

(optional)

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Exercise V Prove that τ_u is A -bilinear, and the last sentence of the proof.

is here 22.11

29.11.17

Thm Let B, C, D be A -rings, and let $g: B \rightarrow D$ and $h: C \rightarrow D$ be A -ring hom's. Then there's a unique A -ring hom.

$$g \otimes h: B \otimes_A C \rightarrow D$$

s.t.

$$(g \otimes h)(b \otimes c) = g(b) \cdot h(c) \quad (\dagger)$$

for all $b \in B$ and $c \in C$.

(def. on p. 52)

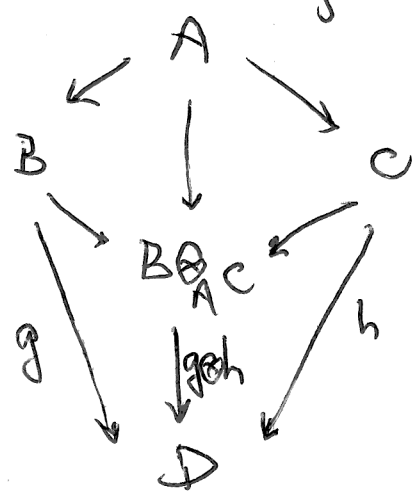
Proof. In the situation of Thm. [p. 50], the functions

$$\begin{cases} B \rightarrow B \otimes_A C \\ b \mapsto b \otimes 1_C \end{cases} \quad \text{and} \quad \begin{cases} C \rightarrow B \otimes_A C \\ c \mapsto 1_B \otimes c \end{cases}$$

are A -ring hom's.

The proof is easy (opt. exercise)

in a comm. diag.



pt of thm . Let $p: B \times C \rightarrow D$ be

$$p(b, c) := g(b) \cdot h(c).$$

This is A -bilinear, so $\exists!$ A -lin. hom

$$g \otimes h: B \otimes_A C \rightarrow D, \text{ s.t. (1).}$$

A calculation shows that $g \otimes h$ is an A -ring hom. ◻

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there is a can. ring isom

Exercise Prove that $\forall A[t] \cong A \otimes_{\mathbb{Z}} \mathbb{Z}[t]$.

Here A is a ring and t is a variable.

Exercise Prove that

$$A[s, t] \cong A[s] \otimes_A A[t]$$

A is a ring and s, t are variables.



SS Tensors and Galois Theory (mostly self study)

Two things we will prove.

Def. A ring A is reduced if the only nilpotent element in it is 0.

Exa. $K[E] := K[t] / (t^2)$ is not reduced.

Thm. Let $K \subseteq L$ be a finite field extension.

TFAF:

(i) L is separable ext. of K .

(ii) For ~~every~~ ^{any} finite field ext. $K \subseteq K'$ the ring $K' \otimes_K L$ is reduced.

For a proof see Matsumura "Comm. Alg." Ch. 27.

→ Exercise ** (=hard)

Consider the field $K := K(t)$, the field of rational functions in the variable t , over a field K of characteristic $p > 0$.

Let $L := K[s] / (s^p - t)$, with $u \in L$ the class of s .

Since the polynomial $s^p - t \in K[s]$

is irreducible, L is a field. In L we have

$$u = \sqrt[p]{t}.$$

The extension $K \subseteq L$ is not separable. Find a nonzero nilpotent element

$$\varepsilon \in L \otimes_K L.$$

[Hint: in char p ,
 $(a+b)^p = a^p + b^p$.]



Suppose $K \subseteq L$ is a finite Galois ^{field} extension, with group G .

Consider the ring

$$L \times \dots \times L = F(G, L)$$

of functions $\varphi: G \rightarrow L$. It has an action by G (permuting the source). There's a ring hom

$$\alpha: L \rightarrow F(G, L) \quad \text{the "const. functions"}$$

$$\alpha(a)(g) = a \quad \forall a \in L, g \in G$$

and also

$$\beta: L \rightarrow F(G, L), \quad \beta(a)(g) = g(a).$$

α & β are K -ring homs so get ring hom

$$\alpha \otimes \beta: L \otimes_K L \rightarrow F(G, L)$$

This $\times_{\mathbb{R}} \mathbb{P}$ is an isomorphism.
(proof: ? discussion in Milne's "Étale Cohomology")

→ Exercise ^{**} Consider the Galois ext. $\mathbb{R} \subseteq \mathbb{C}$.

The group is $G = \{id, \sigma\}$ $\sigma =$ conjugation.

$\mathbb{P} \simeq \sqrt{F(G, \mathbb{C})} = \mathbb{C} \times \mathbb{C} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$

by the theorem

there are \simeq idempotents $e_1 = (1, 0)$ $e_2 = (0, 1)$

$e_1 + e_2 = 1, e_1 \cdot e_2 = 0, e_2 = (id \otimes \sigma)(e_1)$

$e_1^2 = e_1, e_2^2 = e_2$

Find the tensors e_1 and e_2 as elements of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$.



Flatness

A is a comm. ring, Given a module M
and an A -mod. hom $\psi: N_1 \rightarrow N_2$,
there's an induced A -mod. hom

$$\text{id}_M \otimes \psi: M \otimes_A N_1 \rightarrow M \otimes_A N_2.$$

Def An A -module M is called flat if
for every injective A -mod. hom.

$$\psi: N_1 \rightarrow N_2,$$

the hom.

is also injective

$$\text{id}_M \otimes \psi: M \otimes_A N_1 \rightarrow M \otimes_A N_2$$

In terms of exact seq.: M is flat if
for every ex. seq.

$$0 \rightarrow N_1 \xrightarrow{\psi} N_2$$

the seq.

$$0 \rightarrow M \otimes_A N_1 \xrightarrow{\text{id} \otimes \psi} M \otimes_A N_2$$

is also exact.

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Lem. Suppose M is a free A -mod with basis $\underline{m} = \{m_i\}_{i \in I}$. For any A -mod N there's a unique A -mod. isomorphism

$$\Psi: M \otimes_A N \xrightarrow{\cong} F_{\text{fin}}(I, N)$$

st.

$$\Psi(m_i \otimes n)(j) = \begin{cases} n & \text{if } j=i \\ 0 & \text{ow.} \end{cases}$$

using a bilinear function

pf. Very similar to the pf. of Thm [p. 43],
The inverse hom. Φ is

$$\Phi(v) := \sum_{i \in I} m_i \otimes v(i), \quad v \in F_{\text{fin}}(I, N).$$

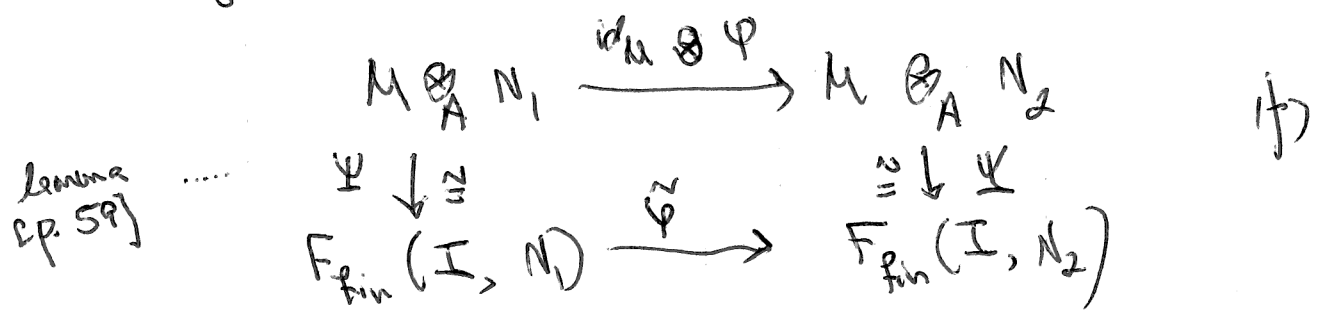
The full details
are an opt. exercise.

□

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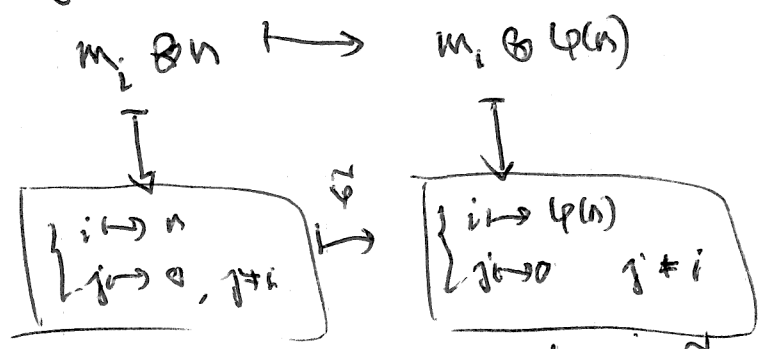
Thm: If M is a free A -module, then it is flat.

prf Let $\underline{m} = \{m_i\}_{i \in I}$ be a basis of M . Given an injective A -mod. hom. $\varphi: N_1 \rightarrow N_2$ we look at this diagram



where $\tilde{\varphi}(v) := \varphi \circ v$

The diag. (1) is commutative, since



It suffices to prove that $\tilde{\varphi}$ is injective. This is easy: if $v \neq 0$ then $v(i) \neq 0$ for some $i \in I$. Then, since φ is inj. we have

$$\begin{aligned}
 \varphi(v(i)) &\neq 0. & \text{So } \tilde{\varphi}(v) &\neq 0. \\
 \parallel & & & \\
 \tilde{\varphi}(v)(i) & & &
 \end{aligned}$$

□

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Thm Let M be a fin. gen. abelian group. TFAE:

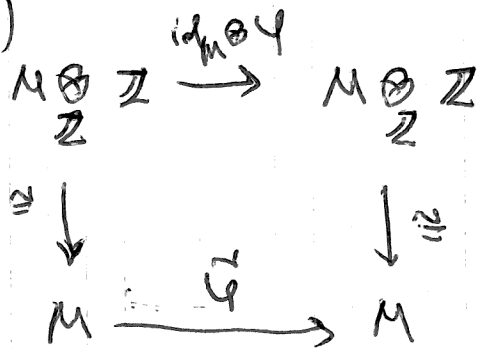
- (i) M is free
- (ii) M is a flat \mathbb{Z} -module.

pf. We already know that (i) \Rightarrow (ii).
 For the other direction, if M is not free, then there is some $m \in M$, $m \neq 0$, but $a \cdot m = 0$ for some nonzero integer a .

Consider $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}$, $\varphi(b) = a \cdot b$.

This is injective. Like in the pt. of last time, \wedge we have a comm. diagram

(using Lem. p. 57)



$\varphi(n) = a \cdot n$.

But $m \neq 0$, $m \in \text{Ker}(\varphi)$.

So M not flat.

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Categories and Functors

Def A category \underline{C} consists of:

- A set $Ob(\underline{C})$, whose elements are the objects of \underline{C} .
- For every pair of object $C, D \in Ob(\underline{C})$ there is a set $Hom_{\underline{C}}(C, D)$, whose elements are called morphisms. They are denoted by $f: C \rightarrow D$.
- For every triple $C, D, E \in Ob(\underline{C})$ there is a function $Hom_{\underline{C}}(D, E) \times Hom_{\underline{C}}(C, D) \rightarrow Hom_{\underline{C}}(C, E)$ called composition. $(g, f) \mapsto g \circ f$.
- For every object C there is a morphism $id_C \in Hom_{\underline{C}}(C, C)$, called the identity morphism.

The axioms are:

- Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$ for composable morphisms f, g, h .
- Units: $id_D \circ f = f = f \circ id_C$ for all $f: C \rightarrow D$.

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Remark on set theory: We fix a universe \underline{U} , a set large enough to contain as elements all mathematical objects (groups, rings etc) that we are interested in, their cartesian products, etc.

A set $S \in \underline{U}$ is called small.
We always assume that

$$\text{Ob}(\underline{C}) \subseteq \underline{U} \quad \text{and} \quad \text{Hom}_{\underline{C}}(C, D) \in \underline{U}$$

[For deeper discussion see Mac Lane, "Categories for the Working Mathematician", I.6 - to here 29.11]

Example 1. The category Set. Its objects are the small sets. $\text{Ob}(\underline{\text{Set}}) = \underline{U}$.
The morphisms $f: S \rightarrow T$ are the functions.

Example 2. Fix a ^(small) ring A . The cat. Mod A has as objects the ^(small) A -modules.
The morphisms $\varphi: M \rightarrow N$ are the A -mod. homs.