73.1

13.12.2017

More Properties of $\otimes$

Prop (symmetry) Let M, N be A-mods. There's a unique A-mod isom

$$M \otimes_A N \xrightarrow{\sim} N \otimes_A M$$

s.t. $m \otimes n \mapsto n \otimes m$

for all $m \in M$ and $n \in N$.

Pf The function $\beta : M \times N \to N \otimes_A M$, $\beta(m,n) := n \otimes m$, is bilinear. etc.... $\square$

Prop There's a unique A-module isom $A \otimes_A M \xrightarrow{\sim} M$,

$$a \otimes m \mapsto a \cdot m.$$

Pf This easy. (We used something like this on page 47.1.) $\square$

<u>Thm</u> (Associativity) Let $f: A \to B$ be a ring hom, let $L, M \in \underline{\mathrm{Mod}}\, B$ and let $N \in \underline{\mathrm{Mod}}\, A$. There's a unique B-mod hom

$$(L \otimes_B M) \otimes_A N \xrightarrow{\sim} L \otimes_B (M \otimes_A N)$$

s.t. $\quad (l \otimes m) \otimes n \mapsto l \otimes (m \otimes n)$

on pure tensors.

<u>pf.</u> Uniqueness is clear. For existence, this is similar to pf. of Thm [p. 50]. I'll just give a partial proof.

<u>Step 1</u>. Fix $n \in N$. Define

$$\beta_n : L \times M \to L \otimes_B (M \otimes_A N) =: P$$
$$(l, m) \mapsto l \otimes (m \otimes n)$$

β_n is B-bilinear. Induces B-lin. hom (*)

$$\varphi_n : L \otimes_B M \to \underline{P}$$

(*) will prove also A-lin. in n, so A-bilinear.

<u>Step 2</u> Define the function

$$\gamma: (L \otimes_B M) \times N \to P, \quad \gamma(u, n) := \varphi_n(u).$$

Clearly it's A-linear in u. (*) write $u = \sum l_i \otimes m_i$

$$\gamma(u, n_1 + n_2) = \sum_i \gamma(l_i \otimes m_i, n_1 + n_2) = \sum_i \varphi_{n_1+n_2}(l_i \otimes m_i)$$
$$= \sum_i l_i \otimes (m_i \otimes (n_1 + n_2)) = \left(\sum_i l_i \otimes (m_i \otimes n_1)\right) + \left(\sum_i l_i \otimes (m_i \otimes n_2)\right)$$
$$= \gamma(u, n_1) + \gamma(u, n_2)$$

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Likewise for $a \cdot n$. We see that γ is A-bilinear. Get A-lin. hom:

$\varphi: (L \otimes_B M) \otimes_A N \to P$. It is B-linear hom. calculation!

Steps 3-4: Use same trick to construct $\psi: P \to Q$,

$\psi(l \otimes (m \otimes n)) = (l \otimes m) \otimes n$

Then $\psi \circ \varphi = \mathrm{id}$, $\varphi \circ \psi = \mathrm{id}$. so φ is an isom.

Δ

Solution of exercise [p. 69.1]

Thm. Let $A \to B$ be a ring hom, and let M be a flat A-mod. Then $B \otimes_A M$ is a flat B-module.

Pf. Consider $\varphi: N' \to N$, an injective B-mod. hom. We have to show that

$$\varphi \otimes \mathrm{id}: N' \otimes_B (B \otimes_A M) \to N \otimes_B (B \otimes_A M)$$

is injective.

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We have this diagram

$$\begin{tikzcd}
N' \otimes_B (B \otimes_A M) \arrow[r, "\varphi \otimes id"] \arrow[d, "\cong", "\substack{\text{assoc} \\ \text{Thm [p 73.2]}}"'] & N \otimes_B (B \otimes_A M) \arrow[d, "\cong"] \\
(N' \otimes_B B) \otimes_A M \arrow[r, "(\varphi \otimes id) \otimes id"] \arrow[d, "\cong", "\substack{\text{prep} \\ {[d, p. 73.1]}}"'] & (N \otimes_B B) \otimes_A M \arrow[d, "\cong"] \\
N' \otimes_A M \arrow[r, "\varphi \otimes id_M"] & N \otimes_A M
\end{tikzcd}$$

the diag is commutative - just track pure tensors:

$$\begin{tikzcd}
n' \otimes (b \otimes m) \arrow[r, mapsto] \arrow[d, mapsto] & \varphi(n') \otimes (b \otimes m) \arrow[d, mapsto] \\
(n' \otimes b) \otimes m \arrow[r, mapsto] \arrow[d, mapsto] & (\varphi(n') \otimes b) \otimes m \arrow[d, mapsto] \\
(b \cdot n') \otimes m \arrow[r, mapsto] & \varphi(b \cdot n') \otimes m = (b \cdot \varphi(n')) \otimes m
\end{tikzcd}$$

Since M is flat over A, $\varphi \otimes id_M$ is injective. So $\varphi \otimes id_{B \otimes_A M}$ (top row) is injective.

(bottom row of diagram)

$\square$

~~13.12.17~~

Localization

[this is a commutative theory]

Def. Let A be a ring. A multiplicatively closed subset S in A is a subset S s.t. $1_A \in S$, and $(s,t \in S \Rightarrow st \in S)$.

Example. A is an integral domain ($A \neq \{0\}$, and every nonzero elt. is not a zero divisor, i.e. $[a,b \neq 0 \Rightarrow a \cdot b \neq 0]$). Then $S := A - \{0\}$ is a m.c.s.

Exa. X a top space, $A = \{\text{cont. funct } f : X \to \mathbb{R}\}$. $x_0 \in X$. $S := \{f \in A \mid f(x_0) \neq 0\}$. This is a m.c.s.

Exa. $s \in A$ an elt. $S := \{s^n \mid n \in \mathbb{N}\}$ is m.c.s.

Given a m.c.s. $S \subseteq A$, consider the set $A \times S$. Define a relation $\sim$ on $A \times S$ like this:

$(a,s) \sim (b,t)$ if $\exists\, u \in S$ s.t. $u \cdot (a \cdot t - b \cdot s) = 0$.

Lemma. $\sim$ is an equivalence relation.

$\to$ Exercise: prove the lemma.

(*) We let $\boxed{A_S := \frac{A \times S}{\sim}}$, the set of equiv. classes, and $\boxed{\frac{a}{s} := \overline{(a,s)}}$ the equiv. class of (a,s)

<u>Thm.</u> Let $S \subseteq A$ be a m.c.s.

The set A_S is an A-ring, with these operations:

- addition: $\frac{a}{s} + \frac{b}{t} := \frac{a \cdot t + b \cdot s}{s \cdot t}$
- multiplication: $\frac{a}{s} \cdot \frac{b}{t} := \frac{a \cdot b}{s \cdot t}$
- zero $0 := \frac{0_A}{1_A}$
- unit: $1 := \frac{1_A}{1_A}$
- struct. hom: $a \mapsto \frac{a}{1_A}$

Pf. We'll only prove that $+$ is well-defined. The proof that $\cdot$ is w-d is similar. Rest is easy.

Say $\frac{a'}{s'} = \frac{a}{s}$ & $\frac{b'}{t'} = \frac{b}{t}$ (i.e. $(a', s') \sim (a, s)$ etc.)

So $\exists\, u, v \in S$ s.t. $u \cdot (a' s - a \cdot s') = 0$ &
$v \cdot (b' \cdot t - b \cdot t') = 0$

then

$$\underset{\substack{\uparrow \\ S}}{\underline{u \cdot v}} \cdot \left(s' t' \cdot (a \cdot t + b \cdot s) - s \cdot t \cdot (a' \cdot t' - b' s') \right)$$

$$= (v \cdot t \cdot t') \cdot \underbrace{u \cdot (s' a - s \cdot a')}_{0} + \cdots \qquad = 0.$$

$\square$

$\boxed{\text{We call } A_S \text{ the } \underline{\text{localization}} \text{ of } A \text{ w.r.t. } S.}$

Note under str. hom $f_{A_S} : A \to A_S$, $f(S) \in (A_S)^\times$.

$\frac{s}{1} \cdot \frac{1}{s} = 1$

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<u>Thm</u> (Universal property)
Let $S \subseteq A$ be a m.c.s., and let B be an A-ring s.t. $f_B(S) \subseteq B^\times$.
Then there is a unique A-ring hom.
$$g: A_S \to B.$$

Pf. Define (†) $\boxed{g\left(\frac{a}{s}\right) := f_B(a) \cdot f_B(s)^{-1} \in B.}$
It is easy to see that this indep. of reps:
$$\frac{a}{s} = \frac{a'}{s'} \Rightarrow \exists\, u \in S \text{ s.t. } u \cdot (a \cdot s' - a' \cdot s) = 0$$
$$\Rightarrow \underbrace{f_B(u)}_{\in B^\times} \underbrace{(f_B(a) \cdot f_B(s') - f_B(a') \cdot f_B(s))}_{=0} = 0$$

Uniqueness is clear, since $\frac{a}{s} = f(a) \cdot f(s)^{-1} \in A_S$. $\square$

<u>Exa</u>. A integ. domain $S := A - \{0\}$
$\Rightarrow A_S =$ field of fractions of A.

→ <u>Exercise</u>. Prove that
$\ker(f_{A_S}: A \to A_S)$ is the ideal
$\{a \in A \mid s \cdot a = 0 \text{ for some } s \in S\}$.

Given a m.c.s. $S \subseteq A$ and an A-module M, let $\sim$ be the relation on $M \times S$:

$(m,s) \sim (m',s')$ if $\exists\ u \in S$ s.t. $u \cdot (s \cdot m' - s' \cdot m) = 0$.

Like Lem. [p 74], this is an equiv. rel.
We write

(**) $\boxed{M_S := \frac{M \times S}{\sim}, \quad \frac{m}{s} := \text{class of } (m,s)}$

<u>Thm.</u> Let $S \subseteq A$ be a m.c.s., and let $M \in \underline{\mathrm{Mod}}\ A$.

(1) the set M_S is an A_S-module, with operations

<u>addition</u> $\frac{m}{s} + \frac{n}{t} := \frac{t \cdot m + s \cdot n}{s \cdot t}$

<u>zero</u> $0 := \frac{0_M}{1}$

<u>multip.</u> $\frac{a}{s} \cdot \frac{m}{t} := \frac{a \cdot m}{s \cdot t}$

(2) There is a unique A_S-mod. isom

$$A_S \otimes_A M \xrightarrow{\simeq} M_S$$

$$\frac{a}{s} \otimes m \mapsto \frac{a \cdot m}{s}$$

<u>Pf.</u> (1) Like pf. of Thm. [p. 75].

(cont. ↓)

② There's a function $\beta: A_S \times M \to M_S$,

$$\beta\left(\frac{a}{s}, m\right) := \frac{a \cdot m}{s}.$$

It is well defined & A-bilinear, (an easy calc.) So get A-lin. hom. $\varphi: A_S \underset{A}{\otimes} M \to M_S$,

$$\varphi\left(\frac{a}{s} \otimes m\right) = \frac{a \cdot m}{s}.$$

Let $\gamma: M \times S \to A_S \underset{A}{\otimes} M$, $\gamma(m, s) := \frac{1}{s} \otimes m$.

γ respects relation $\sim_1$, so get function (again, an easy calc.)

$$\psi: M_S \to A_S \underset{A}{\otimes} M, \quad \psi\left(\frac{m}{s}\right) = \frac{1}{s} \otimes m. \quad (\dagger)$$

Now

$$(\psi \circ \varphi)\left(\frac{a}{s} \otimes m\right) = \psi\left(\frac{a \cdot m}{s}\right) = \frac{1}{s} \otimes (am) = \frac{a}{s} \otimes m$$

$$(\varphi \circ \psi)\left(\frac{m}{s}\right) = \varphi\left(\frac{1}{s} \otimes m\right) = \frac{m}{s}.$$

So φ is $\cong$. $\triangle$

Prop. ① There's an A-mod. hom. $M \to M_S$, $m \mapsto \frac{m}{1}$

② $\operatorname{Ker}(M \to M_S) = \{ m \in M \mid s \cdot m = 0, \text{ some } s \in S \}$.

Pf. ① Clear.

② Same as Exer. [p. 76].

$\triangle$

(†) The function ψ is actually A_S-linear. (easy calc.)

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<u>Thm.</u> Let $S \subseteq A$ (be a m.c.s.) with localization A_S.
Then the ring A_S is a flat A-module.

<u>pf.</u> We must show that if

$$0 \to N' \xrightarrow{\varphi} N$$ is an ex. seq. of A-mods,

then

$$0 \to A_S \otimes_A N' \xrightarrow{\mathrm{id} \otimes \varphi} A_S \otimes_A N$$

is exact. By prev. thm:

$$\begin{array}{ccc} A_S \otimes_A N' & & A_S \otimes_A N \\ \wr\wr & & \wr\wr \\ N'_S & \xrightarrow{\psi} & N_S \end{array}$$

The formula for ψ is

$$\psi\left(\frac{n}{s}\right) = \frac{\varphi(n)}{s} = \frac{1}{s} \cdot \frac{\varphi(n)}{1} \quad \forall\ n \in N',\ s \in S.$$

Let's prove that ψ is injective.

<u>If</u> $\psi\left(\frac{n}{s}\right) = 0$ then $\frac{1}{s} \cdot \frac{\varphi(n)}{1} = 0 \Rightarrow \frac{\varphi(n)}{1} = 0$

$\Rightarrow \varphi(n) \in \mathrm{Ker}\,(N \to N_S)$

$\Rightarrow$ (by prop [p. 77]) $\exists\ t \in S$

s.t. $t \cdot \varphi(n) = 0$

But $t \cdot \varphi(n) = \varphi(t \cdot n)$, φ inj. $\Rightarrow t \cdot n = 0$

$\Rightarrow \frac{n}{1} = 0$ in N'_S. $\Rightarrow \frac{n}{s} = \frac{1}{s} \cdot \frac{n}{1} = 0$ in N'_S. $\square$

(to here 13.12)