page 42.9

(cont exer [p 37])

Exercise (1)

A is a ring.

$$\text{s.e.s. } \Sigma = \left(0 \longrightarrow M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0 \right), \quad \tau : M \to M', \quad \sigma : M'' \to M$$

TFAE:

A-mod. hom

(i) $\exists\ \sigma : M'' \longrightarrow M$ s.t. $\psi \circ \sigma = id_{M''}$

(ii) $\exists\ \tau : M \longrightarrow M'$ s.t. $\tau \circ \varphi = id_{M'}$.

$M \cong M' \oplus M''$

$id_M = \psi \circ \sigma + \tau \circ$

Def Given Σ, if either (i) or (ii) then Σ is called a split short exact seq.

$\langle a,b \rangle$

Exer 2. If, in Exer 1, the A-mod. M'' is free, then a splitting hom. σ exists.

<u>Cor.</u> If M and N are fin. gen. A-mods, then so is $M \otimes_A N$.

<u>pf</u> Immed. from last prop. Δ

Tensor products are usually hard to understand. Often we need to use their univ. properties. This will be done in the next thm.

<u>Exercise.</u> $A = \mathbb{Z}$, $M := \mathbb{Z}/(m)$, $N := \mathbb{Z}/(n)$. Calculate $M \otimes_{\mathbb{Z}} N$. When is it zero?

start with $m=5$, $n=6$.

hint: 6 is invertible mod 5

<u>Thm.</u> If M and N are free A-modules, with bases $\underline{m} = \{m_i\}_{i \in I}$ and $\underline{n} = \{n_j\}_{j \in J}$ resp., then $M \otimes_A N$ is a free A-mod, with basis

$$\{m_i \otimes n_j\}_{(i,j) \in I \times J}.$$

<u>pf.</u> — — — — — 22 Nov 17 — to here 15.11 —

Let $\varphi : F_{fin}(I \times J, A) \to M \otimes_A N$ be the unique A-lin. hom. s.t.

$$\varphi(\delta_{i,j}) := m_i \otimes n_j.$$

We will show that φ is an isom, by producing an inverse ψ.

(44)

Let $\boxed{\beta : M \times N \to F_{fin}(I \times J, A)}$

be the following function. For $(m,n) \in M \times N$ there are unique $\underline{\underline{a}} \in F_{fin}(I, A)$ and $\underline{\underline{b}} \in F_{fin}(J, A)$ s.t.

$$m = \underline{\underline{a}} \cdot \underline{\underline{m}} \quad \text{and} \quad n = \underline{\underline{b}} \cdot \underline{\underline{n}}.$$

We let

$$\beta(m,n) : I \times J \to A$$

be $(\heartsuit)$ $\boxed{\beta(m,n)(i,j) := a_i \cdot b_j \in A.}$

The function β is A-bilinear (see exer. below).
Hence there is a unique A-lin. hom.

$$\Psi : M \otimes_A N \to F_{fin}(I \times J, A)$$

s.t. $\Psi(m \otimes n) = \beta(m,n)$.

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Let's write the function β a bit differently. For $\underline{\underline{a}} \in F_{fin}(I, A)$ let's write

$$\langle \underline{\underline{a}}, \underline{\underline{m}} \rangle := \underline{\underline{a}} \cdot \underline{\underline{m}} = \sum_i a_i \cdot m_i \in M$$

We can do the same for any $\underline{\underline{c}} \in F(I, A)$:

$$\langle \underline{\underline{a}}, \underline{\underline{c}} \rangle \in A.$$

Since $\langle \underline{\underline{a}}, \delta_i \rangle = a_i$ and $\langle \underline{\underline{b}}, \delta_j \rangle = b_j$, we have

(*) $$\beta(m,n)(i,j) = \langle \underline{\underline{a}}, \delta_i \rangle \cdot \langle \underline{\underline{b}}, \delta_j \rangle$$

for

$$m = \langle \underline{\underline{a}}, \underline{\underline{m}} \rangle \text{ and } n = \langle \underline{\underline{b}}, \underline{\underline{n}} \rangle.$$

Thus, for $m := m_k = \langle \delta_k, \underline{\underline{m}} \rangle$ and
$n := n_l = \langle \delta_l, \underline{\underline{n}} \rangle$

we have

$$\Psi(m_k \otimes n_l)(i,j) = \beta(m_k, n_l)(i,j)$$

(*) $$= \langle \delta_k, \delta_i \rangle \cdot \langle \delta_l, \delta_j \rangle$$

$$= \delta_{k,l}(i,j)$$

Therefore

(5) $$\boxed{\Psi(m_k \otimes n_l) = \delta_{k,l}.}$$

Let's calculate $\psi\circ\varphi$, using $(\ast)$:

$$(\psi\circ\varphi)(\delta_{i,j}) = \psi(m_i \otimes n_j) = \delta_{ij}$$

[equal on generators]

$\Rightarrow \psi\circ\varphi = \mathrm{id}$.

Finally

$$(\varphi\circ\psi)(m_i \otimes n_j) = \varphi(\delta_{ij}) = m_i \otimes n_j.$$

So $\varphi\circ\psi = \mathrm{id}$. $\square$

(optional)

__Exercise__: Prove that β on p. 44 is A-bilinear.

__Prop.__ Let $\varphi: M \to M'$ and $\psi: N \to N'$ be A-mod. homs. There is a unique A-mod hom

$$\varphi\otimes\psi : M \otimes_A N \to M' \otimes_A N'$$

s.t.

$$(\varphi\otimes\psi)(m\otimes n) = \varphi(m) \otimes \psi(n)$$

for all $m \in M$ and $n \in N$.

$\longrightarrow$ __Exer.__ prove this prop. (Hint: find $\beta: M\times N \to M' \otimes_A N'$ bilinear.)

Thm. Let M and N be A-mods. (1) There is a unique A-lin. hom.

$$\Psi : \mathrm{Hom}_A(M, A) \otimes_A N \to \mathrm{Hom}_A(M, N)$$

s.t. $\Psi(\chi \otimes n)(m) = \chi(m) \cdot n$
for all $\chi \in \mathrm{Hom}_A(M, A)$ and $n \in N$.

(2) If M is a free A-mod of fin. rank, then Ψ is an isomorphism.

Pf. (1) Define the function $M^* :=$

$$\beta : \mathrm{Hom}_A(M, A) \times N \to \mathrm{Hom}_A(M, N)$$

to be

$$\beta(\chi, n)(m) := \chi(m) \cdot n.$$

It is easy to see that it bilinear. eq.

$$\beta(a\cdot\chi, n)(m) = (a \cdot \chi(m)) \cdot n = a \cdot (\chi(m) \cdot n) = (a \cdot \beta(\chi, n))(m)$$

$$\Rightarrow \quad \beta(a \cdot \chi, n) = a \cdot \beta(\chi, n).$$

Hence there's a unique A-lin. hom. Ψ s.t.
$\Psi(\chi \otimes n) = \beta(\chi, n)$.

4.7.1

Prop M free rank $r < \infty$
$\Rightarrow M^*$ " " ".

(2) Say $M = A^r$ for some $r \in \mathbb{N}$, with basis $\underline{\underline{m}} = (m_1, \dots, m_r)$. Then $\mathrm{Hom}_A(M, A)$ is free too. For each i let $\mu_i : M \to A$ be the hom.

$$\mu_i(m_j) := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{o.w.} \end{cases}$$

Every $\mu : M \to A$ is uniquely of the form

$$\mu = \sum_i a_i \cdot \mu_i, \qquad a_i \in A.$$

So

$$\underline{\underline{\mu}} = (\mu_1, \dots, \mu_r) \text{ is a basis of}$$

$M^* = \mathrm{Hom}_A(M, A)$. It's called the basis dual to $\underline{\underline{m}}$.

Given $\varphi : M \to N$ let $n_i := \varphi(m_i) \in N$. Consider the tensor

$$\Phi(\varphi) := \sum_{i=1}^{r} \mu_i \otimes n_i \in \underbrace{\mathrm{Hom}_A(M, A)}_{= M^*} \otimes_A N.$$

This is an A-lin funct.

$$\Phi : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, A) \otimes_A N,$$

We will prove that Φ is the inverse of Ψ.

47.2

Let calculate $(\Phi \circ \Psi)(\mu_i \otimes n)$.

Since $\Psi(\mu_i \otimes n)(m_j) = \begin{cases} n & \text{if } i = j \\ 0 & \text{o.w.} \end{cases}$

we have
$$n_j = \begin{cases} n & \text{if } i = j \\ 0 & \text{o.w.} \end{cases}$$

so $\Phi(\Psi(\mu_i \otimes n)) = \mu_i \otimes n$.

Thus
$$\Phi \circ \Psi = \mathrm{id}.$$

Conversely, given $\varphi \in \mathrm{Hom}_N(M, N)$

we have $(\Psi \circ \Phi)(\varphi) = \Psi(\sum_i \mu_i \otimes n_i)$

so $(\Psi \circ \Phi)(\varphi)(m_j) = n_j$

so $(\Psi \circ \Phi)(\varphi) = \varphi$.

$\square$

Example. Are there tensors that are not pure?

Yes. We shall use the prev. thm. Let k be a field and $M = N := k^r$ for an integer $r \geq 2$.

Write $M^* := \mathrm{Hom}_k(M, k)$. The thm. gives an isomorphism

$$\Psi : M^* \otimes_k M \xrightarrow{\sim} \mathrm{End}_k(M) \underset{\cong}{} \mathrm{Mat}_{r\times r}(k).$$

For $m \in M$ & $\mu \in M^*$ the endom= matrix $\psi = \Psi(\mu \otimes m)$ is

$$\psi(n) = \Psi(\mu \otimes m)(n) = \mu(n) \cdot m.$$

Thus the rank of ψ (the rank of its image) is ≤ 1. (if $\mu \neq 0$ & $m \neq 0$ then im. is $k \cdot m$.)

But there are matrices = endos of rank r; e.g. $\psi = \mathrm{id}_M$. They correspond under Ψ to non-pure tensors.

If $(m_1, \dots, m_r)$ is a basis of M, with dual basis $(\mu_1, \dots, \mu_r)$, then

$$\mathrm{id}_M = \Psi\left(\sum_{i=1}^{r} \mu_i \otimes m_i \right).$$

Tensor Products of Rings

Recall that all rings are commutative by default.

Def. Let A be a ring.

(1) An <u>A-ring</u> is a pair (B, f_B), where B is a ring and $f_B: A \to B$ is a ring hom (called the structure hom.)

(2) Suppose (B, f_B) and (C, f_C) are A-rings. An <u>A-ring hom.</u> $h: (B, f_B) \to (C, f_C)$ is a ring hom. $h: B \to C$ s.t.

$$f_C = h \circ f_B$$

in a comm. diag:

$$\begin{tikzcd} & B \arrow[d, "h"] \\ A \arrow[ur, "f_B"] \arrow[r, "f_C"'] & C \end{tikzcd}$$

We usually just say that B is an A-ring, leaving f_B implicit.

Note that such B is an A-module, with action

$$a \cdot b := f_B(a) \cdot b.$$

Thm Let A be a ring, and let B and C be A-rings. Then the A-mod. $B \otimes_A C$ has a unique A-ring structure, s.t.:

(unit) – $1_{B \otimes_A C} = 1_B \otimes 1_C$.

(mult) – $(b_1 \otimes c_1) \cdot (b_2 \otimes c_2) = (b_1 \cdot b_2) \otimes (c_1 \cdot c_2)$

(str. hom) – $f_{B \otimes_A C}(a) = a \cdot 1_{B \otimes_A C}$.

Pf. Step 1. Given $(b_2, c_2) \in B \times C$,

Consider the function

$$\beta_{b_2,c_2} : B \times C \to B \otimes_A C$$

$$\beta_{b_2,c_2}(b_1, c_1) := (b_1 \cdot b_2) \otimes (c_1 \cdot c_2)$$

This is A-bilinear, so $\exists!$ A-lin. hom

$$\psi_{b_2,c_2} : B \otimes_A C \to B \otimes_A C$$

s.t.

$$\psi_{b_2,c_2}(b_1 \otimes c_1) = (b_1 \cdot b_2) \otimes (c_1 \cdot c_2).$$

(S1)

(cont.)

Step 2. For any $u \in B \otimes_A C$ consider the function $\gamma_u : B \times C \to B \otimes_A C$

$$\gamma_u(b_2, c_2) := \Psi_{b_2, c_2}(u)$$

This an A-bilin. function (see exer. below). Hence there's an A-lin. func.

$$\theta_u : B \otimes_A C \to B \otimes_A C$$

s.t.

$$\theta_u(b_2 \otimes c_2) = \gamma_u(b_2, c_2).$$

Step 3

For $u, v \in B \otimes_A C$ define the multiplication

(♡) $\boxed{u \cdot v := \theta_u(v) \in B \otimes_A C.}$

Note that on pure tensors we have

$$(\diamond) \left\{ \begin{aligned} (b_1 \otimes c_1) \cdot (b_2 \otimes c_2) &= \theta_{b_1 \otimes c_1}(b_2 \otimes c_2) \\ &= \gamma_{b_1 \otimes b_2}(b_2, c_2) \\ &= \Psi_{b_2, c_2}(b_1 \otimes b_2) \\ &= (b_1 \cdot b_2) \otimes (c_1 \cdot c_2) \end{aligned} \right.$$

Eq. (◊) implies that "·" makes $B \otimes_A C$ into a ring, with the properties stated. (exer.) □

(optional)

Exercise Prove that γ_u is A-bilinear, and the last sentence of the proof.

to here 22.11

Thm Let B, C, D be A-rings, and let $g: B \to D$ and $h: C \to D$ be A-ring hom's. Then there's a unique A-ring hom.

$$g \otimes h : B \underset{A}{\otimes} C \to D$$

s.t.

$$(g \otimes h)(b \otimes c) = g(b) \cdot h(c) \qquad (*)$$

for all $b \in B$ and $c \in C$.

(pf. on p. 52)

Prop. In the situation of Thm. [p. 50], the functions

$$\begin{cases} B \to B \underset{A}{\otimes} C \\ b \mapsto b \otimes 1_C \end{cases} \quad \text{and} \quad \begin{cases} C \to B \underset{A}{\otimes} C \\ c \mapsto 1_B \otimes c \end{cases}$$

are A-ring hom's.

The proof is easy (opt. exercise)