ - & \delta_{m, n_1+n_2} - (\delta_{m, n_1} + \delta_{m, n_2}) \\ - & \delta_{a \cdot m, n} - a \cdot \delta_{m,n} \\ - & \delta_{m, a \cdot n} - a \cdot \delta_{m,n} \end{cases}$$

for all $m, m_1, m_2 \in M$; $n, n_1, n_2 \in N$; and $a \in A$.

Let $K \subseteq F$ be the A-submodule generated by the special elements $(\diamond)$. Define the A-mod

$$\underline{P} := F / K$$

and the function $\beta : M \times N \to \underline{P}$,
$$\beta(m,n) := \delta_{m,n} + K.$$

The function β is A-bilinear, because all conditions are satisfied.

It remains to show that (P, β) has the universal property of a tensor product.

Suppose $\beta' : M \times N \to P'$ is some A-bilin. funct. The universal property of a free module says that there's an A-mod. hom.

$$\tilde{\varphi} : F \to P', \quad \tilde{\varphi}(\delta_{m,n}) := \beta'(m,n).$$

Because β' is A-bilinear and $\tilde{\varphi}$ is A-linear, $\tilde{\varphi}$ sends all the special elements $(*)$ to zero. Thus $\tilde{\varphi}(K) = 0$, and there's an induced A-lin hom.

$$\varphi : P \to P' \quad \text{s.t. } \varphi(\beta(m,n)) = \beta'(m,n).$$

So $\varphi \circ \beta = \beta'$.

The uniqueness of this φ is because the elements $\beta(m,n) = \delta_{m,n} + K$ generate P.

$\square$

$\otimes$

It would seem from the proof that P is very big. This is just an illusion! See Cor. [P 4.3]

<u>Notation</u>. Let (P, β) be the tensor product of the A-mod's M and N. We write

$$M \otimes_A N := P$$

and $m \otimes n := \beta(m,n)$ for $m \in M$, $n \in N$.

called pure tensor

<u>Prop</u>. The A-module $M \otimes_A N$ is generated by the collection $\{m \otimes n\}_{(m,n) \in M \times N}$.

<u>Pf</u>. From the proof of the thm, $M \otimes_A N$ is a quotient of $F_{fin}(M \times N, A)$, and this mod. is gen. by $\{\delta_{m,n}\}_{(m,n) \in M \times N}$. $\square$

<u>Prop</u>. Suppose M and N are generated by $\{m_i\}_{i \in I}$ and $\{n_j\}_{j \in J}$. Then $M \otimes_A N$ is gen. by $\{m_i \otimes n_j\}_{(i,j) \in I \times J}$.

→ <u>Exercise</u>. Prove this prop.

<u>Cor.</u> If M and N are fin. gen. A-mods, then so is $M \otimes_A N$.

pf. Immed. from last prop. Δ

Tensor products are usually hard to understand. Often we need to use their univ. properties. This will be done in the next thm.

$\boxed{\text{Exercise. } A := \mathbb{Z},\ M := \mathbb{Z}/(m),\ N := \mathbb{Z}/(n). \text{ Calculate } M \otimes_{\mathbb{Z}} N. \text{ When is it zero?}}$

$\boxed{\text{start with } m=5,\ n=6.}$

$\boxed{\text{hint: 6 is invertible mod 5}}$

<u>Thm.</u> If M and N are free A-modules, with bases $\underline{m} = \{m_i\}_{i \in I}$ and $\underline{n} = \{n_j\}_{j \in J}$ resp., then $M \otimes_A N$ is a free A-mod, with basis

$$\{m_i \otimes n_j\}_{(i,j) \in I \times J}.$$

- - - - - - to here - 15.11

pf.

Let $\varphi : F_{fin}(I \times J, A) \to M \otimes_A N$ be the unique A-lin. hom. s.t.

$$\varphi(\delta_{i,j}) := m_i \otimes n_j.$$

We will show that φ is an isom, by producing an inverse ψ.